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Strategy-Proofness and Efficiency of Probabilistic Mechanisms for Excludable Public Good

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We study strategy-proof probabilistic mechanisms in a binary excludable public good model. We construct a new class of probabilistic mechanisms satisfying strategy-proofness, called α-mechanisms.

We first show that the α-mechanisms are second-best efficient. Next, we identify the optimal α-mechanism with respect to the supremal welfare loss, and show that it improves the inefficiency of the equal cost sharing with maximal participation mechanism [Moulin (1994)] and the anonymous augmented serial mechanisms [Ohseto (2005)].

Keywords: Strategy-proofness; Probabilistic mechanism; Excludable public good; Second-best efficiency; Supremal welfare loss.

JEL codes: D61; D71; H41.

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1 Introduction

We study a mechanism design problem in a binary excludable public good model. Each agent has a quasi-linear preference. A (direct) mechanism determines the set of agents who consume the public good and the cost share depending on agents' preferences.

We focus on mechanisms satisfying *strategy-proofness*, which requires that it be a dominant strategy for any agent to report his true preference. We evaluate strategy-proof mechanisms from the point of view of efficiency. However, it is well known that there exists no mechanism satisfying strategy-proofness and Pareto-efficiency [Holmström (1979)]. Hence, our aim is to design the best mechanism possible.

Moulin (1994) has constructed a basic mechanism called the *equal cost sharing with maximal participation mechanism* (henceforth, the Moulin mechanism). It is the unique mechanism satisfying strategy-proofness, individual rationality, budget-balance, anonymity, and other desirable properties.[1] Furthermore, it minimizes the *supremal welfare loss* among the set of all mechanisms satisfying strategy-proofness, individual rationality, and the auxiliary axiom [Massó et al. (2015)]. The supremal welfare loss is the supremum of the welfare loss that occurs as a result of applying the mechanism, and is used as a measure of inefficiency.[2]

Ohseto (2005) has generalized the Moulin mechanism and constructed attractive mechanisms called the *anonymous augmented serial mechanisms* (henceforth, the Ohseto mechanisms). They are characterized by strategy-proofness, budget-balance, anonymity, and other desirable properties.[3] Furthermore, the optimal Ohseto mechanism achieves the smaller supremal welfare loss than that of the Moulin mechanism.[4] These facts mean that in order to improve the inefficiency of the mechanism, we will need to forgo a desirable property[5] or expand the scope of the mechanisms from being deterministic to being probabilistic.[6]

[1]See Moulin (1999), Deb and Razzolini (1999a, b), and Ohseto (2000).

[2]For example, see Juarez (2008), Moulin and Shenker (2001), and Ohseto (2010).

[3]See Ohseto (2005) and Hashimoto and Saitoh (2016).

[4]The Ohseto mechanism, other than the Moulin mechanism, do not satisfy individual rationality. Hence, the optimal Ohseto mechanism improve the inefficiency of the Moulin mechanism, but forgo individual rationality. Because a provider of a public good, such as a government, might be able to enforce participation in a mechanism, individual rationality may not be indispensable.

[5]It is known that even if we give up anonymity, we cannot improve the inefficiency of the optimal Ohseto mechanism. See Ohseto (2005).

[6]In the two-agent case, Dobzinski et al. (2017) has studied a probabilistic mechanism satisfying strategy-proofness, individual rationality, budget-balance, and anonymity, and

We take the latter approach and introduce a new class of strategy-proof mechanisms, called *α-mechanisms*, each of which is a linear combination of the Ohseto mechanisms. We first show that the α-mechanisms are *second-best efficient*. Second best efficiency requires[7] that the mechanism be on the Pareto-frontier of the set of strategy-proof mechanisms. Next, we identify the optimal α-mechanism with respect to the supremal welfare loss, and show that it improves the inefficiency of the Moulin mechanism and the Ohseto mechanisms.

The remainder of the paper is organized as follows. In Section 2, we set up the model. In Section 3, we define the basic properties. In Section 4, we introduce the mechanisms. In Section 5, we state our results. In Section 6, we provide the proofs.

2 Model

Let $N = \{1, 2, \ldots, n\}$ be the set of agents. We consider the provision of a binary excludable public good $y \in \{0, 1\}$. The cost function $c(\cdot)$ of the excludable public good is normalized as follows: $c(0) = 0$ and $c(1) = 1$.

Each agent $i \in N$ has a preference for bundles consisting of a consumption level of the excludable public good $s_i \in \{0, 1\}$ and a cost share $t_i \in \mathbb{R}$. We assume that this preference is represented by a quasi-linear utility function, i.e., if agent i's valuation of the excludable public good is $v_i \in \mathbb{R}_+$, then his utility for $(s_i, t_i) \in \{0, 1\} \times \mathbb{R}$ is

$$u_i((s_i, t_i); v_i) = s_i v_i - t_i.$$

A list $v \equiv (v_i)_{i\in N} \in \mathbb{R}^n_+$ is a valuation profile. Given $v \in \mathbb{R}^n_+$ and $N' \subset N$, $v_{N'} \in \mathbb{R}^{\#N'}_+$ and $v_{-N'} \in \mathbb{R}^{\#N\setminus N'}_+$ denote $(v_j)_{j\in N'}$ and $(v_j)_{j\notin N'}$, respectively.

The set of feasible allocations is

$$Z \equiv \Big\{(s_i, t_i)_{i\in N} \in \big(\{0, 1\} \times \mathbb{R}\big)^n : \sum_{i\in N} t_i \geq \max_{i\in N} s_i\Big\}.$$

A deterministic mechanism is a function $f : \mathbb{R}^n_+ \to Z$. Given a deterministic mechanism f and a valuation profile $v \in \mathbb{R}^n_+$, we denote agent i's assignment under $f(v)$ as $f_i(v) \equiv (s_i(v), t_i(v)) \in \{0, 1\} \times \mathbb{R}$.

Let ΔZ be the set of all probability distributions on Z. Given a probability distribution, we denote by $\sigma_i \in [0, 1]$ and $\tau_i \in \mathbb{R}$ the probability that agent

has showed that it improves the inefficiency of the Moulin mechanism.

[7]In other words, it requires that the mechanism be undominated by the other strategy-proof mechanisms.

i consumes the excludable public good and the expected value that agent i pays under the probabilistic distribution, respectively. Each agent i evaluates (σ_i, τ_i) using his expected utility, i.e., if agent i's valuation of the excludable public good is $v_i \in \mathbb{R}_+$, then his expected utility for $(\sigma_i, \tau_i) \in [0,1] \times \mathbb{R}$ is

$$u_i((\sigma_i, \tau_i); v_i) = \sigma_i v_i - \tau_i.$$

A probabilistic mechanism is a function $\varphi : \mathbb{R}^n_+ \to \Delta Z$. Given a probabilistic mechanism φ and a valuation profile $v \in \mathbb{R}^n_+$, we denote by $\sigma_i(v) \in [0,1]$ and $\tau_i(v) \in \mathbb{R}$ the probability that agent i consumes the excludable public good and the expected value that agent i pays under the probability distribution $\varphi(v)$, respectively.

3 Axioms

We define the basic properties. Since the deterministic mechanisms are special cases of the probabilistic mechanisms, we define the properties for the probabilistic mechanisms only.

Strategy-proofness states that it is a dominant strategy for any agent to report his true valuation.

Definition 1. A probabilistic mechanism φ is **strategy-proof** if for any $i \in N$, any $v \in \mathbb{R}^n_+$, and any $v'_i \in \mathbb{R}_+$, it holds that

$$\sigma_i(v)v_i - \tau_i(v) \geq \sigma_i(v'_i, v_{-i})v_i - \tau_i(v'_i, v_{-i}).$$

Second-best efficiency states that the mechanism is on the Pareto-frontier of the set of strategy-proof mechanisms.

Definition 2. A probabilistic mechanism φ^* is **second-best efficient** if there exists no strategy-proof probabilistic mechanism φ such that for any $v \in \mathbb{R}^n_+$ and any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) \geq \sigma^*_i(v)v_i - \tau^*_i(v),$$

and for some $\hat{v} \in \mathbb{R}^n_+$ and some $j \in N$,

$$\sigma_j(\hat{v})\hat{v}_j - \tau_j(\hat{v}) > \sigma^*_j(\hat{v})\hat{v}_j - \tau^*_j(\hat{v}).$$

The *welfare loss* of a mechanism is the difference between the welfare of the first-best mechanism and that of the mechanism we consider.

Definition 3. Given a probabilistic mechanism φ and a valuation $v \in \mathbb{R}^n_+$, the **welfare loss** of φ at v is defined as follows:

$$WL(v;\varphi) \equiv \max\Big\{\sum_{i \in N} v_i - 1, 0\Big\} - \left(\sum_{i \in N}\Big(\sigma_i(v)v_i - \tau_i(v)\Big)\right).$$

The *supremal welfare loss* is the supremum of the welfare loss over all $v \in \mathbb{R}^n_+$.

Definition 4. Given a probabilistic mechanism φ, the **supremal welfare loss** of φ is

$$\sup_{v \in \mathbb{R}^n_+} WL(v;\varphi).$$

4 Mechanisms

4.1 Moulin mechanism

We first define the Moulin mechanism. To do so, we identify the largest set of agents whose valuations are greater than or equal to the equal cost share in that set.

Definition 5. For any $k \in \{1, \dots, n\}$, let $M_k(v) = \{i \in N | v_i \geq \frac{1}{k}\}$. The **largest unanimous coalition** at $v \in \mathbb{R}^n_+$, denoted by $M(v)$, is defined as follows:

1. if there exists $k^* \in \{1, \dots, n\}$ such that $\#M_{k^*}(v) = k^*$, and for any integer k ($k^* < k \leq n$), $\#M_k(v) < k$, then $M(v) = M_{k^*}(v)$, and
2. $M(v) = \emptyset$ otherwise.

Definition 6. A deterministic mechanism f^M is the Moulin mechanism if for any $v \in \mathbb{R}^n_+$ and any $i \in N$,

$$f^M_i(v) = \begin{cases} (1, \frac{1}{\#M(v)}) & \text{if } i \in M(v), \\ (0,0) & \text{otherwise.} \end{cases}$$

Example 1. Let $n = 3$. Let $v_1 = v_2 = \frac{2}{3}$ and $v_3 = \frac{1}{4}$. Then, $M(v) = \{1, 2\}$. Thus, it holds that

$$f^M(v) = \Big((1, \frac{1}{2}), (1, \frac{1}{2}), (0, 0)\Big).$$

Remark 1. For any $i \in N$ and any sufficiently small $\varepsilon > 0$, let $v_i^\varepsilon = \frac{1}{i} - \varepsilon$. Then, for any $i \in N$, $f_i^M(v^\varepsilon) = (0, 0)$. Hence, the welfare loss at v^ε is

$$\sum_{i=1}^{n} \frac{1}{i} - n\varepsilon - 1.$$

As $\varepsilon \to 0$, we have[8] the supremal welfare loss of the Moulin mechanism, which is

$$\sum_{i=1}^{n} \frac{1}{i} - 1.$$

4.2 Ohseto mechanisms

Next, we define the Ohseto mechanisms, which are a generalization of the Moulin mechanism. When the number of agents whose valuations are greater than $\frac{1}{n}$ is smaller than the given number w, the allocation is determined by the Moulin mechanism. On the other hand, when the number of agents whose valuations are greater than $\frac{1}{n}$ is larger than or equal to w, all agents share the cost $\frac{1}{n}$.

Definition 7. Let $w = 1, 2, \ldots, n$. A deterministic mechanism f^w is the w-Ohseto mechanism if for any $v \in \mathbb{R}^n_+$ and any $i \in N$,

1. when $\#\{j \in N | v_j > \frac{1}{n}\} < w$,

$$f_i^w(v) = \begin{cases} (1, \frac{1}{\#M(v)}) & \text{if } i \in M(v), \\ (0, 0) & \text{otherwise,} \end{cases}$$

2. when $\#\{j \in N | v_j > \frac{1}{n}\} \geq w$,

$$f_i^w(v) = \begin{cases} (1, \frac{1}{n}) & \text{if } v_i > 0, \\ (0, \frac{1}{n}) & \text{if } v_i = 0. \end{cases}$$

We denote $f_i^w(v) = (s_i^w(v), t_i^w(v))$.

Remark 2. When $w = n$, the w-Ohseto mechanism coincides with the Moulin mechanism.

[8]See Moulin and Shenker (2001) for a detailed analysis.

Example 2. Let $n = 3$. Let $v_1 = v_2 = \frac{2}{3}$ and $v_3 = \frac{1}{4}$. Since $\#\{j \in N | v_j > \frac{1}{3}\} = 2$, it holds that

$$\begin{aligned} f^1(v) &= \Big((1, \frac{1}{3}), (1, \frac{1}{3}), (1, \frac{1}{3})\Big) \\ f^2(v) &= \Big((1, \frac{1}{3}), (1, \frac{1}{3}), (1, \frac{1}{3})\Big) \\ f^3(v) &= \Big((1, \frac{1}{2}), (1, \frac{1}{2}), (0, 0)\Big). \end{aligned}$$

Remark 3. Among the Ohseto mechanisms, the 1-Ohseto mechanism achieves the smallest supremal welfare loss.[9]

Remark 4. Let $w = 1$. For any sufficiently small $\varepsilon > 0$, let $v_1^\varepsilon = \frac{1}{n} + \varepsilon$. For any $i \neq 1$, let $v_i = 0$. Then, we have $f_1^w(v_1^\varepsilon, v_{-1}) = (1, \frac{1}{n})$, and for any $i \neq 1$, $f_i^w(v_1^\varepsilon, v_{-1}) = (0, \frac{1}{n})$. Hence, the welfare loss at $(v_1^\varepsilon, v_{-1})$ is

$$-(\frac{1}{n} + \varepsilon - 1).$$

As $\varepsilon \to 0$, we have[10] the supremal welfare loss of the 1-Ohseto mechanism, which is given by

$$1 - \frac{1}{n}.$$

4.3 New mechanisms

Finally, we define our new mechanism, which is a linear combination of Ohseto mechanisms. Define $\Delta \equiv \{(\alpha_1, \ldots, \alpha_n) \in [0, 1]^n : \sum_{i=1}^n \alpha_i = 1\}$.

Definition 8. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Delta$. A probabilistic mechanism φ^α is the α-**mechanism** if for any $v \in \mathbb{R}_+^n$,

$$\varphi^\alpha(v) = \Big[\alpha_1 \circ f^1(v), \alpha_2 \circ f^2(v), \ldots, \alpha_n \circ f^n(v)\Big],$$

where for any $k = 1, \ldots, n$, $\alpha_k \circ f^k(v)$ means that the allocation $f^k(v)$ occurs with probability α_k.

Remark 5. When $\alpha_k = 1$, the α-mechanism coincides with the k-Ohseto mechanism. Thus, when $\alpha_n = 1$, the α-mechanism coincides with the Moulin mechanism.

Example 3. Let $n = 3$. Let $v_1 = v_2 = \frac{2}{3}$ and $v_3 = \frac{1}{4}$. Then, $\varphi^\alpha(v)$ generates the allocation $\big((1, \frac{1}{3}), (1, \frac{1}{3}), (1, \frac{1}{3})\big)$ with probability $\alpha_1 + \alpha_2$, and the allocation $\big((1, \frac{1}{2}), (1, \frac{1}{2}), (0, 0)\big)$ with probability α_3.

[9]See Proposition 1 in Ohseto (2005).

[10]See Ohseto (2005) for a detailed analysis.

5 Results

We state our results. All proofs are given in the final section. The first result states that the α-mechanisms satisfy strategy-proofness.

Theorem 1. For any $\alpha \in \Delta$, the α-mechanism is strategy-proof.

The second result states that each α-mechanism is on the Pareto-frontier of the set of strategy-proof mechanisms.

Theorem 2. For any $\alpha \in \Delta$, the α-mechanism is second-best efficient.

Next, we state the third result, which is related to the supremal welfare loss of the α-mechanisms. To do so, we need the following notation. Define $D \subset \mathbb{R}^n_+$ as follows:

$$D \equiv \{v \in \mathbb{R}^n_+ : v_1 \geq v_2 \geq \cdots \geq v_n\}.$$

For any $k = 0, 1, \ldots, n$ and any $m = 0, 1, \ldots, n$, we also define $D(k, m) \subset D$ as follows:

$$D(k,m) \equiv \Big\{v \in D : \#\{i \in N : v_i > \frac{1}{n}\} = k \text{ and } \#M(v) = m\Big\}.$$

Remark 6. When $k < m$, $v \in D(k, m)$ means[11] that for any $i \in N$, $v_i \geq \frac{1}{n}$. Hence, the welfare loss of any Ohseto mechanism at v is zero. Thus, the welfare loss of any α-mechanism at v is also zero. Therefore, we only consider the case $k \geq m$.

Remark 7. When $k = 0$, $v \in D(k, m)$ means that for any $i \in N$, $v_i \leq \frac{1}{n}$. Hence, the welfare loss of any Ohseto mechanism at v is zero. Thus, the welfare loss of any α-mechanism at v is also zero. Therefore, we only consider the case $k \geq 1$.

Proposition 1. Let $\alpha \in \Delta$. The supremal welfare loss of the α-mechanism is given by

$$\max_{k \geq 1} \sup_{v \in D(k,0)} WL(v; \varphi^\alpha).$$

Furthermore, it holds that

$$\sup_{v \in D(k,0)} WL(v; \varphi^\alpha) = \max\Big\{(1 - \sum_{h=1}^{k} \alpha_h)\big(\sum_{h=1}^{k} \frac{1}{h} + (n-k)\frac{1}{n} - 1\big), \sum_{h=1}^{k} \alpha_h (1 - \frac{k}{n})\Big\}.$$

[11]If $v_n < \frac{1}{n}$, then we must have $m < n$. This means that $v_m \geq \frac{1}{m} > \frac{1}{n}$, which implies that $k \geq m$. This is a contradiction.

Using Proposition 1, we identify the optimal α-mechanism. For any $k = 1, \ldots, n$, define $\bar{\alpha}_k$ as follows:

$$\bar{\alpha}_k \equiv 1 - \sum_{h=1}^{k-1} \bar{\alpha}_h - \frac{1 - \dfrac{k}{n}}{\sum_{h=1}^{k} \dfrac{1}{h} + 1 - \dfrac{2k}{n}}. \tag{1}$$

Denote $\bar{\alpha} \equiv (\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$.

Proposition 2. $\bar{\alpha}$ is well-defined, i.e., for any $k = 1, \ldots, n$, $\bar{\alpha}_k \in [0, 1]$ and $\sum_{h=1}^{n} \bar{\alpha}_h = 1$.

Proposition 3. The supremal welfare loss of the $\bar{\alpha}$-mechanism is given by

$$\max_{k \geq 1} \Big\{ \sum_{h=1}^{k} \bar{\alpha}_h (1 - \frac{k}{n}) \Big\}.$$

Theorem 3. Among the α-mechanisms, the $\bar{\alpha}$-mechanism achieves the smallest supremal welfare loss.

The following result states that our new mechanism improves the supremal welfare loss over that of the Ohseto mechanisms.

Corollary 1. The supremal welfare loss of the $\bar{\alpha}$-mechanism is strictly less than that of any Ohseto mechanism.

Example 4. Let $n = 5$. Then, $\bar{\alpha}_1 = \frac{1}{2}$, $\bar{\alpha}_2 = \frac{5}{2 \cdot 17}$, $\bar{\alpha}_3 = \frac{90}{17 \cdot 49}$, $\bar{\alpha}_4 = \frac{12 \cdot 40}{49 \cdot 89}$, and $\bar{\alpha}_5 = \frac{12}{89}$. Note that

$$\begin{aligned}
\bar{\alpha}_1 (1 - \frac{1}{5}) &= \frac{1}{2} \cdot \frac{4}{5} = 0.4 \\
\sum_{h=1}^{2} \bar{\alpha}_h (1 - \frac{2}{5}) &= \frac{11}{17} \cdot \frac{3}{5} \approx 0.3882 \\
\sum_{h=1}^{3} \bar{\alpha}_h (1 - \frac{3}{5}) &= \frac{37}{49} \cdot \frac{2}{5} \approx 0.3020 \\
\sum_{h=1}^{4} \bar{\alpha}_h (1 - \frac{4}{5}) &= \frac{77}{89} \cdot \frac{1}{5} \approx 0.1730 \\
\sum_{h=1}^{5} \bar{\alpha}_h (1 - \frac{5}{5}) &= 0.
\end{aligned}$$

Then, the supremal welfare loss of the $\bar{\alpha}$-mechanism is 0.4, which is half that of the 1-Ohseto mechanism (i.e., the optimal Ohseto mechanism).

Example 5. Figure 1 compares the numerical results of the supremal welfare loss of the $\bar{\alpha}$-mechanism, the 1-Ohseto mechanism, and the Moulin mechanism.

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6 Proofs

6.1 Proof of Theorem 1

Let $\alpha \in \Delta$. Let $i \in N$. Let $v \in \mathbb{R}^n_+$ and $v'_i \in \mathbb{R}_+$. For any $w = 1, \dots, n$, the w-Ohseto mechanism is strategy-proof, that is, it holds that

$$s_i^w(v)v_i - t_i^w(v) \geq s_i^w(v'_i, v_{-i})v_i - t_i^w(v'_i, v_{-i}).$$

Note that

$$\sigma_i^\alpha(v) = \sum_{w=1}^n \alpha_w s_i^w(v)$$

and

$$\tau_i^\alpha(v) = \sum_{w=1}^n \alpha_w t_i^w(v).$$

Thus, we have

$$\sigma_i^\alpha(v)v_i - \tau_i^\alpha(v) \geq \sigma_i^\alpha(v'_i, v_{-i})v_i - \tau_i^\alpha(v'_i, v_{-i}).$$

Therefore, Theorem 1 is valid.

6.2 Myerson's Lemma

To prove Theorem 2, we use the following Lemma, proved by Myerson (1981) in a similar model.

Lemma (Myerson 1981)**.** If a mechanism φ satisfies strategy-proofness, then for any $i \in N$, any $v_i, v'_i \in \mathbb{R}_+$ such that $v_i \leq v'_i$, and any $v_{-i} \in \mathbb{R}^{n-1}_+$, it holds that

$$\sigma_i(v_i, v_{-i}) \leq \sigma_i(v'_i, v_{-i})$$

and

$$\sigma(v_i', v_{-i})v_i' - \tau_i(v_i', v_{-i}) = \sigma(v_i, v_{-i})v_i - \tau_i(v_i, v_{-i}) + \int_{v_i}^{v_i'} \sigma(x_i, v_{-i})dx_i.$$

Myerson's Lemma states that if the mechanism is strategy-proof, then (i) the probability that the agent consumes the excludable public good is a non-decreasing function of his valuation, and (ii) his utilities when he reports truthfully are related by the integral of the probability.

6.3 Proof of Theorem 2

Let $\alpha \in \Delta$. Let φ be a strategy-proof probabilistic mechanism such that for any $v \in \mathbb{R}^n_+$ and any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) \geq \sigma_i^{\alpha}(v)v_i - \tau_i^{\alpha}(v). \tag{2}$$

Note that[12] $\sum_{i\in N} \tau_i(v) \geq \max_{i\in N} \sigma_i(v)$. Then, it follows that

$$\begin{aligned} \max_{i\in N} \sigma_i(v) \sum_{i\in N} v_i - \max_{i\in N} \sigma_i(v) &\geq \sum_{i\in N} \sigma_i(v)v_i - \max_{i\in N} \sigma_i(v) \\ &\geq \sum_{i\in N} \sigma_i(v)v_i - \sum_{i\in N} \tau_i(v). \end{aligned} \tag{3}$$

Lemma 1. For any $v \in \mathbb{R}^n_+$ such that $\sum_{i\in N} v_i < 1$ and any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^{\alpha}(v)v_i - \tau_i^{\alpha}(v).$$

Proof of Lemma 1. We prove Lemma 1 by mathematical induction.

Claim 1. For any $v \in \mathbb{R}^n_+$ such that $\sum_{i\in N} v_i < 1$ and $\#\{i \in N : v_i > \frac{1}{n}\} = 0$, it holds that for any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^{\alpha}(v)v_i - \tau_i^{\alpha}(v).$$

Proof of Claim 1. Let $v \in \mathbb{R}^n_+$ be such that $\sum_{i\in N} v_i < 1$ and $\#\{i \in N : v_i > \frac{1}{n}\} = 0$. Note that[13] for any $i \in N$,

$$\sigma_i^{\alpha}(v) = 0 \text{ and } \tau_i^{\alpha}(v) = 0.$$

Sub-claim 1 - 1. $\max_{i\in N} \sigma_i(v) = 0$.

[12] This follows from the feasibility.

[13] Since $\sum_{i\in N} v_i < 1$, $M(v) = \emptyset$.

Proof of Sub-claim 1 - 1. By adding (2) for all agents, we have

$$\sum_{i\in N}\sigma_i(v)v_i - \sum_{i\in N}\tau_i(v) \geq 0.$$

Then, by (3), it holds that

$$\begin{aligned}\max_{i\in N}\sigma_i(v)\sum_{i\in N}v_i - \max_{i\in N}\sigma_i(v) &\geq \sum_{i\in N}\sigma_i(v)v_i - \sum_{i\in N}\tau_i(v)\\ &\geq 0,\end{aligned}$$

that is,

$$\max_{i\in N}\sigma_i(v)(\sum_{i\in N}v_i - 1) \geq 0.$$

Since $\sum_{i\in N} v_i - 1 < 0$, we have

$$\max_{i\in N}\sigma_i(v) \leq 0,$$

which implies the desired result. □

Sub-claim 1 - 2. For any $i \in N$, it holds that

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^\alpha(v)v_i - \tau_i^\alpha(v).$$

Proof of Sub-claim 1 - 2. Suppose to the contrary that for some $j \in N$,

$$\sigma_j(v)v_j - \tau_j(v) > \sigma_j^\alpha(v)v_j - \tau_j^\alpha(v).$$

Then, by adding (2) for all agents, we have

$$\sum_{i\in N}\sigma_i(v)v_i - \sum_{i\in N}\tau_i(v) > 0.$$

Since, by Sub-claim 1 - 1, $\max_{i\in N}\sigma_i(v) = 0$, by (3), it follows that

$$\begin{aligned}0 &= \max_{i\in N}\sigma_i(v)\sum_{i\in N}v_i - \max_{i\in N}\sigma_i(v)\\ &\geq \sum_{i\in N}\sigma_i(v)v_i - \sum_{i\in N}\tau_i(v)\\ &> 0,\end{aligned}$$

which is a contradiction. □

Thus, Claim 1 is valid. □

Claim 2. Suppose that for any $v \in \mathbb{R}^n_+$ such that $\sum_{i \in N} v_i < 1$ and $\#\{i \in N : v_i > \frac{1}{n}\} \leq k-1$, it holds that for any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^\alpha(v)v_i - \tau_i^\alpha(v).$$

Then, for any $v \in \mathbb{R}^n_+$ such that $\sum_{i \in N} v_i < 1$ and $\#\{i \in N : v_i > \frac{1}{n}\} = k$, it holds that for any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^\alpha(v)v_i - \tau_i^\alpha(v).$$

Proof of Claim 2. Let $v \in \mathbb{R}^n_+$ be such that $\sum_{i \in N} v_i < 1$ and $\#\{i \in N : v_i > \frac{1}{n}\} = k$. Note that for any $i \in N$,

$$\sigma_i^\alpha(v) = \begin{cases} \sum_{h=1}^k \alpha_h & \text{if } v_i > 0 \\ 0 & \text{if } v_i = 0 \end{cases}$$

and

$$\tau_i^\alpha(v) = \frac{1}{n}\sum_{h=1}^k \alpha_h.$$

Sub-claim 2 - 1. $\max_{i \in N} \sigma_i(v) \leq \sum_{h=1}^k \alpha_h$.

Proof of Sub-claim 2 - 1. By adding (2) for all agents, we have

$$\sum_{i \in N} \sigma_i(v)v_i - \sum_{i \in N} \tau_i(v) \geq \sum_{h=1}^k \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^k \alpha_h.$$

Then, by (3), it holds that

$$\begin{aligned} \max_{i \in N} \sigma_i(v) \sum_{i \in N} v_i - \max_{i \in N} \sigma_i(v) &\geq \sum_{i \in N} \sigma_i(v)v_i - \sum_{i \in N} \tau_i(v) \\ &\geq \sum_{h=1}^k \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^k \alpha_h, \end{aligned}$$

that is,

$$(\max_{i \in N} \sigma_i(v) - \sum_{h=1}^k \alpha_h)(\sum_{i \in N} v_i - 1) \geq 0.$$

Since $\sum_{i \in N} v_i - 1 < 0$, we have

$$\max_{i \in N} \sigma_i(v) \leq \sum_{h=1}^k \alpha_h.$$

□

Sub-claim 2 - 2. $\max_{i\in N}\sigma_i(v)=\sum_{h=1}^k\alpha_h$.

Proof of Sub-claim 2 - 2. Let $i\in N$ be such that $v_i>\frac{1}{n}$. Then, by Myerson's Lemma, it holds that

$$\sigma_i(v)v_i-\tau_i(v)=\sigma_i(\frac{1}{n},v_{-i})\frac{1}{n}-\tau_i(\frac{1}{n},v_{-i})+\int_{\frac{1}{n}}^{v_i}\sigma_i(w_i,v_{-i})dw_i.$$

By the assumption of Claim 2 , we have

$$\sigma_i(\frac{1}{n},v_{-i})\frac{1}{n}-\tau_i(\frac{1}{n},v_{-i})=\sum_{h=1}^{k-1}\alpha_h\frac{1}{n}-\frac{1}{n}\sum_{h=1}^{k-1}\alpha_h=0.$$

Hence, by (2), it holds that

$$\begin{aligned}\int_{\frac{1}{n}}^{v_i}\sigma_i(w_i,v_{-i})dw_i &= \sigma_i(v)v_i-\tau_i(v)\\ &\geq \sigma_i^\alpha(v)v_i-\tau_i^\alpha(v)\\ &= \sum_{h=1}^k\alpha_hv_i-\frac{1}{n}\sum_{h=1}^k\alpha_h.\end{aligned}$$

Since, by Sub-claim 2 - 1, $\sigma_i(v)\leq\sum_{h=1}^k\alpha_h$, by Myerson's Lemma, for any $w_i\leq v_i$, it holds that

$$\sigma_i(w_i,v_{-i})\leq\sum_{h=1}^k\alpha_h.$$

Hence, it follows that

$$\begin{aligned}\int_{\frac{1}{n}}^{v_i}\sigma_i(w_i,v_{-i})dw_i &\leq \int_{\frac{1}{n}}^{v_i}\sum_{h=1}^k\alpha_hdw_i\\ &= \sum_{h=1}^k\alpha_hv_i-\frac{1}{n}\sum_{h=1}^k\alpha_h.\end{aligned}$$

Thus, we have

$$\int_{\frac{1}{n}}^{v_i}\sigma_i(w_i,v_{-i})dw_i=\sum_{h=1}^k\alpha_hv_i-\frac{1}{n}\sum_{h=1}^k\alpha_h,$$

which implies that

$$\sigma_i(v)=\sum_{h=1}^k\alpha_h.$$

Since, by Sub-claim 2 - 1, $\max_{i \in N} \sigma_i(v) \leq \sum_{h=1}^{k} \alpha_h$, it follows that

$$\max_{i \in N} \sigma_i(v) = \sum_{h=1}^{k} \alpha_h.$$

$\square$

Sub-claim 2 - 3. For any $i \in N$, it holds that

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^{\alpha}(v)v_i - \tau_i^{\alpha}(v).$$

Proof of Sub-claim 2 - 3. Suppose to the contrary that for some $j \in N$,

$$\sigma_j(v)v_j - \tau_j(v) > \sigma_j^{\alpha}(v)v_j - \tau_j^{\alpha}(v).$$

Then, by adding (2) for all agents, we have

$$\sum_{i \in N} \sigma_i(v)v_i - \sum_{i \in N} \tau_i(v) > \sum_{h=1}^{k} \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^{k} \alpha_h.$$

Since, by Sub-claim 2 - 2, $\max_{i \in N} \sigma_i(v) = \sum_{h=1}^{k} \alpha_h$, by (3), it follows that

$$\begin{aligned}
\sum_{h=1}^{k} \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^{k} \alpha_h &= \max_{i \in N} \sigma_i(v) \sum_{i \in N} v_i - \max_{i \in N} \sigma_i(v) \\
&\geq \sum_{i \in N} \sigma_i(v)v_i - \sum_{i \in N} \tau_i(v) \\
&> \sum_{h=1}^{k} \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^{k} \alpha_h,
\end{aligned}$$

which is a contradiction. $\square$

Thus, Claim 2 is valid. $\square$

Therefore, Lemma 1 is valid. $\square$

Lemma 2. For any $v \in \mathbb{R}_+^n$ such that $\sum_{i \in N} v_i > 1$ and any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^{\alpha}(v)v_i - \tau_i^{\alpha}(v).$$

Proof of Lemma 2. We prove Lemma 2 by mathematical induction.

Claim 1. For any $v \in \mathbb{R}^n_+$ such that $\sum_{i\in N} v_i > 1$ and $\#\{i \in N : v_i > \frac{1}{n}\} = n$, it holds that for any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^\alpha(v)v_i - \tau_i^\alpha(v).$$

Proof of Claim 1. Let $v \in \mathbb{R}^n_+$ be such that $\sum_{i\in N} v_i > 1$ and $\#\{i \in N : v_i > \frac{1}{n}\} = n$. Note that for any $i \in N$,

$$\sigma_i^\alpha(v) = 1 \text{ and } \tau_i^\alpha(v) = \frac{1}{n}.$$

Sub-claim 1 - 1. $\max_{i\in N} \sigma_i(v) = 1$.

Proof of Sub-claim 1 - 1. By adding (2) for all agents, we have

$$\sum_{i\in N} \sigma_i(v)v_i - \sum_{i\in N} \tau_i(v) \geq \sum_{i\in N} v_i - 1.$$

Then, by (3), it holds that

$$\begin{aligned} \max_{i\in N} \sigma_i(v) \sum_{i\in N} v_i - \max_{i\in N} \sigma_i(v) &\geq \sum_{i\in N} \sigma_i(v)v_i - \sum_{i\in N} \tau_i(v) \\ &\geq \sum_{i\in N} v_i - 1, \end{aligned}$$

that is,

$$(\max_{i\in N} \sigma_i(v) - 1)(\sum_{i\in N} v_i - 1) \geq 0.$$

Since $\sum_{i\in N} v_i - 1 > 0$, we have

$$\max_{i\in N} \sigma_i(v) \geq 1,$$

which implies the desired result. $\square$

Sub-claim 1 - 2. For any $i \in N$, it holds that

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^\alpha(v)v_i - \tau_i^\alpha(v).$$

Proof of Sub-claim 1 - 2. Suppose to the contrary that for some $j \in N$,

$$\sigma_j(v)v_j - \tau_j(v) > \sigma_j^\alpha(v)v_j - \tau_j^\alpha(v).$$

Then, by adding (2) for all agents, we have

$$\sum_{i\in N} \sigma_i(v)v_i - \sum_{i\in N} \tau_i(v) > \sum_{i\in N} v_i - 1.$$

Since, by Sub-claim 1 - 2, $\max_{i \in N} \sigma_i(v) = 1$, by (3), it follows that

$$\begin{aligned}
\sum_{i \in N} v_i - 1 &= \max_{i \in N} \sigma_i(v) \sum_{i \in N} v_i - \max_{i \in N} \sigma_i(v) \\
&\geq \sum_{i \in N} \sigma_i(v) v_i - \sum_{i \in N} \tau_i(v) \\
&> \sum_{i \in N} v_i - 1,
\end{aligned}$$

which is a contradiction. $\square$

Thus, Claim 1 is valid. $\square$

Claim 2. Suppose that for any $v \in \mathbb{R}^n_+$ such that $\sum_{i \in N} v_i > 1$ and $\#\{i \in N : v_i > \frac{1}{n}\} \geq k + 1$, it holds that for any $i \in N$,

$$\sigma_i(v) v_i - \tau_i(v) = \sigma_i^\alpha(v) v_i - \tau_i^\alpha(v).$$

Then, for any $v \in \mathbb{R}^n_+$ such that $\sum_{i \in N} v_i > 1$, $\#\{i \in N : v_i > \frac{1}{n}\} = k$, and $\sum_{i \in N \setminus M(v)} v_i = 0$, it holds that for any $i \in N$,

$$\sigma_i(v) v_i - \tau_i(v) = \sigma_i^\alpha(v) v_i - \tau_i^\alpha(v).$$

Proof of Claim 2. Let $v \in \mathbb{R}^n_+$ be such that $\sum_{i \in N} v_i > 1$, $\#\{i \in N : v_i > \frac{1}{n}\} = k$, and $\sum_{i \in N \setminus M(v)} v_i = 0$. Without loss of generality, we assume that $v_1 \geq v_2 \geq \cdots \geq v_n$. Denote $m = \#M(v)$. Since $\sum_{i=m+1}^n v_i = 0$, for any $i = m+1, \ldots, n$, $v_i = 0$. Note that for any $i = 1, \ldots, m$,

$$\sigma_i^\alpha(v) = 1 \text{ and } \tau_i^\alpha(v) = \frac{1}{n} \sum_{h=1}^k \alpha_h + \frac{1}{m}(1 - \sum_{h=1}^k \alpha_h),$$

and for any $i = m+1, \ldots, n$,

$$\sigma_i^\alpha(v) = 0 \text{ and } \tau_i^\alpha(v) = \frac{1}{n} \sum_{h=1}^k \alpha_h.$$

Sub-claim 2 - 1. $\max_{i \in N} \sigma_i(v) = 1$.

Proof of Sub-claim 2 - 1. By adding (2) for all agents, we have

$$\sum_{i=1}^m \sigma_i(v) v_i - \sum_{i \in N} \tau_i(v) \geq \sum_{i=1}^m v_i - 1.$$

Then, by (3), it holds that

$$\begin{aligned}\max_{i\in N}\sigma_i(v)\sum_{i=1}^m v_i-\max_{i\in N}\sigma_i(v) &\geq \sum_{i=1}^m \sigma_i(v)v_i-\sum_{i\in N}\tau_i(v)\\ &\geq \sum_{i=1}^m v_i-1,\end{aligned}$$

that is,

$$(\max_{i\in N}\sigma_i(v)-1)(\sum_{i=1}^m v_i-1)\geq 0.$$

Since $\sum_{i=1}^m v_i-1>0$, we have

$$\max_{i\in N}\sigma_i(v)\geq 1,$$

which implies the desired result. □

Sub-claim 2 - 2. For any $i\in N$, it holds that

$$\sigma_i(v)v_i-\tau_i(v)=\sigma_i^\alpha(v)v_i-\tau_i^\alpha(v).$$

Proof of Sub-claim 2 - 2. Suppose to the contrary that for some $j\in N$,

$$\sigma_j(v)v_j-\tau_j(v)>\sigma_j^\alpha(v)v_j-\tau_j^\alpha(v).$$

Then, by adding (2) for all agents, we have

$$\sum_{i=1}^m \sigma_i(v)v_i-\sum_{i\in N}\tau_i(v)>\sum_{i=1}^m v_i-1.$$

Since, by Sub-claim 2 - 1, $\max_{i\in N}\sigma_i(v)=1$, by (3), it follows that

$$\begin{aligned}\sum_{i=1}^m v_i-1 &= \max_{i\in N}\sigma_i(v)\sum_{i=1}^m v_i-\max_{i\in N}\sigma_i(v)\\ &\geq \sum_{i=1}^m \sigma_i(v)v_i-\sum_{i\in N}\tau_i(v)\\ &> \sum_{i=1}^m v_i-1,\end{aligned}$$

which is a contradiction. □

Thus, Claim 2 is valid. $\square$

Claim 3. Suppose that for any $v \in \mathbb{R}^n_+$ such that $\sum_{i \in N} v_i > 1$ and $\#\{i \in N : v_i > \frac{1}{n}\} \geq k+1$, it holds that for any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^\alpha(v)v_i - \tau_i^\alpha(v).$$

Then, for any $v \in \mathbb{R}^n_+$ such that $\sum_{i \in N} v_i > 1$, $\#\{i \in N : v_i > \frac{1}{n}\} = k$, and $\sum_{i \in N \setminus M(v)} v_i > 0$, it holds that for any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^\alpha(v)v_i - \tau_i^\alpha(v).$$

Proof of Claim 3. Let $v \in \mathbb{R}^n_+$ be such that $\sum_{i \in N} v_i > 1$, $\#\{i \in N : v_i > \frac{1}{n}\} = k$, and $\sum_{i \in N \setminus M(v)} v_i > 0$. Without loss of generality, we assume that $v_1 \geq v_2 \geq \cdots \geq v_n$. Denote $m = \#M(v)$. Note that for any $i = 1, \ldots, m$,

$$\sigma_i^\alpha(v) = 1 \text{ and } \tau_i^\alpha(v) = \frac{1}{n}\sum_{h=1}^k \alpha_h + \frac{1}{m}(1 - \sum_{h=1}^k \alpha_h).$$

Note also that for any $i = m+1, \ldots, n$,

$$\sigma_i^\alpha(v) = \begin{cases} \sum_{h=1}^k \alpha_h & \text{if } v_i > 0, \\ 0 & \text{if } v_i = 0, \end{cases}$$

and

$$\tau_i^\alpha(v) = \frac{1}{n}\sum_{h=1}^k \alpha_h.$$

Sub-claim 3 - 1. $\max_{i=m+1,\ldots,n} \sigma_i(v) \geq \sum_{h=1}^k \alpha_h$.

Proof of Sub-claim 3 - 1. We divide the argument into two cases.

Case 1. $m = 0$.

By adding (2) for all agents, we have

$$\sum_{i \in N} \sigma_i(v)v_i - \sum_{i \in N} \tau_i(v) \geq \sum_{h=1}^k \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^k \alpha_h.$$

Then, by (3), it holds that

$$\begin{aligned} \max_{i \in N} \sigma_i(v) \sum_{i \in N} v_i - \max_{i \in N} \sigma_i(v) &\geq \sum_{i \in N} \sigma_i(v)v_i - \sum_{i \in N} \tau_i(v) \\ &\geq \sum_{h=1}^k \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^k \alpha_h, \end{aligned}$$

that is,

$$(\max_{i\in N}\sigma_i(v)-\sum_{h=1}^{k}\alpha_h)(\sum_{i\in N}v_i-1)\geq 0.$$

Since $\sum_{i\in N} v_i - 1 > 0$, we have

$$\max_{i\in N}\sigma_i(v)\geq\sum_{h=1}^{k}\alpha_h,$$

which implies the desired result.

Case 2. $m \geq 1$.

By adding (2) for all agents, we have

$$\sum_{i\in N}\sigma_i(v)v_i-\sum_{i\in N}\tau_i(v)\geq\sum_{i=1}^{m}v_i+\sum_{h=1}^{k}\alpha_h\sum_{i=m+1}^{n}v_i-1.$$

Note that $\sum_{i\in N}\tau_i(v)\geq\max_{i\in N}\sigma_i(v)$ and $\sum_{i=1}^{m}v_i\geq 1$. Then, it follows that

$$\begin{aligned}(\sum_{i=1}^{m}v_i-1) &+ \max_{i=m+1,\ldots,n}\sigma_i(v)\sum_{i=m+1}^{n}v_i\\ &\geq \max_{i\in N}\sigma_i(v)(\sum_{i=1}^{m}v_i-1)+\max_{i=m+1,\ldots,n}\sigma_i(v)\sum_{i=m+1}^{n}v_i\\ &\geq \max_{i\in N}\sigma_i(v)\sum_{i=1}^{m}v_i+\max_{i=m+1,\ldots,n}\sigma_i(v)\sum_{i=m+1}^{n}v_i-\max_{i\in N}\sigma_i(v)\\ &\geq \sum_{i\in N}\sigma_i(v)v_i-\max_{i\in N}\sigma_i(v)\\ &\geq \sum_{i\in N}\sigma_i(v)v_i-\sum_{i\in N}\tau_i(v)\\ &\geq \sum_{i=1}^{m}v_i+\sum_{h=1}^{k}\alpha_h\sum_{i=m+1}^{n}v_i-1,\end{aligned}$$

that is,

$$\max_{i=m+1,\ldots,n}\sigma_i(v)\sum_{i=m+1}^{n}v_i\geq\sum_{h=1}^{k}\alpha_h\sum_{i=m+1}^{n}v_i.$$

Since $\sum_{i=m+1}^{n} v_i > 0$, we have

$$\max_{i=m+1,\ldots,n}\sigma_i(v)\geq\sum_{h=1}^{k}\alpha_h,$$

which implies the desired result. $\square$

Sub-claim 3 - 2. $\max_{i=m+1,\dots,n} \sigma_i(v) = \sum_{h=1}^{k} \alpha_h$.

Proof of Sub-claim 3 - 2. Let $i \in \arg\max_{i=m+1,\dots,n} \sigma_i(v)$. For any sufficiently small $\varepsilon > 0$, let $\hat{v}_i = \frac{1}{n} + \varepsilon$. Then, by Myerson's Lemma, it holds that

$$\sigma_i(\hat{v}_i, v_{-i})\hat{v}_i - \tau_i(\hat{v}_i, v_{-i}) = \sigma_i(v)v_i - \tau_i(v) + \int_{v_i}^{\hat{v}_i} \sigma_i(w_i, v_{-i})dw_i.$$

By the assumption of Claim 3, we have

$$\sigma_i(\hat{v}_i, v_{-i})\hat{v}_i - \tau_i(\hat{v}_i, v_{-i}) = \sum_{h=1}^{k+1} \alpha_h \hat{v}_i - \frac{1}{n}\sum_{h=1}^{k+1} \alpha_h.$$

Hence, by (2), it holds that

$$\begin{aligned}
\sum_{h=1}^{k+1} \alpha_h \hat{v}_i - \frac{1}{n}\sum_{h=1}^{k+1} \alpha_h \quad - & \quad \int_{v_i}^{\hat{v}_i} \sigma_i(w_i, v_{-i})dw_i \\
= & \quad \sigma_i(\hat{v}_i, v_{-i})\hat{v}_i - \tau_i(\hat{v}_i, v_{-i}) - \int_{v_i}^{\hat{v}_i} \sigma_i(w_i, v_{-i})dw_i \\
= & \quad \sigma_i(v)v_i - \tau_i(v) \\
\geq & \quad \sigma_i^{\alpha}(v)v_i - \tau_i^{\alpha}(v) \\
= & \quad \sum_{h=1}^{k} \alpha_h v_i - \frac{1}{n}\sum_{h=1}^{k} \alpha_h.
\end{aligned}$$

Thus, as $\varepsilon \to 0$, we have

$$-\int_{v_i}^{\frac{1}{n}} \sigma_i(w_i, v_{-i})dw_i \geq \sum_{h=1}^{k} \alpha_h v_i - \frac{1}{n}\sum_{h=1}^{k} \alpha_h,$$

that is,

$$\int_{v_i}^{\frac{1}{n}} \sigma_i(w_i, v_{-i})dw_i \leq \frac{1}{n}\sum_{h=1}^{k} \alpha_h - \sum_{h=1}^{k} \alpha_h v_i.$$

Since, by Sub-claim 3 - 1, $\sigma_i(v) \geq \sum_{h=1}^{k} \alpha_h$, by Myerson's Lemma, for any $w_i \geq v_i$, it holds that

$$\sigma_i(w_i, v_{-i}) \geq \sum_{h=1}^{k} \alpha_h.$$

Hence, it follows that

$$\begin{aligned}\int_{v_i}^{\frac{1}{n}} \sigma_i(w_i, v_{-i}) dw_i &\geq \int_{v_i}^{\frac{1}{n}} \sum_{h=1}^{k} \alpha_h dw_i \\ &= \frac{1}{n} \sum_{h=1}^{k} \alpha_h - \sum_{h=1}^{k} \alpha_h v_i.\end{aligned}$$

These mean that

$$\int_{v_i}^{\frac{1}{n}} \sigma_i(w_i, v_{-i}) dw_i = \frac{1}{n} \sum_{h=1}^{k} \alpha_h - \sum_{h=1}^{k} \alpha_h v_i,$$

which implies that

$$\sigma_i(v) = \sum_{h=1}^{k} \alpha_h.$$

□

Sub-claim 3 - 3. For any $i \in N$, it holds that

$$\sigma_i(v) v_i - \tau_i(v) = \sigma_i^{\alpha}(v) v_i - \tau_i^{\alpha}(v).$$

Proof of Sub-claim 3 - 3. Suppose to the contrary that for some $j \in N$,

$$\sigma_j(v) v_j - \tau_j(v) > \sigma_j^{\alpha}(v) v_j - \tau_j^{\alpha}(v).$$

We divide the argument into two cases.

Case 1. $m = 0$.

By adding (2) for all agents, we have

$$\sum_{i \in N} \sigma_i(v) v_i - \sum_{i \in N} \tau_i(v) > \sum_{h=1}^{k} \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^{k} \alpha_h.$$

Since, by Sub-claim 3 - 2, $\max_{i \in N} \sigma_i(v) = \sum_{h=1}^{k} \alpha_h$, by (3), it follows that

$$\begin{aligned}\sum_{h=1}^{k} \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^{k} \alpha_h &= \max_{i \in N} \sigma_i(v) \sum_{i \in N} v_i - \max_{i \in N} \sigma_i(v) \\ &\geq \sum_{i \in N} \sigma_i(v) v_i - \sum_{i \in N} \tau_i(v) \\ &> \sum_{h=1}^{k} \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^{k} \alpha_h,\end{aligned}$$

which is a contradiction.

Case 2. $m \geq 1$.

By adding (2) for all agents, we have

$$\sum_{i \in N} \sigma_i(v) v_i - \sum_{i \in N} \tau_i(v) > \sum_{i=1}^{m} v_i + \sum_{h=1}^{k} \alpha_h \sum_{i=m+1}^{n} v_i - 1.$$

Note that $\sum_{i \in N} \tau_i(v) \geq \max_{i \in N} \sigma_i(v)$ and $\sum_{i=1}^{m} v_i \geq 1$. Then, it follows that

$$\begin{aligned}
(\sum_{i=1}^{m} v_i - 1) \quad &+ \max_{i=m+1,\ldots,n} \sigma_i(v) \sum_{i=m+1}^{n} v_i \\
&\geq \max_{i \in N} \sigma_i(v) (\sum_{i=1}^{m} v_i - 1) + \max_{i=m+1,\ldots,n} \sigma_i(v) \sum_{i=m+1}^{n} v_i \\
&\geq \max_{i \in N} \sigma_i(v) \sum_{i=1}^{m} v_i + \max_{i=m+1,\ldots,n} \sigma_i(v) \sum_{i=m+1}^{n} v_i - \max_{i \in N} \sigma_i(v) \\
&\geq \sum_{i \in N} \sigma_i(v) v_i - \max_{i \in N} \sigma_i(v) \\
&\geq \sum_{i \in N} \sigma_i(v) v_i - \sum_{i \in N} \tau_i(v) \\
&> \sum_{i=1}^{m} v_i + \sum_{h=1}^{k} \alpha_h \sum_{i=m+1}^{n} v_i - 1,
\end{aligned}$$

that is,

$$\max_{i=m+1,\ldots,n} \sigma_i(v) \sum_{i=m+1}^{n} v_i > \sum_{h=1}^{k} \alpha_h \sum_{i=m+1}^{n} v_i.$$

Since, by Sub-claim 3 - 2, $\max_{i \in N} \sigma_i(v) = \sum_{h=1}^{k} \alpha_h$, we have

$$\sum_{h=1}^{k} \alpha_h \sum_{i=m+1}^{n} v_i > \sum_{h=1}^{k} \alpha_h \sum_{i=m+1}^{n} v_i.$$

Since $\sum_{i=m+1}^{n} v_i > 0$, it is a contradiction. $\square$

Thus, Claim 3 is valid. $\square$

Therefore, Lemma 2 is valid. $\square$

Lemma 3. For any $v \in \mathbb{R}_+^n$ such that $\sum_{i \in N} v_i = 1$ and any $i \in N$,

$$\sigma_i(v) v_i - \tau_i(v) = \sigma_i^{\alpha}(v) v_i - \tau_i^{\alpha}(v).$$

Proof of Lemma 3. Let $v \in \mathbb{R}^n_+$ be such that $\sum_{i \in N} v_i = 1$.

Claim 1. $\sum_{i \in N} \sigma_i^\alpha(v) v_i - \sum_{i \in N} \tau_i^\alpha(v) = 0$.

Proof of Claim 1. Without loss of generality, we assume that $v_1 \geq v_2 \geq \cdots \geq v_n$. Denote $k = \#\{i \in N : v_i > \frac{1}{n}\}$ and $m = \#M(v)$. We divide the argument into two cases.

Case 1. $m = 0$.

Note that for any $i \in N$,

$$\sigma_i^\alpha(v) = \begin{cases} \sum_{h=1}^k \alpha_h & \text{if } v_i > 0, \\ 0 & \text{if } v_i = 0, \end{cases}$$

and

$$\tau_i^\alpha(v) = \frac{1}{n} \sum_{h=1}^k \alpha_h.$$

Hence, it holds that

$$\begin{aligned} \sum_{i \in N} \sigma_i^\alpha(v) v_i - \sum_{i \in N} \tau_i^\alpha(v) &= \sum_{h=1}^k \alpha_h \sum_{i \in N} v_i - \sum_{h=1}^k \alpha_h \\ &= \sum_{h=1}^k \alpha_h (\sum_{i \in N} v_i - 1) \\ &= 0. \end{aligned}$$

Case 2. $m \geq 1$.

Since $\sum_{i \in N} v_i = 1$, for any $i = 1, \ldots, m$, $v_i = \frac{1}{m}$, and for any $i = m+1, \ldots, n$, $v_i = 0$. Note that for any $i = 1, \ldots, m$,

$$\sigma_i^\alpha(v) = 1 \text{ and } \tau_i^\alpha(v) = \frac{1}{n} \sum_{h=1}^k \alpha_h + \frac{1}{m}(1 - \sum_{h=1}^k \alpha_h),$$

and for any $i = m+1, \ldots, n$,

$$\sigma_i^\alpha(v) = 0 \text{ and } \tau_i^\alpha(v) = \frac{1}{n} \sum_{h=1}^k \alpha_h.$$

Hence, it holds that

$$\begin{aligned} \sum_{i \in N} \sigma_i^\alpha(v) v_i - \sum_{i \in N} \tau_i^\alpha(v) &= \sum_{i=1}^m v_i - 1 \\ &= 0. \end{aligned}$$

Thus, Claim 1 is valid. □

Claim 2. For any $i \in N$, it holds that

$$\sigma_i(v)v_i - \tau_i(v) = \sigma_i^\alpha(v)v_i - \tau_i^\alpha(v).$$

Proof of Claim 2. Suppose to the contrary that for some $j \in N$,

$$\sigma_j(v)v_j - \tau_j(v) > \sigma_j^\alpha(v)v_j - \tau_j^\alpha(v).$$

Then, by adding (2) for all agents, we have

$$\sum_{i\in N} \sigma_i(v)v_i - \sum_{i\in N} \tau_i(v) > \sum_{i\in N} \sigma_i^\alpha(v)v_i - \sum_{i\in N} \tau_i^\alpha(v).$$

Since $\sum_{i\in N} v_i = 1$, by (3) and Claim 1, it follows that

$$\begin{aligned} 0 &= \max_{i\in N} \sigma_i(v) \sum_{i\in N} v_i - \max_{i\in N} \sigma_i(v) \\ &\geq \sum_{i\in N} \sigma_i(v)v_i - \sum_{i\in N} \tau_i(v) \\ &> \sum_{i\in N} \sigma_i^\alpha(v)v_i - \sum_{i\in N} \tau_i^\alpha(v) \\ &= 0, \end{aligned}$$

which is a contradiction. Thus, Claim 2 is valid. □

Therefore, Lemma 3 is valid. □

By Lemmas 1, 2, and 3, Theorem 2 is valid.

6.4 Proof of Proposition 1

We calculate the welfare loss of the α-mechanism.

Lemma 4. Let $k = 1, \ldots, n$ and $m = 0, 1, \ldots, n$. For any $v \in D(k, m)$, it holds that

$$\sigma^\alpha(v) = (\underbrace{1, \ldots, 1}_{m}, \underbrace{\sum_{h=1}^{k} \alpha_h, \ldots, \sum_{h=1}^{k} \alpha_h}_{n-m-\ell}, \underbrace{0, \ldots, 0}_{\ell}),$$

where $\ell = \#\{i \in N : v_i = 0\}$.

Proof of Lemma 4. Since $\#\{i \in N : v_i > \frac{1}{n}\} = k$, for any $w \leq k$ and any $i \in N$, it holds that

$$s_i^w(v) = \begin{cases} 1 & \text{if } v_i > 0, \\ 0 & \text{if } v_i = 0. \end{cases}$$

Since $\#M(v) = m$, for any $w > k$ and any $i, j \in N$ such that $i \leq m < j$, it holds that

$$s_i^w(v) = 1 \text{ and } s_j^w(v) = 0.$$

Hence, we have

$$\sigma^\alpha(v) = (\underbrace{1, \ldots, 1}_{m}, \underbrace{\sum_{h=1}^{k} \alpha_h, \ldots, \sum_{h=1}^{k} \alpha_h}_{n-m-\ell}, \underbrace{0, \ldots, 0}_{\ell}),$$

$\square$

Lemma 5. For any $k = 1, \ldots, n$ and any $m = 0, 1, \ldots, n$, it holds that

$$\sup_{v \in D(k,0)} WL(v) \geq \sup_{v \in D(k,m)} WL(v),$$

and

$$\sup_{v \in D(k,0)} WL(v) = \max\Big\{(1 - \sum_{h=1}^{k} \alpha_h)\big(\sum_{h=1}^{k} \frac{1}{h} + (n-k)\frac{1}{n} - 1\big), \sum_{h=1}^{k} \alpha_h (1 - \frac{k}{n})\Big\}.$$

Proof of Lemma 5. Let $k = 1, \ldots, n$ and $m = 0, 1, \ldots, n$. First, consider $v \in D(k, 0)$. When $\sum_{i \in N} v_i \geq 1$,

$$\begin{aligned} WL(v) &= (\sum_{i \in N} v_i - 1) - (\sum_{i \in N} \sum_{h=1}^{k} \alpha_h v_i - \sum_{h=1}^{k} \alpha_h) \\ &= (1 - \sum_{h=1}^{k} \alpha_h)(\sum_{i \in N} v_i - 1). \end{aligned}$$

Since we must have $v_1 < 1, \ldots, v_k < \frac{1}{k}, v_{k+1} \leq \frac{1}{n}, \ldots, v_n \leq \frac{1}{n}$, the supremal value on this case is

$$(1 - \sum_{h=1}^{k} \alpha_h)\big(\sum_{h=1}^{k} \frac{1}{h} + (n-k)\frac{1}{n} - 1\big).$$

When $\sum_{i \in N} v_i < 1$,

$$\begin{aligned} WL(v) &= 0 - (\sum_{i \in N} \sum_{h=1}^{k} \alpha_h v_i - \sum_{h=1}^{k} \alpha_h) \\ &= \sum_{h=1}^{k} \alpha_h (1 - \sum_{i \in N} v_i). \end{aligned}$$

Since we must have $\frac{1}{n} < v_1, \ldots, \frac{1}{n} < v_k, 0 \leq v_{k+1}, \ldots, 0 \leq v_n$, the supremal value on this case is

$$\sum_{h=1}^{k} \alpha_h (1 - \frac{k}{n}).$$

Thus, we have

$$\sup_{v \in D(k,0)} WL(v) = \max\Big\{(1 - \sum_{h=1}^{k} \alpha_h)\big(\sum_{h=1}^{k} \frac{1}{h} + (n-k)\frac{1}{n} - 1\big), \sum_{h=1}^{k} \alpha_h (1 - \frac{k}{n})\Big\}.$$

Next, consider $v \in D(k, m)$ such that $m \geq 1$. Since $m \geq 1$, we must have

$$\sum_{i \in N} v_i \geq 1.$$

Hence, it holds that

$$\begin{aligned} WL(v) &= (\sum_{i \in N} v_i - 1) - (\sum_{i=1}^{m} v_i + \sum_{i=m+1}^{n} \sum_{h=1}^{k} \alpha_h v_i - 1) \\ &= \sum_{i=m+1}^{n} v_i - \sum_{i=m+1}^{n} \sum_{h=1}^{k} \alpha_h v_i \\ &= (1 - \sum_{h=1}^{k} \alpha_h) \sum_{i=m+1}^{n} v_i. \end{aligned}$$

Since we must have $v_{m+1} < \frac{1}{m+1}, \ldots, v_k < \frac{1}{k}, v_{k+1} \leq \frac{1}{n}, \ldots, v_n \leq \frac{1}{n}$, we have

$$\sup_{v \in D(k,m)} WL(v) = (1 - \sum_{h=1}^{k} \alpha_h)\big(\sum_{h=m+1}^{k} \frac{1}{h} + (n-k)\frac{1}{n}\big).$$

Since $\sum_{h=1}^{k} \frac{1}{h} + (n-k)\frac{1}{n} - 1 \geq \sum_{h=m+1}^{k} \frac{1}{h} + (n-k)\frac{1}{n}$, we have

$$\sup_{v \in D(k,0)} WL(v) \geq \sup_{v \in D(k,m)} WL(v).$$

$\square$

Therefore, Proposition 1 is valid.

6.5 Proof of Proposition 2

Let $k = 1, \ldots, n$. We first establish that $\bar{\alpha}_k \geq 0$. Notice that

$$\bar{\alpha}_k \equiv 1 - \sum_{h=1}^{k-2} \bar{\alpha}_h - \bar{\alpha}_{k-1} - \frac{1 - \dfrac{k}{n}}{\sum_{h=1}^{k} \dfrac{1}{h} + 1 - \dfrac{2k}{n}}$$

and

$$\bar{\alpha}_{k-1} \equiv 1 - \sum_{h=1}^{k-2} \bar{\alpha}_h - \frac{1 - \dfrac{k-1}{n}}{\sum_{h=1}^{k-1} \dfrac{1}{h} + 1 - \dfrac{2(k-1)}{n}}.$$

Hence, it holds that

$$\begin{aligned}
\bar{\alpha}_k &= \frac{1 - \dfrac{k-1}{n}}{\sum_{h=1}^{k-1} \dfrac{1}{h} + 1 - \dfrac{2(k-1)}{n}} - \frac{1 - \dfrac{k}{n}}{\sum_{h=1}^{k} \dfrac{1}{h} + 1 - \dfrac{2k}{n}} \\
&= \frac{(1 - \dfrac{k-1}{n})(\sum_{h=1}^{k} \dfrac{1}{h} + 1 - \dfrac{2k}{n}) - (1 - \dfrac{k}{n})(\sum_{h=1}^{k-1} \dfrac{1}{h} + 1 - \dfrac{2(k-1)}{n})}{(\sum_{h=1}^{k-1} \dfrac{1}{h} + 1 - \dfrac{2(k-1)}{n})(\sum_{h=1}^{k} \dfrac{1}{h} + 1 - \dfrac{2k}{n})}.
\end{aligned}$$

Note that

$$\sum_{h=1}^{k-1} \frac{1}{h} + 1 - \frac{2(k-1)}{n} > 0$$

and

$$\sum_{h=1}^{k} \frac{1}{h} + 1 - \frac{2k}{n} > 0.$$

Hence, to establish $\bar{\alpha}_k \geq 0$, it is sufficient to show that the following equation is non-negative:

$$(1 - \frac{k-1}{n})(\sum_{h=1}^{k} \frac{1}{h} + 1 - \frac{2k}{n}) - (1 - \frac{k}{n})(\sum_{h=1}^{k-1} \frac{1}{h} + 1 - \frac{2(k-1)}{n}). \tag{4}$$

Note that

$$
\begin{aligned}
(1-\frac{k-1}{n})(\sum_{h=1}^{k}\frac{1}{h}+1-\frac{2k}{n}) &= (1-\frac{k}{n}+\frac{1}{n})(\sum_{h=1}^{k-1}\frac{1}{h}+1-\frac{2(k-1)}{n}+\frac{1}{k}-\frac{2}{n}) \\
&= (1-\frac{k}{n})(\sum_{h=1}^{k-1}\frac{1}{h}+1-\frac{2(k-1)}{n}) \\
&\quad +(1-\frac{k}{n})(\frac{1}{k}-\frac{2}{n})+\frac{1}{n}(\sum_{h=1}^{k}\frac{1}{h}+1-\frac{2k}{n}).
\end{aligned}
$$

Hence, (4) is equivalent to

$$(1-\frac{k}{n})(\frac{1}{k}-\frac{2}{n})+\frac{1}{n}(\sum_{h=1}^{k}\frac{1}{h}+1-\frac{2k}{n}),$$

which is equal to

$$\frac{1}{k}-\frac{2}{n}+\frac{1}{n}\sum_{h=1}^{k}\frac{1}{h}. \tag{5}$$

Since $\frac{1}{k}-\frac{1}{n}\geq 0$ and $\sum_{h=2}^{k}\frac{1}{h}\geq 0$, (5) is non-negative. Thus, (4) is also non-negative.

By (1), it is obvious that

$$\sum_{h=1}^{n}\bar{\alpha}_h = 1.$$

Then, since for any $h=1,\ldots,n$, $\bar{\alpha}_h\geq 0$, we also have that

$$\bar{\alpha}_k \leq 1.$$

Therefore, Proposition 2 is valid.

6.6 Proof of Proposition 3

By Proposition 1, it is sufficient to show the following Lemma.

Lemma 6. For any $k=1,\ldots,n$, it holds that

$$(1-\sum_{h=1}^{k}\bar{\alpha}_h)\big(\sum_{h=1}^{k}\frac{1}{h}+(n-k)\frac{1}{n}-1\big)=\sum_{h=1}^{k}\bar{\alpha}_h(1-\frac{k}{n}).$$

Proof of Lemma 6. Let $k = 1, \ldots, n$. Note that (1) is equivalent to

$$1 - \sum_{h=1}^{k} \bar{\alpha}_h = \frac{1 - \dfrac{k}{n}}{\sum_{h=1}^{k} \dfrac{1}{h} + 1 - \dfrac{2k}{n}},$$

which is also equivalent to

$$(1 - \sum_{h=1}^{k} \bar{\alpha}_h)(\sum_{h=1}^{k} \frac{1}{h} + 1 - \frac{2k}{n}) = 1 - \frac{k}{n}.$$

Note that the left-hand side of the above equation is equal to

$$(1 - \sum_{h=1}^{k} \bar{\alpha}_h)(\sum_{h=1}^{k} \frac{1}{h} + (n-k)\frac{1}{n} - 1 + 1 - \frac{k}{n}).$$

Hence, we have

$$(1 - \sum_{h=1}^{k} \bar{\alpha}_h)(\sum_{h=1}^{k} \frac{1}{h} + (n-k)\frac{1}{n} - 1) = \sum_{h=1}^{k} \bar{\alpha}_h(1 - \frac{k}{n}).$$

□

Therefore, Proposition 3 is valid.

6.7 Proof of Theorem 3

Let $k^* = 1, \ldots, n$ be such that

$$k^* \in \arg\max_{k \geq 1} \Big\{ \sum_{h=1}^{k} \bar{\alpha}_h (1 - \frac{k}{n}) \Big\}.$$

Notice that $k^* \neq n$. For simplicity of notation, denote $A^* \equiv \sum_{h=1}^{k^*} \bar{\alpha}_h$. Note that

$$\sum_{h=1}^{k^*} \frac{1}{h} + (n - k^*)\frac{1}{n} - 1 > 0 \text{ and } 1 - \frac{k^*}{n} > 0.$$

Hence, $(1 - A)\big(\sum_{h=1}^{k^*} \frac{1}{h} + (n-k^*)\frac{1}{n} - 1\big)$ is a strictly decreasing function of A, and $A(1 - \frac{k^*}{n})$ is a strictly increasing function of A. Since these are equal at A^*, $\max\{(1-A)\big(\sum_{h=1}^{k^*} \frac{1}{h} + (n-k^*)\frac{1}{n} - 1\big), A(1 - \frac{k^*}{n})\}$ has the smallest value at A^*. Thus, the supremal welfare loss of any α-mechanism is larger than or equal to that of the $\bar{\alpha}$-mechanism. Therefore, Theorem 3 is valid.

6.8 Proof of Corollary 1

As mentioned in Remarks 3 and 4, the 1-Ohseto mechanism achieves the smallest supremal welfare loss among the Ohseto mechanisms, with a value of

$$1 - \frac{1}{n}.$$

Note that

$$\bar{\alpha}_1 = \frac{1}{2}.$$

Hence, it holds that

$$\bar{\alpha}_1 (1 - \frac{1}{n}) < 1 - \frac{1}{n}.$$

For any $k = 2, \ldots, n$, we also have

$$\sum_{h=1}^{k} \bar{\alpha}_h (1 - \frac{k}{n}) \leq 1 - \frac{k}{n} < 1 - \frac{1}{n}.$$

Thus, for any $k = 1, \ldots, n$, it holds that

$$\sum_{h=1}^{k} \bar{\alpha}_h (1 - \frac{k}{n}) < 1 - \frac{1}{n}.$$

Therefore, Corollary 1 is valid.

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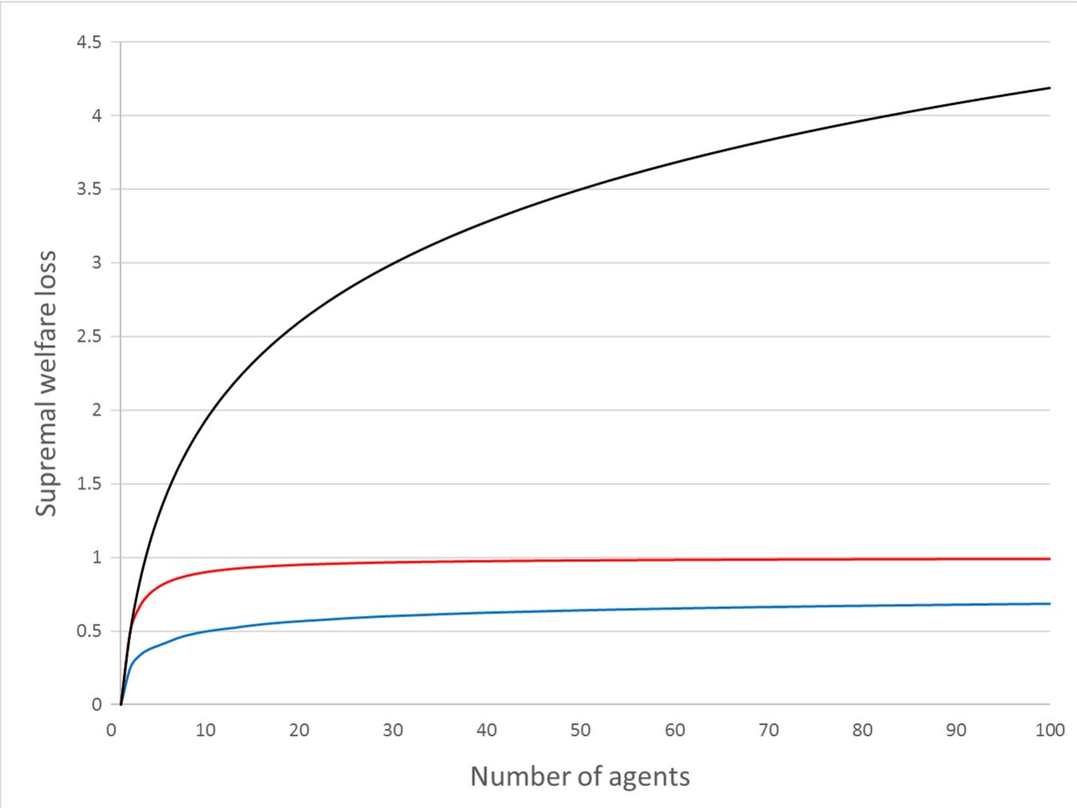

Figure 1: The supremal welfare loss of three mechanisms: the Moulin mechanism (Black), the 1-Ohseto mechanism (Red), and the $\bar{\alpha}$-mechanism (Blue).