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Abstract

This article reexamines optimality of the Friedman rule in an economy, wherein (i) spatial separation and limited communication create a transactions role for money (ii) banks arise to provide liquidity, and (iii) agents are nonsmooth ambiguity aversion. It is shown that the structure of the set of “second-best” monetary policies crucially depends on the relation between the set of beliefs and the marginal productivity of capital. Especially, in order for the Friedman rule to be suboptimal, it is necessary for the maximum of subjective probabilities of realizing liquidity events to be sufficiently small.

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Optimality of the Friedman rule under ambiguity

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Keywords: Friedman rule; Money; Ambiguity; Spatial separation; Overlapping generations model.

JEL Classification Numbers: D81; E40; E41; E42; E43; E50.

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1 Introduction

Characterizing optimal monetary policy is one of central subjects in monetary theory. One of the most celebrated propositions in classical monetary theory is provided by Friedman (1969) as the doctrine on the "optimum quantity of money." He proposed that a desirable monetary policy is one which equates the private opportunity cost of holding money [the nominal interest rate] to its social opportunity cost [which is zero]. His assessment on optimal monetary policy is now called the *Friedman rule*.[1] Its optimality has been verified in the vast theoretical literature including cash-in-advance and money-in-utility models (Lucas and Stokey, 1983; Kimbrough, 1986; Chari, Christiano, and Keheo, 1996),[2] the overlapping generations (OLG) model (Wallace, 1980; McCallum, 1987; Smith, 1991), and the money search model (Aruoba and Wright, 2003; Lagos and Wright, 2005; Williamson and Wright, 2010).

In practice, however, there seems no central bank stating that its object is to implement the Friedman rule. Actually, Walsh (2003) reported that several countries have set their long-run target on the inflation rate between zero and three percent, which is inconsistent with the Friedman rule. In order to resolve this disparity between theory and practice, recent theoretical studies have tried to identify under what circumstances the Friedman rule is suboptimal. Especially, Champ, Smith, and Williamson (1996) and Smith (2002) produced one of important streams in the literature. Champ, Smith, and Williamson (1996) first embedded banks playing a role of the financial intermediary described as Diamond and Dybvig (1983) to a Wallacian OLG model (Wallace, 1980).[3] Then, in such a model, Smith (2002) showed suboptimality of the Friedman rule.

In their models, a sudden *liquidity event*, an event wherein some agents face the need to monetize their illiquid asses, plays an important role. Agents who face a liquidity event might withdraw their deposits from banks and thus banks must hold sufficient liquidity (money) in order to meet the needs of depositors. The previous studies then considered that agents (and banks) can summarize their belief on likelihood of liquidity events by a *single* probability measure. However, it might be difficult to forecast the occurrence of some of serious liquidity events. Who did anticipate, for example, the Great Depression observed in around 1930 and the world-wide financial crisis observed in and after 2008? Also, it seems hard to summarize agents' belief

[1]Friedman's assessment on optimal monetary policy is also called the *Chicago rule*.

[2]See also Woodford (1990) for an excellent survey.

[3]The Wallacian OLG model is further discussed in an stochastic environment. Interested readers might found, for example, Koda (1984) and Kitagawa (1994).

on realizations of such serious liquidity events by a unique probability measure. Therefore, it is variable to consider the situation that agents cannot summarize their belief on occurrence of liquidity events by a single probability measure.

According to Knight (1921) and Keynes (1921), we should now distinguish "risk" and "uncertainty" from each other. In their terminologies, *risk* means a situation that agents can summarize theier belief by a *single* probability measure and *uncertainty* means a situation that they cannot. The importance of such a distinction has been pointed out by the work of Ellsberg (1961). Since the pioneering work by Savage (1954), however, most of studies in traditional economics under uncertainty considered agents who choose their actions to maximize its expected utility with a *unique* probability measure. Methods for explaining the distinction between risk and uncertainty did not appear until the beginning of 80s, in the seminal papers by Schmeidler (1982, 1989) and Gilboa and Schmeidler (1989). They axiomatized the maxim expected utility (MEU) preference, under which agents behave as if they maximize the expected utility minimized over a *set* of probability measures. Although the class of preferences with multiple priors includes the class of those with a single prior as a special case, the situation that agents summarize their belief by multiple priors is now called *ambiguity*.

The aim of this article is to extend an environment, as considered by Champ, Smith, and Williamson (1996) and Smith (2002), by introducing the MEU preference and to study the conduct of monetary policy. More precisely, this article embeds the MEU preference in a variant of the overlapping generations model with special separation, developed by Champ, Smith, and Williamson (1996) and Smith (2002). In the economy, two islands exist and there exists no communication between them. Time is divided into discrete periods and runs from minus infinity to plus infinity. In each period, a single perishable commodity is available and there exists a simple production technology, often called a storage technology. Also, a new generation consisting of finite agents living for two periods is born in each period. At the end of each period, some fraction of agents born in that period faces the liquidity event, i.e., some fraction of agents in each island is randomly selected and moves to the other island. Here, movers need to hold money in order to prepare for the liquidity event and withdraw their deposits from the bank. On the other hand, the bank faces the *liquidity constraint*, i.e., it must prepare sufficient liquidity in order to meet the movers' need. Then, it is assumed that each agent ranks their contingent consumption plans according to the MEU preference.

This article provides two significant observations. First, for any nonnegative nominal interest rates, a monetary steady states exists. Especially, for sufficiently small interest rates, we can observe the existence of *fully-insured* monetary steady state, wherein agents enjoy fully-insured second-period consumption plans. Moreover, it is shown that on the range of interest rates generating fully-insured monetary steady states, marginal changes in nominal interest rates do not affect equilibrium welfare. Our second observation is that the set of second-best monetary policies crucially depends on the relation between the structure of beliefs and the marginal productivity of capital. As a special case, we verify that the Friedman rule can be optimal under sufficiently great ambiguity, whereas it is suboptimal under relatively small ambiguity. Therefore, one might be able to consider that this article provides one of models which justifies the zero-interest-rate policy adopted in several developed countries after the financial crisis in 2008.

The organization of this article is as follows: Section 2 presents some relevant results from the existing literature. Section 3 provides ingredients of the model considered in the present paper. Section 4 introduces money, banks, and firms and define a monetary equilibrium. Section 5 shows that the negative interest rates does not arrow in monetary equilibrium. Section 6 characterizes monetary equilibrium by a difference inclusion. Section 7 shows existence of monetary steady state. Section 8 examines the optimum quantity of money. Section 9 provides a closed solution of monetary steady states and some numerical examples on the equilibrium welfare. Some of proofs are provided in the last section.

2 Related Literature

This article builds on a number of contributions. As mentioned in the Introduction, an overlapping generations (OLG) model with spatial separation is developed by Champ, Smith, and Williamson (1996). In the model, some fraction of agents in each island faces a liquidity event, i.e., some of agents is randomly chosen, moves to the other island, and movers lose their connection to their bank. In such a circumstance, banks should hold sufficient liquidity (money) in order to prevent the liquidity shortage caused by withdrawal by depositors facing the liquidity event. As a result, banks lose some opportunity to invest in assets with higher rates of return and equilibrium will be suboptimal. A contractional policy then increases the rate of return of money and grows such inefficiency. This is a reason that the Friedman rule becomes suboptimal. Lots of works such as Smith (2002), Schreft and Smith (2002), Haslag and Martin (2007), Bhattacharya, Haslag, and Martin (2009), Paal and Smith (2013), Ohtaki

(2013), and related references therein joined this stream and argued on circumstances in which the Friedman rule becomes suboptimal.

This article has three contributions to this line of the literature. First, this article is the first that introduces the MEU preference to the OLG model with spatial separation. Second, this article considers general utility index functions, which satisfies the standard assumptions such as monotonicity, concavity, and so on, whereas almost all of the previous studies restricted index functions to the class of preferences exhibiting constant relative risk aversion of which coefficient is less than or equal to one. Therefore, the analysis of this article is more general than those of the previous studies. Third, this article adds a new circumstance which shows suboptimality of the Friedman rule to the literature. To be more presice, we observe that the Friedman rule can be suboptimal *when* ambiguity on the likelihood on the liquidity event is sufficiently small. This observation also implies that spatial separation and limited communication, emphasized in the previous studies, are *not* sufficient condition for suboptimality of the Friedman rule. Our observation indicates that, in order to imply its suboptimality, small ambiguity is also required.

As mentioned in the Introduction, the MEU preference is first axiomatized by Schmeidler (1982, 1989) and Gilboa and Schmeidler (1989). To be more precise, however, their axiomatizations of the MEU preference were demonstrated in the framework of Anscombe and Aumann (1963), not in that of Savage (1954). By relaxing Savage's postulates, Casadesus-Masanell, Klibanoff, and Ozdenoren (2000) and Alon and Schmeidler (2014) axiomatized the MEU preference in Savage's framework.[4] We should mention that there exists an ever-growing literature on applications of decision making under ambiguity (including the MEU preference, of course) to economics and finance. Especially, decision making under ambiguity is already applied to a wide range of intertemporal macroeconomic models: asset pricing as in Epstein and Wang (1994, 1995), search theory as in Nishimura and Ozaki (2004), real option as in Nishimura and Ozaki (2007), learning as in Epstein and Schneider (2007), and growth model as in Fukuda (2008) are such examples. Good literature surveys have been already provided by Epstein and Schneider (2010), Gilboa and Marinacci (2013), and Guidolin and Rinaldi (2013). However, there seems few studies applying ambiguity to models for examining optimal monetary policy. This article adds a new contribution to this line of the literature.

Finally, our results also contributes to studies on the monetary OLG model with ambiguity.

[4]Etner, Jeleva, and Tallon (2012) surveyed an ever-growing literature regarding decision theory under ambiguity after the papers by Schmeidler (1989) and Gilboa and Schmeidler (1989). They also provided a brief survey on applications on ambiguity to economics.

We should compare our model with that of Ohtaki and Ozaki (2015). Their work is the first to introduce ambiguity to a pure-endowment stochastic OLG model, wherein uncertainty enters the economy through the initial endowments, and demonstrated indeterminacy and optimality of stationary monetary equilibria.[5] In our model, on the other hand, uncertainty enters the economy through the realization of liquidity events and it is shown that a monetary steady state exists uniquely, i.e., there is no indeterminacy of equilibria. One of reasons of this difference might be a lack of aggregate uncertainty. As argued by Ohtaki and Ozaki, in order to generate equilibrium indeterminacy, an OLG environment with ambiguity might require aggregate uncertainty, which is not introduced to our model. Moreover, sub/optimality of the Friedman rule is argued in this article, whereas it is not in Ohtaki and Ozaki (2015). As a summary, our model and that of Ohtaki and Ozaki (2015) started from distinct motivations, studied distinct models, and reached distinct conclusions, although those models have a very little intersection.

3 Ingredients of the Model

This article considers a variant of the overlapping generations model with spatial separation developed by Champ, Smith, and Williamson (1996) and Smith (2002), wherein there exist a finite number of agents, endowed with the maxmin expected utility (MEU) preferences, in each generation.[6]

Time is indexed by t and runs discretely from minus infinity to plus infinity. As distinct locations, two islands exist and there exits no communication among them. In each period, there exist a single perishable commodity, called the consumption good, and one new generation, consisting of finite ex-ante identical agents, appears on each island and lives for two periods. We denote by H the number of agents per generation and let $\mathcal{H} := \{1, \ldots, H\}$. Agents are said to be *young* in the first period of their lives and *old* in the second period.

In this economy, there also exists a simple production technology which yields $F(\ell_t, k_t) = \omega\ell_t + \rho k_t$ units of the consumption good in period t when the labor and the capital stock in that period are ℓ_t and k_t, where $\omega > 0$ and $\rho > 0$ are marginal productivities of labor and capital, respectively, and each of them is a known constant. We denote by f the per-capita production function, i.e., $f(k) := F(1, k)$ for each $k \geq 0$. It is assumed that there exists no depreciation of capital stock, i.e., the depreciation rate of capital is assumed to be zero. We also impose a

[5] Ohtaki and Ozaki (2014) also considered a similar environment and characterizes optimality of stationary allocations.

[6] To be more precise, our model is close to that of Haslag and Martin (2007) except for preferences.

technical assumption that the domains of $F(\ell, \bullet)$ and f are equal to $[0, \omega]$.[7]

The stochastic environment is introduced as follows. In each island, J young agents are randomly chosen at the end of each period and move to the other island before the next period begins, where J is a positive integer being less than H. The set of possible *states* is then denoted by $\mathcal{S} := \{s|\ s \subset \mathcal{H} \text{ and } |s| = J\}$.[8] For each state $s \in \mathcal{S}$, we call an agent $h \in \mathcal{H}$ a *mover* if $h \in s$ and otherwise a *nonmover*. We denote by $\hat{\lambda}_m$ and $\hat{\lambda}_n$ fractions of movers and nonmovers, i.e., $\hat{\lambda}_m := J/H$ and $\hat{\lambda}_n := (H-J)/H = 1-\hat{\lambda}_m$, respectively. Let $\hat{\lambda} := (\hat{\lambda}_m, \hat{\lambda}_n)$. For each $h \in \mathcal{H}$, we also denote by $\mathcal{S}_h$ and $\mathcal{S}_h^c$ the set of states in which agent h becomes a mover and its complement in $\mathcal{S}$, i.e., $\mathcal{S}_h := \{s \in \mathcal{S}|\ h \in s\}$ and $\mathcal{S}_h^c := \mathcal{S} \setminus \mathcal{S}_h$, respectively.[9] Note that uncertainty described as above is purely *idiosyncratic* in the sense that it does not affect aggregate variables such as the total endowment, the marginal productivity of capital and labor, or the numbers of movers and nonmovers. Let $\Delta_{\mathcal{S}}$ be the set of probability measures on the measurable space $(\mathcal{S}, 2^{\mathcal{S}})$.

We then introduce endowments and preference structures. Agents are endowed with one unit labor, which will be inelastically provided in the market, in the first period of their lives and none at the second date. Then, the agent $h \in \mathcal{H}$ of each generation born in period t aims to maximize their utility derived from the second-period consumptions. Since each agent $h \in \mathcal{H}$ is uncertain in their first period about realizations of states in the second period, his/her preference might be defined over the set of consumptions contingent upon the realizations of states in the second period, $c^h = (c^h(s))_{s\in\mathcal{S}}) \in \Re_+^{\mathcal{S}}$. It is then assumed that the agent $h \in \mathcal{H}$ of each generation assigns the *set* of probability measures on $(\mathcal{S}, 2^{\mathcal{S}})$, $\Lambda_h \subset \Delta_{\mathcal{S}} \cap \Re_{++}^{\mathcal{S}}$, to uncertainty and aims to maximize the maxmin expected utility

$$(\forall c^h \in \Re_+^{\mathcal{S}}) \quad U(c^h) := \min_{\lambda\in\Lambda_h} \sum_{s\in\mathcal{S}} u(c^h(s))\lambda(s),$$

where u is a real-valued function being strictly increasing, strictly concave, and continuously differentiable on the interiors of their domains, respectively, and Λ_h is also assumed to be convex and compact in $\Re^{\mathcal{S}}$.[10] Because u is strictly monotone increasing and strictly concave, U is also strictly monotone increasing and strictly concave.

Remark 1 The present model is very close to those in previous works such as Smith (2002), Haslag and Martin (2007), and Bhattacharya, Haslag, and Martin (2009). However, there

[7] Our production technology is often called a *storage technology*.

[8] For each set X, we denote by $|X|$ its cardinal number.

[9] Obviously, $|\mathcal{S}| := H!/(J!(H-J)!)$ and, for each $h \in \mathcal{H}$, $|\mathcal{S}_h| = (J/H)|\mathcal{S}|$ and $|\mathcal{S}_h^c| = (1 - J/H)|\mathcal{S}|$.

[10] The restriction on agents beliefs also assumes that agents believe that the realization of states is independent of time or past histories of states.

exist two differences between our model and theirs. First, the previous studies assumed that the lifetime utility function exhibits constant relative risk aversion (CRRA) with its coefficient being less than or equal to one. On the other hand, we consider a more general lifetime utility function, which is not restricted in the class of CRRA preferences, although it is assumed to satisfy the standard assumptions such as strict monotonicity, strict concavity, and so on. Second, the interpretations on the agents' beliefs in this article and in previous works are different. For example, Haslag and Martin implicitly required that the (unique) subjective probability measure λ should coincide with the true probability measure $\hat{\lambda}$. On the other hand, in the present model, subjective beliefs are not necessarily consistent with the true (or objective) probability measure $\hat{\lambda}$. In other words, the formation of the agent's belief which is represented by a set of priors, whether it is a singleton or not, is totally subjective and it could be totally irrelevant to the true or objective probability measures. This interpretation on the agents' beliefs might be considered as a kind of bounded rationality and is different from those in the existing literature.

In order to guarantee the ex-ante identicality of agents, we assume that

$$(\forall h, i \in \mathcal{H}) \quad \{(\lambda(\mathcal{S}_h), \lambda(\mathcal{S}_h^c)) |\ \lambda \in \Lambda_h\} = \{(\lambda(\mathcal{S}_i), \lambda(\mathcal{S}_i^c)) |\ \lambda \in \Lambda_i\} =: \Lambda,$$

i.e., agents have the same set of probabilities with respect to being movers and nonmovers. Obviously, $\Lambda \subset \Delta_2 \cap \Re^2_{++}$, where Δ_2 is the 1-dimensional unit simplex. We will denote by $\lambda = (\lambda_m, \lambda_n)$ a representative element of Λ. Moreover, let $\underline{\lambda} = (\underline{\lambda}_m, \underline{\lambda}_n)$ and $\overline{\lambda} = (\overline{\lambda}_m, \overline{\lambda}_n)$ be elements of Λ, defined by $\underline{\lambda}_m := \min\{\lambda_m |\ \lambda \in \Lambda\}$ and $\overline{\lambda}_m := \max\{\lambda_m |\ \lambda \in \Lambda\}$, respectively. We also assume that the true probability measure belongs to the set of beliefs, i.e., $\hat{\lambda} \in \Lambda$. Note that this assumption implies that $\underline{\lambda}_m \leq \hat{\lambda}_m \leq \overline{\lambda}_m$. The last assumption holds, for example, when we consider the ε-contamination.

Example 1 Suppose that there exists some $\varepsilon \in [0, 1]$ such that $\Lambda = \{(1-\varepsilon)\hat{\lambda} + \varepsilon\lambda |\ \lambda \in \Delta_2\} =: \Lambda^\varepsilon$, which is often called the *ε-contamination* of $\hat{\lambda}$.[11] This captures a situation that each agent is $(1-\varepsilon) \times 100\%$ certain that she becomes a mover [nonmover] with probability $\hat{\lambda}_m$ $[\lambda_n]$, but she has a fear that, with $\varepsilon \times 100\%$ chance, her conviction is completely wrong and she is left perfectly ignorant about the true probability in the present as well as in the future. Under the ε-contamination, it follows obviously that $\hat{\lambda} \in \Lambda$.

4 Definition of Equilibria

[11] The ε-contamination was first axiomatized by Nishimura and Ozaki (2006).

4.1 Introducing Firms, Banks, and Money

This section defines a monetary equilibrium, which is an equilibrium with circulating money. First, we introduce several economic institutes. In each island and each period t, a firm and a bank are established by the young agents in that period. The production technology of each firm is given by F defined in the previous section.

We also introduce a central bank which issues a durable and intrinsically useless object, called *money*. The per-capita money stock of money in period t is denoted by M_t, which is common to all islands. The stock of money follows the equation $M_t = (1 + \mu)M_{t-1}$ for each period t, where μ is the (constant) growth rate of the per-capita money stock and chosen by the central bank. Young agents in each period t receives the newly issued money, $Z_t := M_t - M_{t-1} = [\mu/(1+\mu)]M_t$, as lump-sum money tax/transfer when they are young.

4.2 Timing of Trades

Timing of trades is as follows. At the beginning of each period t, each young agent meets the firm (established in the previous period) in a local spot market of labor and earn their income. After that, young agents deposit some of their after-tax/transfer income with their bank. The bank then enters local spot markets of capital and money. It meets the firm (established in period t) in the capital market and the old agents in the money market, and chooses its portfolio. Before ending the period t, young agents learn their types (i.e., the liquidity event occurs) and movers will withdraw their deposits. In the following period $t + 1$, the firm and the bank are liquidated. We will consider that all of spot markets in each period t are competitive and denote by $w_t > 0$, $r_{t+1} > 0$, and $P_t > 0$ the wage rate, the real interest rate, and the nominal price of the consumption good in period t markets, respectively. We also denote by q_t the per-capita real money balance in period t, i.e., $q_t := M_t/P_t$. Also let π_{t+1} be the inflation rate, i.e., $\pi_{t+1} := P_{t+1}/P_t - 1$. Furthermore, define the nominal interest rate by $i_{t+1} := (1 + r_{t+1})(1 + \pi_{t+1}) - 1$ for each t.

4.3 Behavior of Firms

Agents' economic activities are now considered. First, the behavior of each firm is considered. The firm established in period t enters the period-$(t + 1)$ labor market and the period-t capital market, wherein the wage and the real interest rates are given by w_{t+1} and r_{t+1}, respectively.

The firm then chooses a labor, L_{t+1}, and a capital stock, k_{t+1}, in order to maximize its profit, $F(L_{t+1}, k_{t+1}) - w_{t+1}L_{t+1} - r_{t+1}k_{t+1}$, given w_{t+1} and r_{t+1}. Under the current setting of the production technology, r_{t+1} and w_{t+1} are determined according to $r_{t+1} = \rho$ and $w_{t+1} = \omega$. We describe other economic agents' behaviors given these rates.

4.4 Contracts between Agents and their Banks

Next, the behavior of the bank established in period t is considered. Throughout this paper, it is assumed that each bank distinguishes its depositors by whether they withdraw their deposits, not by which indexes $h \in \mathcal{H}$ are given to them. The bank then proposes a "contract" to its depositors. Here, a *contract* is a tripret $(d_t, c_t, (k_{t+1}, m_t))$ of agents' deposits, d_t, consumption plans, $c_t = (c^m_{t+1}, c^n_{t+1})$, offered to agents, and the portfolio plans, (k_{t+1}, m_t), where the second-period consumptions (c^m_{t+1}, c^n_{t+1}) are contingent upon whether depositors withdraw their deposits (c^m_{t+1}) or not (c^n_{t+1}) and k_{t+1} and m_t are investments of the bank in the storage technology and money, respectively. In order to consider an "optimal" contract, we carefully describe the objective function and constraints of banks as follows.

Objective of Banks. Let $c_t = (c^m_{t+1}, c^n_{t+1}) \in \Re^2_+$ be a consumption plan offered to agents born in period t, where c^y_t is young agents' consumption and c^m_{t+1} and c^n_{t+1} are consumptions offered to agents who withdraw their deposits and agents who not.[12] Then, the bank established in period t is assumed to behave as a welfare-maximizer and adopt

$$\mathcal{U}(c_t) := \sum_{h \in \mathcal{H}} \gamma^h U(c^m_{t+1}\binom{h}{\bullet}c^n_{t+1})$$

as its objective function, where $\gamma : \mathcal{H} \to \Re_{++}$ is a Pareto wight with $\bar{\gamma} := \sum_{h \in \mathcal{H}} \gamma^h < \infty$ and $c^m_{t+1}\binom{h}{\bullet}c^n_{t+1} : \mathcal{S} \to \Re_+$ is defined by

$$c^m_{t+1}\binom{h}{s}c^n_{t+1} := \begin{cases} c^m_{t+1} & \text{if} \quad h \in s, \\ c^n_{t+1} & \text{if} \quad h \notin s \end{cases}$$

for each $h \in \mathcal{H}$, each $s \in \mathcal{S}$, and each $(c^m_{t+1}, c^n_{t+1}) \in \Re^2_+$. Then, the objective function of each bank established at t can be rewritten in a more simple form:

$$\mathcal{U}(c_t) \quad = \quad \sum_{h \in \mathcal{H}} \gamma^h \left(\min_{\lambda \in \Lambda_h} \sum_{s \in \mathcal{S}} u(c^m_{t+1}\binom{h}{s}c^n_{t+1})\lambda(s) \right)$$

[12] Similar to the standard argument in the existing literature, one can consider gross rates of returns of deposits instead of second-period consumptions. Let δ^m_{t+1} and δ^n_{t+1} be the returns of deposits offered to movers and nonmovers born at date $t \geq 1$. Then, it should satisfy that $c^s_{t+1} = (\omega + \tau_t)\delta^s_t$ for $s = m, n$.

$$
\begin{aligned}
&= \sum_{h\in\mathcal{H}} \gamma^h \left(\min_{\lambda\in\Lambda_h} [u(c^m_{t+1})\lambda(\mathcal{S}_h) + u(c^n_{t+1})\lambda(\mathcal{S}^c_h)] \right) \\
&= \bar{\gamma} \left(\min_{\lambda\in\Lambda} [u(c^m_{t+1})\lambda_m + u(c^n_{t+1})\lambda_n] \right).
\end{aligned}
$$

Because $\bar{\gamma}$ is free from the choice of $c_t = (c^m_{t+1}, c^n_{t+1})$, the decision of the bank is not affected by the Pareto weight, γ. Therefore, we assume without loss of generality that $\bar{\gamma} = 1$. Also note that $\mathcal{U}$ is strictly monotone increasing and strictly concave because u satisfies such properties.

Example 2 Recall the ε-contamination presented in Example 1. Under such ambiguity, the objective function of the bank can be rewritten as

$$
\begin{aligned}
\mathcal{U}(c_t) &= \min_{\lambda\in\Lambda^\varepsilon} [u(c^m_{t+1})\lambda_m + u(c^n_{t+1})\lambda_n] \\
&= (1-\varepsilon)[u(c^m_{t+1})\hat{\lambda}_m + u(c^n_{t+1})\hat{\lambda}_n] + \varepsilon \min_{s=m,n} u(c^s_{t+1}).
\end{aligned}
$$

The ε-contamination of confidence is first axiomatized by Nishimura and Ozaki (2006) and often introduced as an example of the decision making under ambiguity (Epstein and Wang, 1994, for example).

We also introduce a new notation. For each $c_t = (c^m_{t+1}, c^n_{t+1}) \in \Re^2_+$, let

$$
\check{\Lambda}(c_t) := \arg\min_{\lambda\in\Lambda} [u(c^m_{t+1})\lambda_m + u(c^n_{t+1})\lambda_n],
$$

which is the set of beliefs minimizing the lifetime expected utility given the contingent consumption plan c_{t+1}. It is immediate to show that $\check{\Lambda}(c_t)$ is convex because of convexity of Λ. Furthermore, $\check{\Lambda}(c_t) = \Lambda$ if $c^m_{t+1} = c^n_{t+1}$.

Constraints of Banks. Recall that a contract is described by $(d_t, c_t, (k_{t+1}, m_t))$, which is the triplet of agents' deposits, d_t, consumption plans, $c_t = (c^y_t, (c^m_{t+1}, c^n_{t+1}))$, offered to agents, and the portfolio plans, (k_{t+1}, m_t). The bank established in period t faces four constraints. The first constraint is a restriction on deposits. Because each agent's after-tax/transfer income is $\omega + \tau_t$, young agents' consumptions, c^y_t, and a deposit, d_t, must satisfy that

$$
d_t \leq \omega + \tau_t, \tag{1}
$$

where $\tau_t := Z_t/P_t = [\mu/(1+\mu)]q_t$.

The second constraint is a restriction on the portfolio of the bank. In period-t local spot markets of capital and money, each bank chooses its portfolio, i.e., the amounts of investment in capital and money. Its per-capita balance sheet constraint is given by

$$k_{t+1} + \frac{m_t}{P_t} \leq d_t, \tag{2}$$

where $k_{t+1} \in \Re_+$ and $m_t/P_t \in \Re_+$ are per-capita real amounts of investments in capital and money, respectively.

The third constraint is a restriction on the second-period consumptions. Each bank must plan agents' consumptions. The returns of investment of the bank must meet the total second-period consumption. This is captured by the per-capita budget constraint that

$$c^m_{t+1}\hat{\lambda}_m + c^n_{t+1}\hat{\lambda}_n \leq (1+\rho)k_{t+1} + \frac{m_t}{P_{t+1}}. \tag{3}$$

This inequality can be rewritten as

$$(1+\rho)k_{t+1} + \frac{m_t}{P_{t+1}} - \left[c^m_{t+1}\hat{\lambda}_m + c^n_{t+1}\hat{\lambda}_n\right] \geq 0,$$

which can be interpreted as the *individual rationality* (or *participation*) *constraint* for the bank.

The last restriction is the *liquidity constraint.* Because movers loose their connection to their banks in the second period and cannot receive the proceeds of capital investment, they withdraw their money after the state realizes. At the end of the first period, therefore, the bank must have sufficient liquidity in order to meet the needs of movers:

$$c^m_{t+1}J \leq \frac{m_t}{P_{t+1}}H,$$

or equivalently,

$$c^m_{t+1}\hat{\lambda}_m \leq \frac{m_t}{P_{t+1}}. \tag{4}$$

In other words, the total consumption of movers is less than or equal to the return of money.

Optimal Contract. A contract $(d_t, c_t, (k_{t+1}, m_t))$ is now said to be *optimal* (given P_s for $s = t, t+1$) if it maximizes $\mathcal{U}(c_{t+1})$ subject to Eqs.(1)–(4). One can immediately observe that, at an optimal contract $(d_t, c_t, (k_{t+1}, m_t))$, Eqs.(1), (2), and (3) hold with equality, because of the strict monotonicity of $\mathcal{U}$.

4.5 Monetary Equilibrium

A *monetary equilibrium given money growth rate* μ is now defined by a sequence $\{q_t, c_t, k_{t+1}\}_{t=-\infty}^{\infty}$ of per-capita real money balances $q_t \in]0, (1+\mu)\omega]$, contingent consumption plans $c_t = (c^m_{t+1}, c^n_{t+1})$, and per-capita capital investments $k_{t+1} \in [0, \omega]$ such that there exist sequences $\{d_t\}_{t=-\infty}^{\infty}$ and $\{m_t\}_{t=-\infty}^{\infty}$, of deposits and money holdings satisfying that: in each period t,

M1 (Optimality of Contract):

$(d_t, c_t, (k_{t+1}, m_t))$ is optimal, i.e., it maximizes $\mathcal{U}(c_t)$ subject to Eqs.(1)–(4) given $P_s = M_s/q_s$ for $s = t, t+1$; and

M2 (Market Clearing Conditions):

it holds that $m_t = M_t$.

It is said to be *fully-insured* (with respect to the second-period consumptions) if its contingent consumption plans $\{c_t\}_{t=-\infty}^{\infty} = \{c_{t+1}^m, c_{t+1}^n\}_{t=-\infty}^{\infty}$ satisfy that $c_{t+1}^m = c_{t+1}^n$ in each period t. Moreover, it is called a *monetary steady state* (*MSS* for short) *given* μ if there exists some $(q, c, k) \in \Re_{++} \times (\Re_+ \times \Re_+^2) \times \Re_+$ satisfying that $(q_t, c_t, k_{t+1}) = (q, c, k)$ in each period t.

Remark 2 In any monetary equilibrium, the per-capita good market clearing condition holds in addition to the money market clearing conditions, $m_t = M_t$ for each t. In order to observe this fact, note that constraints (1), (2), and (3) hold with equality at the equilibrium because of strict monotonicity of $\mathcal{U}$. Then, it follows that, in each period t,

$$\begin{aligned}
& c_t^m \hat{\lambda}_m + c_t^n \hat{\lambda}_n + k_{t+1} - k_t \\
= \; & k_{t+1} + [c_t^m \hat{\lambda}_m + c_t^n \hat{\lambda}_n] - k_t \\
= \; & [\omega + \tau_t - m_t/P_t] + [(1+\rho)k_t + m_{t-1}/P_t] - k_t \\
= \; & [\omega + (M_t - M_{t-1})/P_t - M_t/P_t] + [(1+\rho)k_t + M_{t-1}/P_t] - k_t \\
= \; & \omega + \rho k_t \\
= \; & f(k_t),
\end{aligned}$$

which is the per-capita good market clearing condition in period t.

5 Nonnegativity of Nominal Interest Rates

Here, we examine the possible range of nominal interest rates at each monetary equilibrium (if any) and argue on the role of the liquidity constraint, i.e., Eq.(4). Recall first that the nominal inflation rate, i_{t+1}, is defined by $1 + i_{t+1} = (1 + r_{t+1})(1 + \pi_{t+1})$ for each t. We therefore obtain that, at each monetary equilibrium, $i_{t+1} > -1$ because $1 + r_{t+1} > 0$ and $1 + \pi_{t+1} = P_{t+1}/P_t > 0$.

Furthermore, we can reduce the possible range of nominal interest rates. It holds that $i_{t+1} \geq 0$ in each period t. In fact, it follows from Eqs.(1)–(3) that, in each period t,

$$\frac{c_{t+1}^m \hat{\lambda}_m + c_{t+1}^n \hat{\lambda}_n}{1 + r_{t+1}} \leq \omega + \tau_t + \left(\frac{1}{1+\rho} \frac{P_t}{P_{t+1}} - 1 \right) \frac{m_t}{P_t} = \omega + \tau_t - \frac{i_{t+1}}{1 + i_{t+1}} \frac{m_t}{P_t}, \tag{5}$$

which is the lifetime budget constraint for the bank established in period t. In the last term of the previous inequality, $[i_{t+1}/(1+i_{t+1})]m_t/P_t$ represents the cost of real money holdings. Therefore, when $i_{t+1} = 0$ (i.e., the *Friedman rule*), the cost of money holdings becomes zero. Suppose here that negative interest rates are allowed, i.e., $-1 < i_{t+1} < 0$ in some period t. This implies that $P_t/P_{t+1} > 1 + r_{t+1}$, i.e., the rate of return of real money holdings exceeds that of interbank lending. Then, the bank established in period t chooses $+\infty$ as m_t,[13] which contradicts the fact that $m_t = M_t$ at any monetary equilibrium. Therefore, at each monetary equilibrium, it holds that $i_{t+1} \geq 0$ for each t. This argument from the view point of the no-arbitrage condition establishes the following proposition:

Proposition 1 *At each monetary equilibrium, nominal interest rates are nonnegative.*

We provide several remarks on nonnegativity of nominal interest rates. The following remark argues on the role of the liquidity constraint.

Remark 3 When the nominal interest rates are positive, i.e., $i_{t+1} > 0$, banks wish to make the amount of money holding as small as possible. In such a case, the liquidity constraint (4) plays an important role because it prevents banks from setting the amount of money holding to 0, and guarantees the existence of monetary equilibrium as shown in later. In fact, the liquidity constraint holds with equality because the movers' consumption at a monetary equilibrium is positive due to the boundary condition imposed on u.

The following remark explains a treatment of monetary equilibrium with "zero" nominal interest rate.

Remark 4 When the nominal interest rates are zero, i.e., $(1 + \rho)(1 + \pi_{t+1}) - 1 = i_{t+1} = 0$, investments in the storage technology and money are completely substitutable. This is because the gross rates of return of the storage technology and money become equal to each other. In such a case, we may find indeterminacy of equilibria even in the class of monetary steady states. However, we will show later that the sets of monetary steady states converge to the identical set as $i_{t+1} \downarrow 0$. For this reason, we identify the set of monetary steady states such that $i_{t+1} = 0$, if any, with limiting cases as $i_{t+1} \downarrow 0$.

[13]To be more precise, the bank borrows more and gets more money, and enjoys higher return in its second period.

We then remark on dependency of equilibrium consumption plans and storage investment on equilibrium real money balances.

Remark 5 Consider a monetary equilibrium with positive nominal interest rates, i.e., $i_{t+1} > 0$ for each t. After an equilibrium real money balances $\{q_t\}_{t=-\infty}^{\infty}$ has been chosen, the corresponding equilibrium consumption plan $\{c_t\}_{t=-\infty}^{\infty} = \{c_{t+1}^m, c_{t+1}^n\}_{t=-\infty}^{\infty}$ and the corresponding amount of the storage investment are automatically and uniquely determined from the budget constraints (1)–(4). Actually, those constraints hold with equality at a monetary equilibrium with $i_{t+1} > 0$ and $\{c_t\}_{t=-\infty}^{\infty}$ and $\{k_{t+1}\}_{t=-\infty}^{\infty}$ is calculated as, for each t,

$$c_{t+1}^m = \frac{1}{\hat{\lambda}_m} \frac{q_{t+1}}{1+\mu} \frac{m_t}{M_t}, \tag{6}$$

$$c_{t+1}^n = \frac{1+\rho}{\hat{\lambda}_n} \left(\omega + \frac{\mu}{1+\mu} q_t - q_t \frac{m_t}{M_t} \right), \tag{7}$$

$$k_{t+1} = \omega + \frac{\mu}{1+\mu} q_t - q_t \frac{m_t}{M_t} \tag{8}$$

where m_t/M_t in these equations must be equal to one because of the money market clearing condition $m_t = M_t$. For this reason, we sometimes denote by $\{q_t\}_{t=-\infty}^{\infty}$ a monetary equilibrium instead of $\{q_t, c_t, k_{t+1}\}_{t=-\infty}^{\infty}$ (omit a consumption plan c_t and a storage investment k_{t+1} in a list).

Finally, we remark on the possible range of "money growth rates" in the class of monetary steady states.

Remark 6 In the class of monetary steady states, nonnegativity of nominal interest rates restricts the possible range of money growth rates μ. In fact, it is equivalent to the condition that $\mu \geq -\rho/(1+\rho)$. This is because $0 \leq i_{t+1} = (1+\rho)(1+\mu)q_t/q_{t+1} - 1$ and $q_t = q_{t+1}$ at any monetary steady state. We will simply denote by i the nominal interest rate at a monetary steady state, i.e., $i = (1+\rho)(1+\mu) - 1$. This also implies the presence of the one-to-one relation between nominal interest rates and monetary growth rates in monetary steady states. Therefore, we sometimes say a monetary steady state given nominal interest rate "i" instead of monetary growth rate "μ."

In addition to the assumption that $r_{t+1} = \rho$ in each period t, we consider throughout the rest of this paper the case that $i_{t+1} > 0$ in each period t except for the limiting case that $i_{t+1} \downarrow 0$ (as mentioned in Remark 4).

6 Characterizations

In order to argue existence of monetary equilibrium and its welfare property, it is reasonable to identify the definition of monetary equilibrium with other tractable conditions. So, this section tries to provide a characterization of monetary equilibrium.

As mentioned in Remark 5 in the previous section, equilibrium consumption plans $\{c_t\}_{t=-\infty}^{\infty}$ and equilibrium storage investment $\{k_{t+1}\}_{t=-\infty}^{\infty}$ are automatically and uniquely determined by real money balances $\{q_t\}_{t=-\infty}^{\infty}$ through Eqs.(6)–(8) with $m_t = M_t$. By using this convenient property, the optimization problem of each bank (see ME1 in the definition of monetary equilibrium) can be simplified to the one of choosing optimal investments in the storage technology and money. For this reason, we define the objective function V^t of storage investment k_{t+1} and money holding m_t for the bank established in period t facing real money balances q_t and q_{t+1}. Formally, let

$$\begin{aligned} V_t(m_t) &:= \mathfrak{U}\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}, \frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right) \\ &= \min_{\lambda\in\Lambda}\left[u\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\lambda_m + u\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\lambda_n\right]. \end{aligned}$$

Note that, for each t, V_t is concave because of strict concavity of u.

6.1 Non-differentiability of V^t

In the standard optimization program with a concave and differentiable objective function, its solutions are characterized by conditions on derivatives of the objective function. We should note here that the objective function V_t of the bank established in period t might not be differentiable at some points due to the MEU preference. In order to verify this observation, define

$$\begin{aligned} V_t^-(m_t) &= \lim_{h\uparrow 0, h\neq 0}\frac{V_t(m_t+h)-V_t(m_t)}{h}, \\ V_t^+(m_t) &= \lim_{h\downarrow 0, h\neq 0}\frac{V_t(m_t+h)-V_t(m_t)}{h}. \end{aligned}$$

Here, V_t^- and V_t^+ are the left- and right-sided derivatives of V_t, respectively. The following proposition describes these one-sided derivatives.

Proposition 2 *For each t, each $q_\tau > 0$ with $\tau = t, t+1$, and each $m_t > 0$,*

$$\begin{aligned} V_t^-(m_t) &= \max_{\lambda\in\check{\Lambda}(c_t)}\frac{1}{M_t}\left[\frac{q_{t+1}}{1+\mu}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\frac{\lambda_m}{\hat{\lambda}_m} - (1+\rho)q_t u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega+\frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\frac{\lambda_n}{\hat{\lambda}_n}\right], \\ V_t^+(m_t) &= \min_{\lambda\in\check{\Lambda}(c_t)}\frac{1}{M_t}\left[\frac{q_{t+1}}{1+\mu}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\frac{\lambda_m}{\hat{\lambda}_m} - (1+\rho)q_t u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega+\frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\frac{\lambda_n}{\hat{\lambda}_n}\right], \end{aligned}$$

where $c_t = (c_t^y, (c_{t+1}^m, c_{t+1}^n))$ *is determined by* m_t *through Eqs.(6)–(7) given* q_s *for* $s = t, t+1$.

When $V_t^-(m_t) = V_t^+(m_t)$ for every m_t, V_t is differentiable. This is true, for example, when the set of beliefs, Λ, is a singleton. There also exists other situation in which V_t is differentiable. Suppose that m_t satisfies that

$$c_{t+1}^n = \frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right] \neq \frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t} = c_{t+1}^m.$$

Then, it holds that $V_t^-(m_t) = V_t^+(m_t)$ because $\check{\Lambda}(c_t)$ degenerates into a singleton. Actually, one can easily verify that $\check{\Lambda}(c_t) = \{\underline{\lambda}\}$ if $c_{t+1}^m > c_{t+1}^n$ and $\check{\Lambda}(c_t) = \{\overline{\lambda}\}$ if $c_{t+1}^m < c_{t+1}^n$. At such a m_t, therefore, V_t is differentiable. Given Proposition 2, however, we can argue on nondifferentiability of V_t. On the other hand, consider pairs of k_{t+1} and m_t, which achieve consumption plans being fully-insured with respect to the second-period consumptions, i.e., $c_{t+1}^m = c_{t+1}^n$. Given Eqs.(7) and (8), this implies that

$$\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right] = \frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}. \tag{9}$$

Obviously, there is a continuum of m_t satisfying this equation. Given such m_t, it holds that $\check{\Lambda}(c_t) = \Lambda$. Therefore, it follows immediately from Proposition 2 that

$$\begin{aligned}
& V_t^-(m_t) \\
= \;& \frac{q_{t+1}}{1+\mu}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\frac{\overline{\lambda}_m}{\hat{\lambda}_m} - (1+\rho)q_t u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\frac{\overline{\lambda}_n}{\hat{\lambda}_n} \\
> \;& \frac{q_{t+1}}{1+\mu}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\frac{\underline{\lambda}_m}{\hat{\lambda}_m} - (1+\rho)q_t u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\frac{\underline{\lambda}_n}{\hat{\lambda}_n} \\
= \;& V_t^+(m_t),
\end{aligned}$$

provided that $\underline{\lambda}_m < \overline{\lambda}_m$ (which is equivalent to the condition that $\underline{\lambda}_n > \overline{\lambda}_n$). This shows that V_t is *not* differentiable at any m_t with $c_{t+1}^m = c_{t+1}^n$. Our nondiferentiability comes obviously from ambiguity (i.e., multiplicity of beliefs) and is often observed in the existing literature on applications of ambiguity to economics and finance.[14]

6.2 Equilibrium System of Difference Inclusions

Although the objective function V_t of the bank established in period t might not be differentiable as discussed in the previous subsection, we can characterize solutions for the optimization

[14]See, for example, Dow and Werlang (1992), Epstein and Wang (1994, 1995), Chateauneuf, Dana, and Tallon (2000), Dana (2004), Fukuda (2008), Mandler (2013), Ohtaki and Ozaki (2015), and so on.

problem by one-sided derivatives of V_t because of its concavity. To be more precise, an interior solution m_t of the problem of the bank established period t is completely characterized by

$$0 \in \partial V_t(m_t),$$

or equivalently,

$$V_t^-(m_t) \geq 0 \geq V_t^+(m_t), \tag{10}$$

where $\partial V_t(m_t)$ is the superdifferential of V_t at m_t.[15] Combining the above argument with the money market clearing condition, $m_t = M_t$ for each $t \geq 1$, one can obtain the following proposition for characterization of monetary equilibrium:

Proposition 3 *A monetary equilibrium given μ, $\{q_t, c_t, k_{t+1}\}_{t=-\infty}^{\infty}$, is characterized by*

$$\min_{\lambda \in \check{\Lambda}(c_t)} \frac{u'(c_{t+1}^m)\lambda_m}{u'(c_{t+1}^n)\lambda_n} \leq (1+\rho)(1+\mu)\frac{q_t}{q_{t+1}}\frac{\hat{\lambda}_m}{\hat{\lambda}_n} \leq \max_{\lambda \in \check{\Lambda}(c_t)} \frac{u'(c_{t+1}^m)\lambda_m}{u'(c_{t+1}^n)\lambda_n}, \tag{11}$$

where $c_t = (c_{t+1}^m, c_{t+1}^n)$ and k_{t+1} are determined by $m_t = M_t$ through Eqs.(6)–(8) given q_s for $s = t, t+1$.

This proposition can be interpreted as an analogue of a standard risk sharing condition at a monetary equilibrium. In order to verify this observation, combine the lifetime budget constraint (5) with the liquidity constraint (4). We can then obtain that

$$\begin{aligned} & \frac{c_{t+1}^m \hat{\lambda}_m + c_{t+1}^n \hat{\lambda}_n}{1+\rho} \\ \leq \quad & \omega + \tau_t - \frac{i_{t+1}}{1+i_{t+1}} \frac{m_t}{P_t} \\ \leq \quad & \omega + \tau_t - \frac{i_{t+1}}{1+i_{t+1}} (1+\pi_{t+1}) c_t^m \hat{\lambda}_m, \end{aligned}$$

which is equivalent to

$$\frac{(1+i_{t+1})c_{t+1}^m \hat{\lambda}_m + c_{t+1}^n \hat{\lambda}_n}{1+\rho} \leq \omega + \tau_t. \tag{12}$$

Therefore, equilibrium relative prices between movers' and nonmovers' consumptions are $(1 + i_{t+1})\hat{\lambda}_m/\hat{\lambda}_n$. On the other hand, the marginal rate of substitution of movers' consumption for nonmovers' is calculated as

$$\begin{aligned} MRS_{mn}(c_t) \quad &:= \quad \frac{\mathcal{U}_1(c_t)}{\mathcal{U}_2(c_t)} \\ &= \quad \begin{cases} \dfrac{u'(c_{t+1}^m)\underline{\lambda}_m}{u'(c_{t+1}^n)\underline{\lambda}_n} & \text{if} \quad c_{t+1}^m > c_{t+1}^n, \\ \dfrac{u'(c_{t+1}^m)\overline{\lambda}_m}{u'(c_{t+1}^n)\overline{\lambda}_n} & \text{if} \quad c_{t+1}^m < c_{t+1}^n \end{cases} \end{aligned}$$

[15]See Definition A and Theorems B and C in the Appendix for the definition and calculations of superdifferentials.

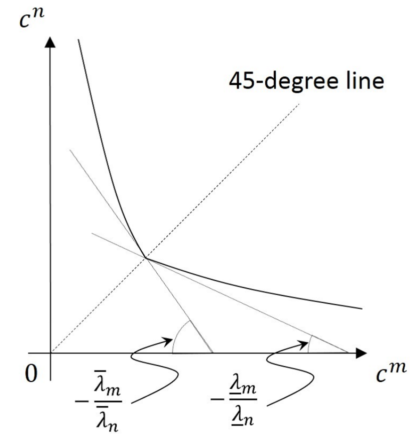


Figure 1: Indifference Curve on the c^m-c^n plain

if $c^m_{t+1} \neq c^n_{t+1}$ and might not be calculated at c_t with $c^m_{t+1} = c^n_{t+1}$. However, because $\mathcal{U}$ is concave, the indifference surfaces are convex. So, indifference "curves" on the c^m-c^n plain can be depicted as in Figure 1. As shown in this figure, at the 45-degree line, each indifference curve has a kink and the marginal rate of substitution between movers' and nonmovers' consumptions is not determined *uniquely* but it is restricted in the certain range. Let

$$\mathcal{M}(c_t) = \left\{ \frac{u'(c^m_{t+1})\lambda_m}{u'(c^n_{t+1})\lambda_n} \middle| \lambda \in \check{\Lambda}(c_t), \right\}$$

which is the set of possible marginal rates of substitutions. Then, the inequality (12) can be rewritten as

$$(1 + i_{t+1})\frac{\hat{\lambda}_m}{\hat{\lambda}_n} \in \mathcal{M}(c_t) \tag{13}$$

which can be interpreted as a natural extension of a standard risk sharing condition.

We also provide a characterization of monetary equilibrium *without* ambiguity.

Corollary 1 *When the set of belief degenerates into the true probability, i.e.,* $\Lambda = \{\hat{\lambda}\}$*, a monetary equilibrium given* μ*,* $\{q_t, c_t, k_{t+1}\}_{t=-\infty}^{\infty}$*, is characterized by*

$$\frac{u'(c^m_{t+1})}{u'(c^n_{t+1})} = 1 + i_{t+1}$$

where $c_t = (c^y_t, (c^m_{t+1}, c^n_{t+1}))$ *and* k_{t+1} *are determined by* $m_t = M_t$ *through Eqs.(6)–(8) given* q_s *for* $s = t, t+1$.

Therefore, when agents' belief λ is equal to the true probability $\hat{\lambda}$, the marginal rate of substitution between contingent consumptions, $u'(c^m_{t+1})/u'(c^n_{t+1})$, must be equal to the gross nominal interest rate, $1 + i_{t+1}$, at a monetary equilibrium.

As shown in the last corollary, when there exists no ambiguity, i.e., the lifetime utility function is differentiable, a monetary equilibrium can be represented by a difference equation. Turning to Proposition 3, we can say that a special feature of the presence of ambiguity aversion is that monetary equilibrium can be characterized by a *difference inclusion*, not an equation.

Remark 7 Our characterizations of monetary equilibrium is different from those in the previous studies such as Champ, Smith, and Williamson (1996) and Smith (2002) in the two points. First, in the previous studies, monetary equilibrium is characterized by conditions on the reserve-deposit ratios, $\gamma_t := (m_t/P_t)/(\omega + \tau_t)$. On the other hand, our characterization is given by the conditions on the marginal rates of substitutions. Although our characterization is new in the above sense, these two types of characterizations of monetary equilibrium are essentially same with each other. The second and critical difference between the previous studies and ours is monetary equilibrium is characterized by a difference inclusion, not an equation. This difference is obviously due to nonsmooth ambiguity aversion.

7 Structure of Monetary Steady States

Because we have obtained a characterization of monetary equilibrium, we should now argue its existence. In order to guarantee the existence of (especially) monetary steady states, we assume throughout this section that $0 < i = (1+\rho)(1+\mu) - 1$, i.e., the central bank chooses $\mu > -\rho/(1+\rho)$ as a money growth rate constant over periods. Also, we consider that $i = 0$ is a limiting case that $i \downarrow 0$. We sometimes say a monetary steady state given the nominal interest rate "i" instead of the money growth rate "μ" because i and μ have a one-to-one relation at any monetary steady state as mentioned in Remark 4. This section then explores a monetary steady state given i. It will be shown that the structure of monetary steady states depends on the structures of Λ (relative to $\hat{\lambda}$) and the nominal interest rate i.

7.1 Existence of MSS with $c^m < c^n$

First, we consider a necessary condition for existence of monetary steady state (q, c, k) given i with $c^m < c^n$. Suppose that there exists such a monetary steady state. By Proposition 3, such a monetary steady state is characterized by

$$\frac{u'(c^m)}{u'(c^n)} = (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}, \tag{14}$$

where $c = (c^m, c^n)$ and k_{t+1} are determined by $m_t = M_t$ through Eqs.(6)–(8) given $q_s = q$ for $s = t, t+1$. Because $c^m < c^n$, we can obtain that

$$1 < \frac{u'(c^m)}{u'(c^n)} = (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n},$$

which implies that

$$i > \frac{\overline{\lambda}_m}{\hat{\lambda}_m}\frac{\hat{\lambda}_n}{\overline{\lambda}_n} - 1 =: \iota.$$

Note that $\iota \geq 0$ (with equality if and only if $\overline{\lambda}_m = \hat{\lambda}_m$) because $\overline{\lambda}_m \geq \hat{\lambda}_m$. Therefore, a monetary steady state with $c^m < c^n$ exists only when $i > \iota$. Conversely, we can show the existence of monetary steady state with $c^m < c^n$ for each $i > \iota$.

In order to present our results, we introduce several notations. For each $i \geq 0$, define $q(i)$ as a solution of the equation:

$$u'\left(\frac{1}{\hat{\lambda}_m}\frac{1+\rho}{1+i}q\right) = (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega - \frac{1+\rho}{1+i}q\right]\right).$$

As shown in Lemma 1 in Section 10, $q(\bullet)$ is well-defined and continuous. Also define $c(i) = (c^m(i), c^n(i))$ and $k(i)$ as a unique solution of Eqs.(6)–(8) given $m_t = M_t$ and $q_s = q(i)$ for $s = t, t+1$. Note that

$$q(\iota) = \frac{(1+\iota)\hat{\lambda}_m}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n}\omega,$$

which is equal to $q_f(\iota)$ defined in the following subsection. This is because, when $i = \iota$, it holds that $u'(c^m(\iota)) = u'(c^n(\iota))$, which implies that

$$\frac{1}{\hat{\lambda}_m}\frac{1+\rho}{1+\iota}q(\iota) = c^m(\iota) = c^n(\iota) = \frac{1+\rho}{\hat{\lambda}_n}\left[\omega - \frac{1+\rho}{1+\iota}q(\iota)\right].$$

Then, we obtain the following proposition.

Proposition 4 *For each $i \geq 0$, a triplet $(q(i), c(i), k(i))$ is a unique monetary steady state satisfying that $c^m(i) < c^n(i)$ if and only if $i > \iota$.*

In our model, the bank faces the liquidity event, i.e., movers' withdrawal of deposits. So, the bank holds some of money in order to meet movers' needs. When $i > \iota$, i.e., the nominal interest rate is sufficiently large, the relative price of movers' consumption compared to nonmovers', $(1+i)\hat{\lambda}_m/\hat{\lambda}_n$, exceeds $\overline{\lambda}_m/\overline{\lambda}_n$, which is the highest marginal rate of substitution of movers' consumption for nonmovers' at fully-insured consumption plans. This implies that, on the lifetime budget line induced from Eq.(12), the bank can improve agents' lifetime utility by a marginal increase in nonmovers' consumption from the fully insured consumption plan. Therefore, movers' consumption is less than nonmovers' when $i > \iota$.

7.2 Existence of MSS with $c^m = c^n$

Next, we consider a necessary condition for existence of fully-insured monetary steady state (q, c, k) given i, one with $c^m = c^n$. Suppose that there exists such a monetary steady state. By Proposition 3, such a monetary steady state is characterized by

$$\min_{\lambda\in\Lambda}(1+i)\frac{\hat{\lambda}_m}{\lambda_m}\frac{\lambda_n}{\hat{\lambda}_n} \leq \frac{u'(c^m)}{u'(c^n)} \leq \max_{\lambda\in\Lambda}(1+i)\frac{\hat{\lambda}_m}{\lambda_m}\frac{\lambda_n}{\hat{\lambda}_n}, \tag{15}$$

where $c = (c^m, c^n)$ and k are determined by $m_t = M_t$ through Eqs.(6)–(8) given $q_s = q$ for $s = t, t+1$. Because $c^m = c^n$, it follows from Eq.(16) that

$$\underline{i} := \frac{\underline{\lambda}_m}{\hat{\lambda}_m}\frac{\hat{\lambda}_n}{\underline{\lambda}_n} - 1 \leq i \leq \frac{\overline{\lambda}_m}{\hat{\lambda}_m}\frac{\hat{\lambda}_n}{\overline{\lambda}_n} - 1 = \iota.$$

Note that $\underline{i} \leq 0$ (with equality if and only if $\underline{\lambda}_m = \hat{\lambda}_m$) because $\underline{\lambda}_m \leq \hat{\lambda}_m$. Therefore, a monetary steady state with $c^m = c^n$ exists only when $0 \leq i \leq \iota$. Furthermore, by the condition that $c^m = c^n$ and the budget constraints, we can easily calculate a candidate of a *fully-insured* monetary steady state as follows:

$$\begin{aligned}
q_f(i) &= \frac{(1+i)\hat{\lambda}_m}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n}\omega, \\
c_f^m(i) = c_f^n(i) &= \frac{1+\rho}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n}\omega =: \bar{c}_f, \quad \text{and} \\
k_f(i) = \omega - \frac{1+\rho}{1+i}q_f(i) &= \frac{\hat{\lambda}_n}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n}\omega =: \bar{k}_f.
\end{aligned}$$

Note that $q_f(i)$ is a unique solution of the system of (6), (7), and equations that $c_{t+1}^m = c_{t+1}^n$ and $q_t = q_{t+1}$. Also note that $c_f(i) = (c_f^m(i), c_f^n(i))$ and $k_f(i)$ are unique solutions of Eqs.(6)–(8) and $c^m = c^n$ given $q_s = q_f(i)$ for $s = t, t+1$.

We then obtain the following proposition.

Proposition 5 *For each $i \geq 0$, a triplet $(q_f(i), c_f(i), k_f(i))$ is a unique monetary steady state satisfying that $c_f^m(i) = c_f^n(i) = \bar{c}_f$ if and only if $i \leq \iota$.*

Note in this proposition, the equilibrium consumption plan is constant over $[0, \iota]$.

In our model, the bank faces the liquidity event, i.e., movers' withdrawal of deposits. So, the bank holds some of money in order to meet movers' needs. When $0 \leq i \leq \iota$, i.e., the nominal interest rate belongs to an appropriate interval, the relative price of movers' consumption compared to nonmovers', $(1+i)\hat{\lambda}_m/\hat{\lambda}_n$, belongs to $[\underline{\lambda}_m/\underline{\lambda}_n, \overline{\lambda}_m/\overline{\lambda}_n]$, which is the set of marginal rates of substitution of movers' consumption for nonmovers', admissible at fully-insured consumption plans. This implies that, on the lifetime budget line induced from Eq.(12), the bank cannot improve agents' lifetime utility by a marginal increase in movers' or nonmovers' consumptions from the fully insured consumption plan. Therefore, movers' consumption coincide with nonmovers' when $0 \leq i \leq \iota$.

7.3 Nonexistence of MSS with $c^m > c^n$

We finally consider the possibility of monetary steady state (q, c, k) given $i > 0$ with $c^m \geq c^n$. Suppose that there exists such a monetary steady state (q, c, k). By Proposition 3, the monetary steady state is characterized by

$$\frac{u'(c^m)}{u'(c^n)} = (1+i)\frac{\hat{\lambda}_m}{\underline{\lambda}_m}\frac{\underline{\lambda}_n}{\hat{\lambda}_n}, \tag{16}$$

where $c = (c^m, c^n)$ and k_{t+1} are determined by $m_t = M_t$ through Eqs.(6)–(8) given $q_s = q$ for $s = t, t+1$. Because $c^m \geq c^n$, we can obtain that

$$1 \geq \frac{u'(c^m)}{u'(c^n)} = (1+i)\frac{\hat{\lambda}_m}{\underline{\lambda}_m}\frac{\underline{\lambda}_n}{\hat{\lambda}_n},$$

which implies that

$$0 < i \leq \frac{\underline{\lambda}_m}{\hat{\lambda}_m}\frac{\hat{\lambda}_n}{\underline{\lambda}_n} - 1 \leq 0$$

because $\underline{\lambda}_m \leq \hat{\lambda}_m$ (and $\underline{\lambda}_n \geq \hat{\lambda}_n$). This is a contradiction. As a result, we obtain the following proposition.

Proposition 6 *For each $i \geq 0$, there exists no monetary steady state, wherein movers' consumption exceeds nonmovers'.*

7.4 Structure of MSSs

Summarizing Propositions 4–6, we can obtain the following theorem on the structure of MSSs.

Theorem. *For each $i \geq 0$, a unique monetary steady state given i is characterized by*

- *$(q_f(i), c_f(i), k_f(i))$, which satisfies that $c_f^m(i) = c_f^n(i) = \bar{c}_f$, if $i \leq \iota$ and*
- *$(q(i), c(i), k(i))$, which satisfies that $c^m(i) < c^n(i)$, if $i > \iota$.*

When there exists no ambiguity and the agents' belief can be summarized by the true probability $\hat{\lambda}$, it holds that $\iota = 0$. As a corollary of the last theorem, therefore, we can obtain a monetary steady state when $\Lambda = \{\hat{\lambda}\}$.

Corollary 2 *When $\Lambda = \{\hat{\lambda}\}$, a unique monetary steady state given $i \geq 0$ is characterized by*

- *$(q_f(i), c_f(i), k_f(i))$, which satisfies that $c_f^m(i) = c_f^n(i)$, if $i = 0$ and*
- *$(q(i), c(i), k(i))$, which satisfies that $c^m(i) < c^n(i)$, if $i > 0$.*

Remark 8 In previous studies such as Haslag and Martin (2007), agents' belief is assumed to coincide with the true probability measure. In such a situation, as shown in Corollary 2, full insurance with respect to the second-period consumptions can be achieved only when the nominal interest rate is zero. In other words, full insurance can occur only when the central bank adopts the Friedman rule. On the other hand, this article shows that, in the presence of ambiguity, full insurance can be realized even when nominal interest rate is positive. This might be a remarkable difference between this article and previous studies.

8 Optimum Quantity of Money

8.1 First-best Monetary Policy

This section examines optimal monetary policy. First, we define allocation and a criterion of optimality. A *stationary feasible allocation* is a pair (c, k) of contingent consumption plan $c = (c^m, c^n)$ and the storage investment $k \in [0, \omega]$ satisfies the resource constraint:[16]

$$c^m \hat{\lambda}_m + c^n \hat{\lambda}_n \leq f(k) = \omega + \rho k.$$

A stationary feasible allocation (c, k) is *golden rule optimal* if there exists no stationary feasible allocation $(\tilde{c}, \tilde{k})$ such that $U(\tilde{c}^m \binom{h}{\bullet} \tilde{c}^n) \geq U(c^m \binom{h}{\bullet} c^n)$ for each $h \in \mathcal{H}$ and $U(\tilde{c}^m \binom{j}{\bullet} \tilde{c}^n) > U(c^m \binom{j}{\bullet} c^n)$ for some $j \in \mathcal{H}$. One can interpret golden rule optimality as (standard) Pareto optimality *restricted* in the space of stationary feasible allocation.

It can be shown easily that $U(c^m \binom{h}{\bullet} c^n) = \min_{\lambda \in \Lambda}[u_o(c^m)\lambda_m + u_o(c^n)\lambda_n]$, i.e., the lifetime utility is independent of agents' indices h. Therefore, a golden rule optimal allocation coincides with a stationary feasible one which maximizes $\min_{\lambda \in \Lambda}[u_o(c^m)\lambda_m + u_o(c^n)\lambda_n]$ subject to the resource constraint holding with equality. Then, the following proposition characterizes a golden rule optimal allocation.

Proposition 7 *A stationary feasible allocation (c, k) is golden rule optimal if and only if it satisfies that $c^m \hat{\lambda}_m + c^n \hat{\lambda}_n = f(k)$, $k = \omega$, and*

$$\min_{\lambda \in \check{\Lambda}(c)} \frac{u'(c^m)\lambda_m}{u'(c^n)\lambda_n} \leq \frac{\hat{\lambda}_m}{\hat{\lambda}_n} \leq \max_{\lambda \in \check{\Lambda}(c)} \frac{u'(c^m)\lambda_m}{u'(c^n)\lambda_n}.$$

[16] Recall that the depreciation rate is zero.

When there exists no ambiguity and agents' belief can be summarized by the true probability measure $\hat{\lambda}$, i.e., $\Lambda = \{\hat{\lambda}\}$, the inequality in the previous proposition can be rewritten as

$$\frac{u'(c^m)}{u'(c^n)} = 1.$$

This can be interpreted as a standard risk-sharing condition. Therefore, the inequality in Proposition 7 can be interpreted as a natural extension of the risk-sharing condition in the absence of ambiguity.

We now turn to consider optimal monetary policy. We say that a nominal interest rate $i \geq 0$ is *first-best* if it generates a monetary steady state, the allocation of which is golden rule optimal. One should note here that it holds that $k = \omega - (1+\rho)q/(1+i) < \omega$ at any monetary steady state (q, c, k) given i. Unfortunately, therefore, we obtain the following corollary.

Corollary 3 *A monetary steady state given any $i \geq 0$ cannot generate golden rule optimal allocation, i.e., there exists no first-best nominal interest rate.*

As shown in this corollary, there exists no first-best monetary policy, i.e., the central bank cannot achieve a golden rule optimal allocation by controlling nominal interest rates i through money growth rates μ. Therefore, we should explore "second-best" monetary policies.

8.2 Second-best Monetary Policy

In order to study "second-best" monetary policy, we consider equilibrium welfare. Because changes in nominal interest rates affect relative prices as argued in the previous section, it might also affect equilibrium welfare. So, a natural definition of a *second-best* nominal interest rate is such that it maximizes equilibrium welfare.[17] For each nominal interest rate $i \geq 0$, define

$$W(i) := \begin{cases} \min\limits_{\lambda \in \Lambda}[u(c_f^m(i))\lambda_m + u(c_f^n(i))\lambda_n] & \text{if} \quad i \leq \iota, \\ \min\limits_{\lambda \in \Lambda}[u(c^m(i))\lambda_m + u(c^n(i))\lambda_n] & \text{if} \quad i > \iota, \end{cases}$$

which represents the equilibrium welfare given i by Theorem. Note that W is constant over the interval $[0, \iota]$ because $c_f^m(i) = c_f^n(i)$ is independent of i on the interval. Also note that it is continuous, especially at $i = \iota$, because it holds that $q(\bullet)$ is continuous and satisfies that $q(\iota) = q_f(\iota)$, which implies that $c(\iota) = c_f(\iota)$ and therefore, $\lim_{i \uparrow \iota} W(i) = W(\iota) = \lim_{i \downarrow \iota} W(i)$.

The following proposition characterizes the set of second-best nominal interest rates.

[17]The second-best optimality argued here is often called *constrained optima*. See, for example, Bhattacharya and Singh (2008) and Matsuoka (2012).

Proposition 8 *The set of second-best nominal interest rates is*

- *singleton and its unique element is equal to* ρ *if* $\iota < \rho$ *and*
- $[0, \iota]$ *if* $\iota \geq \rho$.

As shown in this proposition, when ι is less than ρ, the unique second-best nominal interest rate is ρ, and then the central bank chooses $\mu = 0$ as the money growth rate. On the other hand, when ι is greater than or equal to ρ, there exists a continuum of second-beset nominal interest rates and its set is given by $[0, \iota]$. In this case, the range of money growth rates μ corresponding to second-best nominal interest rates is given by

$$-\frac{\rho}{1+\rho} \leq \mu \leq \frac{\iota - \rho}{1+\rho}.$$

Here, we should remark that the Friedman rule, $i = 0$, is one of second-best policies when $\iota \geq \rho$. Also note that the inflation policy $\mu > 0$ can also be second-best when $\iota > \rho$.

Proposition 8 also implies that the set of second-best nominal interest rates crucially depends on the relation between ι and ρ. More precisely, it depends on the relation between $\overline{\lambda}_m$ and ρ. Actually, it holds that

$$\iota = \left(\frac{\overline{\lambda}_m}{\hat{\lambda}_m}\right)\left(\frac{\overline{\lambda}_n}{\hat{\lambda}_n}\right)^{-1} - 1,$$

which implies that

$$\iota \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \rho \quad \Leftrightarrow \quad \overline{\lambda}_m \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \frac{(1+\rho)\hat{\lambda}_m}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n} < 1.$$

Therefore, we can say that the condition $\iota \geq \rho$ can hold if $\overline{\lambda}_m$ is sufficiently large, i.e., agents believe that they tend to be movers. Conversely, the condition $\iota < \rho$ can hold if $\overline{\lambda}_m$ is sufficiently small.

Proposition 8 has two remarkable differences from the results of the previous studies. In order to observe such differences, we consider the case without ambiguity. Recall that, when $\Lambda = \{\hat{\lambda}\}$, $\iota = 0$ and therefore it follows that $\iota < \rho$. As a corollary of Proposition 8, therefore, we can also characterize a second-best nominal interest rate when there is no ambiguity.

Corollary 4 *When* $\Lambda = \{\hat{\lambda}\}$, *a second-best interest rate is unique and equal to* ρ.

This result consists with those of the previous studies such as Haslag and Martin (2007), Matsuoka (2011), and so on. This corollary implies that (i) the central bank adopts zero inflation policy, i.e., it chooses $\mu = 0$ as the money growth rate and (ii) the Friedman rule is not second-best (of course, it is not first-best). On the other hand, Proposition 8 says that, when $\overline{\lambda}_m$ is sufficiently large, (i′) the central bank can choose $\mu > 0$ as a second-best monetary policy (if $\iota > \rho$) and (ii′) the Friedman rule can also be second best. These are remarkable difference from results of the previous studies.

9 Closed Solution under CRRA Utility Indices

In this section, we provide a closed solution of monetary steady state and some numerical examples on the equilibrium welfare. For this aim, suppose that the utility indices are specified by those with constant relative risk aversion (CRRA), i.e., assume that

$$u(c) = \begin{cases} \dfrac{c^{1-\theta}}{1-\theta} & \text{if} \quad \theta \neq 1, \\ \ln c & \text{if} \quad \theta = 1, \end{cases}$$

where $\theta \geq 0$ is a constant coefficient of relative risk aversion.

First, we find a closed solution of monetary steady state. We can easily obtain a monetary steady states given $i \leq \iota$, $(q_f(i), c_f(i), k_f(i))$, because it is given by

$$\begin{aligned} q_f(i) &= \frac{(1+i)\hat{\lambda}_m}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n}\omega, \\ c_f^m(i) = c_f^n(i) &= \frac{1+\rho}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n}\omega, \quad \text{and} \\ k_f(i) = \omega - \frac{1+\rho}{1+i}q_f(i) &= \frac{\hat{\lambda}_n}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n}\omega \end{aligned}$$

as argued in Subsection 7.1. Therefore, we should specify a monetary steady state given $i > \iota$, $(q(i), c(i), k(i))$. As argued in Subsection 7.1, the real money balance in such a monetary steady state is characterized by a solution of the equation that

$$u'\left(\frac{1}{\hat{\lambda}_m}\frac{1+\rho}{1+i}q(i)\right) = (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega - \frac{1+\rho}{1+i}q(i)\right]\right).$$

Under the assumption of the CRRA utility function, this can be rewritten as

$$\left(\frac{1}{\hat{\lambda}_m}\frac{1+\rho}{1+i}q(i)\right)^{-\theta} = (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega - \frac{1+\rho}{1+i}q(i)\right]\right)^{-\theta}.$$

Solving this equation with respect to $q(i)$, we can obtain that

$$q(i) = \frac{(1+i)A(i)}{\hat{\lambda}_n + (1+\rho)A(i)\hat{\lambda}_m}\omega,$$

where

$$A(i) = \left[(1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}\right]^{-\frac{1}{\theta}}.$$

Then, a monetary steady state given $i > \iota$, $(q(i), c(i), k(i))$ is given by

$$\begin{aligned} c^m(i) &= \frac{1}{\hat{\lambda}_m}\frac{1+\rho}{1+i}q(i), \\ c^n(i) &= \frac{1+\rho}{\hat{\lambda}_n}\left[\omega - \frac{1+\rho}{1+i}q(i)\right], \\ k(i) &= \omega - \frac{1+\rho}{1+i}q(i), \end{aligned}$$

where $q(i)$ is calculated as above.

Given a closed solution of monetary steady state, the equilibrium welfare can be written as

$$W(i) = \begin{cases} \frac{1}{1-\theta}\left(\frac{1+\rho}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n}\omega\right)^{1-\theta} & \text{if} \quad i \in [0,\iota], \\ \frac{1}{1-\theta}\left[\left(\frac{1}{\hat{\lambda}_m}\frac{1+\rho}{1+i}q(i)\right)^{1-\theta}\overline{\lambda}_m + \left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega - \frac{1+\rho}{1+i}q(i)\right]\right)^{1-\theta}\overline{\lambda}_n\right] & \text{if} \quad i \in [\iota,\infty[\end{cases}$$

if $\theta \neq 1$ and otherwise

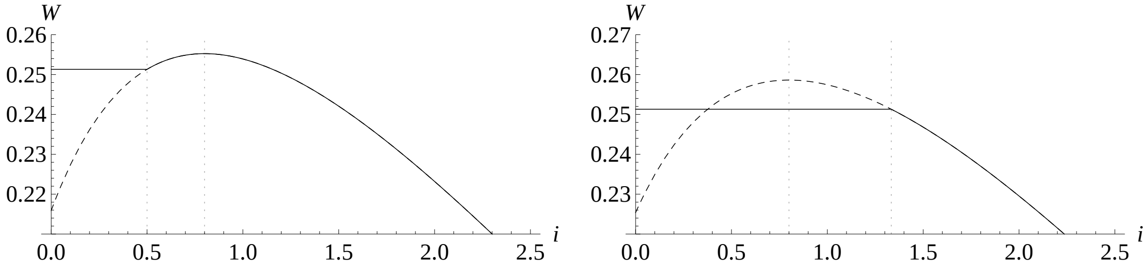
$$W(i) = \begin{cases} \ln\left(\frac{1+\rho}{(1+\rho)\hat{\lambda}_m + \hat{\lambda}_n}\omega\right) & \text{if} \quad i \in [0,\iota], \\ \left[\overline{\lambda}_m \ln\left(\frac{1}{\hat{\lambda}_m}\frac{1+\rho}{1+i}q(i)\right) + \overline{\lambda}_n \ln\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega - \frac{1+\rho}{1+i}q(i)\right]\right)\right] & \text{if} \quad i \in [\iota,\infty[\,. \end{cases}$$

We can then provide a graph of the equilibrium welfare W. Figure 2 depicts W in the economy specified as $u(c) = \ln c$, $\omega = 1$, $\rho = 0.8$, and $\hat{\lambda}_m = \hat{\lambda}_n = 0.5$. Moreover, Figure 2.a considers the case that $\overline{\lambda}_m = 0.6$. In this case, $\iota = 0.5$ and a second-best interest rate is unique and equal to $\rho = 0.8$. On the other hand, Figure 2.b considers the case that $\overline{\lambda}_m = 0.7$. In this case, $\iota = 4/3 \approx 1.3$ and the set of second-best interest rate is given by $[0, 4/3]$. As argued in Section 8, when $\overline{\lambda}_m$ is sufficiently large, both the Friedman rule, $i = 0$, and an inflation policy, $i > 0$, can be second-best.

10 Proofs of Propositions

Proof of Proposition 1. The proof is given before stating Proposition 1. Q.E.D.

Proof of Proposition 2. We calculate V_t^+ as an example. Fix m_t and let $c_t = (c_{t+1}^m, c_{t+1}^n)$ and k_{t+1} be the consumption plan and the storage investment determined by m_t through Eqs.(6)–(8)



a. $\overline{\lambda}_m = 0.6$ b. $\overline{\lambda}_m = 0.7$

Figure 2: Relationship between the nominal interest rate and the equilibrium welfare

given q_s for $s = t, t+1$. Then, for any *sufficiently* small $h > 0$,

$$
\begin{aligned}
&V_t(m_t + h) \\
= &\left\{ \begin{array}{ll} u\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t+h}{M_t}\right)\underline{\lambda}_m + u\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t+h}{M_t}\right]\right)\underline{\lambda}_n & \text{if} \quad c^m_{t+1} \geq c^n_{t+1}, \\ u\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t+h}{M_t}\right)\overline{\lambda}_m + u\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t+h}{M_t}\right]\right)\overline{\lambda}_n & \text{if} \quad c^m_{t+1} < c^n_{t+1}. \end{array} \right.
\end{aligned}
$$

Therefore, for any sufficiently small $h > 0$,

$$
\begin{aligned}
&V_t(m_t + h) - V_t(m_t) \\
= &\left[u\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t+h}{M_t}\right) - u\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\right]\underline{\lambda}_m \\
&+ \left[u\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t+h}{M_t}\right]\right) - u\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\right]\underline{\lambda}_n
\end{aligned}
$$

if $c^m_{t+1} \geq c^n_{t+1}$ and otherwise

$$
\begin{aligned}
&V_t(m_t + h) - V_t(m_t) \\
= &\left[u\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t+h}{M_t}\right) - u\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\right]\overline{\lambda}_m \\
&+ \left[u\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t+h}{M_t}\right]\right) - u\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\right]\overline{\lambda}_n.
\end{aligned}
$$

Because

$$
\check{\Lambda}(c_t) = \left\{ \begin{array}{ll} \{\underline{\lambda}\} & \text{if} \quad c^m_{t+1} > c^n_{t+1}, \\ \Lambda & \text{if} \quad c^m_{t+1} = c^n_{t+1}, \\ \{\overline{\lambda}\} & \text{if} \quad c^m_{t+1} < c^n_{t+1} \end{array} \right.
$$

and $\underline{\lambda}_m \leq \lambda_m \leq \overline{\lambda}_m$ (equivalently, $\overline{\lambda}_n \leq \lambda_n \leq \underline{\lambda}_n$) for each $\lambda \in \Lambda$, we can obtain that

$$
\begin{aligned}
&V_t^+(m_t) \\
= &\left\{ \begin{array}{ll} \frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{M_{t+1}}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\underline{\lambda}_m - \frac{1+\rho}{\hat{\lambda}_n}\frac{q_t}{M_t}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\underline{\lambda}_n & \text{if} \quad c^m_{t+1} \geq c^n_{t+1}, \\ \frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{M_{t+1}}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\overline{\lambda}_m - \frac{1+\rho}{\hat{\lambda}_n}\frac{q_t}{M_t}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\overline{\lambda}_n & \text{if} \quad c^m_{t+1} < c^n_{t+1} \end{array} \right. \\
= &\min_{\lambda \in \check{\Lambda}(c_t)} \frac{1}{M_t}\left[\frac{q_{t+1}}{1+\mu}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\frac{\lambda_m}{\hat{\lambda}_m} - (1+\rho)q_t u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega + \frac{\mu}{1+\mu}q_t - q_t\frac{m_t}{M_t}\right]\right)\frac{\lambda_n}{\hat{\lambda}_n}\right].
\end{aligned}
$$

Similar to the calculation of V_t^+, one can easily calculate V_t^-. This completes the proof of Proposition 2.[18] Q.E.D.

Proof of Proposition 3. By Proposition 2, the inequality (10) is equivalent to

$$\begin{aligned}&\begin{cases}\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{M_{t+1}}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\underline{\lambda}_m-\frac{1+\rho}{\hat{\lambda}_n}\frac{q_t}{M_t}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega+\frac{\mu}{1+\mu}q_t-q_t\frac{m_t}{M_t}\right]\right)\underline{\lambda}_n & \text{if} \quad c_{t+1}^m>c_{t+1}^n\\ \frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{M_{t+1}}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\overline{\lambda}_m-\frac{1+\rho}{\hat{\lambda}_n}\frac{q_t}{M_t}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega+\frac{\mu}{1+\mu}q_t-q_t\frac{m_t}{M_t}\right]\right)\overline{\lambda}_n & \text{if} \quad c_{t+1}^m\le c_{t+1}^n\end{cases}\\ \geq\ & 0\\ \geq\ &\begin{cases}\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{M_{t+1}}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\underline{\lambda}_m-\frac{1+\rho}{\hat{\lambda}_n}\frac{q_t}{M_t}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega+\frac{\mu}{1+\mu}q_t-q_t\frac{m_t}{M_t}\right]\right)\underline{\lambda}_n & \text{if} \quad c_{t+1}^m\ge c_{t+1}^n\\ \frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{M_{t+1}}u'\left(\frac{1}{\hat{\lambda}_m}\frac{q_{t+1}}{1+\mu}\frac{m_t}{M_t}\right)\overline{\lambda}_m-\frac{1+\rho}{\hat{\lambda}_n}\frac{q_t}{M_t}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega+\frac{\mu}{1+\mu}q_t-q_t\frac{m_t}{M_t}\right]\right)\overline{\lambda}_n & \text{if} \quad c_{t+1}^m< c_{t+1}^n\end{cases}\end{aligned}$$

where $c_t = (c_{t+1}^m, c_{t+1}^n)$ is determined by m_t through Eqs.(6)–(8) given q_s for $s = t, t+1$. Obviously, this is equivalent to the inequality (11). This completes the proof of Proposition 3. Q.E.D.

Lemma 1 *For each $i \geq 0$, the equation,*

$$u'\left(\frac{1}{\hat{\lambda}_m}\frac{1+\rho}{1+i}q\right) = (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega-\frac{1+\rho}{1+i}q\right]\right), \tag{17}$$

has a unique solution $q(i)$, which is continuous with respect to i. Furthermore, it holds that

$$\frac{q'(i)(1+i)-q(i)}{(1+i)^2} < 0$$

for each $i > 0$.

Proof of Lemma 1. Let

$$\Phi(q,i) \quad := \quad u'\left(\frac{1}{\hat{\lambda}_m}\frac{1+\rho}{1+i}q\right) - (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}u'\left(\frac{1+\rho}{\hat{\lambda}_n}\left[\omega-\frac{1+\rho}{1+i}q\right]\right)$$

for each $q \in [0, (1+i)\omega/(1+\rho)]$ given $i \geq 0$. Note that Φ is continuous and $\Phi_1(q,i) < 0$ for each q and each i because of strict concavity of u. A solution q of Eq.(17) is then identical to that of the equation that $\Phi(q,i) = 0$.

First, we show existence. As argued above, $\Phi(\bullet, i)$ is continuous. Moreover, $\lim_{q\downarrow 0}\Phi(q,i) = +\infty > 0$ and $\lim_{q\uparrow(1+i)\omega/(1+\rho)}\Phi(q,i) = -\infty < 0$. Therefore, the intermediate value theorem ensured the unique existence of $q(i) \in]0, (1+i)\omega/(1+\rho)[$ such that $\Phi(q(i), i) = 0$ for each $i \geq 0$.

Next, we show uniqueness. Suppose the contrary that there exist two distinct solutions q and q' for the equation in Lemma 1. It is assumed without loss of generality that $q < q'$. Then,

[18] One can also prove Proposition 2 by applying Proposition 6 of Aubin (1979, p.118).

it follows from the fact that Φ is monotone decreasing with respect to the first argument that

$$0 = \Phi(q, i) > \Phi(q', i) = 0,$$

which is a contradiction. Hence, a solution of the equation in Lemma 1 is unique.

We then show that

$$\frac{q'(i)(1+i) - q(i)}{(1+i)^2} < 0$$

for each $i > 0$. Because $\Phi_1(q, i) \neq 0$, the implicit function theorem ensures that $q(\bullet)$ is continuously differentiable on $\Re_{++}$. Let $\xi(i) := q(i)/(1+i)$ and $\Psi(i) := \Phi(q(i), i) \equiv 0$ for each $i \geq 0$. Then, we can obtain that

$$\begin{aligned} 0 &= \Psi'(i) \\ &= \frac{1+\rho}{\hat{\lambda}_m}\xi'(i)u''(c^m) - \frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}u'(c^n) + (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}\frac{(1+\rho)^2}{\hat{\lambda}_m}\xi'(i)u''(c^n), \end{aligned}$$

where c^m and c^n are determined by $m_t = M_{t+1}$ through Eqs.(6)–(8) given $a_s = q(i)$ for each $s = t, t+1$. This implies that

$$\begin{aligned} &\frac{q'(i)(1+i) - q(i)}{(1+i)^2} \\ =\ & \xi'(i) \\ =\ & \frac{\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}u'(c^n)}{\frac{1+\rho}{\hat{\lambda}_m}u''(c^m) + (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}\frac{(1+\rho)^2}{\hat{\lambda}_m}u''(c^n)} \\ <\ & 0. \end{aligned}$$

This completes the proof of Lemma 1. Q.E.D.

Proof of Proposition 4. It is enough to show the *if* part. By Lemma 1, we can find a unique $q^* := q(i)$, under which

$$\frac{u'(c^m)}{u'(c^n)} = (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}$$

hold for each $i \geq \underline{i}$, where $c = (c^y, (c^m, c^n))$ is determined by $m_t = M_t$ through Eqs.(6)–(8) given $q_s = q^*$ for $s = t, t+1$. Moreover, it follows from $i \geq \underline{i}$ that

$$\frac{u'(c^m)}{u'(c^n)} = (1+i)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n} > 1,$$

which implies that $c^m < c^n$. This completes the proof of Proposition 4. Q.E.D.

Proof of Proposition 5. Although it is enough to show the *if* part, $(q_f(i), c_f(i), k_f(i))$ obviously satisfies Eq.(15). Therefore, $(q_f(i), c_f(i), k_f(i))$ is a unique monetary steady state given $i \leq \iota$. Q.E.D.

Proof of Proposition 6. This is given before stating Proposition 6. Q.E.D.

Proof of Proposition 7. Because $\rho > 0$, it follows that $k = \omega$. Moreover, because of strict monotonicity of the lifetime utility function U, the resource constraint holds with equality. Thus, we can observe that a stationary feasible allocation (c, k) is golden rule optimal if and only if it maximizes U subject to $c^m\hat{\lambda}_m + c^n\hat{\lambda}_n = (1+\rho)\omega$. Therefore, by Theorems C and B in the Appendix, a golden rule allocation (c, k) is characterized by $c^m\hat{\lambda}_m + c^n\hat{\lambda}_n = (1+\rho)\omega$ and

$$0 \in \left\{ u'(c^m)\lambda_m - \frac{\hat{\lambda}_m}{\hat{\lambda}_n} u'\left(\frac{1}{\hat{\lambda}_n}\left[(1+\rho)\omega - c^m\hat{\lambda}_m\right]\right)\lambda_n \,\middle|\, \lambda \in \check{\Lambda}(c) \right\},$$

which is equivalent to

$$\min_{\lambda \in \check{\Lambda}(c)} \frac{u'(c^m)\lambda_m}{u'(c^n)\lambda_n} \leq \frac{\hat{\lambda}_m}{\hat{\lambda}_n} \leq \max_{\lambda \in \check{\Lambda}(c)} \frac{u'(c^m)\lambda_m}{u'(c^n)\lambda_n}.$$

This completes the proof of Proposition 7. Q.E.D.

In order to prove Proposition 8, we prepare several notations. Let

$$\begin{aligned} W_f(i) &:= u(c_f^m(i))\lambda_m + u(c_f^n(i))\lambda_n, \\ W_0(i) &:= u(c^m(i))\overline{\lambda}_m + u(c^n(i))\overline{\lambda}_n \end{aligned}$$

for each $i \geq 0$. It follows immediately that

$$W(i) = \begin{cases} W_f(i) & \text{if} \quad i \in [0, \iota], \\ W_0(i) & \text{if} \quad i \in [\iota, \infty[\ . \end{cases}$$

Note that W_f is constant over $[0, \iota]$ because $c_f^m(i) = c_f^n(i)$ for each $i \in [0, \iota]$. Hence, each $i \in [0, \iota]$ maximizes W_f on $[0, \iota]$. So, we explore i maximizing W_0. The following lemma characterizes a value of i, which maximizes W_0.

Lemma 2 *A unique element maximizing* W_0 *is* $i^* := \rho$.

Proof of Lemma 2. Let $\xi(i) := q(i)/(1+i)$ as defined in Lemma 1. By differentiating W_0, we can obtain that, for each $i \geq 0$,

$$W_0'(i) = (1+\rho)\xi'(i)u'(c^m(i))\frac{\overline{\lambda}_m}{\hat{\lambda}_m} - (1+\rho)^2\xi'(i)u'(c^n(i))\frac{\overline{\lambda}_n}{\hat{\lambda}_n}$$

$$= (1+\rho)\frac{\overline{\lambda}_m}{\hat{\lambda}_m}\xi'(i)\left[u'(c^m(i)) - (1+\rho)\frac{\hat{\lambda}_m}{\overline{\lambda}_m}\frac{\overline{\lambda}_n}{\hat{\lambda}_n}u'(c^n(i))\right]$$

$$= (1+\rho)\xi'(i)\frac{\overline{\lambda}_n}{\hat{\lambda}_n}u'(c^n(i))(i-\rho)\left\{\begin{array}{c} > \\ = \\ < \end{array}\right\}0 \quad \Leftrightarrow \quad i\left\{\begin{array}{c} < \\ = \\ > \end{array}\right\}\rho,$$

where the third equality follows from the definitions of $c^m(i)$ and $c^n(i)$ and the *if and only if* part follows from the fact that $\xi'(i) < 0$ as shown in Lemma 1. This implies that $i^* := \rho$ is a unique element maximizing W_0. Q.E.D.

Proof of Proposition 8. Consider first the case that $\iota < \rho$. In this case, we can obtain that

$$W'(i) = \left\{\begin{array}{ll} W_f'(i) & = \\ W_0'(i) & \left\{\begin{array}{c} > \\ = \\ < \end{array}\right. \end{array}\right\} 0 \quad \text{if} \quad i \in \left\{\begin{array}{c} [0,\iota[, \\ ]\iota,\rho[, \\ \{\rho\}, \\ ]\rho,\infty[, \end{array}\right.$$

where the value of W' on $]\iota,\infty[$ follows from Lemma 2. Because W is continuous, this implies that $i^* = \rho$ maximizes W.

On the other hand, consider the case that $\iota \geq \rho$. In this case, we can obtain that

$$W'(i)\left\{\begin{array}{l} W_f'(i) = \\ W_0'(i) < \end{array}\right\}0 \quad \text{if} \quad i \in \left\{\begin{array}{c} [0,\iota[, \\ ]\iota,\infty[, \end{array}\right.$$

where the value of W' on $]\iota,\infty[$ follows from Lemma 2. Because W is continuous, this implies that each $i \in [0,\iota]$ maximizes W. Q.E.D.

Appendix: Superdifferential and its Calculus

This appendix aims to introduce the definition of superdifferential and its calculus rules. We first define the concept of superdifferential following Rockafellar (1970, p.214) and Hiriart-Urruty and Lemaréchal (2004, Definition D.1.2.1).

Definition A For each concave real-valued function f on $\Re^n$ and each $\rho \in \Re^n$, the set

$$\partial f(x) := \{s \in \Re^n : (\forall y \in \Re^n) \quad f(y) \leq f(x) + \langle s, y-x\rangle\}$$

and each of its elements are called the *superdifferantial* and a *supergradient of f at ρ*, respectively.

The following result follows from Hiriart-Urruty and Lemaréchal (2004, Theorem D.4.1.1).

Theorem A *For each concave real-valued functions f_1 and f_2 on $\Re^n$, each positive numbers a_1 and a_2, and each $\rho \in \Re^n$, $\partial(a_1 f_1 + a_2 f_2)(x) = a_1\partial f_1(x) + a_2\partial f_2(x)$.*

We should note that this observation does not necessarily hold for more general concave functions (Rockafellar, 1970, p.223).

The next result follows from Hiriart-Urruty and Lemaréchal (2004, Corollary D.4.4.4).

Theorem B *Let J be a compact set in some metric space and $\{f_j\}_{j\in J}$ be a family of differentiable concave real-valued functions on $\Re^n$. Define the real-valued function f on $\Re^n$ by*

$$f(x) := \inf_{j\in J} f_j(x)$$

and let $J(x) := \{j \in J : f_j(x) = f(x)\}$ for each $\rho \in \Re^n$. Then, it follows that

$$\partial f(x) = \mathrm{co}\left\{\nabla f_j(x) : j \in J(x)\right\}.$$

Finally, we provide a useful result following from Hiriart-Urruty and Lemaréchal (2004, Theorem D.2.2.1), for optimization.

Theorem C *For each concave real-valued function f on $\Re^n$ and each $\rho \in \Re^n$, $f(x) \geq f(y)$ for each $y \in \Re^n$ if and only if $0 \in \partial f(x)$.*

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