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Keywords. multi-object allocation; strategy-proofness; ex-post revenue maximization; minimum Walrasian equilibrium price rule; non-quasilinear preferences; no wastage; equal treatment of equals.

JEL Code. D82, D47, D71, D63.

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1 Introduction

One of the most challenging problems in microeconomic theory is the design of revenue maximizing mechanism in multi-object allocation problem. Ever since the seminal work of Myerson (1981) for solving the revenue maximizing mechanism for the single object environment, advances in the mechanism design literature have convinced researchers that it is difficult to precisely describe a revenue maximizing mechanism design in multi-object environment. [1] We offer a robust resolution to this difficulty by imposing some additional desiderata that are appealing in many settings.

We study the problem of allocating m indivisible objects to $n > m$ agents, each of whom can be assigned at most one object (unit demand agents) - such unit demand settings are common in selling advertisement slots on internet pages, selling team franchises in professional sports leagues, and even in selling a small number of spectrum licenses. [2] Agents in our model can have non-quasilinear preferences over consumption bundles - (object, transfer) pairs. We impose four desiderata on (allocation) rules [3]: (1) *strategy-proofness* or *dominant strategy incentive compatibility*, (2) *ex-post individual rationality*, (3) *equal treatment of equals* - two agents having identical preferences must be assigned consumption bundles (i.e., (object, payment) pairs) to which they are indifferent, (4) *no wastage* (every object is allocated to some agent). Any rule satisfying these properties is termed *desirable*.

A domain (the admissible class of preferences) is *rich* if it includes enough variety preferences of agents. Our richness requirement is mild enough to be satisfied by various well known classes of preferences. [4] For example, the class of quasilinear preferences, the one containing preferences exhibiting income effects, the one containing only preferences exhibiting positive income effects, satisfy our richness.

If the domain is rich, then our main result says that the *minimum Walrasian equilibrium price (MWEP)* rule is *ex-post revenue optimal* among all desirable rules satisfying *no subsidy*, i.e., for each preference profile, the MWEP rule generates more revenue for the seller than any desirable rule satisfying no subsidy. No subsidy requires that payment of each agent

[1] An extensive literature review is provided in Section 6.

[2] Although modern spectrum auctions involve sale of of *bundles* of spectrum licenses, Binmore and Klemperer (2002) report that one of the biggest spectrum auctions in UK involved selling a fixed number of licenses to bidders, each of whom can be assigned at most one license. The unit demand setting is also one of the few *computationally* tractable model of combinatorial auction studied in the literature (Blumrosen and Nisan, 2007).

[3] Allocation rules are sometimes called direct mechanisms in the literature. Since we impose strategy-proofness, we can restrict our attention to direct mechanisms without loss of generality.

[4] See Section 4.1 for its definition.

is non-negative. Further, we show that if the domain includes all positive income effect preferences, then the MWEP rule is ex-post revenue optimal in the class of all desirable and *no bankruptcy* rules, where no bankruptcy requires that the sum of payments of all agents across all profiles is bounded below. Notice that no bankruptcy is weaker than no subsidy. No bankruptcy is an indispensable requirement since without it, the seller runs the risk of being bankrupt at some profile of preferences. Our revenue maximization result is robust in an ex-post sense. Hence, we can recommend the MWEP rule without resorting to any prior-based maximization.

The MWEP rule is based on a "market-clearing" notion. A price vector on objects is called a *Walrasian equilibrium price vector* if there is an allocation of objects such that each agent gets an object from his demand set. Demange and Gale (1985) showed that the set of Walrasian equilibrium price vectors is always a non-empty compact lattice in our model. This means that there is a unique minimum Walrasian equilibrium price vector. [5] The MWEP rule selects the minimum Walrasian equilibrium price vector at every profile of preferences and uses a corresponding equilibrium allocation. The MWEP rule is desirable (Demange and Gale, 1985) and satisfies no subsidy. In the quasilinear domain of preferences, the MWEP rule coincides with the Vickrey-Clarke-Groves (VCG) rule (Leonard, 1983). We show that in many domains of preferences, the MWEP rule is revenue-optimal among all desirable and no subsidy rules.

Our results stand out in the literature in another important way - ours is the first paper to study revenue maximization in multi-object allocation problem when preferences of agents are not quasilinear. Quasilinearity has been the standard assumption in most of mechanism design. While it allows for analysis of mechanism design problems using standard convex analysis tools (illustrated by the analysis of Myerson (1981)), its practical relevance is debatable in many settings. For instance, in spectrum auctions, the payments of bidders are large sums of money. Firms have limited liquidity to pay these sums and usually borrow from banks at non-negligible interest rates. Since larger amount of borrowings have higher interest rates, it introduces non-quasilinear preferences over consumption bundles. Moreover, income effects are present in many standard settings and should not be overlooked. By analyzing revenue maximization mechanism design without any functional form assumption on preferences, we carry out a "detail-free" mechanism design of our problem. Along with the robustness to distributional assumptions, this brings in another dimension of robustness to our results.

[5] Results of this kind were earlier known for quasilinear preferences (Shapley and Shubik, 1971; Leonard, 1983).

We briefly discuss what drives our surprisingly robust results. The literature on revenue maximization mechanism design (single or multiple objects) considers only incentive and participation constraints: Bayesian incentive compatibility and interim individual rationality. We have departed from this by considering stronger form of incentive and participation constraints: strategy-proofness and ex-post individual rationality. [6] This is consistent with our objective of providing a robust recommendation of allocation rule in our setting. Further, it allows us to stay away from prior-based analysis.

The main drivers for our results are equal treatment of equals, no subsidy, and no wastage. When allocating public assets, governments are supposed to pursuit several goals other than revenue maximization. One such goal is fairness. Though the literature uses a variety of fairness axioms, each differing from the other in the way they treat different agents, they all agree that equals should be treated equally. [7] In this sense, equal treatment of equals is a minimal requirement of fairness. It is also consistent with some fundamental philosophies of equity. [8] Further, Deb and Pai (2016) cite many legal implications of violating such symmetric treatment of bidders in auctions. The no subsidy axiom is standard in almost all allocation rules in practice. Further, we show some possibility to weaken it (by using no bankruptcy) in the positive income effect domain of preferences.

Perhaps the most controversial axiom in our results is no wastage. An important aspect of Myerson's revenue maximization result for single object sale (in quasilinear domain) is that a Vickrey rule with an *optimally* chosen reserve price maximizes expected revenue (Myerson, 1981). In the multi-object environment, the structure of incentive and participation constraints (even in the quasilinear environment) becomes quite messy. Among many other difficulties in extending Myerson's result to the multi-object environment, one major difficulty is finding the *optimal* reserve prices.

Our no wastage axiom avoids this particular difficulty. No wastage is a mild efficiency restriction on the set of allocation rules, and still leaves us with a large set of allocation rules to optimize. Thus, even after imposing no wastage, it is still challenging to find an optimal rule in multi-object environment. To our knowledge, the literature is silent on this issue.

Undoubtedly, reserve prices are used in many auctions in real-life. However, the consequences of such reserve prices are unclear in cases where it is doubtful that a seller can

[6] There is also a large literature (discussed in Section 6) on single agent revenue maximization mechanism design problem, commonly referred to as the screening problem, where the two solution concepts coincide.

[7] See Thomson (2016) for a detailed discussions on other fairness axioms like anonymity in welfare, envy-freeness, egalitarian equivalence, etc.

[8] Aristotle writes in "Politics" that Justice is considered to mean equality. It does not mean equality - but equality for those who are equal, and not for all.

commit to reserve prices. For instance, when governments sell natural resources using auctions, unsold objects and low revenues create a lot of controversies in the public, and often, the unsold objects are resold. As an example, the Indian spectrum auctions reported a large number of unsold spectrum blocks and low revenues in 2016, and all of them are supposed to be re-auctioned. [9] In other words, governments are expected to pursuit revenue maximization without wasting resources. Even if the seller is not a Government, resale of unsold objects in auctions are common - for instance, Ashenfelter and Graddy (2003) analyze art auctions data and find evidence that unsold art objects (due to reserve prices) are often resold. Hence, no wastage seems to be an appropriate requirement in many settings. Our results show the implication of such a minimal form of efficiency on revenue maximization mechanism design in multi-object environment. In Section 4.5, we give two further motivating examples which seem to fit most of our assumptions in the model.

Our results rely on the fact that the rule selects a Walrasian equilibrium allocation. Further, the desirable properties and the no subsidy (or, no bankruptcy) axiom impose nice structure on the set of rules. We exploit these to give elementary proofs of our two main results. This is an added advantage of our results.

Finally, the MWEP rule can be implemented as a simple ascending price auction - for quasilinear domains, see Demange et al. (1986), and for non-quasilinear domains, see Morimoto and Serizawa (2015). Such ascending auctions have distinct advantages of practical implementation and are often used in practice - the main selling point seems to be their efficiency properties (Ausubel and Milgrom, 2002). Our results provide a *revenue maximizing* and robust foundation for such ascending price auctions for the unit demand model.

2 Preliminaries

We now formally define our model. A seller has a m objects to sell, denoted by $M := \{1, \ldots, m\}$. There are $n > m$ agents (buyers), denoted by $N := \{1, \ldots, n\}$. Each agent can receive at most one object (unit-demand preference). Let $L \equiv M \cup \{0\}$, where 0 is the null object, which is assigned to any agent who does not receive any object in M - thus, the null object can be assigned to more than one agent. Note that the unit demand restriction can either be a restriction on preferences or an institutional constraint. For instance, objects may be substitutable for the agents as in the advertisement display slots on an internet page. The unit demand restriction can also be institutional as was the case in the spectrum license

[9] See the following news article: http://www.livemint.com/Industry/xt5r4Zs5RmzjdwuLUdwJMI/Spectrum-auction-ends-after-lukewarm-response-from-telcos.html

auction in UK in 2000 (Binmore and Klemperer, 2002). As long as the mechanism designer restricts messages in the mechanisms to *only* use information on preferences over individual objects, our results apply.

The (consumption) bundles of every agent is the set $L \times \mathbb{R}$, where a typical element $z \equiv (a,t)$ corresponds to object $a \in L$ and transfer $t \in \mathbb{R}$. Throughout the paper, t will be interpreted as the amount *paid* by an agent to the designer, i.e., a negative t will indicate that the agent receives a transfer of $-t$.

Now, we formally introduce preferences of agents and the notion of a desirable rule.

2.1 The preferences

A preference ordering R_i (of agent i) over $L \times \mathbb{R}$, with strict part P_i and indifference part I_i, is **classical** if it satisfies the following assumptions:

1. **Money monotonicity.** for every $t > t'$ and for every $a \in L$, we have $(a,t')\ P_i\ (a,t)$.
2. **Desirability of objects.** for every t and for every $a \in M$, $(a,t)\ P_i\ (0,t)$.
3. **Continuity.** for every $z \in L \times \mathbb{R}$, the sets $\{z' : z'\ R_i\ z\}$ and $\{z' : z\ R_i\ z'\}$ are closed.
4. **Possibility of compensation.** for every $z \in L \times \mathbb{R}$ and for every $a \in L$, there exists t and t' such that $z\ R_i\ (a,t)$ and $(a,t')\ R_i\ z$.

A *quasilinear* preference is classical. In particular, a preference R_i is quasilinear if there exists $v \in \mathbb{R}^{|L|}$ such that for every $a,b \in L$ and $t,t' \in \mathbb{R}$, $(a,t)\ R_i\ (b,t')$ if and only if $v_a - t \geq v_b - t'$. Usually, v is referred to as the valuation of the agent, and v_0 is normalized to 0. The idea of valuation may be generalized as follows for non-quasilinear preferences.

Definition 1 *The* **valuation** *at a classical preference R_i for object $a \in L$ with respect to bundle z is defined as $V^{R_i}(a,z)$, which uniquely solves $(a, V^{R_i}(a,z))\ I_i\ z$.*

A straightforward consequence of our assumptions is that for every $a \in L$, for every $z \in L \times \mathbb{R}$, and for every classical preference R_i, the valuation $V^{R_i}(a,z)$ exists. For any R and for any $z \in L \times \mathbb{R}$, the valuations at bundle z with preference R is a vector in $\mathbb{R}^{|L|}$.

An illustration of the valuation is shown in Figure 1. In the figure, the horizontal lines correspond to objects: $L = \{0,a,b,c\}$. The horizontal lines indicate transfer amounts. Hence, the four lines are the entire set of consumption bundles of the agent. For example, z denotes the bundle consisting of object b and the payment equal to the distance of z from the vertical dotted line. Money monotonicity implies that bundles to the left of z (on

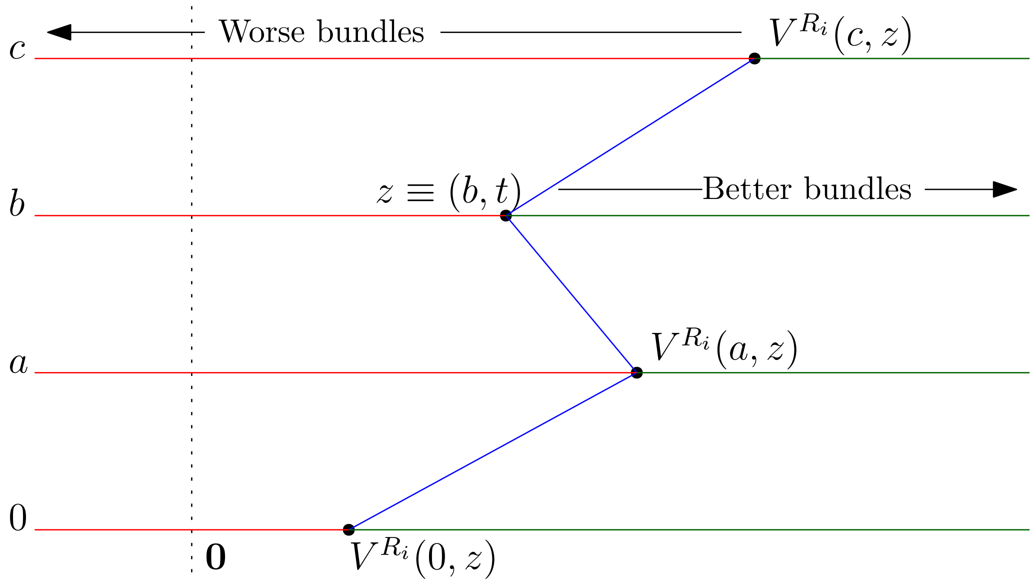

Figure 1: Valuation at a preference

the same horizontal line) are better than z. A preference R_i can be described by drawing (non-intersecting) indifference vectors through these consumption bundles (lines). One such indifference vector passing through z is shown in Figure 1. This indifference vector actually consists of four points: $V^{R_i}(0, z), V^{R_i}(a, z), z \equiv (b, t), V^{R_i}(c, z)$ as shown. Parts of the curve in Figure 1 which lie between the consumption bundle lines is useless and has no meaning - it is only displayed for convenience. As we go to the left along the horizontal lines starting from any bundle, we get worse bundles (due to money monotonicity). Similarly, bundles to the right of a particular bundle are better than that bundle. This is shown in Figure 1 with respect to the indifference vector.

Our modeling of preferences captures income effects even though we do not model income explicitly. Indeed, as transfer changes, the income levels of agents change and this is automatically reflected in the preferences.

2.2 Desirable rules

Let $\mathcal{R}^C$ denote the set of all classical preferences and $\mathcal{R}^Q$ denote the set of all quasilinear preferences. We will consider an arbitrary class of classical domain $\mathcal{R} \subseteq \mathcal{R}^C$ - we will put specific restrictions on $\mathcal{R}$ later. A preference of agent i is denoted by $R_i \in \mathcal{R}$. A preference profile is a list of preferences $R \equiv (R_1, \ldots, R_n)$. The usual notations R_{-i} and $R_{-N'}$ will denote a preference profile without the preference of agent i and without the preferences of agents in $N' \subseteq N$ respectively.

An *object allocation* is an n-tuple $(a_1, \ldots, a_n) \in L^n$ such that no real (non-null) object is assigned to two agents, i.e., $a_i \neq a_j$ for all i, j with $a_i, a_j \neq 0$. The set of all object allocations is denoted by A. A (feasible) allocation is an n-tuple $((a_1, t_1), \ldots, (a_n, t_n)) \in A \times \mathbb{R}$, where (a_i, t_i) is the allocation of agent i. Let Z denote the set of all feasible allocations. For every allocation $(z_1, \ldots, z_n) \in Z$, we will denote by z_i the allocation of any agent i.

An **allocation rule** or a **rule** for short is a map $f : \mathcal{R}^n \to Z$. Notice that we focus attention to deterministic rules. A recent paper by Chen et al. (2016) has shown that in quasilinear domains, there is no *loss of generality* in restricting attention to deterministic rules if the seller wants to maximize expected revenue. However, (a) we consider preferences which are not necessarily quasilinear and (b) we impose extra conditions beyond incentive compatibility. Hence, it is not clear if the robustness of deterministic rules proved in Chen et al. (2016) extends to our setting. Our restriction to deterministic rules is purely driven by simplicity of analysis.

At a preference profile $R \in \mathcal{R}^n$, we denote the allocation of agent i in rule f as $f_i(R) \equiv (a_i(R), t_i(R))$, where $a_i(R)$ and $t_i(R)$ are respectively the object allocated to agent i and the transfer paid by agent i at preference profile R. We call $a(\cdot)$ and $p(\cdot)$ an object allocation rule and a payment rule, respectively.

Definition 2 *A rule $f : \mathcal{R}^n \to Z$ is* **desirable** *if it satisfies the following properties:*

1. **Strategy-proofness.** *for every $i \in N$, for every $R_{-i} \in \mathcal{R}^{n-1}$, and for every $R_i, R'_i \in \mathcal{R}$, we have*
$$f_i(R_i, R_{-i}) \ R_i \ f_i(R'_i, R_{-i}).$$
2. **Ex-post individual rationality (IR).** *for every $i \in N$, for every $R \in \mathcal{R}^n$, we have $f_i(R) \ R_i \ (0, 0)$.*
3. **Equal treatment of equals (ETE).** *for every $i, j \in N$, for every $R \in \mathcal{R}^n$ with $R_i = R_j$, we have $f_i(R) \ I_i \ f_j(R)$.*
4. **No wastage (NW).** *for every $R \in \mathcal{R}^n$ and for every $a \in M$, there exists some $i \in N$ such that $a_i(R) = a$.*

Out of the four properties of a desirable rule, strategy-proofness and IR are standard constraints imposed on a rule. Most of the literature considers Bayesian incentive compatibility and interim individual rationality. As a consequence, one ends up working in the "reduced-form" problems (Border, 1991), and one needs to put additional constraints, commonly referred to as "Border constraints", in the optimization program. The multi-object analogues of the Border constraints are difficult to characterize (Che et al., 2013) - also see Gopalan et al. (2015) for a computational impossibility of extending the Border inequalities to our problem. Working with strategy-proof and ex-post IR, we get around these problems. [10]

ETE is a very mild form of fairness requirement. It states that two agents with identical preferences must be assigned bundles to which they should be indifferent. As argued in the introduction, such minimal notion of fairness is often required by law. The desirability of NW is debatable, and the readers are referred back to the Introduction section for more discussions on this. Besides desirability, for some of our results, we will require some form of restrictions on payments.

Definition 3 *A rule $f : \mathcal{R}^n \to Z$ satisfies* **no subsidy** *if for every $R \in \mathcal{R}^n$ and for every $i \in N$, we have $t_i(R) \geq 0$.*

No subsidy can be considered desirable to exclude "fake" agents, who participate in mechanisms just to take away available subsidy. As was discussed earlier, it is an axiom satisfied by most standard mechanisms in practice. No subsidy is motivated by the fact that in many settings, the seller may not have any means to finance any agents.

3 The minimum Walrasian equilibrium price rule

In this section, we define the notion of a Walrasian equilibrium, and use it to define a desirable rule. A price vector $p \in \mathbb{R}_+^{|L|}$ defines a price for every object with $p_0 = 0$. At any price vector p, let $D(R_i, p) := \{a \in L : (a, p_a)\ R_i(b, p_b)\ \forall\ b \in L\}$ denote the demand set of agent i with preference R_i at price vector p. [11]

Definition 4 *An object allocation $(a_1, \ldots, a_n)$ and a price vector p is a* **Walrasian equilibrium** *at a preference profile $R \in \mathcal{R}^n$ if*

1. *$a_i \in D(R_i, p)$ for all $i \in N$ and*
2. *for all $a \in M$ with $a_i \neq a$ for all $i \in N$, we have $p_a = 0$.*

[10] On a related note, in the single object case, there is strong equivalence between the set of strategy-proof and Bayesian incentive compatible rules (Mookherjee and Reichelstein, 1992; Manelli and Vincent, 2010; Gershkov et al., 2013). But this equivalence is lost in the multi-object problem.

[11] A more traditional definition of demand set using the notion of a budget set is also possible. Here, we define the budget set of each agent at price vector p as $B(p) := \{(a, p_a) : a \in L\}$ and the demand set of agent i is just the maximal bundles in the budget set according to preference R_i.

We refer to p and $\{z_i \equiv (a_i, p_{a_i})\}_{i\in N}$ defined above as a **Walrasian equilibrium price vector** *and a* **Walrasian equilibrium allocation** *at R respectively.*

Since we assume $n > m$, the conditions of Walrasian equilibrium implies that for all $a \in M$, we have $a_i = a$ for some $i \in N$. [12]

A Walrasian equilibrium price vector p is a **minimum Walrasian equilibrium price vector** at preference profile R if for every Walrasian equilibrium price vector p' at R, we have $p_a \leq p'_a$ for all $a \in L$. Demange and Gale (1985) prove that if R is a profile of classical preferences, then a Walrasian equilibrium exists at R, and the set of Walrasian equilibrium price vectors forms a lattice with a unique minimum and a unique maximum. We denote the minimum Walrasian equilibrium price vector at R as $p^{min}(R)$. Notice that if $n > m$, then for every $a \in A$, we have $p^{min}_a(R) > 0$. [13]

We give an example to illustrate the notion of minimum Walrasian equilibrium price vector. Suppose $N = \{1, 2, 3\}$ and $M = \{a, b\}$. Figure 2 shows some indifference vectors of a preference profile $R \equiv (R_1, R_2, R_3)$ and the corresponding minimum Walrasian equilibrium price vector $p^{min}(R) \equiv p^{min} \equiv (p^{min}_0 = 0, p^{min}_a, p^{min}_b)$.

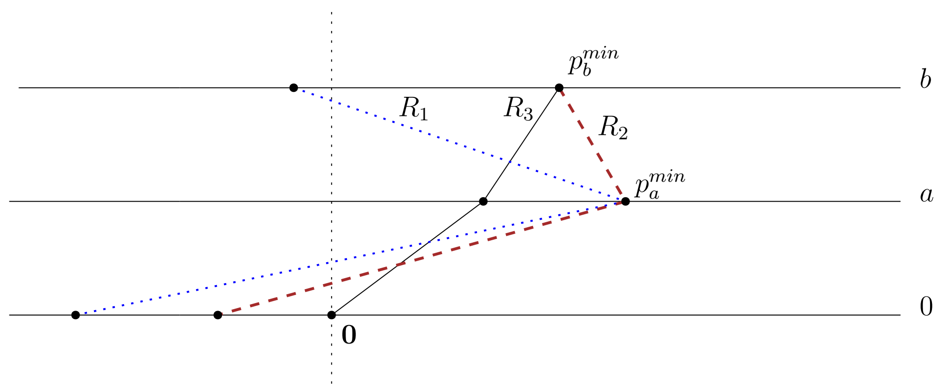

Figure 2: The minimum Walrasian equilibrium price vector

First, note that

$$D(R_1, p^{min}) = \{a\}, D(R_2, p^{min}) = \{a, b\}, D(R_3, p^{min}) = \{0, b\}.$$

[12] To see this, suppose that there is $a \in M$ such that $a_i \neq a$ for each $i \in N$. Then, by the second condition of Walrasian equilibrium, $p_a = 0$. By $n > m$, $a_i = 0$ for some $i \in N$. By desirability of objects, $(a, 0)\ P_i\ (a_i, 0)$, contradicting the first condition of Walrasian equilibrium.

[13] To see this, suppose $p^{min}_a(R) = 0$, then any agent $i \in N$ who is not assigned in the Walrasian equilibrium will prefer $(a, 0)$ to $(0, 0)$ contradicting the fact that he is assigned a bundle from his demand set. Indeed, this argument holds for any Walrasian equilibrium price vector.

Hence, a Walrasian equilibrium is the allocation where agent 1 gets object a, agent 2 gets object b, and agent 3 gets the null object at the price vector p^{min}. Also, p^{min} is the minimum such Walrasian equilibrium price vector. To see this, let p be any other Walrasian equilibrium price vector. If $p_a < p_a^{min}$ and $p_b < p_b^{min}$, then no agent demands the null object, contradicting Walrasian equilibrium. Thus, $p_a \geq p_a^{min}$ or $p_b \geq p_b^{min}$. If $p_b < p_b^{min}$, then by $p_a \geq p_a^{min}$, both agents 2 and 3 will demand only object b, contradicting Walrasian equilibrium. Thus, $p_b \geq p_b^{min}$. But, if $p_a < p_a^{min}$, both agents 1 and 2 will demand only object a, a contradiction to Walrasian equilibrium. Hence, $p \geq p^{min}$.

We now describe a desirable rule satisfying no subsidy. The rule picks a minimum Walrasian equilibrium allocation at every profile of preferences. Although the minimum Walrasian equilibrium price vector is unique at every preference profile, there may be multiple supporting object allocation - all these object allocations must be indifferent to all the agents. To handle this multiplicity problem, we introduce some notation. Let $Z^{min}(R)$ denote the set of all allocations at a minimum Walrasian equilibrium at preference profile R. Note that if $((a_1, \ldots, a_n), p) \in Z^{min}(R)$ then $p = p^{min}(R)$.

Definition 5 *A rule* $f^{min} : \mathcal{R}^n \to Z$ *is a* **minimum Walrasian equilibrium price (MWEP)** *rule if*

$$f^{min}(R) \in Z^{min}(R) \;\forall\; R \in \mathcal{R}^n.$$

Demange and Gale (1985) showed that every MWEP rule is strategy-proof. [14] Clearly, it also satisfies individual rationality, no subsidy, and ETE. We document this fact below.

Fact **1 (Demange and Gale (1985); Morimoto and Serizawa (2015))** *Every MWEP rule is desirable and satisfies no subsidy.*

4 The results

In this section, we formally state our results. The proofs of our results will be presented in Section 5. Before we state our result, we define some extra notations and the richness in domain necessary for our results.

[14]The MWEP rule satisfies a stronger incentive property called *group-strategy-proofness*, which means that no coalition of agents can manipulate this rule.

4.1 Richness and ex-post revenue maximization

The domain of preferences that we consider for our first result is the following. [15]

Definition 6 *A domain of preferences $\mathcal{R}$ is* **rich** *if for all $a \in M$ and for every price vector $\hat{p}$ with $\hat{p}_a > 0, \hat{p}_b = 0$ for all $b \neq a$ and for every price vector $p > \hat{p}$, there exists $R_i \in \mathcal{R}$ such that*

$$D(R_i, p) = \{0\} \text{ and } D(R_i, \hat{p}) = \{a\}.$$

Figure 3 illustrates this notion of richness with two objects a and b - two possible price vectors p and $\hat{p}$ are shown and two indifference vectors of a preference R_i are shown such that $D(R_i, p) = \{0\}$ and $D(R_i, \hat{p}) = \{a\}$.

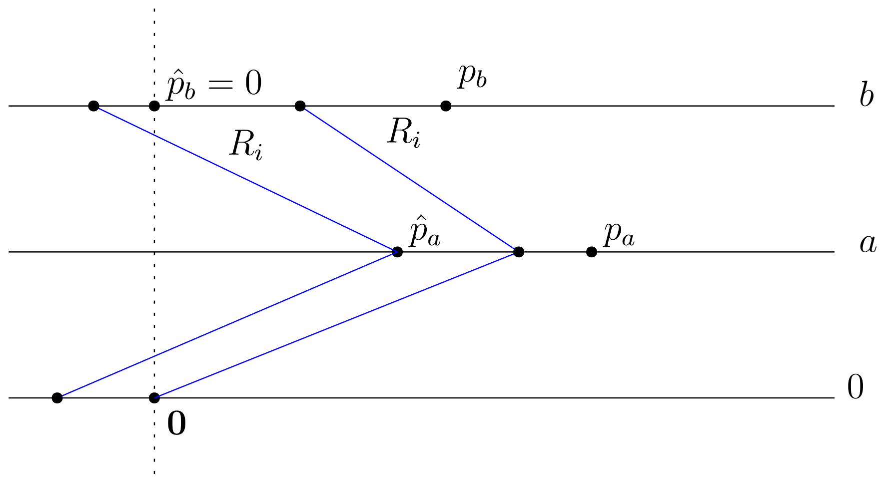

Figure 3: Illustration of richness

Richness requires that if there are two price vectors $p > \hat{p}$, where the only positive price object at $\hat{p}$ is object a, then there is a preference ordering where the agent only demands a at $\hat{p}$ and demands nothing at p. In quasilinear domain, if the set of valuations is the set of all positive real numbers, then our richness condition is satisfied - for instance, consider a quasilinear preference where we pick a value for object a between $\hat{p}_a$ and p_a and value for all other objects arbitrarily close to zero. Later, we show that this richness condition can be satisfied for many non-quasilinear preferences also.

A closer inspection of the richness reveals that if p is too small, then richness requires the existence of a preference where the "value" for real objects is very small. In quasilinear

[15]For every price vector $p \in \mathbb{R}_+^{|L|}$, we assume that $p_0 = 0$. Further, for any pair of price vectors $p, \hat{p} \in \mathbb{R}_+^{|L|}$, we write $p > \hat{p}$ if $p_a > \hat{p}_a$ for all $a \in M$.

domain, we can weaken this richness to a weaker condition which requires valuations in an interval of the form (v^{min}, ∞), where $v^{min} \geq 0$ is any lower bound on the valuation of the objects. A formal definition and proof is available upon request.

We now formally state our first main result. For any rule $f : \mathcal{R}^n \to Z$, we define the **revenue** at preference profile $R \in \mathcal{R}^n$ as

$$\textsc{Rev}^f(R) := \sum_{i \in N} t_i(R).$$

Definition 7 *A rule $f : \mathcal{R}^n \to Z$ is* **ex-post revenue optimal** *among a class of rules defined on $\mathcal{R}^n$ if for every rule g in this class, we have*

$$\textsc{Rev}^f(R) \geq \textsc{Rev}^g(R) \quad \forall\ R \in \mathcal{R}^n.$$

It is not clear that an ex-post revenue optimal rule exists. Our main result shows that the MWEP rule is ex-post revenue optimal among the class of desirable rules satisfying no subsidy.

Theorem **1** *Suppose $\mathcal{R}$ is a rich domain of preferences. Every MWEP rule is ex-post revenue optimal among the class of desirable rules satisfying no subsidy defined on $\mathcal{R}^n$.*

Theorem 1 clearly implies that even if we do *expected* revenue maximization with respect to *any* prior on the preferences of agents, we will only get an MWEP rule among the class of desirable and no subsidy rules.

Although it is difficult to describe the set of desirable rules satisfying no subsidy, such rules exist even in the domain of quasilinear preference (which is a rich domain) which are different from the MPWE rules. We include an example of Tierney (2016) in the supplementary appendix at the end of this manuscript for completeness. Indeed, the set of all desirable rules satisfying no subsidy seems quite complicated to describe in the quasilinear domain of preferences. Our main result shows that every MWEP rule is revenue-optimal in a strong sense in the class of desirable and no subsidy rules.

4.2 Income effects and no bankruptcy

We now discuss some specific domains where our richness condition holds. We also show how Theorem 1 can be strengthened in some specific rich domains.

Definition 8 *A preference R_i satisfies* **positive income effect** *if for every $a, b \in L$ and for every t, t' with $t < t'$ and $(b, t')\ I_i\ (a, t)$, we have*

$$(b, t' - \delta)\ P_i\ (a, t - \delta) \quad \forall\ \delta > 0.$$

A preference R_i satisfies **non-negative income effect** *if for every $a, b \in L$ and for every t, t' with $t < t'$ and $(b, t')\ I_i\ (a, t)$, we have*

$$(b, t' - \delta)\ R_i\ (a, t - \delta) \quad \forall\ \delta > 0.$$

Let $\mathcal{R}^{++}$ and $\mathcal{R}^{+}$ denote the sets of all positive income effect and non-negative income effect domain of preferences respectively.

A standard definition of positive income effect will say that a preferred object is more preferred as income increases. In our model, when income increases by $\delta > 0$, the origin of consumption space moves to right by δ. This movement is equivalent to a decrease of prices by δ. In the above definition, $(b, t')\ I_i\ (a, t)$ and $t' > t$ imply that object b is strictly preferred to object a at any common payment levels $t'' \in [t, t')$. Then, positive income effect requires that consumption bundle $(b, t' - \delta)$ is strictly preferred to consumption bundle $(a, t - \delta)$.

Positive (non-negative) income effects are natural restrictions to impose in settings where the objects are normal goods. Our next claim shows that the richness condition is satisfied in a variety of domains containing positive income effect preferences. Since the proof is straightforward, we skip it.

Claim 1 *A domain of preferences $\mathcal{R}$ satisfies richness if any of the following conditions holds: (1) $\mathcal{R} \supseteq \mathcal{R}^Q$; (2) $\mathcal{R} \supseteq \mathcal{R}^+$; (3) $\mathcal{R} \supseteq \mathcal{R}^{++}$; (4) $\mathcal{R} \supseteq \mathcal{R}^C \setminus \mathcal{R}^Q$.*

Next, we show that if the domain contains all the positive income effect preferences, then our result can be strengthened - we can replace no subsidy in Theorem 1 by the following no bankruptcy condition.

Definition 9 *A rule $f : \mathcal{R}^n \to Z$ satisfies* **no bankruptcy** *if there exists $\ell \leq 0$ such that for every $R \in \mathcal{R}^n$, we have $\sum_{i \in N} t_i(R) \geq \ell$.*

Obviously, no bankruptcy is a weaker property than no subsidy. No bankruptcy is motivated by settings where the seller has limited means to finance the auction participants. Theorem 1 can now be strengthened in the positive income effect domain.

Theorem 2 *Suppose $\mathcal{R} \supseteq \mathcal{R}^{++}$. Every MWEP rule is ex-post revenue optimal among the class of desirable rules satisfying no bankruptcy defined on $\mathcal{R}^n$.*

4.3 Pareto efficiency

Since no wastage is a minimal form of efficiency axiom, it is natural to explore the implications of stronger forms of efficiency. We now discuss the implications of Pareto efficiency in our problem and relate it to our results. Before we formally define it, we must state the obvious fact that no wastage is a much weaker but more testable axiom in practice than Pareto efficiency. Our results establish that even if a seller maximizes her revenue with this weak form of efficiency, it will be forced to use a Pareto efficient rule.

Definition **10** *A rule $f : \mathcal{R}^n \to Z$ is* **Pareto efficient** *if at every preference profile $R \in \mathcal{R}^n$, there exists no allocation $((\hat{a}_1, \hat{t}_1), \dots, (\hat{a}_n, \hat{t}_n))$ such that*

$$(\hat{a}_i, \hat{t}_i) \ R_i \ f_i(R) \qquad \forall \ i \in N$$
$$\sum_{i \in N} \hat{t}_i \geq \textsc{Rev}^f(R),$$

with either the second inequality holding strictly or some agent i strictly preferring $(\hat{a}_i, \hat{t}_i)$ to $f_i(R_i)$.

The above definition is the appropriate notion of Pareto efficiency in this setting: (a) the first set of inequalities just say that no agent i prefers the allocation $(\hat{a}_i, \hat{t}_i)$ to that of the rule and (b) the second inequality ensures that the seller's revenue is not better in the proposed allocation. Without the second inequality, there is always an allocation where some money is distributed to all the agents to make them better off than the allocation in the rule.

The MWEP rule is Pareto-efficient (Morimoto and Serizawa, 2015). An immediate corollary of our results is the following.

Corollary **1** *Let $f : \mathcal{R}^n \to Z$ be a desirable rule. If $\mathcal{R}$ is rich and f satisfies no subsidy, then consider the following statements.*

1. *$f = f^{min}$.*
2. *$\textsc{Rev}^f(R) \geq \textsc{Rev}^{f'}(R)$ for any desirable rule $f' : \mathcal{R}^n \to Z$ satisfying no subsidy.*
3. *f is Pareto efficient.*

Statements (1) and (2) are equivalent, and each of them imply Statement (3).

If $\mathcal{R} \supseteq \mathcal{R}^+$ and f satisfies no bankruptcy, then the same equivalence between (1) and (2) holds with no subsidy weakened to no bankruptcy in (2), and each of them still imply (3).

Proof: The MWEP rule is Pareto efficient - first welfare theorem, see also Morimoto and Serizawa (2015). This implies (1) $\Rightarrow$ (3). The equivalence of (1) and (2) follows from Theorem 1. Hence, (2) $\Rightarrow$ (3).

Similarly, the implications with no bankruptcy follows from Theorem 2. ■

In other words, even if the seller maximizes her revenue among the set of all desirable rules satisfying no subsidy (or no bankruptcy in the positive income effect domain), it will be forced to use a Pareto efficient rule. Hence, we get Pareto efficiency as a corollary without imposing it explicitly.

If Pareto efficiency is explicitly imposed, then the following two results are known in the literature, and using them, we can strengthen Corollary 1 further.

1. In the *quasilinear* domain, every strategy-proof and Pareto efficient rule is a Groves rule (Holmstrom, 1979). [16] Imposing individual rationality and no subsidy immediately implies that the pivotal or the Vickrey-Clarke-Groves (VCG) rule is the unique strategy-proof rule satisfying Pareto efficiency, individual rationality, and no subsidy - notice that equal treatment of equals is not needed for this result and no wastage is implied by Pareto efficiency. The MWEP rule coincides with the VCG rule in the quasilinear domain.

2. In the classical domain $\mathcal{R}^C$ (containing *all* classical preferences), the MWEP rule is the unique rule satisfying strategy-proofness, individual rationality, Pareto efficiency, and no subsidy (Morimoto and Serizawa, 2015) - again, equal treatment of equals is not needed for this result and no wastage is implied by Pareto efficiency.

Both these results imply the following strengthening of Corollary 1 in quasilinear and classical domains - notice that the corollary may not hold in *every* rich classical domain.

Corollary 2 *Let $f : \mathcal{R}^n \rightarrow Z$ be a desirable rule. If $\mathcal{R} \in \{\mathcal{R}^Q, \mathcal{R}^C\}$ and f satisfies no subsidy, then the following statements are equivalent.*

1. $f = f^{min}$.

2. $\text{Rev}^f(R) \geq \text{Rev}^{f'}(R)$ *for any desirable rule $f' : \mathcal{R}^n \rightarrow Z$ satisfying no subsidy.*

3. *f is Pareto efficient.*

[16] Such revenue equivalence results usually require some richness in domain, for instance, space of valuations must be (topologically) *connected* (Chung and Olszewski, 2007). Such results and our results will fail if we do not have enough richness - for instance, in discrete domains.

4.4 Some examples illustrating necessity of additional axioms

In this section, we give some examples to illustrate the implications of our axioms on the result.

Notion of incentive compatibility and IR. Consider a rule that chooses the maximum Walrasian equilibrium allocation at every profile. Such a rule will satisfy no subsidy and all the properties of desirability except strategy-proofness. Similarly, an MWEP rule supplemented by a *participation fee* satisfies no subsidy and all the properties of desirability except ex-post IR. Both these rules generate more revenue than the MWEP rule. Hence, strategy-proofness and ex-post IR are necessary for our results to hold.

What is less clear is if we can relax the notion of incentive compatibility to Bayesian incentive compatibility in our results. For this, consider an example with a single object and quasilinear preferences. With *symmetric* agents (i.e., agents having independent and identical distribution of values), a symmetric Bayesian Nash equilibrium strategy of the first price auction is increasing and continuous function $b(\cdot)$ of valuations - for an exact expression of this function, see Krishna (2009). Consider the rule such that for each valuation profile $v = (v_1, \ldots, v_n)$, the outcome of the bid profile $(b(v_1), \ldots, b(v_n))$ of the first price auction is chosen. Call this rule *the first-price based rule*. It is Bayesian incentive compatible. Though, the first-price based rule satisfies no subsidy, ex-post individual rationality, and no wastage, it fails to satisfy ETE (unless, we break ties using uniform randomization). To see this, if two agents have same value, they bid the same amount in the first-price based rule. If there is no randomization to break ties, only one of those agents wins the object at his bid amount, whereas the other agent gets zero payoff. Since bid amount is less than the value in the first-price based rule, the winner gets positive payoff, and this violates ETE.

However, this can be rectified in two ways. First, whenever there is tie for the winning bid, all the winning agents get the object with equal probability. This introduces uniform randomization, and ETE is now satisfied. Hence, the *randomized* first-price based rule is Bayesian incentive compatible, satisfies ex-post IR, ETE, no wastage, and no subsidy. Obviously, there are profiles of values where such a first-price based rule generates more revenue than the Vickrey rule - winning bid in the first-price auction may be higher than the second highest value. [17]

An alternate approach to restoring ETE in the first-price based rule is to modify it

[17] It is well known that the expected revenue from both the auctions is the same. Also, as we discussed earlier, interim equivalence of strategy-proof and Bayesian incentive compatible rules are known for single object quasilinear models.

in a deterministic manner whenever there is a tie in the winning bids. Consider a profile of values $(v_1, \ldots, v_n)$ such that more than one agent has bid the highest amount, say, B. Note that this bid B corresponds to value $b^{-1}(B)$. In such a case, we break the winning agent tie deterministically by giving the object (with probability 1) to one of the winning agents. Further, we ask him to pay his value $b^{-1}(B)$. This ensures that the winner and the losing agents all get a payoff of zero, and thus, it restores ETE. More formally, the rule corresponding to this *modified first-price based rule* is the following.

1. Agents submit their values $(v_1, \ldots, v_n)$.

2. If there is a unique highest valued agent i, he is given the object and he pays $b(v_i)$, where b is the unique symmetric Bayesian equilibrium bidding function of the first-price auction.

3. If there are more than one highest valued agents, then *any* one of them is given the object and is asked to pay his value.

Notice that this only modifies the rule corresponding to the first-price auction at zero measure profiles of values. Hence, the modified first-price based rule is Bayesian incentive compatible. Further, it is deterministic, satisfies ETE, no wastage, no subsidy, and ex-post IR. Because of the same reasons given for first-price auction, there are profiles of values where such a modified first-price based rule generates more revenue than the Vickrey rule.

This illustrates that we cannot relax strategy-proofness to Bayesian incentive compatibility in our results.

No wastage. It is easy to see that no wastage is required for our result - in the quasilinear domain of preferences with one object, Myerson (1981) shows that Vickrey rule with an *optimally* chosen reserve price maximizes expected revenue for independent and identically distributed values of agents. Such a rule wastes the object and generates more revenue than the Vickrey rule, which is also the MWEP rule, at some profiles of preferences.

No wastage is also necessary in a more indirect manner. Consider the domain of quasilinear preferences with two objects $M \equiv \{a, b\}$ and $N = \{1, 2, 3\}$. We show that the seller may increase her revenue by *not* selling all the objects. Consider a profile of valuations as follows:

$$v_1(a) = v_1(b) = 5$$

$$v_2(a) = v_2(b) = 4$$

$$v_3(a) = v_3(b) = 1.$$

The MWEP price at this profile is $p_a^{min} = p_b^{min} = 1$, which generates a revenue of 2 to the seller. On the other hand, suppose the seller conducts a Vickrey rule of object a only. Then, he generates a revenue of 4. Hence, the seller can increase her revenue at some profiles of valuations by withholding objects. Notice that withholding objects is a stronger violation of efficiency, and is easier to detect than misallocating the objects among agents.

In allocating public assets, governments are supposed to pursue several goals such as revenue and efficiency. Usually, revenue and efficiency are not compatible. No wastage is a mild requirement on efficiency and our result shows how revenue maximization can be reconciled with efficiency using no wastage.

Equal treatment of equals. Consider an example with one object and two agents in the quasilinear domain of preferences. Hence, the preference of each agent $i \in \{1, 2\}$ can be described by his *valuation* for the object v_i. Note that the MWEP rule collapses to the Vickrey rule for this problem.

We define the following rule: the object is first offered to agent 1 at price $p > 0$; if agent 1 accepts the offer, then he gets the object at price p and agent 2 does not get anything and does not pay anything; else, agent 2 is given the object for free.

This rule generates a revenue of p whenever $v_1 \geq p$ (but generates zero revenue otherwise). However, note that the Vickrey rule generates a revenue of v_2 when $v_1 > v_2$. Hence, if $p > v_2$, then this rule generates more revenue that the Vickrey rule. Also, notice that this rule satisfies no subsidy and all the properties of desirability except equal treatment of equals.

No subsidy. It is tempting to conjecture that no subsidy can be relaxed in quasilinear domain of preferences. A natural approach to prove this is to use Theorem 1, which applies to the quasilinear domain, in the following way: (1) For every desirable rule, we construct another desirable rule which satisfies no subsidy and generates more revenue; (2) Use Theorem 1 to arrive at the conclusion that the MWEP rule is revenue-optimal in the class of desirable rule. The first step does not quite work. In the quasilinear domain, every desirable rule can be converted to a strategy-proof, individually rational, and no subsidy rule using "multidimensional" versions of revenue equivalence formula (Jehiel et al., 1999; Krishna and Maenner, 2001; Milgrom and Segal, 2002; Chung and Olszewski, 2007; Heydenreich et al., 2009). But such a transformation may not preserve equal treatment of equals. As a result, we cannot apply Step (2) any more. We now give a concrete example to illustrate that our result does not hold without no subsidy.

For the example, consider one object and two agents in the quasilinear domain - hence, preferences of agents can be represented by their valuations v_1 and v_2. Further, assume that valuations lie in $\mathbb{R}_{++}$. Choose $k \in (0,1)$ and define the rule $f \equiv (a,t)$ as follows: for every (v_1, v_2)

$$a(v_1,v_2) = \begin{cases} (1,0) & \text{if } kv_1 > v_2, \\ (0,1) & \text{otherwise,} \end{cases}$$

$$t_1(v_1,v_2) = \begin{cases} -(v_2 - kv_2) & \text{if } a_1(v_1,v_2) = 0, \\ \frac{v_2}{k} - (v_2 - kv_2) & \text{if } a_1(v_1,v_2) = 1, \end{cases}$$

$$t_2(v_1,v_2) = \begin{cases} 0 & \text{if } a_2(v_1,v_2) = 0, \\ kv_1 & \text{if } a_2(v_1,v_2) = 1. \end{cases}$$

It is straightforward to check that the object allocation rule a is monotone (i.e., fixing the valuation of one agent, if valuation of the other agent is increased, his allocation probability increases) and payments satisfy the revenue equivalence formula, and hence, the rule is strategy-proof (a more direct proof is also possible). It is also not difficult to see that utilities of the agents are always non-negative, and hence, individual rationality holds. Finally, if $v_1 = v_2$, we have

$$a_1(v_1,v_2) = 0, a_2(v_1,v_2) = 1, \quad t_1(v_1,v_2) = -(v_2 - kv_2), t_2(v_1,v_2) = kv_1.$$

Hence, net utility of agent 1 is $v_2 - kv_2$ and that of agent 2 is $v_1 - kv_1$, which are equal since $v_1 = v_2$. This shows that the rule satisfies equal treatment of equals.

However, the rule pays agent 1 when he does not get the object. Thus, it violates no subsidy. The revenue from this rule when $kv_1 > v_2$ is

$$v_2\Big(\frac{1}{k} + k - 1\Big) \geq v_2.$$

The Vickrey rule generates a revenue of v_2 when $kv_1 > v_2$. Hence, this rule generates more revenue than the Vickrey rule when $kv_1 > v_2$. This shows that we cannot drop no subsidy from Theorem 1. [18]

4.5 Discussions on applicability of the results

As discussed in the introduction, our results are driven by a particular set of assumptions we have made in the paper, which are different from the literature. Here, we give two real-life

[18] Further inspection reveals that the revenue from this rule when $v_1 = v_2 = v$ is $kv - v(1-k) = v(2k-1)$. So, if $k < \frac{1}{2}$, this revenue approaches $-\infty$ as $v \to \infty$. Hence, this rule even violates no bankruptcy.

examples of allocation problems, where most of the assumptions made in the paper appear to make sense.

Indian Premier League auctions. A professional cricket league, called the *Indian Premier League (IPL)* was started in India in 2007. [19] Eight Indian cities were chosen and it was decided to have a team from each of those cities (i.e., eight heterogeneous objects were sold). An auction was held to sell these teams to interested owners (bidders). The auctions, whose details are not available in public domain, fetched more than 700 million US Dollars in revenue to IPL. Clearly, it does not make sense for two teams to have the the same owner - so, the unit demand assumption in our model is satisfied in this problem. The huge sums of bids implied that most of these teams were financed out of loans from banks, which implies non-quasilinear preferences of bidders. Further, when IPL was starting out, it must be interested in starting with teams in as many cities as possible - else, it would have sent a wrong signal to its future prospects. Indeed, all the teams were sold with high bid prices. So, a natural objective for IPL seems to be revenue maximization with no wastage. Finally, as is common in such settings, IPL did not subsidize any bidders.

Online advertisement auctions. Google sells billions of dollars worth of keywords using auctions for advertisement slots (Edelman et al., 2007; Varian, 2009). Many other search engines also sell *display advertisement* slots on webpages, which are auctioned as soon as web pages are displayed (Lahaie et al., 2008; Ghosh et al., 2009). Usually, each advertisement slot is awarded a unique bidder - so, the unit demand assumption is satisfied. [20] It is not clear whether Google uses reserve prices or not, but there is widespread belief that Google aims to be efficient. [21] However, it is fair to say that Google aims to maximize revenue from its sale of advertisement slots. The bidders are usually given a fixed budget to work with, and this results in an extreme form of non-quasilinearity. This has started a big literature on auctions with budget constraints in the computer science community (Ashlagi et al., 2010; Dobzinski et al., 2012; Lavi and May, 2012). Finally, Google does not subsidize any of its bidders.

[19] Interested readers can read the Wiki entry for IPL: https://en.wikipedia.org/wiki/Indian_Premier_League and a news article here: http://content-usa.cricinfo.com/ipl/content/current/story/333193.html.

[20] The analysis of this problem has been done under the unit demand assumption in the literature (Edelman et al., 2007; Varian, 2009).

[21] See this issue being discussed in a blog post by Noam Nisan: https://agtb.wordpress.com/2009/06/09/revenue-vs-efficiency-in-auctions/

Another example that fits our model is the allocation of public housing to citizens in different countries (Andersson and Svensson, 2014; Andersson et al., 2016), where houses are allocated to agents with unit demand constraint. These examples reinforce the fact that even though a precise description to revenue maximizing multi-object auction is impossible in many settings, for a variety of problems where no wastage makes sense, the MWEP rule is a strong candidate.

In the two examples above, the seller is not the Government. It makes more sense for such a seller to maximize her revenue. Corollaries 1 and 2 establish that even if such a seller maximizes her revenue, under desirability and no subsidy she would be forced to pick an MWEP rule, which is Pareto efficient.

5 The proofs

In this section, we present all the proofs. The proofs, though tedious and far from trivial, do not require any sophisticated mathematical tool. This is an added advantage of our approach, and makes the results even more surprising. The proofs use the following fact very crucially: the MWEP rule chooses a Walrasian equilibrium outcome. [22]

Intuitions. Before diving into the proofs, we want to stress here that a greedy approach of proving our results would be to first prove that any desirable rule satisfying no subsidy and maximizing revenue must be Pareto efficient. In the quasilinear domain, using revenue equivalence will then pin down the MWEP (VCG) rule. This approach will fail in our setting because our results work even without quasilinearity and revenue equivalence does not hold in such domains. Our proofs work by showing various implications of desirability and no subsidy on consumption bundles of agents. It uses richness of the domain to derive these implications. In that sense, it departs from traditional Myersonian techniques, where revenue maximization is a programming problem with object allocation rules as decision variables. This also means that our proof is less intuitive than standard approaches for quasilinear preference domain.

We start off by showing an elementary lemma which shows that if a desirable rule gives every agent weakly better consumption bundles than an MWEP rule at every preference

[22] With quasilinear preferences, if valuations of agents satisfy the *gross substitutes* property, then a minimum Walrasian equilibrium price vector exists, but it is no longer strategy-proof (Gul and Stacchetti, 1999). Hence, it is not clear if our results can be extended to such a model. We leave this as an open question.

profile, then its revenue is less than the MWEP rule. This lemma will be used to prove both our results.

Lemma **1** *For every desirable rule $f : \mathcal{R}^n \to Z$, where $\mathcal{R}$ is a rich domain, and for every $R \in \mathcal{R}^n$, the following holds:*

$$\big[f_i(R)\ R_i\ f_i^{min}(R)\ \forall\ i \in N\big] \Rightarrow \big[\textsc{Rev}^{f^{min}}(R) \geq \textsc{Rev}^f(R)\big],$$

where f^{min} is an MWEP rule.

Proof: Fix a profile of preferences R and denote $f^{min}(R) \equiv (z_1, \ldots, z_n)$, where for each $i \in N$, $z_i \equiv (a_i, p^{min}_{a_i}(R))$. Now, for every $i \in N$, we have $f_i(R) \equiv (a_i(R), t_i(R))\ R_i\ (a_i, p^{min}_{a_i}(R))$ and by the Walrasian equilibrium property, $(a_i, p^{min}_{a_i}(R))\ R_i\ (a_i(R), p_{a_i(R)})$. This gives us $t_i(R) \leq p_{a_i(R)}$ for each $i \in N$. Hence,

$$\textsc{Rev}^f(R) = \sum_{i \in N} t_i(R) \leq \sum_{i \in N} p_{a_i(R)} = \textsc{Rev}^{f^{min}}(R),$$

where the last equality follows from the fact that all the objects with positive price are allocated in a Walrasian equilibrium and f also allocates all the objects (because of no wastage). ■

5.1 Proof of Theorem 1

We start with a series of Lemmas before providing the main proof. Throughout, we assume that $\mathcal{R}$ is a rich domain of preferences and f is a desirable rule satisfying no subsidy on $\mathcal{R}^n$. For the lemmas, we need the following definition. A preference R_i is (a,t)-**favoring** for $t > 0$ and $a \in M$ if for price vector p with $p_a = t, p_b = 0$ for all $b \neq a$, we have $D(R_i, p) = \{a\}$. An equivalent way to state this is that R_i is (a,t)-favoring for $t > 0$ and $a \in M$ if $V^{R_i}(b, (a,t)) < 0$ for all $b \neq a$.

Lemma **2** *For every preference profile R, for every $i \in N$ with $f_i(R) \neq 0$, and for every R'_i such that R'_i is an $f_i(R)$-favoring preference, we have $f_i(R'_i, R_{-i}) = f_i(R)$.*

Proof: If $a_i(R'_i, R_{-i}) = a_i(R)$, then strategy-proofness implies $t_i(R'_i, R_{-i}) = t_i(R)$, and we are done. Suppose $a = a_i(R) \neq a_i(R'_i, R_{-i}) = b$. By strategy-proofness,

$$\big[(b, t_i(R'_i, R_{-i}))\ R'_i\ (a, t_i(R))\big] \Rightarrow\ \big[t_i(R'_i, R_{-i}) \leq V^{R'_i}(b, (a, t_i(R)))\big].$$

Since R'_i is $(a, t_i(R))$-favoring, we must have $V^{R'_i}(b, (a, t_i(R))) < 0$. This implies that $t_i(R'_i, R_{-i}) < 0$, which is a contradiction to no subsidy. ■

Lemma **3** *For every preference profile R and for every $i \in N$ with $f_i(R) \neq 0$, there is no $j \neq i$ such that R_j is $f_i(R)$-favoring.*

Proof: Assume for contradiction that there is $j \neq i$ such that R_j is $f_i(R)$-favoring. Consider $R'_i \equiv R_j$. By equal treatment of equals $f_i(R'_i, R_{-i})\ I_j\ f_j(R'_i, R_{-i})$. Also, by Lemma 2, $f_i(R'_i, R_{-i}) = f_i(R)$. Hence, $f_i(R)\ I_j\ f_j(R'_i, R_{-i})$. Note that $a = a_i(R) = a_i(R'_i, R_{-i}) \neq a_j(R'_i, R_{-i}) = b$. Then, $t_j(R) = V^{R_j}(b, f_i(R)) < 0$, where the strict inequality followed from the fact that R_j is $f_i(R)$-favoring and $b \neq a_i(R)$. But this contradicts no subsidy. ■

Lemma **4** *For every preference profile R, for every $i \in N$, for every (a,t) with $a = a_i(R) \neq 0$ and $t > 0$, if there exists $j \neq i$ such that R_j is (a,t)-favoring, then $t_i(R) > t$.*

Proof: Suppose $t_i(R) \leq t$. Since R_j is (a,t)-favoring, $t_i(R) \leq t$ implies that R_j is also $f_i(R) \equiv (a, t_i(R))$-favoring. This is a contradiction to Lemma 3. ■

For the proof, we use a slightly stronger version of (a,t)-favoring preference.

Definition **11** *For every bundle (a,t) with $t > 0$ and for every $\epsilon > 0$, a preference $R_i \in \mathcal{R}$ is a $(a,t)^\epsilon$-favoring preference if it is a (a,t)-favoring preference and*

$$V^{R_i}(a,(0,0)) < t + \epsilon$$
$$V^{R_i}(b,(0,0)) < \epsilon \ \forall\ b \in M \setminus \{a\}.$$

The following lemma shows that if $\mathcal{R}$ is rich, then $(a,t)^\epsilon$-favoring preferences exist for every (a,t) and ϵ.

Lemma **5** *Suppose $\mathcal{R}$ is rich. Then, for every bundle (a,t) with $t > 0$ and for every $\epsilon > 0$, there exists a preference $R_i \in \mathcal{R}$ such that it is $(a,t)^\epsilon$-favoring.*

Proof: Define $\hat{p}$ as follows:

$$\hat{p}_a = t, \quad \hat{p}_b = 0 \ \ \forall\ b \neq a.$$

Define p as follows:

$$p_a = t + \epsilon, \quad p_0 = 0, \quad p_b = \epsilon \ \ \forall\ b \in M \setminus \{a\}.$$

By richness, there exists R_i such that $D(R_i, \hat{p}) = \{a\}$ and $D(R_i, p) = \{0\}$. But this implies that R_i is (a,t)-favoring and

$$V^{R_i}(a,(0,0)) < t + \epsilon$$
$$V^{R_i}(b,(0,0)) < \epsilon \ \forall\ b \in M \setminus \{a\}.$$

Hence, R_i is $(a,t)^\epsilon$-favoring. ■

We will now prove Theorem 1 using these lemmas.

Proof of Theorem 1

Proof: Fix a desirable rule $f : \mathcal{R}^n \to Z$ satisfying no subsidy, where $\mathcal{R}$ is a rich domain of preferences. Fix a preference profile $R \in \mathcal{R}^n$. Let $(z_1, \ldots, z_n) \equiv f^{min}(R)$ be the allocation chosen by an MWEP rule f^{min} at R. For simplicity of notation, we will denote $z_j \equiv (a_j, p_j)$, where $p_j \equiv p^{min}_{a_j}(R)$, for all $j \in N$. We prove that $f_i(R)\ R_i\ z_i$ for all $i \in N$, and by Lemma 1, we will be done.

To prove that $f_i(R)\ R_i\ z_i$ for all $i \in N$, assume for contradiction that there is some agent, without loss of generality agent 1, such that $z_1\ P_1\ f_1(R)$. We first construct a finite sequence of agents and preferences $(i_1, R'_{i_1}), (i_2, R'_{i_2}), \ldots, (i_n, R'_{i_n})$ satisfying certain properties. For notational convenience, we denote this sequence as $(1, R'_1), \ldots, (n, R'_n)$. This sequence satisfies the properties that for every $k \in \{1, \ldots, n\}$,

1. $z_k\ P_k\ f_k(R)$ if $k = 1$ and $z_k\ P_k\ f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ if $k > 1$, where $N_{k-1} \equiv \{1, \ldots, k-1\}$.
2. $a_k \neq 0$,
3. R'_k is z^ϵ_k-favoring for some $\epsilon > 0$ but arbitrarily close to zero.

Now, we construct this sequence inductively.

Step 1 - Constructing $(1, R'_1)$. Pick $\epsilon > 0$ but arbitrarily close to zero and consider a z^ϵ_1-favoring preference R'_1 - by Lemma 5, such R'_1 can be constructed. By our assumption, $z_1\ P_1\ f_1(R)$. Suppose $a_1 = 0$. Then, $z_1 = (0,0)\ P_1\ f_1(R)$, which contradicts individual rationality. Hence, $a_1 \neq 0$.

Step 2 - Constructing (k, R'_k) for $k > 1$. We proceed inductively - suppose, we have already constructed $(1, R'_1), \ldots, (k-1, R'_{k-1})$ satisfying Properties (1), (2), and (3). Consider agent j such that $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$.

If $j = k-1$, then individual rationality implies that

$$t_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_{k-1}}(a_{k-1}, (0,0)) < p_{k-1} + \epsilon,$$

where the last inequality followed from the fact that R'_{k-1} is $(z_{k-1})^\epsilon$-favoring. Further, by our induction hypothesis, $z_{k-1}\ P_{k-1}\ f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}})$, and we get

$$p_{k-1} < V^{R_{k-1}}(a_{k-1}, f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}})).$$

Since ϵ is arbitrarily close to zero, we get $t_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}}) < V^{R_{k-1}}(a_{k-1}, f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}}))$. But this implies that $f_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}})\ P_{k-1}\ f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}})$, which contradicts strategy-proofness. Hence, $j \neq k-1$.

If $j \in N_{k-2}$, then by individual rationality, we get $t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_j}(a_{k-1}, (0,0)) < \epsilon$, where the last inequality followed from the fact that R'_j is $(z_j)^\epsilon$-favoring and $j \neq (k-1)$. Since ϵ is arbitrarily close to zero, we get

$$t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) < \epsilon < p_{k-1}. \tag{1}$$

But, notice that agent $(k-1) \neq j$ and R'_{k-1} is z_{k-1}-favoring (since it is $(z_{k-1})^\epsilon$-favoring). Further $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$. Then, Lemma 4 implies that $t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) > p_{k-1}$, which is a contradiction to Inequality 1.

Thus, we have established $j \notin N_{k-1}$. Hence, we denote $j \equiv k$, and note that

$$z_k\ R_k\ z_{k-1}\ P_k\ f_k(R'_{N_{k-1}}, R_{-N_{k-1}}),$$

where the first inequality follows from the Walrasian equilibrium property and the second follows from the fact that $a_k(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$ and $p_{k-1} < t_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}})$ (Lemma 4). Hence Property (1) is satisfied for agent k. Next, if $a_k = 0$, then $(0,0) = z_k\ P_k\ f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ contradicts individual rationality. Hence, Property (2) also holds. Now, we satisfy Property (3) by constructing R'_k, which is z_k^ϵ-favoring for some $\epsilon > 0$ but arbitrarily close to zero - by Lemma 5, such R'_k can be constructed.

Thus, we have constructed a sequence $(1, R'_1), \ldots, (n, R'_n)$ such that $a_k \neq 0$ for all $k \in N$. This is impossible since $n > m$, giving us the required contradiction. ■

5.2 Proof of Theorem 2

We now fix a desirable rule $f : \mathcal{R}^n \to Z$, where $\mathcal{R} \supseteq \mathcal{R}^+$. Further, we assume that f satisfies no bankruptcy, where the corresponding bound as $\ell \leq 0$. We start by proving an analogue of Lemma 4.

Lemma 6 *For every preference profile $R \in \mathcal{R}^n$, for every $i \in N$, and every $(a,t) \in M \times \mathbb{R}_+$ with $a = a_i(R) \neq 0$ and $t > 0$, if there exists $j \neq i$ such that*

$$V^{R_j}(b, (a,t)) < -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \ell,$$

then $t_i(R) > t$.

Proof: Assume for contradiction $t_i(R) \leq t$. Consider $R'_i = R_j$. By strategy-proofness, $f_i(R'_i, R_{-i})\ R'_i\ f_i(R) = (a, t_i(R))$. By equal treatment of equals,

$$f_j(R'_i, R_{-i})\ I_j\ f_i(R'_i, R_{-i})\ R_j\ (a, t_i(R)).$$

Note that either $a_i(R'_i, R_{-i}) \neq a$ or $a_j(R'_i, R_{-i}) \neq a$. Without loss of generality, assume that $a_j(R'_i, R_{-i}) = b \neq a$. Then, using the fact that $(b, t_j(R'_i, R_{-i}))\ R_j\ (a, t_i(R))$ and $t_i(R) \leq t$, we get

$$\begin{aligned} t_j(R'_i, R_{-i}) &\leq V^{R_j}(b, (a, t_i(R))) \\ &\leq V^{R_j}(b, (a, t)) \\ &< -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \ell. \end{aligned}$$

By individual rationality

$$t_i(R'_i, R_{-i}) \leq V^{R'_i}(a_i(R'_i, R_{-i}), (0,0)) \leq \max_{c \in M} V^{R'_i}(c, (0,0)).$$

Further, individual rationality also implies that for all $k \notin \{i, j\}$,

$$t_k(R'_i, R_{-i}) \leq V^{R_k}(a_i(R'_i, R_{-i}), (0,0)) \leq \max_{c \in M} V^{R_k}(c, (0,0)).$$

Adding these three sets of inequalities above, we get

$$\begin{aligned} &\sum_{k \in N} t_k(R'_i, R_{-i}) \\ &< -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \ell + \max_{c \in M} V^{R'_i}(c, (0,0)) + \sum_{k \in N \setminus \{i,j\}} \max_{c \in M} V^{R_k}(c, (0,0)) \\ &= -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \ell + \max_{c \in M} V^{R_j}(c, (0,0)) + \sum_{k \in N \setminus \{i,j\}} \max_{c \in M} V^{R_k}(c, (0,0)) \\ &\leq -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + (n-1)\big(\max_{k \in N \setminus \{i\}} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \ell \\ &\leq -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0)) - \max_{k \in N \setminus \{i\}} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \ell \\ &\leq \ell. \end{aligned}$$

This contradicts no bankruptcy. ■

Using Lemma 6, we can mimic the proof of Theorem 1 to complete the proof of Theorem 2. We start by defining a class of positive income effect preferences by strengthening the notion of $(a, t)^\epsilon$-favoring preference. For every $(a, t) \in M \times \mathbb{R}_+$, for each $\epsilon > 0$, and for each $\delta > 0$, define $\mathcal{R}((a, t), \epsilon, \delta)$ be the set of preferences such that for each $\hat{R}_i \in \mathcal{R}((a, t), \epsilon, \delta)$, the following holds:

1. $\hat{R}_i$ is $(a,t)^\epsilon$-favoring and
2. $V^{\hat{R}_i}(b,(a,t)) < -\delta$ for all $b \neq a$.

A graphical illustration of $\hat{R}_i$ is provided in Figure 4. Since $\delta > 0$, it is clear that a $\hat{R}_i$ can be constructed in $\mathcal{R}((a,t),\epsilon,\delta)$ such that it exhibits positive income effect. Hence, $\mathcal{R}^+ \cap \mathcal{R}((a,t),\epsilon,\delta) \neq \emptyset$.

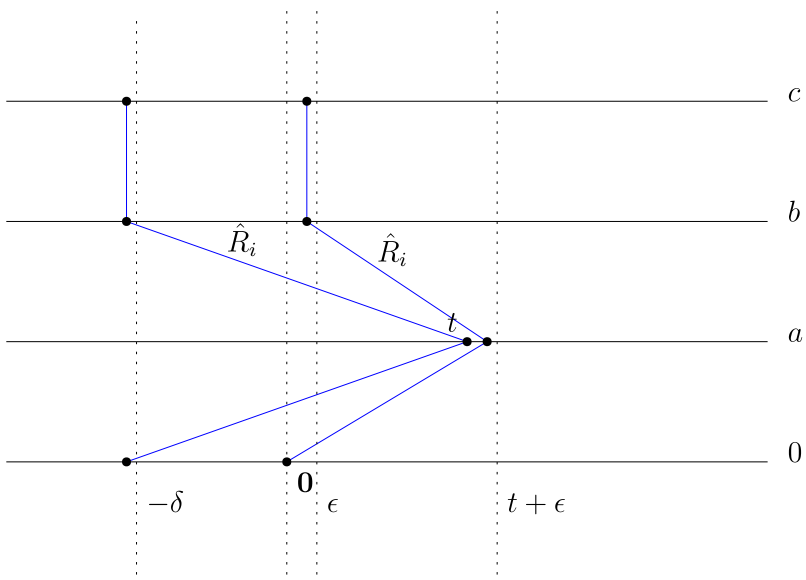

Figure 4: Illustration of $\hat{R}_i$

Proof of Theorem 2

Proof: Now, we can mimic the proof of Theorem 1. We only show parts of the proof that requires some change. As in the proof of Theorem 1, by Lemma 1, we only need to show that for every profile of preferences R and for every $i \in N$, $f_i^{min}(R)\ R_i\ f(R)$, where f^{min} is an MWEP rule. Assume for contradiction that there is some profile of preferences R and some agent, without loss of generality agent 1, such that $z_1\ P_1\ f_1(R)$, where $(z_1,\ldots,z_n) \equiv f^{min}(R)$ be the allocation chosen by the MWEP rule at R. For simplicity of notation, we will denote $z_j \equiv (a_j, p_j)$, where $p_j \equiv p_{a_j}^{min}(R)$, for all $j \in N$.

Define $\bar{\delta} > 0$ as follows:

$$\bar{\delta} := n\big(\max_{k\in N}\max_{c\in M} V^{R_k}(c,(0,0))\big) - \ell.$$

We first construct a finite sequence of agents and preferences: $(1,R'_1),(2,R'_2),\ldots,(n,R'_n)$ such that for every $k \in \{1,\ldots,n\}$,

1. $z_k \; P_k \; f_k(R)$ if $k = 1$ and $z_k \; P_k \; f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ if $k > 1$, where $N_{k-1} \equiv \{1, \ldots, k-1\}$.
2. $a_k \neq 0$,
3. $R'_k \in \mathcal{R}^+ \cap \mathcal{R}(z^k, \epsilon, \bar{\delta})$ for some $\epsilon > 0$ but arbitrarily close to zero.

Now, we can complete the construction of this sequence inductively as in the proof of Theorem 1 (using Lemma 6 instead of Lemma 4), giving us the desired contradiction. ■

6 Relation to the literature

Our paper is related to two strands of literature in mechanism design: (1) multi-object revenue maximization literature and (2) literature on object allocation problem without quasilinearity. We discuss them in some detail below.

Revenue maximization literature. Ever since the work of Myerson (1981), various extensions of his work to multi-object case have been attempted in quasilinear domain. Most of these extensions focus on the single agent (or, screening problem of a monopolist) with additive valuations (value for a bundle of objects is the sum of values of objects). Armstrong (1996, 2000) are early papers that show the difficulty in extending Myerson's optimal mechanisms to multiple objects case - he identifies optimal mechanisms for the cases where agents' preferences are binary, i.e, the valuations of each agent on a given object are only low and high values, but demonstrates that it is too complicated to identify optimal mechanisms for other cases. [23] Rochet and Choné (1998) show how to extend the convex analysis techniques in Myerson's work to multidimensional environment and point out various difficulties in the derivation of an optimal mechanism. These difficulties are more precisely formulated in the following line of work for the single agent additive valuation case: (1) optimal mechanism may require randomization (Thanassoulis, 2004; Manelli and Vincent, 2007); (2) simple mechanism like selling each good separately (Daskalakis et al., 2016) and selling all the goods as a grand bundle (Manelli and Vincent, 2006) are optimal for very specific distributions; (3) there is inherent revenue non-monotonicity of the optimal mechanism - if we take two distributions with one first-order stochastic-dominating the other, the optimal mechanism revenue may not increase (Hart and Reny, 2015); (4) the optimal mechanism may require an infinite menu of prices (Hart and Nisan, 2013).

[23] Whenever we say optimal mechanisms, we mean, like in Myerson (1981), an expected revenue maximizing mechanism under incentive and participation constraints with respect to some prior distribution.

Since these extensions are for a single agent who has *additive* valuation for bundles of objects, this may give the impression that the multi-object optimal mechanism design problem is difficult *only* when agents can be allocated more than one object. However, this impression is not true - the source of the problem is the multiple dimension of private information, which continues to exist even in the unit demand model considered in our paper. In our model, even with quasilinearity, the multiple dimensions of private information will be valuations for each object. As illustrated in Armstrong (1996, 2000), the multiple dimensions of private information implies that the incentive constraints become complicated to handle. In quasi-linear domain, the Myersonian approach to this problem would pin down payments of agents in terms of object allocation rules using the well known revenue equivalence formula (Krishna and Maenner, 2001; Milgrom and Segal, 2002). Then, the objective function (maximizing sum of expected payments) is rewritten in terms of object allocation rule. On the constraint side, necessary and sufficient conditions are identified for the object allocation rule to be implementable (Rochet, 1987; Jehiel et al., 1999; Bikhchandani et al., 2006), and they are put as constraints. Whether agents can be allocated at most one object or multiple objects, the multidimensional nature of private information makes *both* the revenue equivalence formula and the constraints of the optimization problem become extremely difficult to handle. Vohra (2011) provides a linear programming approach to study such multidimensional mechanism design problems and points out similar difficulties.

Further, it is unclear how some of the above single agent results can be extended to the case of multiple agents. In the multiple agent problems, the set of feasible allocations starts interacting with the incentive constraints of the agents. Further, the standard Bayesian incentive compatibility constraints become challenging to handle. Note that in the single agent problem, these notions of incentive compatibility are equivalent, and for one-dimensional mechanism design problems, they are equivalent in a useful sense (Manelli and Vincent, 2010; Gershkov et al., 2013). Because we work in a model without quasilinearity, we are essentially operating in an "infinite" dimensional mechanism design problem. Hence, we should expect the problems discussed in quasilinear environment to appear in an even more complex way in our model.

To circumvent the difficulties from the multidimensional private information and multiple agents, a literature in computer science has developed approximately optimal mechanisms for our model - multiple objects and multiple agents with unit demand agents (but with quasilinearity). Contributions in this direction include Chawla et al. (2010a,b); Briest et al. (2010); Cai et al. (2012). Many of these approximate mechanisms allow for randomization. Further, these approximately optimal mechanisms involve reserve prices and violate no wastage axiom. It is unlikely that these results extend to environments without quasilinearity.

Finally, the Myersonian approach may not work if preferences are not quasilinear. In a companion paper (Kazumura et al., 2017), we investigate mechanism design without quasilinearity more abstractly and illustrate the difficulty of solving the single object optimal mechanism design problem. Hence, solving for full optimality without imposing the additional axioms that we put seems to be even more challenging in our model. In that sense, our results provide a useful resolution to this complex problem.

Our work can be connected to a beautiful result by Bulow and Klemperer (1996) and its extension by Roughgarden et al. (2015). In Bulow and Klemperer (1996), it was shown that (under standard independent and identical agent assumption with *regular* distribution) a single object optimal mechanism (with quasilinear preferences) for n agents generates less expected revenue than a single object Vickrey rule for $(n+1)$ agents. Hence, if the cost of recruiting an agent is small, then the Vickrey rule can be recommended. [24] This result has been extended to our multi-object unit-demand agent setting with quasilinear preferences: the expected revenue maximizing mechanism for n agents generates less expected revenue than the VCG rule for $(m+n)$ agents, where m is the number of objects (Roughgarden et al., 2015) - note that the MWEP rule is the VCG rule in the quasilinear domain. Combined with our result, we can strengthen this result in quasilinear domain as follows. For every desirable and no subsidy rule for $(m+n)$ agents, consider the difference between its expected revenue and the maximum expected revenue for n agents. This difference is non-negative and is maximized by the VCG (MWEP) rule. [25]

Non-quasilinearity literature. There is a short but important literature on object allocation problem with non-quasilinear preferences. Baisa (2016a) considers the single object model and allows for randomization with non-quasilinear preferences. He introduces a novel rule in his setting and studies its optimality properties (in terms of revenue maximization). We do not consider randomization and our solution concept is different from his. Further, ours is a model with multiple objects.

The literature with non-quasilinear preferences and multiple objects have traditionally

[24] Of course, one can argue that if we have $(n+1)$ agents, then the seller must use the *expected revenue maximizing* rule for $(n+1)$ agents. The main point in Bulow and Klemperer (1996) is that the Vickrey rule is a prior-free robust rule, whereas the expected revenue maximizing mechanism requires knowledge of priors.

[25] The computer science literature is interested in such prior-free bounds on optimal multidimensional rules (which is hard to compute) - a recent paper by Eden et al. (2017) provide further extensions of Bulow-Klemperer results in multi-object environments where buyers can consume more than one object but have additive valuations.

looked at Pareto efficient rules. As discussed earlier, the closest paper is Morimoto and Serizawa (2015) who consider the same model as ours. They characterize the MWEP rule using Pareto efficiency, individual rationality, incentive compatibility, and no subsidy when the domain includes *all* classical preferences - see an extension of this characterization in a smaller domain in Zhou and Serizawa (2016). Pareto efficiency and the *complete* class of classical preferences play a critical role in pinning down the MWEP rule in these papers. As Tierney (2016) points out, even in the quasilinear domain of preferences, there are desirable rules satisfying no subsidy which are different from the MWEP rule. By imposing revenue maximization as an objective instead of Pareto efficiency, we get the MWEP rule in our model. Pareto efficiency is obtained as an implication (Corollaries 1 and 2). Finally, our results work for not only the complete class of classical preferences, but for a large variety of domains, such as the class of all quasilinear preferences, one including all non-quasilinear preferences, one including all preferences exhibiting positive income effects, etc.

Tierney (2016) considers axioms like *no discrimination*, *welfare continuity*, and some stronger form of strategy-proofness to give various characterizations of the MWEP rule with reserve prices in the quasilinear domain. Using our result, he shows that in the *quasilinear domain*, the MWEP rule is the unique rule satisfying strategy-proofness, *no-discrimination*, individual rationality, no wastage, and *welfare continuity*.

In the single object model, earlier papers have carried out axiomatic treatment similar to Morimoto and Serizawa (2015) - work along this line includes Saitoh and Serizawa (2008); Sakai (2008, 2013b,a); Adachi (2014); Ashlagi and Serizawa (2012).

When the set of preferences include all or a very rich class of non-quasilinear preferences and we consider multi-object model where agents can consume more than one object, strategy-proofness and Pareto efficiency (along with other axioms) have been shown to be incompatible - (Kazumura and Serizawa, 2016) show this for multi-object allocation problems where agents can be allocated more than one object; (Baisa, 2016b) shows this for homogeneous object allocation problems; and Dobzinski et al. (2012); Lavi and May (2012) show similar results for hard budget-constrained model of a single object. Pareto efficiency along with other axioms play a crucial role in such impossibility results.

There is a literature in mechanism design and algorithmic game theory on single object allocation problem with budget-constrained agents - see Che and Gale (2000); Pai and Vohra (2014); Ashlagi et al. (2010); Lavi and May (2012). The budget-constraint in these papers introduces a particular form of non-quasilinearity in preferences of agents. Further, the budget-constraint in these models is *hard*, i.e., the utility from any payment above the budget is minus infinity. This assumption is not satisfied by the preferences considered in our model since it leads to discontinuities. Further, these papers focus on single object model.

7 Conclusion

We circumvent the technical difficulties of designing optimal rule in multi-object allocation problem by imposing additional axioms on rules. We believe that these additional axioms are appealing in a variety of environment. A consequence of these assumptions is that we provide robust recommendations on revenue maximizing rule: the MWEP rule is revenue-maximal profile-by-profile, and the preferences of agents need not be quasilinear. Our proofs are elementary and without any convex analysis techniques used in the literature. Whether we can weaken some of these axioms and further strengthen our results is a question for future research.

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Supplementary Appendix

A non-MWEP desirable rule

In this appendix, we reproduce an example of a desirable rule for quasilinear preferences from Tierney (2016). This example demonstrates that there is a desirable rule satisfying no subsidy on the quasilinear domain that is not an MWEP rule. It also illustrates that the space of desirable rules satisfying no subsidy may be complex to describe.

The example has three objects: $M := \{a, b, c\}$ and requires the following four *quasilinear* preferences. To remind, a quasilinear preference R_i of agent i can be described by a valuation function $v_i : M \to \mathbb{R}_+$. Hence, we report the valuation functions of these four preferences in Table 1 to describe the respective preferences. Denote the quasilinear preference corresponding to valuation functions $v^\alpha, v^\beta, v^\gamma, v^\lambda$ as $R^\alpha, R^\beta, R^\gamma, R^\lambda$ respectively.

	a	b	c
v^α	2	2	2
v^β	2	2	ϵ
v^γ	2	ϵ	2
v^λ	ϵ	2	2

Table 1: Four quasilinear preferences - $\epsilon > 0$ but arbitrarily close to zero

The example has five agents: $N := \{1, 2, 3, 4, 5\}$. The rule we describe works in the class of all quasilinear preferences, and we denote this domain by $\mathcal{Q}$. For any $i \in N$, we say a profile of preferences $R \equiv (R_1, \ldots, R_5) \in \mathcal{Q}^5$ is **special** for agent i if there exists a bijective map

$$\rho : (N \setminus \{i\}) \to \{\alpha, \beta, \gamma, \lambda\}$$

such that for each $j \in (N \setminus \{i\})$, $R_j = R^{\rho(j)}$. We say a preference profile R is **special** if there is some agent i such that R is special for i.

Before describing the rule, we make a comment about special preference profiles.

Claim **2** *For every special preference profile $R \in \mathcal{Q}^5$, $p_a^{min}(R) = p_b^{min}(R) = p_c^{min}(R) = 2$.*

Proof: Suppose R is special for agent i. Let p^* be the price vector: $p^*_a = p^*_b = p^*_c = 2$. Then for all $j \neq i$, $0 \in D(R_j, p^*)$ and for each $x \in M$, there is $j \neq i$ such that $x \in D(R_j, p^*)$. These properties ensure that p^* is a Walrasian equilibrium price vector at R. To see that it is the minimum Walrasian equilibrium price vector at R, assume for contradiction $p < p^*$ is the minimum Walrasian equilibrium price vector. If price of at least two objects are less than 2 in p, then $0 \notin D(R_j, p)$ for all $j \neq i$. This is impossible since at any Walrasian equilibrium, at least two agents must be allocated the null object. So, assume without loss of generality, $p_a = p_b = 2$ and $p_c < 2$. But then, $|\{j \in N \setminus \{i\} : \{c\} = D(R_j, p)\}| \geq 3$, which contradicts the fact that p is a Walrasian equilibrium price vector. Hence, $p^* = p^{min}(R)$. ■

Now, the rule $f^* : \mathcal{Q}^5 \to Z$ is defined as follows. Let p be a price vector with $p_a = p_b = p_c = 1$. For every preference profile $R \in \mathcal{Q}^5$ and for every $i \in N$, let $f_i^*(R) \equiv (a_i^*(R), p_i^*(R))$ be such that

$$a_i^*(R) \in \begin{cases} D(R_i, p) & \text{if } R \text{ is special for } i, \\ D(R_i, p^{min}(R)) & \text{otherwise,} \end{cases}$$

$$p_i^*(R) = \begin{cases} p_{a_i^*(R)} & \text{if } R \text{ is special for } i, \\ p^{min}_{a_i^*(R)}(R) & \text{otherwise.} \end{cases}$$

Further, f^* must allocate all the objects at R, i.e., for every $x \in M$, there exists $i \in N$ such that $a_i^*(R) = x$.

A clarification regarding the feasibility of f^* is in order. It is not clear that $a^*(R)$ is an object allocation. If R is not special, then by the definition of Walrasian equilibrium, a feasible object allocation can be chosen by $a^*(R)$ such that all the objects are allocated. If R is special, then it is special for either (a) one agent or (b) for two agents. We consider both the cases. Note here that by Claim 2, $p^{min}(R) \equiv (2, 2, 2)$.

Case 1. If it is special for some agent i only, then agent i can be assigned any object in $D(R_i, p)$. Since each agent $j \neq i$ has $0 \in D(R_j, p^{min}(R))$ (due to Claim 2), $a^*(R)$ can be chosen as a feasible object allocation. Moreover, since for each $S \subseteq M$, $|S| \leq |\{j \neq i : D(R_j, p^{min}(R)) \cap S \neq \emptyset\}|$, Hall's marriage theorem implies that $a^*(R)$ can allocate objects in $M \setminus \{a_i^*(R)\}$ to agents in $N \setminus \{i\}$. This implies that $a^*(R)$ can be constructed such that all the objects are assigned at R.

Case 2. If it is special for two agents $\{i, j\}$, then $R_i = R_j \in \{R^\alpha, R^\beta, R^\gamma, R^\lambda\}$. In that case, by the definition of p, $0 \notin D(R_i, p) = D(R_j, p)$ and $|D(R_i, p)| = |D(R_j, p)| \geq 2$. Hence, we can assign $a_i^*(R) \in D(R_i, p)$ and $a_j^*(R) \in D(R_j, p)$ such that $a_i^*(R) \neq a_j^*(R)$. Notice that $a_i^*(R), a_j^*(R) \in M$. Without loss of generality assume that $a_i^*(R) = a, a_j^*(R) = b$. Note that there is some $k \notin \{i, j\}$ such that $c \in D(R_k, p^{min}(R))$. Hence, $a^*(R)$ can be constructed such that all the objects are assigned at R. Also, $0 \in D(R_k, p^{min}(R))$ for all $k \notin \{i, j\}$ (due to Claim 2). As a result, $a^*(R)$ can be constructed as a feasible object allocation.

In principle, f^* is not defined uniquely since $a^*(R)$ can be chosen in various ways at some R by breaking the ties in the demand sets differently. Here, we refer to f^* as *any* one such selection of object allocation. Our next claim argues that f^* is strategy-proof.

Claim **3** *f^* is strategy-proof.*

Proof: Fix $R \in \mathcal{Q}^5$ and $i \in N$. If R is not special for i, then by changing his preference to R_i', (R_i', R_{-i}) is not special for i. In both the preference profiles, we pick the respective minimum Walrasian equilibrium allocation, and by Demange and Gale (1985), i cannot manipulate to R_i'.

If R is special for i, then by changing his preference to R_i', (R_i', R_{-i}) is also special for i. Hence, $a_i^*(R) \in D(R_i, p)$ and $a_i^*(R_i', R_{-i}) \in D(R_i', p)$. Clearly, agent i cannot manipulate to R_i'. ■

Since f^* does not discriminate between agents, it satisfies equal treatment of equals. By construction, it satisfies no subsidy and ex-post individual rationality. It also allocates all the object at every profile of preferences, and hence, satisfies no wastage. As a result, f^* is a desirable rule satisfying no subsidy in the domain of preferences $\mathcal{Q}^5$. However, if R is a special preference profile, revenue from f^* at R can be lower than the revenue from the MWEP rule. In particular, if R is special for i and i is assigned a (real) object in f^*, then the payment of i in f^* is strictly lower than the corresponding payment in the MWEP rule. Thus, f^* is not an MWEP rule.