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Abstract

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Abstract

This paper investigates how and to what extent nonlinearities, including the zero lower bound on the nominal interest rate, affect the estimate of the natural rate of interest in a dynamic stochastic general equilibrium model with sticky prices and wages. The estimated natural rate of interest in a nonlinear model is substantially different from that in its linear counterpart because of a contractionary effect of the zero lower bound. Price and wage dispersion, from which a linear model abstracts, play a minor role in identifying the natural rate.

Keywords: Natural rate of interest, Nonlinearity, Zero lower bound, Particle filter

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1 Introduction

The natural rate of interest—the equilibrium real interest rate that yields price stability (Wicksell, 1898)—has been a key concept for monetary policy analysis. In particular, a modern New Keynesian framework relates the concept of the natural rate to intertemporally optimizing agents and makes it relevant for social welfare (Woodford, 2003; Galí, 2008). The level of the natural interest rate in this framework is a useful indicator for policymakers because it is a benchmark as to whether policy is too tight or too loose from a welfare perspective.[1] However, the natural rate is unobservable and must be estimated. Whereas the literature has developed various empirical methods to infer the natural rate of interest, an increasing number of researchers have estimated the natural rate measures based on New Keynesian dynamic stochastic general equilibrium (DSGE) models.[2] Examples for the U.S. economy include Andrés, López-Salido, and Nelson (2009), Barsky, Justiniano, and Melosi (2014), Cúrdia (2015), Cúrdia, Ferrero, Ng, and Tambalotti (2015), Del Negro, Giannone, Giannoni, and Tambalotti (2017), Edge, Kiley, and Laforte (2008), Justiniano and Primiceri (2010), and Neiss and Nelson (2003).

This paper estimates the natural rate of interest in the U.S. using a nonlinear New Keynesian DSGE model with a zero lower bound (ZLB) constraint on the nominal interest rate and examines how and to what extent nonlinearities affect the estimates of the natural rate and its driving forces. Whereas the previous studies estimate the DSGE-based natural interest rate only in a linear setting that abstracts from the ZLB, this paper is one of the first to estimate the natural rate in a fully nonlinear and stochastic setting that incorporates the ZLB.[3] Our analysis is motivated by the following two strands of literature. First, Fernández-Villaverde and Rubio-Ramírez (2005) and Fernández-Villaverde, Rubio-Ramírez, and Santos (2006) demonstrate that the level of likelihood

[1]Closing the gap between the actual real interest rate and the natural rate is not necessarily optimal in the economy where "divine coincidence" (Blanchard and Galí, 2007) does not hold. However, Barsky, Justiniano, and Melosi (2014) demonstrate that, even in such a circumstance, a central bank would be able to stabilize both inflation and the welfare-relevant output gap to a considerable degree by tracking the natural rate using an estimated New Keynesian model.

[2]Another stream of the literature estimates the long-run natural interest rate based on semi-structural or reduced-form models. See, for instance, Holston, Laubach, and Williams (2017), Johannsen and Mertens (2016), Kiley (2015), Laubach and Williams (2003, 2016), Lubik and Matthes (2015), Pescatori and Turunen (2016), and Williams (2015). Based on a DSGE model, Del Negro, Giannone, Giannoni, and Tambalotti (2017) present the estimates of the 20- and 30-year forward natural rates, which are comparable to the long-run natural rate.

[3]A contemporaneous paper by Iboshi, Shintani, and Ueda (2018), which evolved independently from our work, estimates a nonlinear New Keynesian model for Japan and extracts a sequence of the natural rate.

and parameter estimates based on a linearized model can be significantly different from those based on the original nonlinear model. The same may be true for the estimation of unobservable state variables, including the natural rate. Second, the recent experience of the global financial crisis and the extremely low interest rate period that followed has led researchers to conduct empirical analyses based on nonlinear DSGE models in order to take the ZLB into consideration. For instance, Gust, Herbst, López-Salido, and Smith (2017) incorporate the ZLB into a medium-scale DSGE model similar to those developed by Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007), and estimate the model in a nonlinear setting using U.S. macroeconomic time series. Plante, Richter, and Throckmorton (2018) and Richter and Throckmorton (2016a, 2016b) estimate a nonlinear version of a prototypical New Keynesian model with the ZLB for the U.S. economy, and Iiboshi, Shintani, and Ueda (2018) estimate a similar model for the Japanese economy. Aruoba, Cuba-Borda, and Schorfheide (2018) consider Markov switching between the targeted-inflation and deflation steady states in a New Keynesian framework with the ZLB and estimate the probabilities of the U.S. and Japan having been in either the targeted-inflation or deflation regime using a nonlinear filtering technique. The present paper contributes to this strand of the literature by focusing on the estimation of the natural rate.

In estimating the natural rate of interest, we follow the two-step approach employed by Aruoba, Cuba-Borda, and Schorfheide (2018). First, to parameterize the model, we estimate a linearized version of the model using U.S. data prior to the date when the nominal interest rate was bounded at zero. Hirose and Sunakawa (2015) demonstrate that a linearized DSGE model gives rise to biased estimates of parameters if the ZLB existing in an economy is omitted in estimation but that neglecting the other nonlinearities does not lead to biased estimates for a sample period during which the ZLB is not binding. Thus, this approach enables us not only to avoid a computational burden that would increase exponentially in the estimation of a fully nonlinear model, but also to obtain reliable estimates of parameters.

Next, given the estimated parameters, we solve the model in a fully nonlinear and stochastic setting with the ZLB and apply a nonlinear filter to a full sample to extract the sequence of the natural interest rate. The literature (e.g., Boneva, Braun, and Waki, 2016; Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez, 2015; Gavin, Keen, Richter, and Throckmorton, 2015; Gust, Herbst, López-Salido, and Smith, 2017; Nakata, 2016, 2017; Ngo, 2014; and Richter and Throckmorton, 2016a) has emphasized the importance of considering nonlinearity in assessing the quantitative implications of New Keynesian models that include the ZLB. The natural rate estimated in the present paper takes account of this important feature. Moreover, our analysis is based on an empirically richer DSGE model than the prototypical New Keynesian model. The model features habit persistence in consumption preferences, price and wage stickiness, backward-looking components in price and wage settings, and monetary policy smoothing. Because of the high dimensionality of the state variables, it is computationally challenging to solve such a richer DSGE model in a fully nonlinear setting. To overcome this issue, we employ a projection method that adopts a very efficient Smolyak algorithm developed by Judd, Maliar, Maliar, and Valero (2014). Their solution method is very accurate, albeit with the reduced number of grid points.

The main results are summarized as follows. Comparing the estimated natural interest rate based on the nonlinear model with the rate based on the linear counterpart, we find that the former is higher than the latter to a substantial degree, particularly in the periods when the nominal interest rate is close to or bounded at zero. This difference is ascribed to a contractionary effect arising from the ZLB, which is considered only in the nonlinear model. Although such a contractionary effect lowers expected output and inflation, actual output and inflation are pegged to the corresponding observables in the filtering process. Then, larger positive shocks to aggregate demand must be identified in order to satisfy the optimality conditions of households and firms. As a consequence, the estimated natural rate increases in the nonlinear setting. Although price and wage dispersion potentially affect the identification of shocks and the estimate of the natural rate, their effects turn out to be negligible. These findings allure researchers to use a quasi-linear model, in which the ZLB constraint is imposed but all of the equilibrium conditions are linearized, because such a model is easy to solve. However, we demonstrate that the quasi-linear model cannot be a substitute for the fully nonlinear model in estimating the natural rate.

The remainder of the paper proceeds as follows. Section 2 describes the model used in our analysis and a strategy for estimating the natural rate of interest. Section 3 presents our results. Section 4 is the conclusion.

2 The Model and the Estimation Strategy

This section begins by describing the model used in our analysis. In the model economy, there are households, perfectly competitive final-good firms, monopolistically competitive intermediate-good firms, and a central bank. To ensure a better fit to the macroeconomic time series, the model features habit persistence in consumption preferences, price and wage stickiness, backward-looking components in price and wage settings, and monetary policy smoothing. In the model, the natural rate of interest is defined as the real interest rate that would prevail if prices and wages were fully flexible without any markup shocks.[4]

To obtain the estimates of the natural interest rate, we follow the two-step approach as in Aruoba, Cuba-Borda, and Schorfheide (2018). First, we estimate a linearized version of the model using U.S. data before the period of the global financial crisis and the virtually zero nominal interest rate. Next, given the estimated parameters, we solve the model in a fully nonlinear and stochastic setting with the ZLB constraint on the nominal interest rate and apply a nonlinear filter to extract the sequence of the natural rate.

2.1 The model

2.1.1 Households

Each household $h \in [0,1]$ consumes final goods $C_{h,t}$, supplies labor $l_{h,t} = \int_0^1 l_{f,h,t} df$ to intermediate-good firms $f \in [0,1]$, and purchases one-period riskless bonds $B_{h,t}$ so as to maximize the following utility function

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\prod_{k=1}^{t} d_k \right)^{-1} \left[\log \left(C_{h,t} - \gamma C_{t-1} \right) - \frac{l_{h,t}^{1+\eta}}{1+\eta} \right],$$

subject to the budget constraint

$$P_t C_{h,t} + B_{h,t} = W_{h,t}^n l_{h,t} + R_{t-1}^n B_{h,t-1} + T_{h,t},$$

where $\beta \in (0,1)$ is the subjective discount factor, $\gamma \in [0,1]$ is the degree of external habit persistence in consumption preferences (C_{t-1} is the aggregate consumption in period $t-1$), $\eta \geq 0$ is the inverse of the labor supply elasticity, P_t is the price of final goods, $W_{h,t}^n$ is the nominal wage for household h, R_t^n is the gross nominal interest rate, and $T_{h,t}$ is the sum of a lump-sum public transfer and profits received from firms. Following Eggertsson and Woodford (2003) and Christiano, Eichenbaum, and Rebelo (2011), a shock to the discount factor d_t affects the weight of the utility in period $t+1$ relative to the one in period t. In the present model, this shock is broadly interpreted as a shock to aggregate demand. The log of the discount factor shock follows an AR(1) process

$$\log d_t = \rho_d \log d_{t-1} + \varepsilon_{d,t}, \tag{1}$$

where $\rho_d \in [0,1)$ is an autoregressive coefficient and $\varepsilon_{d,t}$ is a normally distributed innovation with mean zero and standard deviation σ_d. The first-order conditions for optimal decisions on

[4]This definition is the most commonly used in the literature that estimates the natural rate based on DSGE models (Woodford, 2003; Galí, 2008). Cúrdia, Ferrero, Ng, and Tambalotti (2015) estimate the efficient interest rate, which is defined as the real interest rate under perfect competition and, therefore, with zero markups.

consumption and bond-holding are identical among households, and therefore become

$$\Lambda_t = \frac{1}{C_t - \gamma C_{t-1}}, \tag{2}$$

$$\Lambda_t = \frac{\beta}{d_t} R_t \mathbb{E}_t \frac{\Lambda_{t+1}}{\Pi_{t+1}}, \tag{3}$$

where Λ_t is the marginal utility of consumption and $\Pi_t = P_t/P_{t-1}$ denotes gross inflation.

2.1.2 Wage setting

A labor packer collects differentiated labor $\{l_{f,h,t}\}$ from each household h and resells a labor package augmented by a CES aggregator $l_{f,t} = \left[\int_0^1 l_{f,h,t}^{(\theta_w-1)/\theta_w} dh\right]^{\theta_w/(\theta_w-1)}$ to intermediate-good firms f, where $\theta_w > 1$ represents the elasticity of substitution among labor varieties. Given the nominal wage for each household $W_{h,t}^n$, cost minimization yields a set of labor demand schedules $l_{f,h,t} = \left(W_{h,t}^n/W_t^n\right)^{-\theta_w} l_{f,t}$ and the aggregate wage index $W_t^n = \left(\int_0^1 {W_{h,t}^n}^{1-\theta_w} dh\right)^{1/(1-\theta_w)}$.

Given the demand for labor by the labor packers, labor unions representing each household h set nominal wages on a staggered basis, as in Erceg, Henderson, and Levin (2000). In each period, a fraction $1-\xi_w \in (0,1)$ of labor unions reoptimizes their nominal wages, whereas the remaining fraction ξ_w indexes nominal wages to the economy's trend growth γ_a and a weighted average of past inflation Π_{t-1} and steady-state inflation $\bar{\Pi}$. The labor unions that reoptimize their nominal wages in the current period then maximize expected utility as follows

$$\mathbb{E}_t \sum_{j=0}^{\infty} \xi_w^j \beta^j \left(\prod_{k=1}^{j} d_k\right)^{-1} \left[\frac{\gamma_a^j W_{h,t}^n}{P_{t+j}} \prod_{k=1}^{j} \left(\Pi_{t+k-1}^{\iota_w} \bar{\Pi}^{1-\iota_w}\right) \Lambda_{h,t+j} l_{h,t+j} - \frac{l_{h,t+j}^{1+\eta}}{1+\eta}\right],$$

subject to the labor demand

$$l_{f,h,t+j} = \left[\frac{\gamma_a^j W_{h,t}^n}{W_{t+j}^n} \prod_{k=1}^{j} \left(\Pi_{t+k-1}^{\iota_w} \bar{\Pi}^{1-\iota_w}\right)\right]^{-\theta_w} l_{f,t+j},$$

where $l_{h,t} = \int_0^1 l_{f,h,t} df$ is the amount of labor supplied by each household h, and $\iota_w \in [0,1)$ is the weight of wage indexation to past inflation relative to steady-state inflation. The first-order condition for the reoptimized wage $W_t^{n,o}$ is given by

$$\left(\frac{W_t^{n,o}}{W_t^n}\right)^{1+\eta\theta_w} = \frac{\theta_w}{\theta_w - 1} \frac{\mathbb{E}_t \sum_{j=0}^{\infty} \xi_w^j \beta^j \left(\prod_{k=1}^{j} d_k\right)^{-1} \left[\left(\prod_{k=1}^{j} \Pi_{t+k-1}^{\iota_w} \bar{\Pi}^{1-\iota_w} \frac{\gamma_a^j W_t^n}{W_{t+j}^n}\right)^{-(1+\eta)\theta_w} l_{d,t+j}^{1+\eta}\right]}{\mathbb{E}_t \sum_{j=0}^{\infty} \xi_w^j \beta^j \left(\prod_{k=1}^{j} d_k\right)^{-1} \left[\left(\prod_{k=1}^{j} \Pi_{t+k-1}^{\iota_w} \bar{\Pi}^{1-\iota_w} \frac{\gamma_a^j W_t^n}{W_{t+j}^n}\right)^{1-\theta_w} \Lambda_{t+j} \frac{W_{t+j}^n}{P_{t+j}} l_{d,t+j}\right]}, \tag{4}$$

where $l_{d,t} = \int_0^1 l_{f,t} df$ is the total labor demand. The aggregate nominal wage index $W_t^n = \left(\int_0^1 W_{h,t}^{n \ 1-\theta_w} dh \right)^{1/(1-\theta_w)}$ can be written as

$$W_t^n = \left[(1-\xi_w) \left(W_t^{n,o} \right)^{1-\theta_w} + \xi_w \left(\Pi_{t-1}^{\iota_w} \bar{\Pi}^{1-\iota_w} \gamma_a W_{t-1}^n \right)^{1-\theta_w} \right]^{\frac{1}{1-\theta_w}}. \tag{5}$$

2.1.3 Firms

The representative final-good firm produces output Y_t under perfect competition by choosing a combination of intermediate inputs $\{Y_{f,t}\}$ so as to maximize profit $P_t Y_t - \int_0^1 P_{f,t} Y_{f,t} df$, subject to a CES production technology $Y_t = \left[\int_0^1 Y_{f,t}^{(\theta_p - 1)/\theta_p} df \right]^{\theta_p/(\theta_p-1)}$, where $P_{f,t}$ is the price of intermediate good f and $\theta_p > 1$ denotes the elasticity of substitution among the variety of intermediate goods. The first-order condition for profit maximization yields the final-good firm's demand for each intermediate good $Y_{f,t} = (P_{f,t}/P_t)^{-\theta_p} Y_t$ and the aggregate price index $P_t = \left(\int_0^1 P_{f,t}^{1-\theta_p} df \right)^{1/(1-\theta_p)}$.

Each intermediate-good firm f produces a differentiated good $Y_{f,t}$ under monopolistic competition by choosing a labor input $l_{f,t}$, given the real wage $W_t = W_t^n / P_t$, and subject to the production function

$$Y_{f,t} = A_t l_{f,t},$$

where A_t represents total factor productivity. The log of the productivity level follows a nonstationary stochastic process

$$\log A_t = \log \gamma_a + \log A_{t-1} + a_t, \tag{6}$$

where $\log \gamma_a$ represents the steady-state growth rate of productivity and a_t is a shock to the productivity growth. The productivity shock follows an AR(1) process

$$a_t = \rho_a a_{t-1} + \varepsilon_{a,t}, \tag{7}$$

where $\rho_a \in [0, 1)$ is an autoregressive coefficient and $\varepsilon_{a,t}$ is a normally distributed innovation with mean zero and standard deviation σ_a. Assuming the existence of a shock to real marginal cost z_t, which is interpreted as an inefficient cost-push shock, the first-order condition for cost minimization is given by[5]

$$MC_t = \frac{W_t}{A_t} z_t. \tag{8}$$

The log of the cost-push shock follows an AR(1) process

$$\log z_t = \rho_z \log z_{t-1} + \varepsilon_{z,t}, \tag{9}$$

[5]The first-order condition also indicates that the real marginal cost MC_t is identical across the intermediate-good firms.

where $\rho_z \in [0,1)$ is an autoregressive coefficient and $\varepsilon_{z,t}$ is a normally distributed innovation with mean zero and standard deviation σ_z.

In the face of the final-good firm's demand and marginal cost, the intermediate-good firms set the prices of their products on a staggered basis, as in Calvo (1983). In each period, a fraction $1-\xi_p \in (0,1)$ of intermediate-good firms reoptimizes their prices, whereas the remaining fraction ξ_p indexes prices to a weighted average of past inflation Π_{t-1} and steady-state inflation $\bar{\Pi}$. The firms that reoptimize their prices in the current period then maximize expected profit as follows

$$\mathbb{E}_t \sum_{j=0}^{\infty} \xi_p^j \beta^j \left(\prod_{k=1}^{j} d_k \right)^{-1} \frac{\Lambda_{t+j}}{\Lambda_t} \left[\frac{P_{f,t}}{P_{t+j}} \prod_{k=1}^{j} \left(\Pi_{t+k-1}^{\iota_p} \bar{\Pi}^{1-\iota_p} \right) - MC_{t+j} \right] Y_{f,t+j},$$

subject to the final-good firm's demand

$$Y_{f,t+j} = \left[\frac{P_{f,t}}{P_{t+j}} \prod_{k=1}^{j} \left(\Pi_{t+k-1}^{\iota_p} \bar{\Pi}^{1-\iota_p} \right) \right]^{-\theta_p} Y_{t+j},$$

where $\iota_p \in [0,1)$ denotes the weight of price indexation to past inflation relative to steady-state inflation. The first-order condition for the reoptimized price P_t^o is given by

$$\frac{P_t^o}{P_t} = \frac{\theta}{\theta-1} \frac{\mathbb{E}_t \sum_{j=0}^{\infty} \xi_p^j \beta^j \left(\prod_{k=1}^{j} d_k \right)^{-1} \frac{\Lambda_{t+j}}{\Lambda_t} \left[\left(\prod_{k=1}^{j} \left[\left(\frac{\Pi_{t+k-1}}{\bar{\Pi}} \right)^{\iota_p} \frac{\bar{\Pi}}{\Pi_{t+k}} \right] \right)^{-\theta_p} MC_{t+j} Y_{t+j} \right]}{\mathbb{E}_t \sum_{j=0}^{\infty} \xi_p^j \beta^j \left(\prod_{k=1}^{j} d_k \right)^{-1} \frac{\Lambda_{t+j}}{\Lambda_t} \left[\left(\prod_{k=1}^{j} \left[\left(\frac{\Pi_{t+k-1}}{\bar{\Pi}} \right)^{\iota_p} \frac{\bar{\Pi}}{\Pi_{t+k}} \right] \right)^{1-\theta_p} Y_{t+j} \right]}. \tag{10}$$

The final-good's price $P_t = \left(\int_0^1 P_{f,t}^{1-\theta_p} df \right)^{1/(1-\theta_p)}$ can be written as

$$P_t = \left[(1-\xi_p) \left(P_t^o \right)^{1-\theta_p} + \xi_p \left(\Pi_{t-1}^{\iota_p} \bar{\Pi}^{1-\iota_p} P_{t-1} \right)^{1-\theta_p} \right]^{\frac{1}{1-\theta_p}}. \tag{11}$$

2.1.4 Market clearing conditions

The final-good market clearing condition is

$$Y_t = C_t, \tag{12}$$

whereas the labor market clearing condition leads to

$$l_t = \frac{\Delta_{p,t} \Delta_{w,t} Y_t}{A_t}, \tag{13}$$

where $l_t = \int_0^1 \int_0^1 l_{f,h,t} df dh$ is the aggregate labor input, $\Delta_{p,t} = \int_0^1 \left(P_{f,t}/P_t \right)^{-\theta_p} df$ is price dispersion across the intermediate-good firms, and $\Delta_{w,t} = \int_0^1 \left(W_{h,t}^n / W_t^n \right)^{-\theta_w} dh$ is wage dispersion across the labor unions. Equation (13) can be rewritten in terms of $l_{d,t} = \int_0^1 l_{f,t} df$ as

$$l_{d,t} = \frac{\Delta_{p,t} Y_t}{A_t}. \tag{14}$$

In the present model, the price and wage dispersion evolve according to

$$\Delta_{p,t} = (1-\xi_p)\left(\frac{P_t^o}{P_t}\right)^{-\theta_p} + \xi_p \left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\theta_p}\left(\frac{\Pi_{t-1}}{\bar{\Pi}}\right)^{-\iota_p\theta_p}\Delta_{p,t-1}, \tag{15}$$

$$\Delta_{w,t} = (1-\xi_w)\left(\frac{W_t^{n,o}}{W_t^n}\right)^{-\theta_w} + \xi_w \left(\frac{\Pi_t W_t}{\bar{\Pi}\gamma_a W_{t-1}}\right)^{\theta_w}\left(\frac{\Pi_{t-1}}{\bar{\Pi}}\right)^{-\iota_w\theta_w}\Delta_{w,t-1}. \tag{16}$$

2.1.5 Flexible wage and price equilibrium

Natural output Y_t^* and the natural rate of interest R_t^* are defined as the levels that would prevail if both wages and prices were perfectly flexible with no cost-push shocks. Such a flexible wage and price equilibrium is obtained with $\xi_w = \xi_p = 0$, $W_{h,t}^n = W_t^n$, $P_{f,t} = P_t$, and $z_t = 1$ for all h, f, and t in the model above and is characterized by the following equations:

$$\left(Y_t^* - \gamma Y_{t-1}^*\right)\left(\frac{Y_t^*}{A_t}\right)^{\eta} = \mu A_t, \tag{17}$$

$$R_t^* = \frac{d_t}{\beta}\left(\mathbb{E}_t \frac{Y_t^* - \gamma Y_{t-1}^*}{Y_{t+1}^* - \gamma Y_t^*}\right)^{-1}, \tag{18}$$

where $\mu = \frac{\theta_w - 1}{\theta_w}\frac{\theta_p - 1}{\theta_p}$ is the product of price and wage markups. Thus, the law of motion for natural output Y_t^* is determined by (17), given the sequence of total factor productivity A_t. The natural rate of interest R_t^* is determined by (18), with the sequences of natural output Y_t^* and the discount factor shock d_t.

2.1.6 Central bank

A monetary policy rule is specified as

$$R_t^n = \max[\widehat{R}_t^n, 1], \tag{19}$$

where

$$\widehat{R}_t^n = (\widehat{R}_{t-1}^n)^{\phi_r}\left[\bar{R}\bar{\Pi}\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{\gamma_a Y_{t-1}}\right)^{\phi_y}\right]^{1-\phi_r}\exp(\varepsilon_{r,t}). \tag{20}$$

$\widehat{R}_t^n$ denotes the hypothetical nominal interest rate that the central bank would set according to a Taylor (1993) type monetary policy rule, where $\bar{R}$ is the steady-state gross real interest rate, $\phi_r \in [0, 1)$ is the policy-smoothing parameter, and $\phi_\pi \geq 0$ and $\phi_y \geq 0$ are the degrees of the interest rate policy response to inflation and output growth. $\varepsilon_{r,t}$ is a monetary policy shock, which is normally distributed with mean zero and standard deviation σ_r. The max function in (19) constrains the nominal interest rate to be greater than or equal to zero. If $\widehat{R}_t^n > 1$, the ZLB constraint is not imposed, i.e., $R_t^n = \widehat{R}_t^n$. If $\widehat{R}_t^n \leq 1$, the ZLB is binding, i.e., $R_t^n = 1$.

2.1.7 Equilibrium

An equilibrium is given by the sequences $\{Y_t, C_t, \Lambda_t, W_t, W_t^n, W_t^{n,o}, l_t, l_{d,t}, MC_t, \Pi_t, P_t, P_t^o, \Delta_{p,t}, \Delta_{w,t}, Y_t^*, R_t^*, R_t^n, \widehat{R}_t^n, d_t, A_t, a_t, z_t\}_{t=0}^{\infty}$ satisfying the equilibrium conditions (1)–(20) and two definitional equations, $W_t = W_t^n/P_t$ and $\Pi_t = P_t/P_{t-1}$.

Because total factor productivity A_t is nonstationary, as specified by (6), we rewrite the equilibrium conditions in terms of stationary variables detrended by A_t, as follows: $y_t = Y_t/A_t$, $c_t = C_t/A_t$, $\lambda_t = \Lambda_t A_t$, $w_t = W_t/A_t$, $w_t^n = W_t^n/A_t$, $w_t^{n,o} = W_t^{n,o}/A_t$, $mc_t = MC_t/A_t$, and $y_t^* = Y_t^*/A_t$, so that we can derive a nonstochastic steady state for the detrended variables.

2.2 Estimation of parameters

To parameterize the model, we estimate a linearized version of the model using four U.S. quarterly time series: the per capita real GDP growth rate ($100\Delta \log GDP_t$), the inflation rate of the GDP implicit price deflator ($100\Delta \log PGDP_t$), the federal funds rate (FF_t), and the log of hours worked ($100 \log H_t$).[6] Following Wolters (2018), the data on hours worked is adjusted for low-frequency movements due to sectoral and demographic changes so that the data is consistent with the model. The sample period is from 1987:III to 2008:IV. The beginning of the sample period is set at the time when Alan Greenspan became the Chairman of the Fed, because thereafter, the style of the Fed's policy conduct seems consistent and stable. The end of the sample is selected so that the observations exclude the periods of virtually zero nominal interest rates. The linearized equilibrium conditions and observation equations are presented in Appendix A.

The parameters are estimated using Bayesian methods. The prior distributions of the parameters are presented in the second to fourth columns of Table 1. For most of the parameters, each prior mean is set at the corresponding prior mean used in Smets and Wouters (2007). The prior mean of the policy-smoothing parameter ϕ_r is set at 0.5, which is lower than that in Smets and Wouters (2007) because a higher value of the estimated ϕ_r would lead to a nonconvergence problem in solving our nonlinear model.[7] As for the steady-state values of output growth, inflation, and real interest rates and hours worked ($\bar{a}$, $\bar{\pi}$, $\bar{r}$, $\bar{l}$), the priors are centered at the sample mean. The prior mean of the AR(1) coefficient for the discount factor shock ρ_d is 0.75, whereas that for the productivity and cost-push shocks (ρ_a, ρ_z) is 0.5. For the standard deviations of the shocks ($100\sigma_d$,

[6] The series of hours worked is demeaned.

[7] For the same reason, relatively tight priors are used for the parameters that determine the persistency of endogenous variables in the model.

$100\sigma_a$, $100\sigma_z$, $100\sigma_r$), we assign inverse-gamma distributions with a mean of 0.5 and a standard deviation of 2.0.

In the estimation, $200,000$ posterior draws are generated using the Random-Walk Metropolis-Hastings algorithm, and the first $50,000$ draws are discarded. The posterior mean and 90 percent credible interval for each parameter are reported in the last two columns of Table 1. In the subsequent analysis, the parameters are fixed at the posterior mean estimates except for the steady-state values of the output growth, inflation, and real interest rates and hours worked ($\bar{a}$, $\bar{\pi}$, $\bar{r}$, $\bar{l}$), which are set at their respective averages of the extended sample from 1987:III to 2016:III.

2.3 Nonlinear solution and filtering

The model is solved in a fully nonlinear and stochastic setting with the ZLB constraint on the nominal interest rate using a projection method. The model has seven endogenous state variables (output y_{t-1}, inflation Π_{t-1}, the real wage w_{t-1}, the hypothetical nominal interest rate $\widehat{R}^n_{t-1}$, price dispersion $\Delta_{p,t-1}$, wage dispersion $\Delta_{w,t-1}$, and natural output y^*_{t-1}) and four exogenous shocks (the discount factor shock d_t, the productivity shock a_t, the cost-push shock z_t, and the monetary policy shock $\varepsilon_{r,t}$). The policy functions satisfying the detrended equilibrium conditions can be written as

$$\mathbb{S}_t = g(\mathbb{S}_{t-1}, \tau_t),$$

where $\mathbb{S}_{t-1} = [y_{t-1}, \Pi_{t-1}, w_{t-1}, \widehat{R}^n_{t-1}, \Delta_{p,t-1}, \Delta_{w,t-1}, y^*_{t-1}]'$ and $\tau_t = [d_t, a_t, z_t, \varepsilon_{r,t}]'$.

Because of the high dimensionality of the state variables, it is computationally very expensive to apply a conventional projection method that uses the tensor product of one-dimensional polynomials. In this regard, we employ a projection method equipped with a Smolyak algorithm, in which a relatively small number of grid points are selected on the basis of their potential importance for the quality of approximation. Moreover, we adopt a more efficient Smolyak algorithm developed by Judd, Maliar, Maliar, and Valero (2014). In this algorithm, a union of the unidimensional disjoint sets of grid points of the endogenous state variables are constructed instead of the conventional nested sets in order to avoid repetitions of grid points. Then, projection functions onto the grid points are computed with interpolation coefficients and a Chebyshev family of orthogonal basis functions.[8] The grid points obtained by this algorithm are *sparse*; therefore, the algorithm is more

[8]To obtain the interpolation coefficients, we follow Judd, Maliar, Maliar, and Valero (2014) and use a Lagrange interpolation method, whereas Malin, Krueger, and Kubler (2011) use the closed-form formula to obtain approximated functions.

likely to be free from the curse of dimensionality. The details of the solution method are described in Appendix B.

According to an artificial sample of 40,000 periods simulated from the nonlinear solution of the model, the economy is at the ZLB for 12.1 percent of quarters, and the average duration of ZLB spells is 4.3 quarters. These statistics indicate that our model economy is much more frequently constrained by the ZLB and that the average duration of ZLB spells is longer than the simulation results in the previous studies that employ nonlinear New Keynesian models, such as Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015) and Gust, Herbst, López-Salido, and Smith (2017).[9]

We apply a particle filter as developed by Fernández-Villaverde and Rubio-Ramírez (2007) to extract the sequence of the state variables and then compute the estimates of the natural interest rate.[10] The data used for filtering is the same as those used for the parameter estimation in Section 2.2, but the period over which the filter is run is extended to 2016:III. To facilitate the use of the particle filter, measurement errors are added in the observation equations. The measurement errors of output growth, inflation, the nominal interest rate, and hours worked are respectively set to be 20, 20, 10, and 5 percent of their standard deviations in the data over the sample from 1987:III to 2016:III so that the smoothed (two-sided) estimates of the observables can track the data reasonably well, as shown in Figure 1. We use 100,000 particles and confirmed that any further increase in the number of particles delivered almost the same results as those presented below.

3 Results

This section presents the estimate of the natural interest rate based on the nonlinear model and compares it with the estimate based on its linear counterpart. To understand the cause of the difference between the two estimates, we investigate how the natural rate of interest is identified in each case. Moreover, we consider a quasi-linear model, in which the ZLB constraint is imposed but

[9] Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015) simulate a small-scale model calibrated for the U.S. economy and show that the economy spends 5.5 percent of quarters at the ZLB and that the average duration at the ZLB is 2.1 quarters. Gust, Herbst, López-Salido, and Smith (2017) estimate a medium-scale model in a nonlinear setting using U.S. data from 1983:Q1 to 2014:Q1, and the simulation of their estimated model demonstrates that the economy is at the ZLB for about 4 percent of quarters on average and that the average duration of the ZLB spells is just over 3.5 quarters.

[10] For a textbook treatment of a particle filter, see Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) or Herbst and Schorfheide (2015).

all the equilibrium conditions are linearized, and examine whether it can be a useful substitute for estimating the natural rate accurately.

3.1 Estimated natural rate of interest

The solid line in Figure 2 shows the smoothed mean estimate of the natural rate of interest on an annualized basis. The estimated natural rate measure peaked around 8 percent at the end of 1980s and the beginning of 2000, then fell to about -6 percent in the aftermath of the global financial crisis, and thereafter increased to a slightly positive value toward the end of the sample period. The overall cyclical movements in the natural rate are very similar to those estimated by Barsky, Justiniano, and Melosi (2014), who employ a medium-scale New Keynesian DSGE model with capital accumulation, although their estimate of the natural rate exhibits more variability than ours; that is, their estimate peaked at more than 10 percent in 2000 and dropped to around -5 percent in 2003.

The primary objective of this paper is to examine how and to what extent nonlinearity, including the ZLB, affects the estimates of the natural rate of interest. To this end, we estimate the natural rate using a linear counterpart of the model, as in the previous studies, and compare it with the one obtained above. The dotted line in Figure 2 is the smoothed mean estimate of the natural interest rate based on the linearized version of the model with the same parameters and data set as used in the nonlinear case.[11] The figure indicates that the natural rate based on the linearized model is mostly lower than that based on the nonlinear model, except for the period before 1990. In particular, the difference is pronounced in the periods when the actual nominal interest rate (shown in Figure 1) is close to or bounded at zero.

To understand what causes the difference between the two estimates, we consider how the natural rate of interest is identified in each case. As addressed in Section 2.1, equation (17), i.e., $\left(Y_t^* - \gamma Y_{t-1}^*\right)(Y_t^*/A_t)^\eta = \mu A_t$, determines natural output Y_t^*, given the sequence of total factor productivity A_t (or, equivalently, the productivity shock a_t). The natural rate of interest R_t^* can be traced out from equation (18), i.e., $R_t^* = d_t/\beta \left[\mathbb{E}_t(Y_t^* - \gamma Y_{t-1}^*)/(Y_{t+1}^* - \gamma Y_t^*)\right]^{-1}$, with the sequences of natural output Y_t^* and the discount factor shock d_t. Thus, the natural interest rate is pinned down by identifying the two shocks, a_t and d_t.

[11] In the case of the linear model, the smoothed estimates of model variables could be computed with a widely used linear solution method and Kalman filter. However, we apply the same projection method and particle filter as in the nonlinear case to avoid the possibility that differences in the solution and filtering methods could affect the estimate of the natural rate.

In the linear model, the productivity shock a_t is explicitly identified by the data on output and hours worked because detrending and log-linearizing the labor market clearing condition (13) yields $\tilde{y}_t = \tilde{l}_t$ and because the associated observation equations (abstracting from the observation errors) are $100\Delta \log GDP_t = \bar{a} + \tilde{y}_t - \tilde{y}_{t-1} + a_t$ and $100 \log H_t = \bar{l} + \tilde{l}_t$, where $\bar{a}$ and $\bar{l}$ are the steady-state growth rate and hours worked, respectively, and the variables with $\tilde{}$ represent percentage deviations from their steady-state values. In the nonlinear model, however, equation (13) contains the price and wage dispersion, $\Delta_{p,t}$ and $\Delta_{w,t}$, and can be written as $y_t = l_t/(\Delta_{p,t}\Delta_{w,t})$ in detrended terms. These dispersion terms fluctuate so that $\Delta_{p,t} \geq 1$ and $\Delta_{w,t} \geq 1$, as the price and wage, respectively, deviate from the steady state. Thus, y_t becomes lower than in the linear case where the dispersion terms are suppressed. Consequently, to satisfy the observation equation for output growth, a_t can be identified as being larger in the nonlinear case. Higher productivity results in the higher natural rate.

Identification of the discount factor shock d_t is more complicated and influenced by the whole structure of the model. However, taking account of the finding that the two estimates of the natural interest rate differ from each other during the periods when the nominal interest rate is close to or bounded at zero, the existence of the ZLB, from which the linear model abstracts, possibly affects the identification of d_t in the nonlinear model. The literature has established that the ZLB has a contractionary effect on the economy not only when the nominal interest is already binding at zero, but also when uncertainty exists about whether the ZLB will bind in the future.[12] Although such a contractionary effect lowers expected output and inflation, the particle filter pegs actual output and inflation to the corresponding observables, which are the same in the linear and nonlinear cases. Then, in the nonlinear case, the discount factor shock d_t must increase to satisfy the optimality conditions of households and firms. As a result, the natural rate increases.

To quantify the differences in the sequences of identified shocks, Figure 3 shows the smoothed mean estimates of the discount factor shocks d_t and the productivity shocks a_t, in percentage terms, based on the nonlinear model (solid lines) and its linear counterpart (dotted lines). The sequence of d_t identified in the nonlinear model is remarkably different from that identified in the linear model. In particular, the difference remains substantial after the global financial crisis. On the other hand, the movements of a_t are very similar between the two estimates although temporary deviations are occasionally found. Therefore, the difference in d_t is the main source of the different estimates of the natural rate between the two cases.

[12] Hills, Nakata, and Schmidt (2016) quantify such an uncertainty effect on inflation in the face of the interest rate lower bound.

Figure 4 confirms the mechanism behind the difference in the estimates of d_t, which is described above. In the figure, the smoothed estimates of expected inflation $E_t \log \Pi_{t+1}$ and expected output $E_t y_{t+1}$ in detrended terms are compared for the nonlinear case (solid lines) and the linear case (dotted lines), in terms of percentage deviation from the steady state. In the case of the nonlinear setting, $E_t \log \Pi_{t+1}$ shifts downward to a large extent during the periods of the low nominal interest rate, whereas the downward shift in $E_t y_{t+1}$ is limited. The limited shift in $E_t y_{t+1}$ is ascribed to the consequence of the increased d_t, which has a direct positive effect on actual output through the Euler equation and accordingly raises the expected output.

The finding of the small difference in the estimated productivity shocks a_t implies that the price and wage dispersion terms, $\Delta_{p,t}$ and $\Delta_{w,t}$, play a minor role in the nonlinear model. Indeed, as shown in Figure 5, the smoothed estimates of $\Delta_{p,t}$ and $\Delta_{w,t}$ based on the nonlinear model fluctuate little, i.e., less than 0.15 percent at most, even though they exhibit cyclical movements over the sample period.

3.2 The natural rate of interest based on the quasi-linear model

The analysis thus far suggests that the existence of the ZLB constraint plays a crucial role in identifying the natural rate of interest in a nonlinear setting, but that the inclusion of price and wage dispersion is relatively minor. These findings tempt researchers to exploit a quasi-linear model, in which the ZLB constraint is imposed but all the equilibrium conditions are linearized, for estimating the natural rate measures, because such a model is easier to solve than a fully nonlinear model. However, Boneva, Braun, and Waki (2016), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015), Gavin, Keen, Richter, and Throckmorton (2015), Gust, López-Salido, and Smith (2017), Nakata (2016, 2017), Ngo (2014), and Richter and Throckmorton (2016a) argue that the solution of this sort of quasi-linear model can give rise to an inaccurate assessment of the ZLB. Thus, the estimated natural interest rate in a quasi-linear setting may be subject to the same problem. To investigate this point, this subsection estimates the natural rate based on a quasi-linear version of the model and compare it with those obtained in the preceding subsection.

Figure 6 depicts the smoothed mean estimate of the natural interest rate in the quasi-linear setting (dashed line) along with the estimates in the fully nonlinear setting (solid line) and the linear setting (dotted line). While the estimate based on the quasi-linear model almost coincide with that based on the fully nonlinear model during the periods before 1990 and after 2010, these two estimates are still substantially different from each other during the other periods. In particular, the estimated natural rate in the quasi-linear setting remains almost unchanged from its linear counterpart in the early 1990s, the early 2000s, and the aftermath of the global financial crisis, when the nominal interest rate was lowered rapidly. The same is confirmed by the upper panel in Figure 7. It shows that the smoothed mean estimate of the discount factor shock d_t in the quasi-linear model (dashed line) is lower than that in the fully nonlinear model (solid line) during the same periods.

As addressed in the previous subsection, the ZLB has a contractionary effect when there exists uncertainty about whether the ZLB will bind in the future. This effect increases as the nominal interest rate is lowered. Such an uncertainty effect is enhanced in the fully nonlinear setting but not in the quasi-linear setting, which causes the differences in the estimates above. Therefore, the quasi-linear model incorporating the ZLB cannot be a possible substitute for a fully nonlinear model in estimating the natural rate measures.

4 Concluding Remarks

This paper has estimated the natural rate of interest in a nonlinear New Keynesian model using U.S. macroeconomic data and compared it with the rate estimated with the model's linear counterpart. We have found that the natural rate based on the nonlinear model is substantially higher than that based on the linear model, particularly in the periods when the nominal interest rate is close to or bounded at zero. This difference is explained by a contractionary effect of the ZLB, which is considered only in the nonlinear model. Although the existence of the price and wage dispersion terms potentially affects the estimate of the natural rate in the nonlinear setting, we have demonstrated that their effects are relatively minor.

Whereas the present paper employs an empirically richer DSGE model than the prototypical New Keynesian model, existing studies, including Barsky, Justiniano, and Melosi (2014), Cúrdia, Ferrero, Ng, and Tambalotti (2015), Edge, Kiley, and Laforte (2008), Del Negro, Giannone, Giannoni, and Tambalotti (2017), and Justiniano and Primiceri (2010) have estimated the natural interest rate using medium-scale DSGE models with capital accumulation. Our analysis could be extended to exploit such a medium-scale model so that the estimated natural rate would be comparable to the rates obtained in these studies. We conjecture that our results regarding the higher estimate of the natural rate in a nonlinear setting would still hold, even if we extended our model to a larger scale, because the main mechanism through which nonlinearities can affect the identification of the natural rate remains unchanged.

Appendix

A Linearized Equilibrium Conditions and Observation Equations

Log-linearizing the detrended equilibrium conditions around the nonstochastic steady state, and rearranging the resulting equations, yields

$$
\begin{aligned}
\tilde{y}_t &= \frac{\gamma_a}{\gamma_a+\gamma}(\mathbb{E}_t\tilde{y}_{t+1}+\mathbb{E}_t a_{t+1}) + \frac{\gamma}{\gamma_a+\gamma}(\tilde{y}_{t-1}-a_t) - \frac{\gamma_a-\gamma}{\gamma_a+\gamma}(\tilde{R}^n_t - \mathbb{E}_t\tilde{\Pi}_{t+1} - \tilde{d}_t), \\
\tilde{w}_t &= \tilde{w}_{t-1} - \tilde{\Pi}_t + \iota_w\tilde{\Pi}_{t-1} - a_t + \beta(\mathbb{E}_t\tilde{w}_{t+1} - \tilde{w}_t + \mathbb{E}_t\tilde{\Pi}_{t+1} - \iota_w\tilde{\Pi}_t + \mathbb{E}_t a_{t+1}) \\
&\quad + \frac{(1-\xi_w)(1-\xi_w\beta)}{\xi_w(1+\eta\theta_w)}\left[\eta\tilde{l}_t + \frac{1}{\gamma_a+\gamma}(\gamma_a\tilde{y}_t - \gamma\tilde{y}_{t-1} + \gamma a_t) - \tilde{w}_t\right], \\
\tilde{y}_t &= \tilde{l}_t, \\
\tilde{\Pi}_t &= \frac{\beta}{1+\beta\iota_p}\mathbb{E}_t\tilde{\Pi}_{t+1} + \frac{\iota_p}{1+\beta\iota_p}\tilde{\Pi}_{t-1} + \frac{(1-\xi_p)(1-\xi_p\beta)}{\xi_p(1+\beta\iota_p)}(\tilde{w}_t + \tilde{z}_t), \\
\tilde{y}^*_t &= \frac{\gamma}{\gamma_a(1+\eta)-\gamma\eta}(\tilde{y}^*_{t-1} - a_t), \\
\tilde{R}^n_t &= \phi_r\tilde{R}^n_{t-1} + (1-\phi_r)\left[\phi_\pi\tilde{\Pi}_t + \phi_y(\tilde{y}_t - \tilde{y}_{t-1} + a_t)\right] + \varepsilon_{r,t}, \\
\tilde{d}_t &= \rho_d\tilde{d}_{t-1} + \varepsilon_{d,t}, \\
a_t &= \rho_a a_{t-1} + \varepsilon_{a,t}, \\
\tilde{z}_t &= \rho_z\tilde{z}_{t-1} + \varepsilon_{z,t},
\end{aligned}
$$

where the variables with ˜ represent percentage deviations from their steady-state values.

The observation equations are

$$
\begin{bmatrix} 100\Delta\log GDP_t \\ 100\Delta\log PGDP_t \\ FF_t \\ 100\log H_t \end{bmatrix} = \begin{bmatrix} \bar{a} \\ \bar{\pi} \\ \bar{r}+\bar{\pi} \\ \bar{l} \end{bmatrix} + \begin{bmatrix} \tilde{y}_t - \tilde{y}_{t-1} + a_t \\ \tilde{\Pi}_t \\ \tilde{R}^n_t \\ \tilde{l}_t \end{bmatrix},
$$

where $\bar{a} = 100\log\gamma_a$, $\bar{\pi} = 100\log\bar{\Pi}$, $\bar{r} = 100\log\bar{R}(= 100\log(\gamma_a/\beta))$, and $\bar{l}$ are, respectively, the steady-state growth rate, the inflation rate, the real interest rate, and hours worked.

B Nonlinear Solution Method

B.1 Recursive forms of the price and wage setting equations

After detrending, the equilibrium conditions (10) and (4) can be written in the following recursive forms:

$$
\begin{aligned}
\frac{P_t^o}{P_t} &= \frac{S_{p,t}}{F_{p,t}} \\
S_{p,t} &= \theta_p w_t z_t + \xi_p \beta d_t^{-1} \mathbb{E}_t \left[\left(\frac{\Pi_{t+1}}{\bar{\Pi}} \right) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p} \frac{y_{t+1}}{y_t} \frac{\lambda_{t+1}}{\lambda_t} S_{p,t+1}, \\
F_{p,t} &= (\theta_p - 1) + \xi_p \beta d_t^{-1} \mathbb{E}_t \left[\left(\frac{\Pi_{t+1}}{\bar{\Pi}} \right) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p - 1} \frac{y_{t+1}}{y_t} \frac{\lambda_{t+1}}{\lambda_t} F_{p,t+1}, \\
\left(\frac{W_t^{n,o}}{W_t^n} \right)^{1+\eta\theta_w} &= \frac{S_{w,t}}{F_{w,t}} \\
S_{w,t} &= \theta_w l_{d,t}^{\eta} \lambda_t^{-1} + \xi_w \beta d_t^{-1} \mathbb{E}_t \left[\left(\frac{\Pi_{w,t+1}}{\bar{\Pi}} \exp(a_{t+1}) \right) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{-\iota_w} \right]^{(1+\eta)\theta_w} \frac{l_{d,t+1}}{l_{d,t}} \frac{\lambda_{t+1}}{\lambda_t} S_{w,t+1}, \\
F_{w,t} &= (\theta_w - 1) w_t + \xi_w \beta d_t^{-1} \mathbb{E}_t \left[\left(\frac{\Pi_{w,t+1}}{\bar{\Pi}} \exp(a_{t+1}) \right) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{-\iota_w} \right]^{\theta_w - 1} \frac{l_{d,t+1}}{l_{d,t}} \frac{\lambda_{t+1}}{\lambda_t} F_{w,t+1},
\end{aligned}
$$

where $\Pi_{w,t} = \Pi_t w_t / w_{t-1}$ and $l_{d,t} = \Delta_{p,t} y_t$.

B.2 Solution algorithm

In what follows, we drop the time subscript and use -1 and $'$ for previous- and next-period variables, respectively. To solve for the policy functions on each grid point of the state space $(\mathbb{S}_{-1}, \tau)$, where $\mathbb{S}_{-1} = [y_{-1}, \Pi_{-1}, w_{-1}, \widehat{R}_{-1}^{n}, \Delta_{p,-1}, \Delta_{w,-1}, y_{-1}^{*}]'$ and $\tau = [d, a, z, \varepsilon_r]'$, we follow an index-function approach as in Aruoba, Cuba-Borda, and Schorfheide (2018), Gust, Herbst, López-Salido, and Smith (2017) and Nakata (2017).[13] First, regime-specific expectation functions are defined as

[13] See also Hirose and Sunakawa (2019) for details about the solution algorithm with an example of a prototypical New Keynesian model with the ZLB.

follows:

$$
\begin{aligned}
e_{\lambda,s}(\mathbb{S}_{-1},\tau) &\equiv \beta d^{-1} R^n \int_{\tau'} \left\{ \frac{1}{\gamma_a \exp(a')} \frac{\lambda'}{\Pi'} \right\} \Phi(\tau'|\tau) d\tau', \\
e_{sp,s}(\mathbb{S}_{-1},\tau) &\equiv \theta_p w z + \xi_p \beta d^{-1} \int_{\tau'} \left\{ \left[\left(\frac{\Pi'}{\bar{\Pi}} \right) \left(\frac{\Pi}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p} \frac{y' \lambda' S_p'}{y\lambda} \right\} \Phi(\tau'|\tau) d\tau', \\
e_{fp,s}(\mathbb{S}_{-1},\tau) &\equiv \theta_p - 1 + \xi_p \beta d^{-1} \int_{\tau'} \left\{ \left[\left(\frac{\Pi'}{\bar{\Pi}} \right) \left(\frac{\Pi}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p - 1} \frac{y' \lambda' F_p'}{y\lambda} \right\} \Phi(\tau'|\tau) d\tau', \\
e_{sw,s}(\mathbb{S}_{-1},\tau) &\equiv \theta_w l_d^{\eta} \lambda^{-1} + \xi_w \beta d^{-1} \int_{\tau'} \left\{ \left[\left(\frac{\Pi_w' \exp(a')}{\bar{\Pi}} \right) \left(\frac{\Pi}{\bar{\Pi}} \right)^{-\iota_w} \right]^{(1+\eta)\theta_w} \frac{l_d' \lambda' S_w'}{l_d \lambda} \right\} \Phi(\tau'|\tau) d\tau', \\
e_{fw,s}(\mathbb{S}_{-1},\tau) &\equiv (\theta_w - 1) w + \xi_w \beta d^{-1} \int_{\tau'} \left\{ \left[\left(\frac{\Pi_w' \exp(a')}{\bar{\Pi}} \right) \left(\frac{\Pi}{\bar{\Pi}} \right)^{-\iota_w} \right]^{\theta_w - 1} \frac{l_d' \lambda' F_w'}{l_d \lambda} \right\} \Phi(\tau'|\tau) d\tau',
\end{aligned}
$$

where the index $s \in \{\text{NZLB}, \text{ZLB}\}$ is associated with the interest-rate regime in which the hypothetical nominal interest rate $\widehat{R}^n$ implied by its unconstrained policy function $g_{\widehat{R}^n,\text{NZLB}}(\mathbb{S}_{-1},\tau)$ is either above or below the lower bound. Then, the expectation functions are constructed as weighted averages of the regime-specific functions

$$
e_x(\mathbb{S}_{-1},\tau) = e_{x,\text{NZLB}}(\mathbb{S}_{-1},\tau) 1_{\{\widehat{R}^n > 1\}} + e_{x,\text{ZLB}}(\mathbb{S}_{-1},\tau) 1_{\{\widehat{R}^n \leq 1\}},
$$

where $1_{\{D\}}$ is the indicator function that equals one if the condition D is true and zero otherwise.

We obtain the policy functions by a time iteration method, which takes the following steps.

1. Make an initial guess for the expectation functions $e_s^{(0)} = \left(e_{\lambda,s}^{(0)}, e_{sp,s}^{(0)}, e_{fp,s}^{(0)}, e_{sw,s}^{(0)}, e_{fw,s}^{(0)} \right)$ for $s \in \{\text{NZLB}, \text{ZLB}\}$.

2. For $i = 1, 2, ...$ (i is an index for the number of iterations), taking as given the expectation functions previously obtained $e_s^{(i-1)}$, solve the relevant equations to obtain the policy functions $g_s^{(i)} = \left(g_{\Pi,s}^{(i)}, g_{\Delta_p,s}^{(i)}, g_{\Pi_w,s}^{(i)}, g_{\Delta_w,s}^{(i)}, g_{y,s}^{(i)}, g_{w,s}^{(i)}, g_{l_d,s}^{(i)}, g_{\widehat{R}^n,s}^{(i)} \right)$.

3. Update the expectation functions $e_s^{(i)}$ by interpolating the policy functions $g_s^{(i)}$.

4. Repeat Steps 2-3 until $\left\| e_s^{(i)} - e_s^{(i-1)} \right\|$ is small enough.

In Step 2, taking as given the values of $e_{x,s}^{(i-1)}(\mathbb{S}_{-1,j}, \tau_m)$ for $x \in \{\lambda, sp, fp, sw, fw\}$ at each grid point indexed by (j,m) and each regime $s \in \{\mathrm{NZLB}, \mathrm{ZLB}\}$, we have

$$\frac{\Pi_{jms}}{\bar{\Pi}} = \left(\xi_p^{-1} + (1-\xi_p^{-1})\left[\frac{e_{sp,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}{e_{fp,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}\right]^{1-\theta_p}\right)^{\frac{1}{\theta_p-1}} \left(\frac{\Pi_{-1,j}}{\bar{\Pi}}\right)^{\iota_p},$$

$$\Delta_{p,jms} = (1-\xi_p)\left[\frac{e_{sp,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}{e_{fp,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}\right]^{-\theta_p} + \xi_p\left(\frac{\Pi_{jms}}{\bar{\Pi}}\right)^{\theta_p}\left(\frac{\Pi_{-1,j}}{\bar{\Pi}}\right)^{-\iota_p}\Delta_{p,-1,j},$$

$$\frac{\Pi_{w,jms}\exp(a)}{\bar{\Pi}} = \left(\xi_w^{-1} + (1-\xi_w^{-1})\left[\frac{e_{sw,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}{e_{fw,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}\right]^{\frac{1-\theta_w}{1+\eta\theta_w}}\right)^{\frac{1}{\theta_w-1}} \left(\frac{\Pi_{-1,j}}{\bar{\Pi}}\right)^{\iota_w},$$

$$\Delta_{w,jms} = (1-\xi_w)\left[\frac{e_{sw,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}{e_{fw,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}\right]^{\frac{-\theta_w}{1+\eta\theta_w}} + \xi_w\left(\frac{\Pi_{w,jms}\exp(a_m)}{\bar{\Pi}}\right)^{\theta_w}\left(\frac{\Pi_{-1,j}}{\bar{\Pi}}\right)^{-\iota_w}\Delta_{w,-1,j},$$

$$y_{jms} = e_{\lambda,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)^{-1} + \frac{\gamma y_{j,-1}}{\gamma_a \exp(a_m)}$$

$$w_{jms} = w_{-1}\Pi_{w,jms}/\Pi_{jms},$$

$$l_{d,jms} = \Delta_{p,jms} y_{jms},$$

$$\widehat{R}_{jms}^n = (\widehat{R}_{-1,j}^n)^{\phi_r}\left[\bar{R}\bar{\Pi}\left(\frac{\Pi_{jms}}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{y_{jms}\exp(a_m)}{y_{-1}}\right)^{\phi_y}\right]^{1-\phi_r}\exp(\varepsilon_{r,m}).$$

Then, we can evaluate $\left(\Pi_{jms}, \Delta_{p,jms}, \Pi_{w,jms}, \Delta_{w,jms}, y_{jms}, w_{jms}, l_{d,jms}, \widehat{R}_{jms}^n\right)$ at each grid point (j,m) and each regime s and the policy functions $g_{x,s}^{(i)}(\mathbb{S}_{-1},\tau;\boldsymbol{\theta})$ for $x = \{\Pi, \Delta_p, \Pi_w, \Delta_w, y, w, l_d, \widehat{R}^n\}$ parameterized by a vector of polynomial coefficients $\boldsymbol{\theta}$ for computing the values off the grid points. Note that we do not rely on any numerical optimization routines to solve the nonlinear equations.

In Step 3, the expectation functions are updated by

$$e^{(i)}_{\lambda,s}(\mathbb{S}_{j,-1},\tau_m) = \beta d_m^{-1} R^n_{jms} \int_{\tau'} \left\{ \frac{1}{\gamma_a \exp(a')} \frac{e^{(i-1)}_{\lambda}(\mathbb{S},\tau';\boldsymbol{\theta})}{g^{(i)}_{\Pi}(\mathbb{S},\tau';\boldsymbol{\theta})} \right\} \Phi(\tau'|\tau) d\tau',$$

$$\begin{aligned}
e^{(i)}_{sp,s}(\mathbb{S}_{j,-1},\tau_m) &= \theta_p w_{jms} z_m + \xi_p \beta d_m^{-1} \int_{\tau'} \left\{ \left[\left(\frac{g^{(i)}_{\Pi}(\mathbb{S},\tau';\boldsymbol{\theta})}{\bar{\Pi}} \right) \left(\frac{\Pi_{jms}}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p} \right. \\
&\quad \left. \times \frac{g^{(i)}_{y}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{\lambda}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{sp}(\mathbb{S},\tau';\boldsymbol{\theta})}{y_{jms} e^{(i-1)}_{\lambda}(\mathbb{S}_{j,-1},\tau_m;\boldsymbol{\theta})} \right\} \Phi(\tau'|\tau) d\tau',
\end{aligned}$$

$$\begin{aligned}
e^{(i)}_{fp,s}(\mathbb{S}_{j,-1},\tau_m) &= \theta_p - 1 + \xi_p \beta d_m^{-1} \int_{\tau'} \left\{ \left[\left(\frac{g^{(i)}_{\Pi}(\mathbb{S},\tau';\boldsymbol{\theta})}{\bar{\Pi}} \right) \left(\frac{\Pi_{jms}}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p - 1} \right. \\
&\quad \left. \times \frac{g^{(i)}_{y}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{\lambda}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{fp}(\mathbb{S},\tau';\boldsymbol{\theta})}{y_{jms} e^{(i-1)}_{\lambda}(\mathbb{S}_{j,-1},\tau_m;\boldsymbol{\theta})} \right\} \Phi(\tau'|\tau) d\tau',
\end{aligned}$$

$$\begin{aligned}
e^{(i)}_{sw,s}(\mathbb{S}_{j,-1},\tau_m) &= \theta_w l^{\eta}_{d,jms} \lambda^{-1}_{jms} + \xi_w \beta d_m^{-1} \int_{\tau'} \left\{ \left[\left(\frac{g^{(i)}_{\Pi_w}(\mathbb{S},\tau';\boldsymbol{\theta}) \exp(a')}{\bar{\Pi}} \right) \left(\frac{\Pi_{jms}}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p} \right. \\
&\quad \left. \times \frac{g^{(i)}_{l_d}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{\lambda}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{sw}(\mathbb{S},\tau';\boldsymbol{\theta})}{l_{d,jms} e^{(i-1)}_{\lambda}(\mathbb{S}_{j,-1},\tau_m;\boldsymbol{\theta})} \right\} \Phi(\tau'|\tau) d\tau',
\end{aligned}$$

$$\begin{aligned}
e^{(i)}_{fw,s}(\mathbb{S}_{j,-1},\tau_m) &= (\theta_w - 1) w_{jms} + \xi_w \beta d_m^{-1} \int_{\tau'} \left\{ \left[\left(\frac{g^{(i)}_{\Pi_w}(\mathbb{S},\tau';\boldsymbol{\theta}) \exp(a')}{\bar{\Pi}} \right) \left(\frac{\Pi_{jms}}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p} \right. \\
&\quad \left. \times \frac{g^{(i)}_{l_d}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{\lambda}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{fw}(\mathbb{S},\tau';\boldsymbol{\theta})}{l_{d,jms} e^{(i-1)}_{\lambda}(\mathbb{S}_{j,-1},\tau_m;\boldsymbol{\theta})} \right\} \Phi(\tau'|\tau) d\tau',
\end{aligned}$$

where the values $(\Pi_{jms}, \Pi^w_{jms}, y_{jms}, w_{jms}, R^n_{jms})$ and the policy functions $g^{(i)}_x(\mathbb{S},\tau';\boldsymbol{\theta})$ evaluated at the next period's state $(\mathbb{S},\tau')$ are obtained in the previous step. Note that we interpolate $g^{(i)}_x(\mathbb{S},\tau';\boldsymbol{\theta})$ for $x \in \{\Pi, \Pi_w, y, l_d, \widehat{R}^n\}$ off the grid points (or equivalently $e^{(i-1)}_x(\mathbb{S},\tau';\boldsymbol{\theta})$ for $x \in \{\lambda, sp, fp, sw, fw\}$) by piecewise Chebyshev polynomials. Numerical integrals are computed with regard to τ'.

For the interpolation, we adopt a very efficient Smolyak algorithm developed by Judd, Maliar, Maliar, and Valero (2014). Specifically, the unidimensional disjoint sets of grid points of the state variables are constructed instead of the conventional nested sets in order to avoid repetitions of grid points. Then, projection functions onto the grid points are computed with interpolation coefficients and a Chebyshev family of orthogonal basis functions. In the algorithm, the level of approximation is set at 2, following Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015). The integrals over τ' are approximated by the Gauss–Hermite quadrature formula with four nodes.

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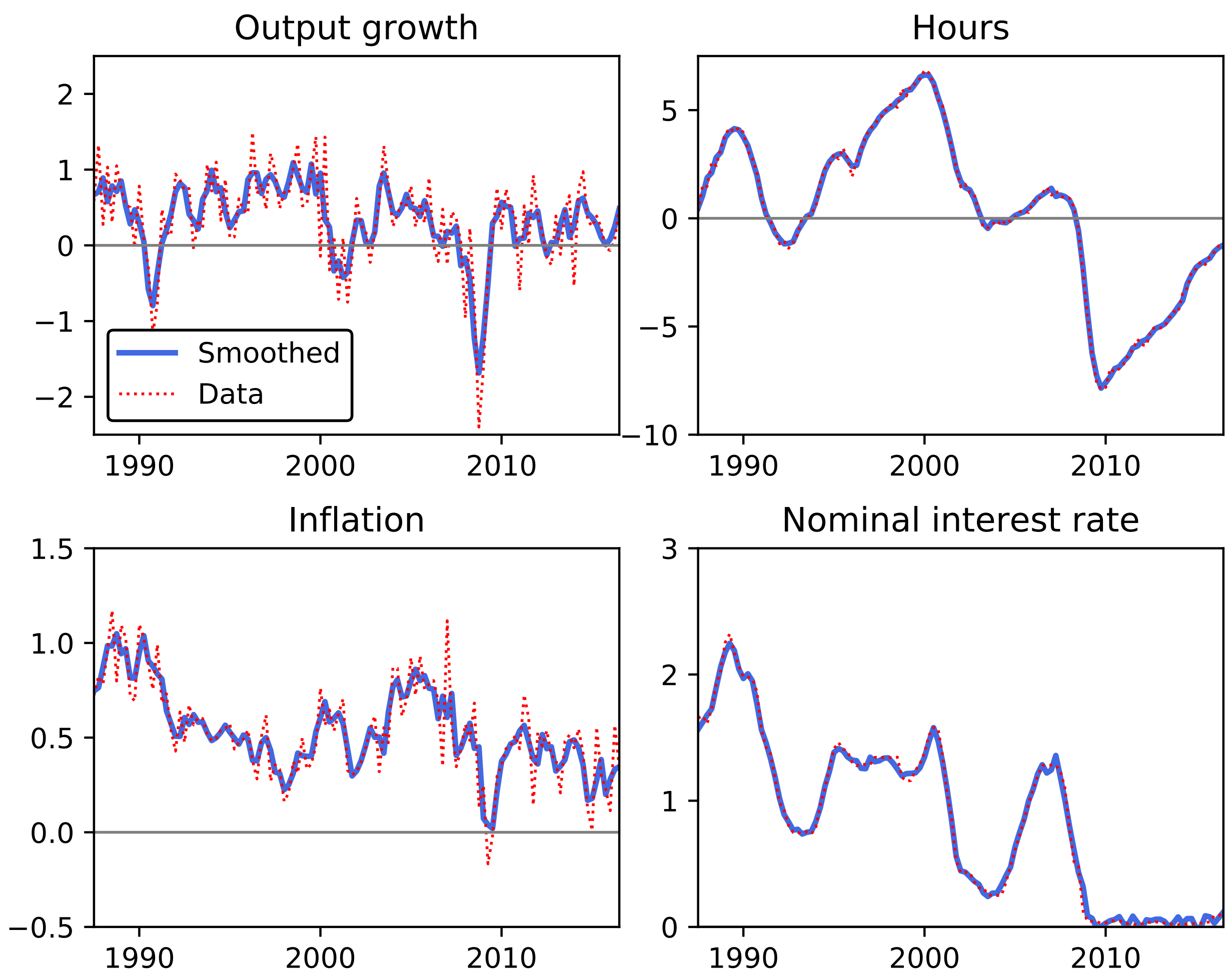
Table 1: Prior and posterior distributions of parameters

Parameter	Prior			Posterior	
	Distribution	Mean	S.D.	Mean	90% interval
γ	Beta	0.500	0.100	0.649	[0.564, 0.731]
η	Gamma	2.000	0.250	1.923	[1.506, 2.319]
ξ_w	Beta	0.500	0.100	0.615	[0.431, 0.800]
ι_w	Beta	0.500	0.100	0.515	[0.347, 0.676]
ξ_p	Beta	0.500	0.100	0.846	[0.789, 0.904]
ι_p	Beta	0.500	0.100	0.377	[0.216, 0.532]
ϕ_π	Gamma	1.500	0.250	1.781	[1.465, 2.078]
ϕ_y	Gamma	0.125	0.050	0.307	[0.174, 0.447]
ϕ_r	Beta	0.500	0.050	0.753	[0.721, 0.790]
$\bar{a}$	Normal	0.390	0.100	0.406	[0.299, 0.515]
$\bar{\pi}$	Normal	0.599	0.100	0.608	[0.501, 0.717]
$\bar{r}$	Gamma	0.544	0.100	0.513	[0.407, 0.610]
$\bar{l}$	Normal	0.000	0.100	-0.006	[-0.163, 0.164]
ρ_d	Beta	0.750	0.100	0.851	[0.794, 0.911]
ρ_a	Beta	0.500	0.100	0.338	[0.228, 0.448]
ρ_z	Beta	0.500	0.100	0.723	[0.574, 0.887]
$100\sigma_d$	Inv. Gamma	0.500	2.000	0.318	[0.192, 0.442]
$100\sigma_a$	Inv. Gamma	0.500	2.000	0.560	[0.486, 0.630]
$100\sigma_z$	Inv. Gamma	0.500	2.000	4.361	[1.529, 7.370]
$100\sigma_r$	Inv. Gamma	0.500	2.000	0.152	[0.128, 0.174]

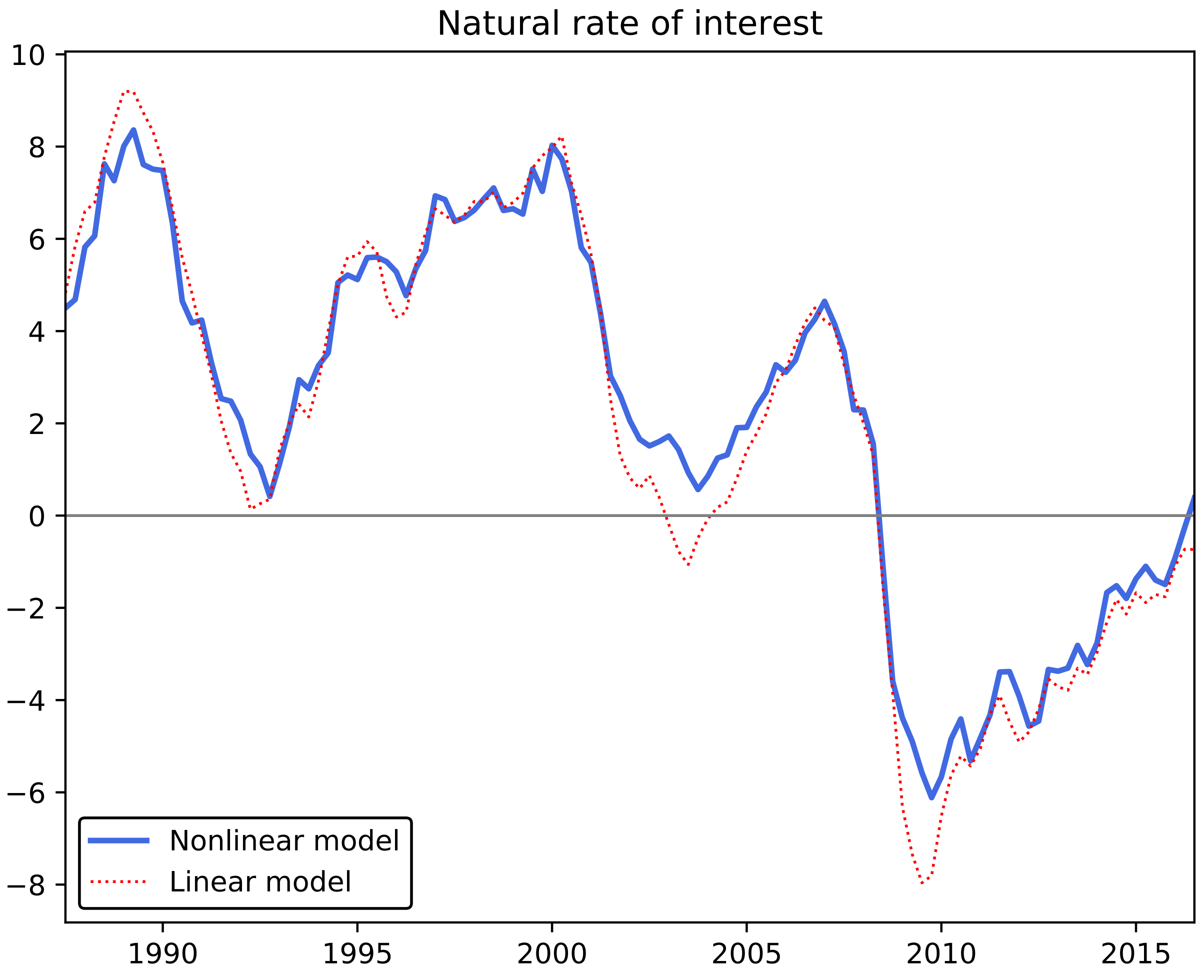
Note: Each posterior mean and 90% credible interval are calculated from $150,000$ draws generated using the Metropolis-Hastings algorithm.

Figure 1: Data and smoothed estimates



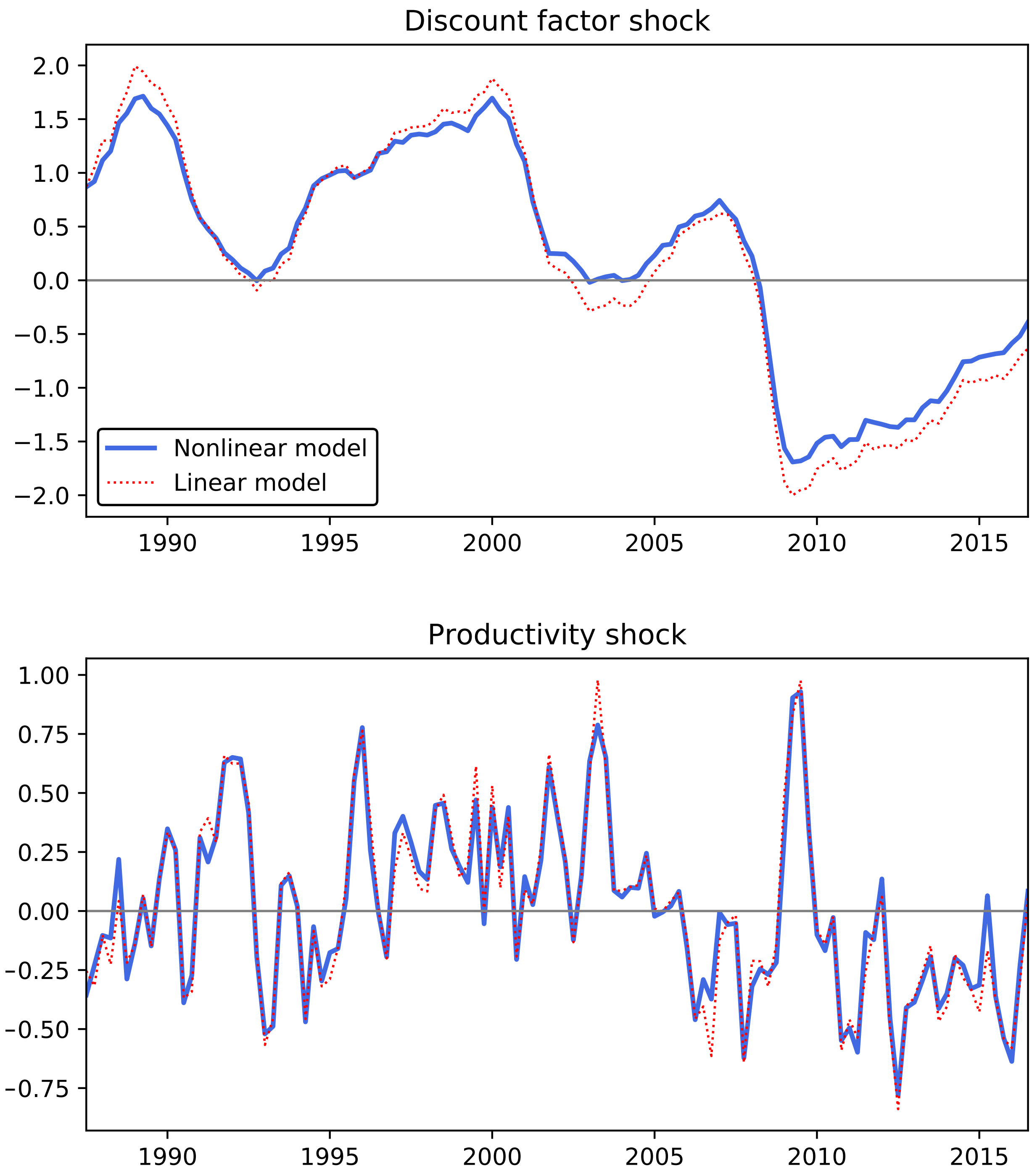
Note: The figure compares the data (dotted lines) on output growth, inflation, hours worked, and the nominal interest rate with the smoothed mean estimates (solid lines) of the corresponding variables.

Figure 2: Natural rate of interest



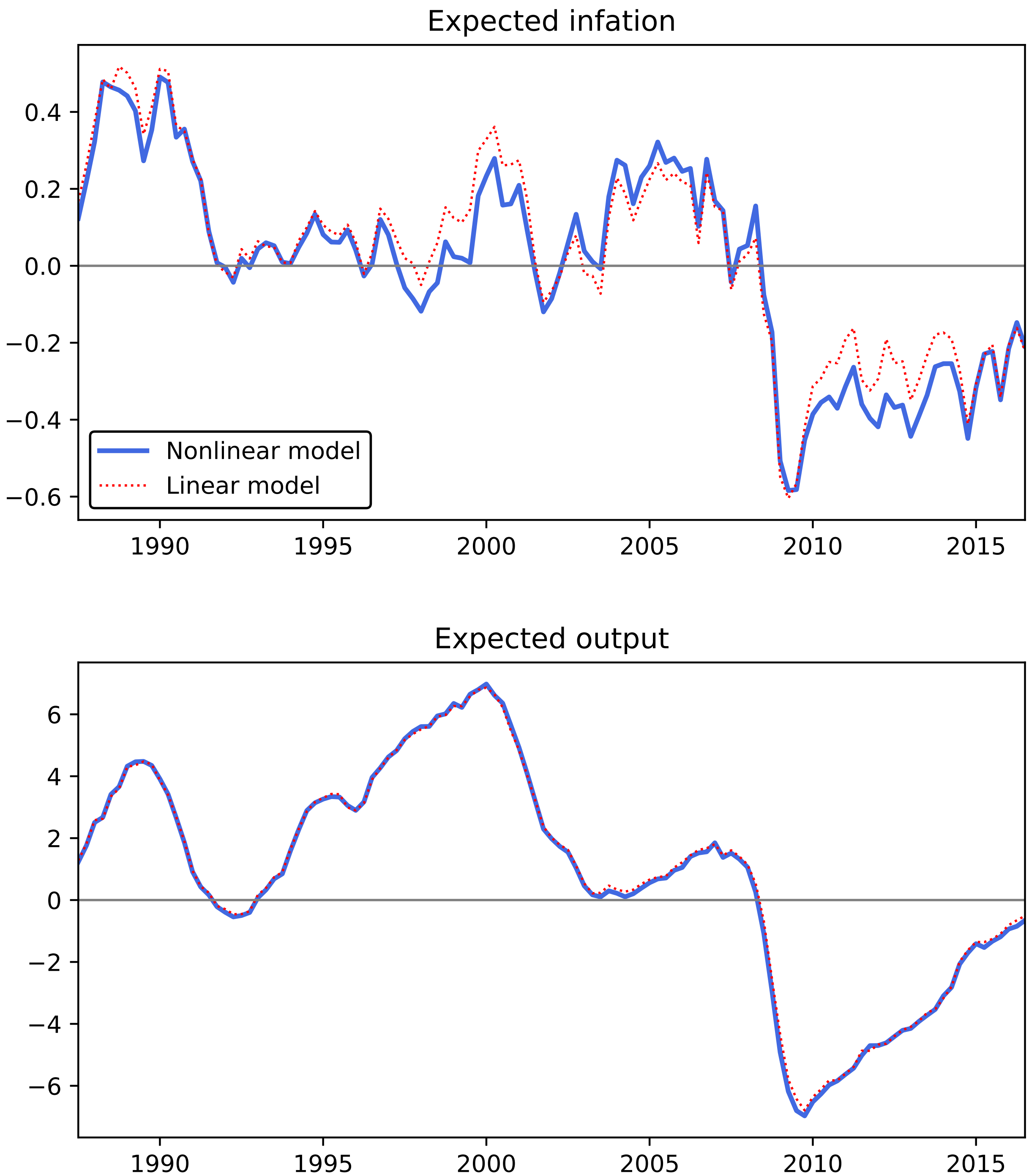
Note: The figure compares the smoothed mean estimate of the natural interest rate, in annualized terms, based on the nonlinear model (solid line) with that based on the linear model (dotted line).

Figure 3: Estimated shocks



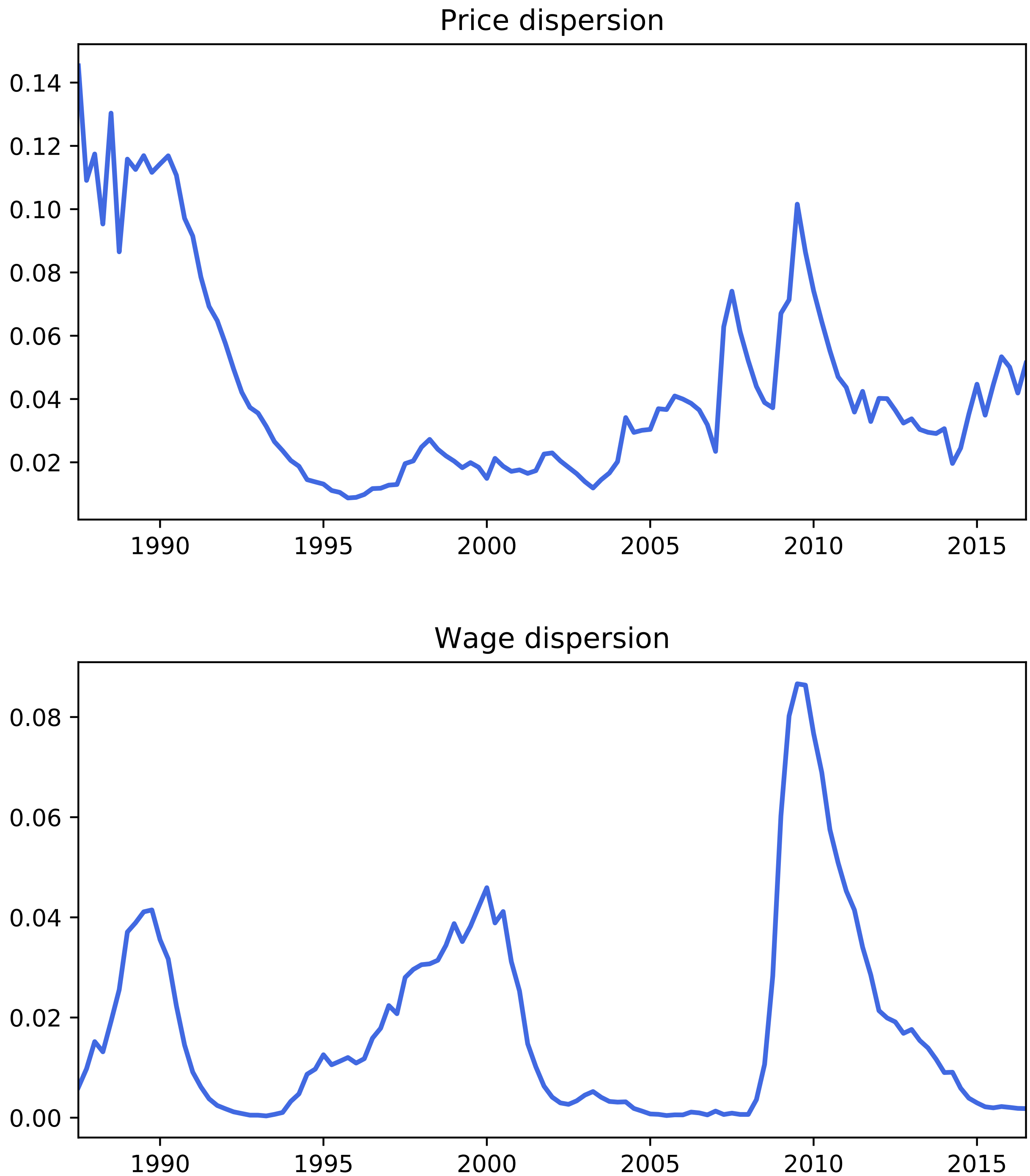
Note: The figure compares the smoothed mean estimates of the discount factor shocks d_t and the productivity shocks a_t, in percentage terms, based on the nonlinear model (solid lines) with those based on the linear model (dotted lines).

Figure 4: Expected inflation and output



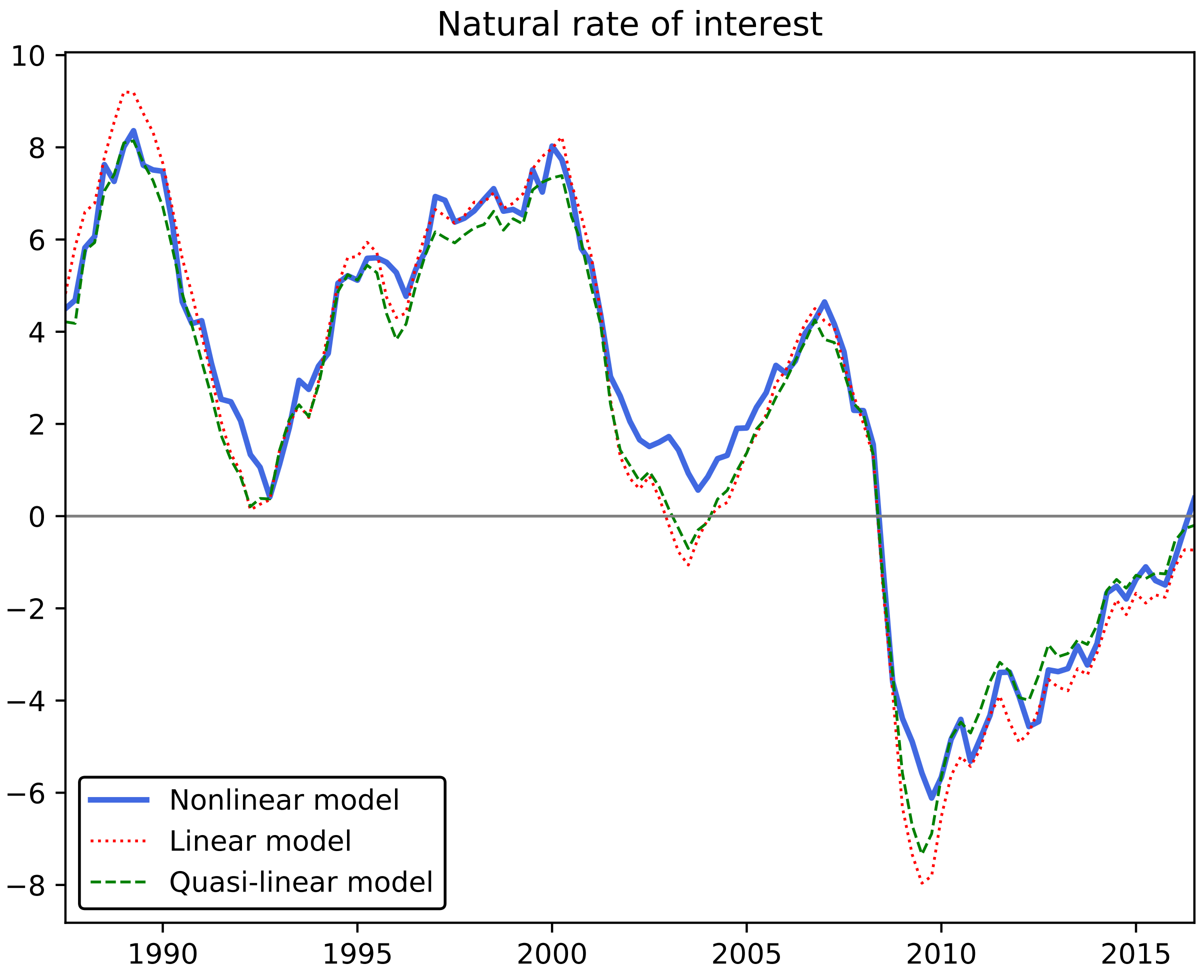
Note: The figure compares the smoothed mean estimates of expected inflation $E_t \log \Pi_{t+1}$ and expected output $E_t y_{t+1}$, in terms of percentage deviation from the steady state, based on the nonlinear model (solid lines) with those based on the linear model (dotted lines).

Figure 5: Price and wage dispersion



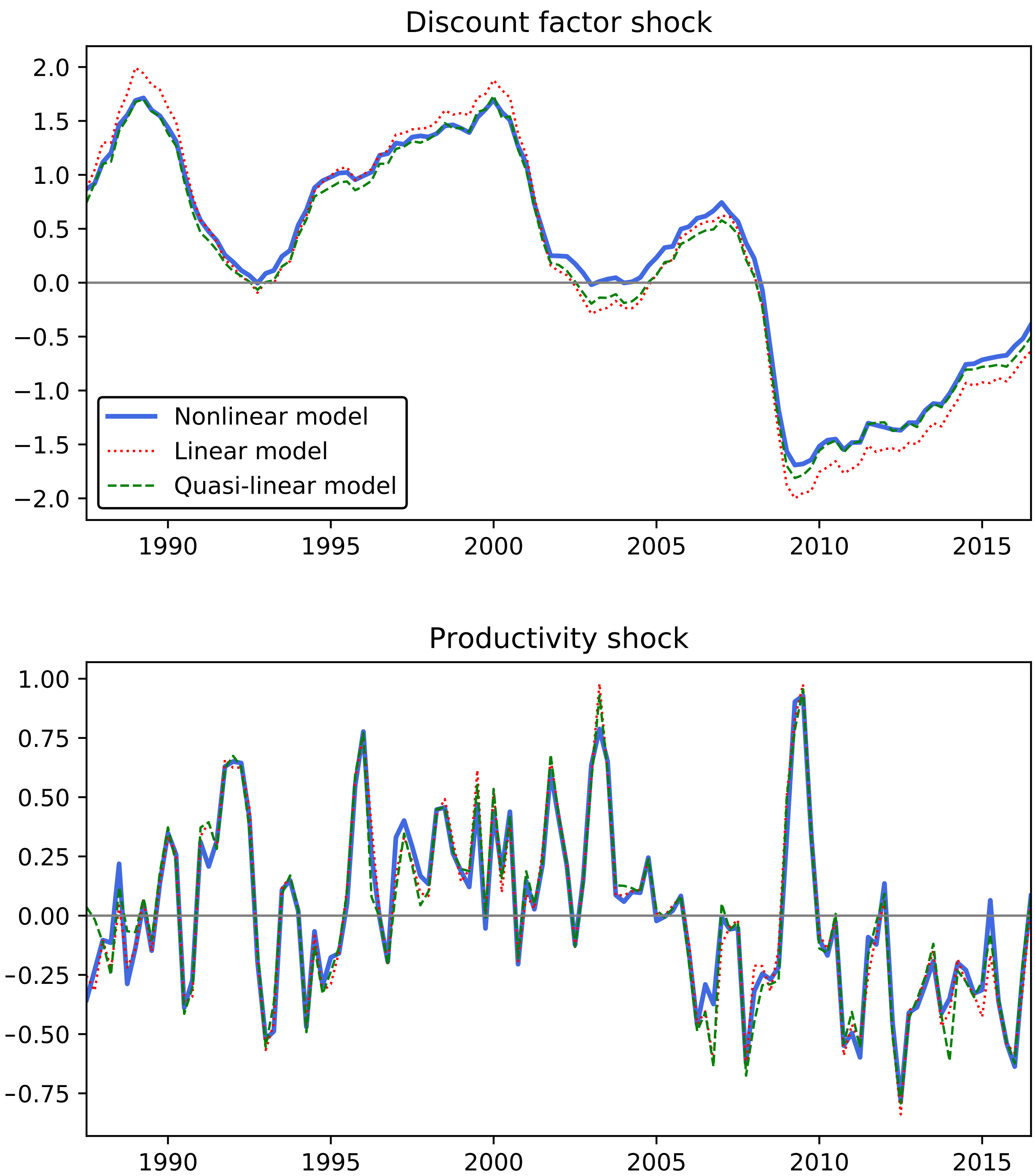
Note: The figure shows the smoothed mean estimates of the price and wage dispersion in terms of percentage deviation from the steady state.

Figure 6: Natural rate of interest (Nonlinear vs. Quasi-linear)



Note: The figure compares the smoothed mean estimates of the natural interest rate, in annualized terms, based on the nonlinear model (solid line), the linear model (dotted line), and the quasi-linear model (dashed line).

Figure 7: Estimated shocks (Nonlinear vs. Quasi-linear)



Note: The figure compares the smoothed mean estimates of the discount factor shocks d_t and the productivity shocks a_t, in percentage terms, based on the nonlinear model (solid lines), the linear model (dotted line), and the quasi-linear model (dashed line).