TCER Working Paper Series

Efficiency and stability in sender-receiver games under the selection-mutation dynamics

Seigo Uchida

April 2019

Working Paper E-132
http://tcer.or.jp/wp/pdf/e132.pdf

TOKYO CENTER FOR ECONOMIC RESEARCH
1-7-10-703 Iidabashi, Chiyoda-ku, Tokyo 102-0072, Japan

Abstract

Our study aims to reveal the relationship between the efficiency of neutrally stable strategies and asymptotic stability of rest points close to those strategies in Lewis-type sender-receiver games under the selection-mutation dynamics. We focus on the game in which the number of states is not equal to that of signals. While no strict Nash strategy exists in our case, we show that there are some neutrally stable strategies that have rest points close to these strategies, and that these rest points can be asymptotically stable under the selection-mutation dynamics. Moreover, those neutrally stable strategies give agents the maximal payoff. We name those neutrally stable strategies the extended signaling system, the unilaterally mixed strategy, and the max hybrid strategy.

Seigo Uchida
Tokyo University and Graduate School of
Social Welfare
Education Center for International Students
1-11 horifune, Kita-ku, Tokyo, 114-0004, Japan
seigo.uchida@gmail.com

Efficiency and stability in sender–receiver games under the selection–mutation dynamics *

Seigo Uchida[†]and Masakazu Fukuzumi[‡]

April 25, 2019

Abstract

Our study aims to reveal the relationship between the efficiency of neutrally stable strategies and asymptotic stability of rest points close to those strategies in Lewis-type sender–receiver games under the selection–mutation dynamics. We focus on the game in which the number of states is not equal to that of signals. While no strict Nash strategy exists in our case, we show that there are some neutrally stable strategies that have rest points close to these strategies, and that these rest points can be asymptotically stable under the selection–mutation dynamics. Moreover, those neutrally stable strategies give agents the maximal payoff. We name those neutrally stable strategies the *extended signaling system*, the *unilaterally mixed strategy*, and the *max hybrid strategy*.

*This is the modified version of the paper published in the TCER Working Paper Series (2017). We are grateful to Nozomu Muto, Yasuhiro Shirata, and the participants at Meeting of Japanese Economic Association and DC conference.

[†]Education Center for International Students, Tokyo University and Graduate School of Social Welfare. 1-11 horifune, Kita-ku, Tokyo, 114-0004, Japan (email: seigo.uchida@gmail.com)

[‡]Faculty of Humanities and Social Science, University of Tsukuba. 1-1-1 Tennoudai Tsukuba City, Ibaraki, 305-8571, Japan (email: fukuzumi.masakazu.fn@u.tsukuba.ac.jp)

JEL classification: C72; C73

Keywords: Sender–receiver game; Selection–mutation dynamics; Neutrally stable strategy; Asymptotic stablity

1 Introduction

In this study, we consider the dynamical behavior for signaling interactions with a common interest of a sender and a receiver under the selection–mutation dynamics. The signaling game with a common interest is a coordination problem among states, and acts through the strategic use of signals. The structure of this game almost does not lead the long–run behavior of each agent through the selection process to the optimal point.

Two approaches are mainly taken to address this problem (Hofbauer and Hatteger, 2008). The first approach is to study the evolution of signaling interactions in a finite population under the frequency-dependent Moran process (Pawlowitsh, 2007). The second approach is to study the dynamic behavior of signaling interactions when the replicator dynamics is perturbed in the restricted case (Hofbauer and Huttegger, 2008, 2015).[1] We take the second approach, where, in the general case, the number of states is not equal to that of the states.

Our benchmark is the classic model of a sender–receiver game as examined by Lewis (1969) or Nowak and Krakauer (1999). In the beginning of this game, a sender observes a state of the world picked up from a set of states by nature. Then, the sender chooses a signal from a set of signals. Next, a

[1] Hofbauer and Hatteger(2008) studied the case of two states and two signals. Hofbauer and Hatteger(2015) studied the case of three states and three signals.

receiver who does not know the state initially is informed of the signal chosen by the sender. The receiver associates the signal with a state in the set of the states. When the state observed by the sender is coincident with the state associated by the receiver, both agents receive a common payoff.[2]

Previous studies illustrate the difficulty of efficient and stable communication through signals. A perfect communication is represented by a strict Nash strategy that guarantees a maximal payoff as well as asymptotic stability. In a strict Nash strategy, every state is bijectively associated with one signal and vice versa. Such an equilibrium is called a *signaling system* (Lewis, 1969). However, signaling systems exist only when the number of states is equal to that of the signals (Wärneryd, 1993).

Moreover, the replicator dynamics almost do not converge to the strict Nash strategy or the evolutionary stable strategy (Pawlowitsch, 2008).[3] Neutrally stable strategies that belong to a continuous strategy space block it. They thus stay in continuos strategy space, that is, the suboptimal point. By perturbing the replicator dynamics with small noises over the strategy distributions uniformlly, most of the rest points are destroyed. Thus, there is no rest point that satisfies asymptotic stability except for a strict Nash strategy in the case of three states and three signals (Hofbauer and Huttegger, 2015).

When the number of states is not equal to that of the signals, the neutrally stable strategies that the continuous strategy space contains gurantee a maximal payoff. On the other hand, each rest point of these neutrally

[2]Sender-receiver games are divided into two classes: asymmetric games between a sender and a receiver, and symmetric games of both the role of a sender and the role of a receiver.

[3]The strict Nash strategies of asymmetric games correspond to the evolutionarily stable strategy of their symmetrization. See section 2.1.

stable strategies is not asymptotically stable under the replicator dynamics. A neighborhood of each rest point of neutrally stable strategies is continuous and not isolated under the replicator dynamics (Pawlowitsh, 2008). In this paper, we study neutrally stable strategies that have maximal payoff, and stability of these strategies under the selection–mutation dynamics especially when the number of signals is larger than that of the states. Our work makes three important contributions:

- We propose three types of neutrally stable strategies that lead to asymptotic stability under the selection–mutation dynamics. We focus on neutrally stable strategies that have a maximal payoff and formalize three types of these neutrally stable strategies in the general form.

- We show that there exists a rest point close to each of three neutrally stable strategies under the selection-mutation dynamics. Most of rest points that exist under the replicator dynamics are destroied under the selection–mutation dynamics. This paper studies the structure of rest points under the selection–mutation dynamics.

- We propose the conditions in which a rest point close to each of the three neutrally stable strategies is asymptotically stable under the selection–mutation dynamics. The conditions are represented by the relations between the number of states, that of signals, and the mutation rates of a sender and a receiver.

The remainder of this paper is organized as follows. Section 2 provides the formal model of the sender–receiver game and the definition of the selection–mutation dynamics. Section 3, 4, and 5 introduces the notions of the *extended-signaling system*, *the unilaterally mixed strategy* and the *max hybrid strategy*, along with the study of the stablity of these strategies. Section 6 compares these strategies, and section 7 concludes the paper.

2 The model

2.1 Sender-receiver games

A sender-receiver game consists of a sender and a receiver. There are n states of the world given by the set $N = \{1, 2, \ldots, n\}, n \geq 2$, and m signals given by the set $M = \{1, 2, , \ldots, m\}, m \geq 2$. Both agents communicate with each other through m signals in n states of the world.

A **sender's mixed strategy** is represented by a stochastic $n \times m$ matrix

$$P \in \mathcal{P}^{\triangle}_{n\times m} = \{P \in \mathbb{R}^{n\times m}_{+} : \sum_{j=1}^{m} p_{ij} = 1, \forall i\},$$

where a sender associates a state $i \in N$ with a signal $j \in M$ on the probability p_{ij} when Nature chooses a state i.

A **receiver's mixed strategy** is represented by a stochastic $m\times n$ matrix

$$Q \in \mathcal{Q}^{\triangle}_{m\times n} = \{Q \in \mathbb{R}^{m\times n}_{+} : \sum_{i=1}^{n} q_{ji} = 1, \forall j\},$$

where a receiver associates a signal $j \in M$ with a state $i \in N$ on the probability q_{ji} when the sender sends a signal j.

We consider the evolutionary dynamics of sender–receiver games by focusing on behavioral strategies (see section 2.3). In an extensive form of this games, when nature chooses a state of the world, a sender belongs to a praticular information set. After the sender sends a signal, a receiver also belongs to a particular information set. A probability measure over strategies of a sender and a receiver at each of their information sets specifies a behavioral strategy (Kuhn, 1953).

For each pair of strategies $(P, Q) \in P^{\triangle}_{n\times m} \times Q^{\triangle}_{m\times n}$, the **payoff of a sender and a receiver** is equally defined by

$$\pi(P, Q) = \sum_{i=1}^{n}(\sum_{j=1}^{m} p_{ij}q_{ji}) = tr(PQ).^{4}$$

Our **game** $\Gamma_{n,m} = \{P^{\triangle}_{n\times m} \times Q^{\triangle}_{m\times n}, \pi(P, Q)\}$ is described as an **asymmetric game** between a sender and a receiver (Lewis, 1969). When each agent plays both the role of a sender and the role of a receiver, the sender–receiver game is called a **symmetric game**. The symmetric game is introduced by Wärneryd (1993).

We study the Nash equilibria of the asymmetric game $\Gamma_{n,m}$. Let $B(Q) \in P^{\triangle}_{n\times m}$ and $B(P) \in Q^{\triangle}_{m\times n}$ denote the **best-response correspondence of** P **and** Q respectively. Following Pawlowitsch (2008), and Hofbauer and Hatteger (2015), we introduce a Nash strategy and a strict Nash strategy in the sender–receiver games.

Lemma 1. A pair $(P, Q) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ is
(i) a **Nash strategy** of $\Gamma_{n,m}$ if and only if $P \in B(Q)$ and $Q \in B(P)$,
(ii) a **strict Nash strategy** of $\Gamma_{n,m}$ if and only if P is a permutation matrix

[4] For simplicity of notation, we have multiplied the payoff of each agent by the number of possible states n.

and $Q = P^{\top}$.

A perfect communication where every state is bijectively associated with one signal and vice versa is called "signaling system" (Lewis, 1969). It is represented by a strict Nash strategy in asymmetric games. Strict Nash strategies of assymetric games correspond to their **evolutionarily stable strategies** of their symmetrization (Selten, 1980). A strict Nash strategy exists only when $n = m$ (Trapa and Nowak, 2000). Hence, we turn to a weaker equilibrium concept (Pawlowitsch, 2008).

Lemma 2. A strategy $(P, Q) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ is a **neutrally stable strategy** if
(i) (P, Q) is a Nash strategy and
(ii) whenever $(P', Q') \in B(Q) \times B(P)\backslash\{(P, Q)\}$, $\pi(P, Q) \geq \pi(P', Q')$.

The useful characterization of a neutrally stable strategy of $\Gamma_{n,m}$ is (Pawlowitsch, 2008):
Proposition 1. Let $(P, Q) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ be a Nash strategy. (P, Q) is neutrally stable if and only if
(i) at least one of the two matrices, P or Q, has no zero column, and
(ii) neither P nor Q has a column with multiple maximal elements that are strictly between 0 and 1.

2.2 Types of neutrally stable strategies

$\Gamma_{n,m}$ has a large set of neutrally stable strategies. We focus on neutrally stable strategies that have a maximal payoff.
Maximal payoff (Nowak, Plotkin, and Krakauer, 1999)

A maximal payoff of sender–receiver games is given by

$$\max_{P,Q} tr(P, Q) = \min\{m, n\}.$$

When $n < m$, there are three types of neutrally stable strategies that have a maximal payoff.

(i) an extended version of a signaling system.[5]

(ii) a mixed strategy, and

(iii) a hybrid strategy.

At each state, a sender and a receiver who choose these strategies necessarily receive one payoff by sharing the information of each state with one signal or multiple signals. Thus, there is no zero column of Q. Proposition 1 indicates that each of these strategies is a neutrally stable strategy.

(i) The first type is an extended version of the signaling system. Both agents use a signaling system for n states and n signals. A sender does not use the othe signals $m - n$. On the other hand, a receiver associates each of these signals (others) $m - n$ with states or a state. This strategy clearly exists if $n < m$. We call this an **extended signaling system**, that is, when a receiver associates each signal of the other signals $m - n$ with each state *equally.*

Example 1 (an extended signaling system).

[5]This strategy is represented by an extended permutation matrix (Trapa and Nowak, 2000).

$$P_1^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Q_1^* = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) Another type is a **unilaterally mixed strategy**. In this strategy, a sender associates each state with multiple signals. Then, a receiver associates these signals with each state. Both agents necessarilly receive one payoff at each state by communicating with multiple signals. This strategy exists if $n \leq \frac{m}{2}$.

Example 2 (a unilaterally mixed strategy).

$$P_2^\star = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, Q_2^\star = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

(iii) The last type is a **hybrid strategy**. A sender and a receiver choose a signaling system for some states and signals, whereas they choose a unilaterally mixed strategy for the others. This strategy clearly exists if $n < m$.

Example 3 (a hybrid strategy).

$$P_3^{*\star} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, Q_3^{*\star} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.3 Selection-mutation dynamics

We now consider the selection–mutation dynamics on the behavioral strategies, as per Hofbauer and Hutteger (2015). In an extensive form game of a sender–receiver game, a behavioral strategy is represented by a probability measure over strategies of a sender and a receiver.

We define an $(m-1)$-dimensional behavioral strategy simplex of a sender when the sender observes a state $i \in N$, defined by S_i, as

$$S_i = \{(p_{i1}, p_{i2}, \dots, p_{im}) | \sum_{j=1}^{m} p_{ij} = 1, p_{ij} \geq 0 \text{ for each } j \in M\}.$$

Similarly, we define an $(n-1)$-dimensional behavioral strategy simplex of a receiver when the receiver observes a signal $j \in M$, defined by S_j, as

$$S_j = \{(q_{j1}, q_{j2}, \dots, q_{jn}) | \sum_{k=1}^{n} q_{jk} = 1, q_{jk} \geq 0 \text{ for each } j \in N\}.$$

The space of behavioral strategies is defined by $S = \Pi_{i \in N} S_i \times \Pi_{j \in M} S_j$.

Our dynamic selection process is described by a dynamical system of differential equations defined for all points in S. The dynamical system is formulated as the following $2mn$ differential equations: For each $i \in N$ and $j \in M$,

$$\dot{p}_{ij} = p_{ij}(q_{ji} - \sum_{s \in M}^{m} p_{is} q_{si}) + \varepsilon(1 - m p_{ij}),$$

$$\dot{q}_{ji} = q_{ji}(p_{ij} - \sum_{t \in N}^{n} q_{jt} p_{tj}) + \delta(1 - n q_{ji}),$$

where ε and δ are small, uniform mutation rates. We denote this system by $\dot{S} = \Phi(S)$. This dynamical system is called the **selection–mutation dynamics** (Hofbauer, 1985). If $\varepsilon = \delta = 0$, the selection–mutation dynamics coincides with the **replicator dynamics**.

3 Extended signaling system

In this section, we define and study an extended signaling system. Given $n < m$, an extended signaling system has m-n zero-columns of P. We define the jth column vector of the matrix P, denoted by $\mathbf{p}_j$, and the ith column vector of the matrix Q, denoted by $\mathbf{q}_i$, that is, $P = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)$, and $Q = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$. Let $\mathbf{Z}_P = \{j \in M \mid \mathbf{p}_j$ is a zero-column of the matrix $P\}$ denote the set of signals that are not employed, and $\mathbf{Z}_Q = \{i \in N \mid \mathbf{q}_i$ is a zero-column of the matrix $Q\}$ the set of states that are not conveyed.

Definition 1. We say that a strategy $(P^*, Q^*) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ is an extended signaling system if the following properties hold:

(i) $|Z_{P^*}| = m - n$,[6]

(ii) $p^*_{ij} = q^*_{ji} = 1$ or $p^*_{ij} = q^*_{ji} = 0$ for each $i \in N$ and each $j \in M \backslash Z^*_P$,

(iii) $q^*_{ji} = \frac{1}{n}$ for each $j \in Z^*_P$ and each $i \in N$,

where $\left(p^*_{ij}, q^*_{ji}\right)_{(i,j)\in N\times M}$ denotes the entries of an extended signaling system (P^*, Q^*).

Example 4.

$$P^*_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; Q^*_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

[6] In what follows, $|X|$ denotes the cardinality of a set X.

where $Z_{P_4^*} = \{3\}$, $Z_{Q_1^*} = \phi$.

We introduce additional notatins to provide clear expositions in this subsection. Given an extended signaling system (P^*, Q^*), we divide the set $N \times M$ into subsets $I_i, i = 1, 2, 3$.

$I_1^* = \{(i,j) \in N \times M \mid p_{ij}^* = 1\}$,

$I_2^* = \{(i,j) \in N \times M \mid j \notin Z_{P^*}, p_{ij}^* = 0\}$,

$I_3^* = \{(i,j) \in N \times M \mid j \in Z_{P^*}\}$.

For (P_4^*, Q_4^*) in Example 4, we can easily check that $I_1 = \{(1,1),(2,2)\}, I_2 = \{(1,2),(2,1)\}, I_3 = \{(1,3),(2,3)\}$.

We find a rest point of the selection-mutation dynamics close to each extended signaling system. The rest point is symmetric for the corresponding extended signaling system. A rest point of our dynamical system, $S' = \Phi(S)$, is generally defined as a point that satisfies $\Phi\big((p_{ij}, q_{ji})_{(i,j)\in N\times M}\big) = \mathbf{0}$, where $\mathbf{0} \in \mathbf{R}^{|N\times M|}$ is a zero-column vector.

Definition 2. We say that a rest point $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta)) \in \mathbf{S}$ has a *symmetric* form for an extended signaling system (P^*, Q^*) if there are real values $\varepsilon_1, \varepsilon_2, \delta_1, q_1$ such that

$$\tilde{p}_{ij}(\varepsilon,\delta) = \begin{cases} 1 - (n-1)\varepsilon_1 - (m-n)\varepsilon_2 & \text{for each } (i,j) \in I_1^*, \\ \varepsilon_1 & \text{for each } (i,j) \in I_2^*, \\ \varepsilon_2 & \text{for each } (i,j) \in I_3^*, \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon,\delta)=\begin{cases}1-(n-1)\delta_1 & \text{for each } (i,j)\in I_1^*,\\ \delta_1 & \text{for each } (i,j)\in I_2^*,\\ q_1 & \text{for each } (i,j)\in I_3^*,\end{cases}$$

where $\big(\tilde{p}_{ij}(\varepsilon,\delta),\tilde{q}_{ji}(\varepsilon,\delta)\big)_{(i,j)\in N\times M}$ are entries of $(\tilde{P}(\varepsilon,\delta),\tilde{Q}(\varepsilon,\delta))$.

Example 5.

For an extended signaling system,

$$P_5^*=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&1&0&0\end{pmatrix},\quad Q_5^*=\begin{pmatrix}1&0&0\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\0&0&1\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\0&1&0\end{pmatrix},$$

the symmetric rest point of the selection–mutation dynamics has a form such that

$$\tilde{P}_5(\varepsilon,\delta)=\begin{pmatrix}1-2\varepsilon_1-2\varepsilon_2&\varepsilon_2&\varepsilon_1&\varepsilon_2&\varepsilon_1\\\varepsilon_1&\varepsilon_2&\varepsilon_1&\varepsilon_2&1-2\varepsilon_1-2\varepsilon_2\\\varepsilon_1&\varepsilon_2&1-2\varepsilon_1-2\varepsilon_2&\varepsilon_2&\varepsilon_1\end{pmatrix},$$

$$\tilde{Q}_5(\varepsilon,\delta)=\begin{pmatrix}1-2\delta_1&\delta_1&\delta_1\\q_1&q_1&q_1\\\delta_1&\delta_1&1-2\delta_1\\q_1&q_1&q_1\\\delta_1&1-2\delta_1&\delta_1\end{pmatrix},$$

where $Z_{P^*}=\{2,4\}, I_1^*=\{(1,1),(3,3),(2,5)\}$,

$I_2^* = \{(2,1),(3,1),(1,3),(2,3),(1,5),(3,5)\}$,
$I_3^* = \{(1,2),(2,2),(3,2),(1,4),(2,4),(3,4)\}$.

Theorem 1. For each pair of the mutation rates (ε, δ), there is a neighborhood of the point (P^*, Q^*) that contains a unique symmetric rest point $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta))$ of the selection–mutation dynamics.

All proofs are relegated to Appendix. Corollary 1 explicitly shows the form of the first-order approximation to the symmetric rest point close to the corresponding exteded signaling systems. This form clearly indicates that an extended signaling system is the limit point of the family of the symmetric rest points as (ε, δ) goes to $(0,0)$. Furthermore, using this form, we derive Thorem 2, which explores the stability of the rest point of our dynamical system.

Corollary 1. The first-order approximated entries $\big(\tilde{p}_{ij}(\varepsilon,\delta), \tilde{q}_{ji}(\varepsilon,\delta)\big)_{(i,j)\in N\times M}$ of the rest point $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta)) \in S$ in a neighborhood of an extended signaling system (P^*, Q^*) are explicitly given as follows:

$$\tilde{p}_{ij}(\varepsilon,\delta) = \begin{cases} 1 - \frac{mn-2n+1}{n-1}\varepsilon & \text{for each } (i,j) \in I_1^*, \\ \varepsilon & \text{for each } (i,j) \in I_2^*, \\ \frac{n}{n-1}\varepsilon & \text{for each } (i,j) \in I_3^*, \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon,\delta) = \begin{cases} 1 - (n-1)\delta & \text{for each } (i,j) \in I_1^*, \\ \delta & \text{for each } (i,j) \in I_2^*, \\ \frac{1}{n} & \text{for each } (i,j) \in I_3^*. \end{cases}$$

Example 6. The first order approximation of the symmetric rest point close to the extended signaling system (P_4^*, Q_4^*) in Example 4 is given by

$$\tilde{P}_4(\varepsilon, \delta) = \begin{pmatrix} 1-3\varepsilon & \varepsilon & 2\varepsilon \\ \varepsilon & 1-3\varepsilon & 2\varepsilon \end{pmatrix}, \; \tilde{Q}_4(\varepsilon, \delta) = \begin{pmatrix} 1-\delta & \delta \\ \delta & 1-\delta \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

where $Z_{P_4^*} = \{3\}$.

Sequentially, we demonstrate how the asymptotic stability of the selection-mutation dynamics at the symmetric rest point depends on the mutation rates (ε, δ) as well as the numbers of states and signals (n, m). Table 1 is an example of the first order approximated Jacobian matrix of our dynamical system evaluated at the symmetric rest point close to the exteded signaling system (P_4^*, Q_4^*) with $(n, m) = (2, 3)$. Each entry of Table 1 is

$$\frac{\left[\partial \text{ a row of Table 1}\right]}{\left[\partial \text{ a column of Table 1}\right]}$$

, for examle, $\frac{\partial \dot{p_{ij}}}{\partial p_{st}}$. We obtain the characteristic equation of the first-order approximated Jacobian matrix evaluated at the symmetric rest point close to the extended-signaling system (P_4^*, Q_4^*) from Table 1.

We turn to the general case, $i = 1, \ldots, n, j = 1, \ldots, m$. Let $J\Phi(\varepsilon, \delta, n, m)$ denote the Jacobian matrix with respect to $\dot{p}_{ij}, \dot{q}_{ji}$, evaluated at the first-order approximated rest point close to the corresponding extended-signaling system of our dynamical system with the mutation rates (ε, δ). $J\Phi(\varepsilon, \delta, n, m)$ has $(2nm)^2$ entries that are completely listed in Table 2.A. The list consists of six sub-lists, rows of the entries as follows: the r-th row comprises the values of $\frac{\partial \dot{p}_{ij}}{\partial p.}$ and $\frac{\partial \dot{p}_{ij}}{\partial q.}$ with $(i, j) \in I_r, r = 1, 2, 3$; $\frac{\partial \dot{q}_{ji}}{\partial p.}$ and $\frac{\partial \dot{q}_{ji}}{\partial q.}$ with $(i, j) \in$

$I_{r-3}, r = 4, 5, 6$. The contents of each cell of the sub-lists are explained in the following table.

	specifying the pair of (s,t)
specifying the pair of (i,j)	value of $\left(\frac{\partial \dot{p}_{ij}}{\partial p_{st}} \text{ or } \frac{\partial \dot{p}_{ji}}{\partial q_{ts}} \text{ or } \frac{\partial \dot{q}_{ji}}{\partial p_{st}} \text{ or } \frac{\partial \dot{q}_{ji}}{\partial q_{ts}}\right)$
	number of such entries above in the Jacobian matrix

Using this list in Table 2.A, we obtain the general formula of the characteristic equation of the first-order approximated Jacobian matrix evaluated at the symmetric rest point close to each extended-signaling system.

Theorem 2. Suppose that $\frac{\varepsilon}{\delta} < \frac{n(n-1)}{mn-n^2-1}$ for sufficiently small ε and δ. Then the symmetric rest point close to the corresponding extended-signaling system is asymptotically stable.

An extended signaling system is coincident with a signaling system when $n = m$. All eigenvalues of the characteristic polynomial from the proof of Theorem 2 are negative when $n = m$. Therefore, a rest point close to a signaling system is asymptotically stable.

4 Unilaterally mixed strategy

For each $i \in N$, let $K_i \subset M$ denote a set of signals that a sender sends after nature chooses a state $i \in N$. For each $i \in N$, let each $k_i \in K_i$ be a signal of $K_l \subset K$.

Definition 4. We say that a pair of strategies $(P^\star, Q^\star) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ is a *unilaterally mixed strategy* if the following properties hold:

(i) $|M| = \sum_i^N |K_i|$ and $K_i \cap K_j = \emptyset$ for each $i, j \in N(i \neq j)$ and $K_i, K_j \subset M$,

(ii) $0 < p_{ik_i} < 1$ with $\sum_{k_{i'}}^{K_i} p_{ik_{i'}} = 1$, and $q^\star_{ji} = q_{k_i i} = 1$ for each $i \in N$, each $K_i \subset M$ and each $k_i \in K_i$,

(iii) $p^\star_{ij} = q^\star_{ji} = 0$ for each $i \in N$ and each $j \in M \backslash K_i$.

Example 7. A unilaterally mixed strategy:

$$P^\star_6 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; Q^\star_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where $K_1 = \{3,4\}, K_2 = \{5,6,7\}, K_3 = \{1,2\}$.

This depends on the relationship of ε and δ, that is, whether a rest point close to a unilaterally mixed strategy exists under the selection–mutation dynamics. For a example, there is always a rest point close to the strategy including a unilaterally mixed strategy as a part of strategy form when $\varepsilon = \delta$(Hatteger and Hofbauer, 2015). We mainly focus on the case $\varepsilon \neq \delta$. Then we find that there exists a rest point close to the unilaterally mixed strategy if $p_{ik_i} = \frac{1}{|K_i|}$ for each $i \in N$, each $K_i \subset M$ and each $k_i \in K_i$.

Definition 5. We say that a rest point $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta)) \in \mathbf{S}$ has a *symmetric* form for $(P^\star, Q^\star)$ if there are real values $\varepsilon_{1_{ij}}, \delta_{1_{ji}}$ such that

$$\tilde{p}_{ij}(\varepsilon,\delta) = \begin{cases} \frac{1}{|K_i|} - \frac{1}{|K_i|}\left(\sum_{i'}^{N\backslash\{i\}} \sum_{k_{i'}}^{K_{i'}} \varepsilon_{1_{ik_{i'}}}\right) & \text{for each } i \in N \text{ and each } j \in K_i, \\ \varepsilon_{1_{ij}} & \text{for each } i \in N \text{ and each } j \in M\backslash K_i, \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon,\delta) = \begin{cases} 1 - \sum_{i'}^{N\setminus\{i\}} \delta_{1_{ji'}} & \text{for each } i \in N \text{ and each } j \in K_i, \\ \delta_{1_{ji}} & \text{for each } i \in N \text{ and each } j \in M\setminus K_i, \end{cases}$$

where $\big(\tilde{p}_{ij}(\varepsilon,\delta), \tilde{q}_{ji}(\varepsilon,\delta)\big)_{(i,j)\in N\times M}$ are entries of $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta))$.

The contenuous strategy space contains a rest point close to a unilaterally mixed strategy under the replicator dynamics. However, the perturbation of the replicator dynamics destroys most of rest points in the case $\varepsilon \neq \delta$ (Hofbauer and Hatteger, 2015). Similarly, the perturbation destroys each rest point except for a rest point close to a unilaterally mixed strategy with $p_{ij} = \frac{1}{|K_i|}$ for each $i \in N$ and each $K_i \subset M$. Thus, the rest point close to the unilaterally stable strategy is isolated

Theorem 3. Suppose that $p_{ik_i} = \frac{1}{|K_i|}$ for each $i \in N$, each $K_i \subset M$, and each $k_i \in K_i$. For each pair of mutation rates (ε, δ), there is a neighborhood of the point $(P^\star, Q^\star)$ that contains a unique symmetric rest point, $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta))$, of the selection–mutation dynamics.

Sequentially, we have the first-order approximated values of $\varepsilon_{1_{ij}}, \delta_{1_{ji}}$ for each $i \in N$, each $j \in M \setminus K_i$.

Corollary 2. The first-order approximated entries $\big(\tilde{p}_{ij}(\varepsilon,\delta), \tilde{q}_{ji}(\varepsilon,\delta)\big)_{(i,j)\in N\times M}$ of the rest point $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta)) \in S$ in a neighborhood of a unilaterally mixed strategy $(P^\star, Q^\star)$ is explicitly given as follows:

$$\tilde{p}_{ij}(\varepsilon,\delta) = \begin{cases} \frac{1}{|K_i|} - \frac{m-|K_i|}{|K_i|}\varepsilon) & \text{for each } i \in N \text{ and each } j \in K_i, \\ \varepsilon & \text{for each } i \in N \text{ and each } j \in M\setminus K_i, \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon,\delta) = \begin{cases} 1 - |K_i|(n-1)\delta & \text{for each } i \in N \text{ and each } j \in K_i, \\ |K_{i'}|\delta & \text{for each } i \in N \text{ and each } j \in K_{i'} (i \neq i'), \end{cases}$$

where $\big(\tilde{p}_{ij}(\varepsilon,\delta), \tilde{q}_{ji}(\varepsilon,\delta)\big)_{(i,j)\in N\times M}$ are entries of $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta))$.

Example 8. A unilaterally mixed strategy:

$$P_6^* = \begin{pmatrix} \varepsilon & \varepsilon & \frac{1}{2}-\frac{5}{2}\varepsilon & \frac{1}{2}-\frac{5}{2}\varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \frac{1}{3}-\frac{4}{3}\varepsilon & \frac{1}{3}-\frac{4}{3}\varepsilon & \frac{1}{3}-\frac{4}{3}\varepsilon \\ \frac{1}{2}-\frac{5}{2}\varepsilon & \frac{1}{2}-\frac{5}{2}\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix},$$

$$Q_6^* = \begin{pmatrix} 2\delta & 2\delta & 1-4\delta \\ 2\delta & 2\delta & 1-4\delta \\ 1-4\delta & 2\delta & 2\delta \\ 1-4\delta & 2\delta & 2\delta \\ 3\delta & 1-6\delta & 3\delta \\ 3\delta & 1-6\delta & 3\delta \\ 3\delta & 1-6\delta & 3\delta \end{pmatrix},$$

where $K_1 = \{3,4\}, K_2 = \{5,6,7\}, K_3 = \{1,2\}$.

We also obtain the characteristic equation of the first-order approximated Jacobian matrix evaluated at the symmetric rest point close to each unilaterally mixed strategy with $|K_i| = 2$ for each $i \in N$.

Theorem 4. Suppose that $|K_i| = 2$ for each $i \in N$ and $\frac{\varepsilon}{\delta} > \frac{m(n-1)}{4}$ for sufficiently small ε and δ. Then, the symmetric rest point close to the corresponding unilaterally mixed strategy is asymptotically stable.

5 Hybrid strategy

Let $L \subset N$ be a subset of N, and $K \subset M$ a subset of M. We define a set of signals after observing a state $l \in L$, denoted by $K_l \subset K$. Let $k_l \in K_l$ be a signal of $K_l \subset K$.

Definition 6. We say that a pair of strategies $(P^{\star\star}, Q^{\star\star}) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ is a *hybrid strategy* if the following properties hold:

(i) $|K| = \sum_l^L |K_l|$ and $m - n = |K| - |L|$,

(ii) $p^{\star\star}_{ij} = q^{\star\star}_{ji} = 1$ or $p^{\star\star}_{ij} = q^{\star\star}_{ji} = 0$ for each $i \in N\backslash L$ and each $j \in M\backslash K$,

(iii) $p^{\star\star}_{ij} = q^{\star\star}_{ji} = 0$ for each $i \in N\backslash L$ and each $j \in K$,

(iv) $p^{\star\star}_{ij} = p_{ik_i}$, $0 < p_{ik_i} < 1$ with $\sum_{k_l}^{K_i} p_{ik_l} = 1$ and $q^{\star\star}_{ji} = q_{k_i i} = 1$ for each $i \in L$ and each $k_i \in K_i$, and

(v) $p^{\star\star}_{ij} = q^{\star\star}_{ji} = 0$ for each $i \in L$ and each $j \in M\backslash K_i$,

where $\left(p^{\star\star}_{ij}, q^{\star\star}_{ji}\right)_{(i,j)\in N\times M}$ denotes the entries of a hybrid strategy $(P^{\star\star}, Q^{\star\star})$.

A hybrid strategy is coincident with a signaling system when $|L| = |K| = 0$. A hybrid strategy is coincident with a unilaterally mixed strategy when $|N\backslash L| = |M\backslash K| = 0$.

We introduce additional notatoins to provide clear expositions in this subsections. Given a hybrid strategy $(P^{\star\star}, Q^{\star\star})$, we divide the set $N \times M$ into subsets, $I_i, i = 1, 2, 3, 4, 5, 6$,

$I^{\star\star}_1 = \{(i,j) \in N \times M \mid p^{\star\star}_{ij} = q^{\star\star} = 1,\ i \in N\backslash L \text{ and } j \in M\backslash K\}$,

$I^{\star\star}_2 = \{(i,j) \in N \times M \mid p^{\star\star}_{ij} = q^{\star\star}_{ji} = 0,\ i \in N\backslash L \text{ and } j \in M\backslash K\}$,

$I^{\star\star}_3 = \{(i,j) \in N \times M \mid p^{\star\star}_{ij} = q^{\star\star}_{ji} = 0,\ i \in N\backslash L \text{ and } j \in K\}$,

$I^{\star\star}_4 = \{(i,j) \in N \times M \mid p^{\star\star}_{ij} = q^{\star\star}_{ji} = 0,\ i \in L \text{ and } j \in M\backslash K\}$,

$I_5^{**} = \{(i,j) \in N \times M \mid p_{ij}^{**} = q_{ji}^{**} = 0,\ i \in L \text{ and } j \in K \backslash K_i\}$,

$I_6^{**} = \{(i,j) \in N \times M \mid p_{ij}^{**} = \frac{1}{|K_i|} \text{ and } q_{ji}^{**} = 1,\ i \in L \text{ and } j \in K_i\}$.

Example 9. A hybrid strategy:

$$P_7^{**} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ Q_7^{**} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $L = \{2,3\}, K = \{1,3,4,5\}, K_2 = \{1,4\}, K_3 = \{3,5\}$,

$I_1^{**} = \{(1,2),(4,6)\},\ I_2^{**} = \{(1,6),(4,2)\}$,

$I_3^{**} = \{(1,1),(1,3),(1,4),(1,5),(4,1),(4,3),(4,4),(4,5),\}$,

$I_4^{**} = \{(2,2),(2,5),(3,2),(3,5)\},\ I_5^{**} = \{(2,3),(2,5),(3,1),(3,4)\}$,

$I_6^{**} = \{(2,1),(2,4),(3,3),(3,5)\}$.

Definition 7. A rest point $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta)) \in \mathbf{S}$ has a symmetric form for (P^{**}, Q^{**}) if there are real values $\varepsilon_1, \varepsilon_{2_j}, \varepsilon_{3_i}, \varepsilon_{4_{ij}}, \delta_1, \delta_{2_i}, \delta_{3_j}, \delta_{4_{ji}}$ such that

$$\tilde{p}_{ij}(\varepsilon,\delta) = \begin{cases} 1 - (m - |K| - 1)\varepsilon_1 - \sum_l^L \sum_{k_l}^{K_l} \varepsilon_{2_{k_l}} & \text{for each } (i,j) \in I_1^{**}, \\ \varepsilon_1 & \text{for each } (i,j) \in I_2^{**}, \\ \varepsilon_{2_j} & \text{for each } (i,j) \in I_3^{**}, \\ \varepsilon_{3_i} & \text{for each } (i,j) \in I_4^{**}, \\ \varepsilon_{4_{ij}} & \text{for each } (i,j) \in I_5^{**}, \\ \frac{1}{|K_i|} - \frac{1}{|K_i|}((m - |K|)\varepsilon_{3_i} + \sum_l^{L\backslash i} \sum_{k_l}^{K_l} \varepsilon_{4_{ik_l}}) & \text{for each } (i,j) \in I_6^{**}, \end{cases}$$

$$
\tilde{q}_{ji}(\varepsilon,\delta) = \begin{cases} 1-(n-|L|-1)\delta_1 - \sum_l^L \delta_{2_l} & \text{for each } (i,j)\in I_1^{**}, \\ \delta_1 & \text{for each } (i,j)\in I_2^{**}, \\ \delta_{2_i} & \text{for each } (i,j)\in I_4^{**}, \\ \delta_{3_j} & \text{for each } (i,j)\in I_3^{**}, \\ \delta_{4_{ji}} & \text{for each } (i,j)\in I_5^{**}, \\ 1-(n-|L|)\delta_{3_j} - \sum_l^{L\setminus\{i\}} \delta_{4_{jl}} & \text{for each } (i,j)\in I_6^{**}, \end{cases}
$$

where $\big(\tilde{p}_{ij}(\varepsilon,\delta),\tilde{q}_{ji}(\varepsilon,\delta)\big)_{(i,j)\in N\times M}$ are entries of $(\tilde{P}(\varepsilon,\delta),\tilde{Q}(\varepsilon,\delta))$.

A rest point close to a hybrid strategy exists, since it has the hybrid form of a signaling system and a unilaterally mixed strategy.

Theorem 5. Suppose that $p_{ik_i} = \frac{1}{|K_i|}$ for each $i \in L$, each $K_i \subset M$ and each $k_i \in K_i$. For each pair of mutation rates (ε,δ), there is a neighborhood of the point (P^{**},Q^{**}) that contains a unique symmetric rest point, $(\tilde{P}(\varepsilon,\delta),\tilde{Q}(\varepsilon,\delta))$, of the selection-mutation dynamics.

Sequentially, we have the explicit form of a rest point.

Corollary 3. Suppose that $p_{ik_i} = \frac{1}{|K_i|}$ for each $i \in N$, each $K_i \subset M$, and each $k_i \in K_i$. The first-order approximated entries $\big(\tilde{p}_{ij}(\varepsilon,\delta),\tilde{q}_{ji}(\varepsilon,\delta)\big)_{(i,j)\in N\times M}$ of a rest point $(\tilde{P}(\varepsilon,\delta),\tilde{Q}(\varepsilon,\delta)) \in S$ in a neighborhood of a hybrid strategy (P^{**},Q^{**}) are explicitly given as follows:

$\cdot$ For real values ε, δ,

$$\tilde{p}_{ij}(\varepsilon,\delta) = \begin{cases} 1-(m-1)\varepsilon & \text{for each } (i,j) \in I_1^{**}, \\ \varepsilon & \text{for each } (i,j) \in I_2^{**}, \\ \varepsilon & \text{for each } (i,j) \in I_3^{**}, \\ \varepsilon & \text{for each } (i,j) \in I_4^{**}, \\ \varepsilon & \text{for each } (i,j) \in I_5^{**}, \\ \frac{1}{|K_i|} - \frac{m-|K_i|}{|K_i|}\varepsilon & \text{for each } (i,j) \in I_6^{**}, \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon,\delta) = \begin{cases} 1-(n-1)\delta & \text{for each } (i,j) \in I_1^{**}, \\ \delta & \text{for each } (i,j) \in I_2^{**}, \\ |K_{i'}|\delta & \text{for each } (i,j) \in I_3^{**} \text{ with } j \in K_{i'}, \\ \delta & \text{for each } (i,j) \in I_4^{**}, \\ |K_{i'}|\delta & \text{for each } (i,j) \in I_5^{**} \text{ with } j \in K_{i'}, \\ 1-|K_i|(n-1)\delta & \text{for each } (i,j) \in I_6^{**} \text{ with } j \in K_{i'}, \end{cases}$$

where $\big(\tilde{p}_{ij}(\varepsilon,\delta), \tilde{q}_{ji}(\varepsilon,\delta)\big)_{(i,j)\in N\times M}$ are entries of $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta))$.

Example 10. The first order approximation of the symmetric rest point close to a hybrid strategy (P_7^{**}, Q_7^{**}) in Example 9 is given by

$$P_7^{**} = \begin{pmatrix} \varepsilon & 1-5\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \frac{1}{2}-2\varepsilon & \varepsilon & \varepsilon & \frac{1}{2}-2\varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \frac{1}{2}-2\varepsilon & \varepsilon & \frac{1}{2}-2\varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1-5\varepsilon \end{pmatrix},$$

$$Q_7^{**} = \begin{pmatrix} 2\delta & 1-6\delta & 2\delta & 2\delta \\ 1-3\delta & \delta & \delta & \delta \\ 2\delta & 2\delta & 1-6\delta & 2\delta \\ 2\delta & 1-6\delta & 2\delta & 2\delta \\ 2\delta & 2\delta & 1-6\delta & 2\delta \\ \delta & \delta & \delta & 1-3\delta \end{pmatrix}.$$

We also obtain the characteristic equation of the first-order approximated Jacobian matrix evaluated at the symmetric rest point close to each unilaterally mixed strategy with $|K_l| = 2$ for each $l \in L$ and each $K_l \subset K$. This condition leads to $|K| = 2|L|$.

Theorem 6. Suppose that $|L| \geq 1$ and $|K_l| = 2$ for each $l \in L$ and each $K_l \subset K$, and $\frac{\varepsilon}{\delta} > \frac{|K|(n-1)}{4}$ for sufficiently small ε and δ. Based on Theorem 5, a first-order approximated Jacobian matrix exists, and it is evaluated at the symmetric rest point close to a hybrid strategy with $|K_l| = 2$ for each $l \in L$ and each $K_l \subset K$. Then, the symmetric rest point close to the hybrid strategy is asymptotically stable.

6 Discussion

6.1 Comparisons among strategies

Our conditions in sections 3, 4 and 5 guarantee asymptotic stability of a rest point close to each of the three neutrally stable strategies for $m > n \geq 2$.

- an extended signaling system:

  $\frac{\varepsilon}{\delta} < \frac{n(n-1)}{mn-n^2-1}$,

- a unilaterally mixed strategy with $|K_i| = 2$ for each $i \in N$ and each $K_i \subset M$:

  $\frac{\varepsilon}{\delta} > \frac{m(n-1)}{4}$, where $m = 2n$,

- a hybrid strategy with $|K_l| = 2$ for each $l \in L$ and each $K_l \subset K$:

  $\frac{\varepsilon}{\delta} > \frac{|K|(n-1)}{4}$, where $m - n = |K| - |L|$, $|K| = 2|L|$, and $|L| \geq 1$.

Note that a unilaterally mixed strategy and a hybrid strategy do not simultaneously exist because both conditions are not compatible for n, m, $|L|$, and $|K|$. By arranging these results, we define three functions from a set N to a set M, denoted by

$$
\begin{aligned}
m_e(n) &\equiv \frac{1}{\frac{\varepsilon}{\delta}}(n-1) + n + \frac{1}{n}, \\
m_u(n) &\equiv \frac{4\frac{\varepsilon}{\delta}}{n-1}, \\
m_h(n) &\equiv n + \frac{2\frac{\varepsilon}{\delta}}{n-1}.
\end{aligned}
$$

where $m_e(n)$ is the function describing the line that divides two regions: a stable parameter region for a rest point close to an extended signaling system and a unstable parameter region for it; m_u for a unilaterally mixed strategy and m_h for a hybrid strategy.

We illustrate the foregoing results by means of the following examples.

Example 8. Consider $\frac{\varepsilon}{\delta} = 1, 10, \frac{1}{10}$. We obtain the following inequalities.

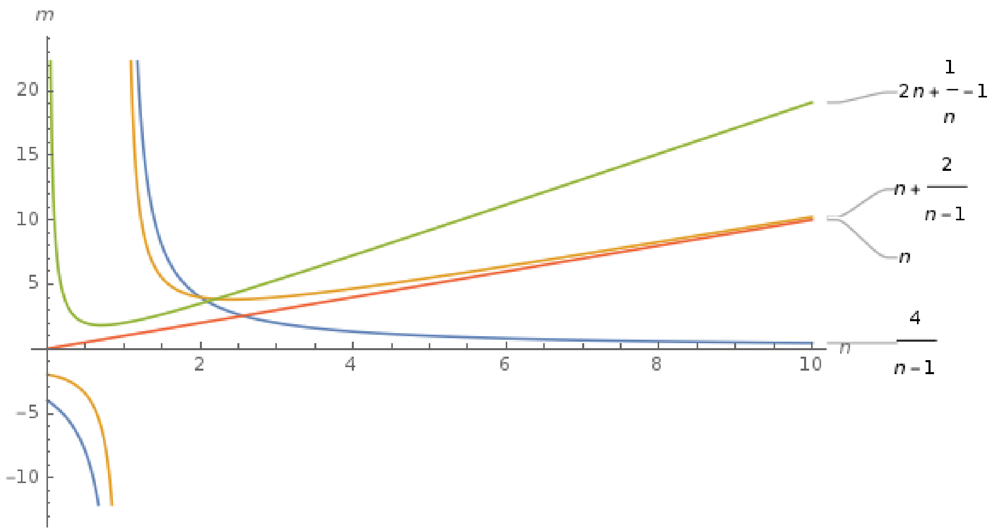

Figure 1: $\frac{\varepsilon}{\delta} = 1$

· $\frac{\varepsilon}{\delta} = 1$ (Figure 1); $m < 2n + \frac{1}{n} - 1$ (an extended signaling system), $m < \frac{4}{n-1}$ (a unilaterally mixed strategy), and $m < n + \frac{2}{n-1}$ (a hybrid strategy) .

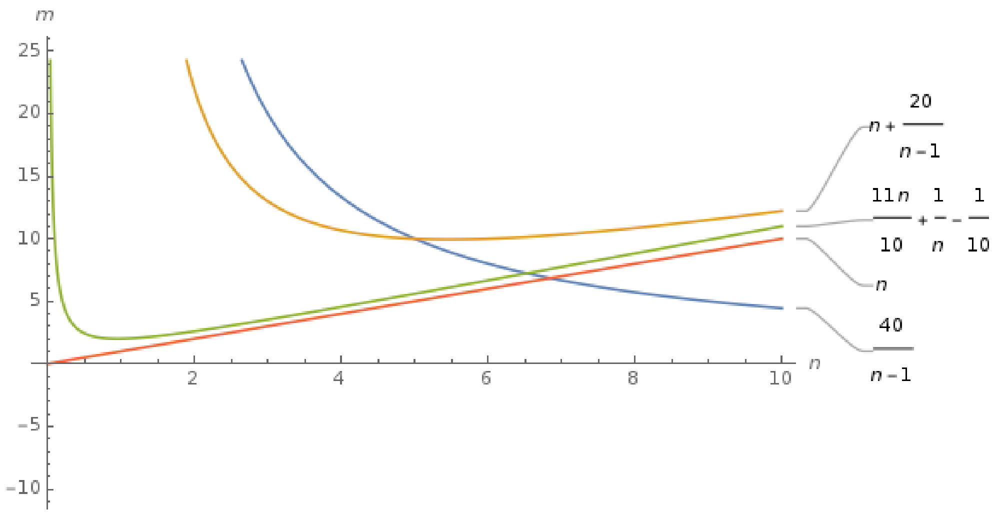

Figure 2: $\frac{\varepsilon}{\delta} = 10$

· $\frac{\varepsilon}{\delta} = 10$ (Figure 2); $m < \frac{11}{10}n + \frac{1}{n} - \frac{1}{10}$ (an extended signaling system), $m < \frac{40}{n-1}$ (a unilaterally mixed strategy), and $m < n + \frac{20}{n-1}$ (a hybrid strategy) .

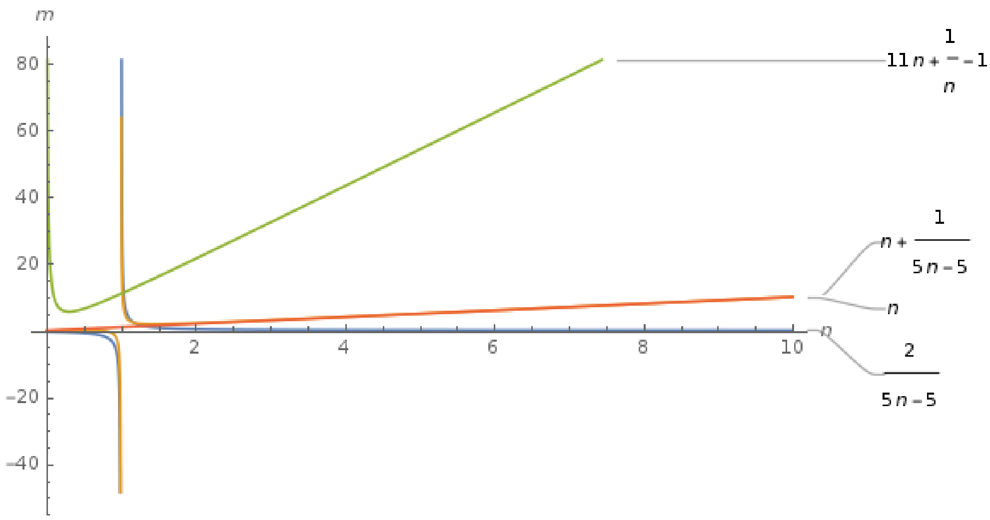

Figure 3: $\frac{\varepsilon}{\delta} = \frac{1}{10}$

· $\frac{\varepsilon}{\delta} = \frac{1}{10}$ (Figure 3); $m < 11n + \frac{1}{n} - 1$ (an extended signaling system), $m < \frac{2}{5n-5}$ (a unilaterally mixed strategy), $m < n + \frac{1}{5n-5}$ (a hybrid strategy) .

We consider the parameter region over the line $m = n$ $(n < m)$. In the parameter regions below each m_e, m_u, and m_h, a rest point close to each strategy is asymptotically stable. When $\frac{\varepsilon}{\delta} = 1$, more rest points close to each extended signaling system is asymptotically stable for $3 \leq n$. When $\frac{\varepsilon}{\delta} = 10$, more rest points close to each unilaterally mixed strategy is asymptotically stable for $2 \leq n \leq 5$, each hybrid strategy for $5 \leq n \leq 14$, each hybrid strategy for $15 \leq n$. When $\frac{\varepsilon}{\delta} = \frac{1}{10}$, more rest points close to each extended signaling system is asymptotically stable. We now turn to the general analysis. Differentiating $m_e(n), m_u(n)$ and $m_h(n)$ for n, we obtain

$$\begin{aligned}
\frac{\partial m_e(n)}{\partial n} &= \frac{1}{\frac{\varepsilon}{\delta}} + 1 - \frac{1}{n^2}, \\
\frac{\partial m_u(n)}{\partial n} &= -\frac{4\frac{\varepsilon}{\delta}}{(n-1)^2}, \\
\frac{\partial m_h(n)}{\partial n} &= 1 - \frac{2\frac{\varepsilon}{\delta}}{(n-1)^2}.
\end{aligned}$$

Thus, the first derivative test gives where m_e, m_u and m_h increase and then decrease.

(i) m_e increases as δ increases. m_e always increases for $n \geq 2$.
(ii) m_u increases as ε increases. m_u always decreases for $n \geq 2$.
(iii) m_h increases as ε increases. m_h increases for $n > 1+\sqrt{\frac{2\varepsilon}{\delta}}$. m_h decreases for $2 \leq n < 1+\sqrt{\frac{2\varepsilon}{\delta}}$.

In sender-receiver games with a common interest, more rest points close to each extended signaling system exist because δ are bigger, whereas more rest points close to each unilaterally mixed strategy and each hybrid strategy exist because ε is bigger.

7 Conclusion

In this paper, we showed that each rest point close to an extended signaling system, a unilaterally mixed strategy and a hybrid strategy can be asymptotically stable under the selection–mutation dynamics when $n < m$. Our results in Theorem 2, Theorem 4 and Theorem 6 show the optimal strategy for stability according to n, m, ε and δ. Under the selection-mutation dynamics, a signaling system or a signaling system is not always the optimal strategy.

We then studied tha case in which $n < m$. From symmetry, we also obtained similiar results when $n > m$. We omited the case in which $n > m$ for want of space.

Nevertheless, our investigation is also beset by two limitations. First, regarding the analysis of a unilaterally mixed strategy and a hybrid strategy, we only analyze the case in which $|K_i| = 2$ for $i \in N$ or $i \in L$.

Second, regarding all strategies except for an extended signaling system, a unilaterally mixed strategy, and a hybrid strategy, we did not study the stability of a rest point close to each strategy except for the aforementioned strategies. We conjecture that there is no strategy, except for the aforementioned, that leads to asymptotic stability.

Appendix

Proof of Theorem 1

Fix a signal j of $(i,j) \in I_3$ and denote it by $\bar{j}$. Since $\sum_{i=1}^{n} q_{\bar{j}i} = nq_1 = 1$, we obtain $q_1 = \frac{1}{n}$.

We sequentially find the values of the entries of the symmetric rest points $\varepsilon_1, \varepsilon_2$, and δ_1. The values of $\varepsilon_1, \varepsilon_2$, and δ_1 are consistent with the conditions, $\dot{p}_{ij} = 0$ and $\dot{q}_{ji} = 0$ for each $(i,j) \in I_1, I_2, I_3$. Our dynamical system $S' = \Phi(S)$ of the selection-mutation dynamics consists of $2mn$ differential equations:

$$\begin{cases} \dot{p}_{11} &= p_{11}(q_{11} - p_{11}q_{11} - p_{12}q_{21} - \cdots - p_{1m}q_{m1}) + \varepsilon(1 - mp_{11}), \\ \dot{p}_{12} &= p_{12}(q_{21} - p_{11}q_{11} - p_{12}q_{21} - \cdots - p_{1m}q_{m1}) + \varepsilon(1 - mp_{12}), \\ &\vdots \\ \dot{p}_{nm} &= p_{nm}(q_{mn} - p_{n1}q_{1n} - p_{n2}q_{2n} - \cdots - p_{nm}q_{mn}) + \varepsilon(1 - mp_{nm}), \\ \dot{q}_{11} &= q_{11}(p_{11} - q_{11}p_{11} - q_{12}p_{21} - \cdots - q_{1n}p_{n1}) + \delta(1 - nq_{11}), \\ \dot{q}_{12} &= q_{12}(p_{21} - q_{11}p_{11} - q_{12}p_{21} - \cdots - q_{1n}p_{n1}) + \delta(1 - nq_{12}), \\ &\vdots \\ \dot{q}_{mn} &= q_{mn}(p_{nm} - q_{m1}p_{1m} - q_{m2}p_{2m} - \cdots - q_{mn}p_{nm}) + \delta(1 - nq_{mn}). \end{cases}$$

By the symmetric form of the rest points, these equations are divided into six equations:

$$\begin{cases} \dot{p}_{ij} = 0 \ \text{ for each } (i,j) \in I_1, \\ \dot{p}_{ij} = 0 \ \text{ for each } (i,j) \in I_2, \\ \dot{p}_{ij} = 0 \ \text{ for each } (i,j) \in I_3, \\ \dot{q}_{ji} = 0 \ \text{ for each } (i,j) \in I_1, \\ \dot{q}_{ji} = 0 \ \text{ for each } (i,j) \in I_2, \\ \dot{q}_{ji} = 0 \ \text{ for each } (i,j) \in I_3. \end{cases}$$

We remove redundant equations, that is. equations $\dot{p}_{ij} = 0$ and $\dot{q}_{ij} = 0$ for each $(i,j) \in I_1$, and $\dot{q}_{ij} = 0$, for each $(i,j) \in I_3$. Thus, we have the following three equations:

$$\begin{aligned} \dot{p}_{ij} &= 0 \ \text{ for each } (i,j) \in I_2, \\ \dot{p}_{ij} &= 0 \ \text{ for each } (i,j) \in I_3, \\ \dot{q}_{ji} &= 0 \ \text{ for each } (i,j) \in I_2. \end{aligned}$$

By substituting the entries $(\tilde{p}_{ij}, \tilde{q}_{ji})$ of Definition 2 and $q_1 = \frac{1}{n}$ into the above equations, we obtain the following reduced system $F = \mathbf{0}$ that consists of $f_I(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = 0, I = 1, 2, 3$:

$$F = \mathbf{0} \Leftrightarrow \begin{cases} f_1(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = 0, \text{for each } (i,j) \in I_2, \\ f_2(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = 0, \text{for each } (i,j) \in I_3, \\ f_3(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = 0, \text{for each } (i,j) \in I_2. \end{cases}$$

$$\Leftrightarrow \begin{cases} \tilde{p}_{ij}[\tilde{q}_{ij} - \sum_{(i,s)\in I_1} \tilde{p}_{is}\tilde{q}_{sj} - \sum_{(i,s)\in I_2} \tilde{p}_{is}\tilde{q}_{sj} - \sum_{(i,s)\in I_3} \tilde{p}_{is}\tilde{q}_{sj}] + \varepsilon(1 - m\tilde{p}_{ij}) = 0 \ \text{ for each } (i,j) \in I_2, \\ \tilde{p}_{ij}[\tilde{q}_{ij} - \sum_{(i,s)\in I_1} \tilde{p}_{is}\tilde{q}_{sj} - \sum_{(i,s)\in I_2} \tilde{p}_{is}\tilde{q}_{sj} - \sum_{(i,s)\in I_3} \tilde{p}_{is}\tilde{q}_{sj}] + \varepsilon(1 - m\tilde{p}_{ij}) = 0 \ \text{ for each } (i,j) \in I_3, \\ \tilde{q}_{ji}[\tilde{p}_{ji} - \sum_{(t,j)\in I_1} \tilde{q}_{jt}\tilde{p}_{tj} - \sum_{(t,j)\in I_2} \tilde{q}_{jt}\tilde{p}_{tj} - \sum_{(t,j)\in I_3} \tilde{q}_{jt}\tilde{p}_{tj}] + \delta(1 - n\tilde{q}_{ji}) = 0 \ \text{ for each } (i,j) \in I_2, \end{cases}$$

$$\Leftrightarrow \begin{cases} \varepsilon_1 \left\{\delta_1 - (n-1)\varepsilon_1\delta_1 - [1-(n-1)\varepsilon_1 - (m-n)\varepsilon_2]\,[1-(n-1)\delta_1] - \frac{1}{n}(m-n)\varepsilon_2\right\} + \varepsilon(1-m\varepsilon_1) = 0, \\ \varepsilon_2 \left\{\frac{1}{n} - (n-1)\varepsilon_1\delta_1 - [1-(n-1)\varepsilon_1 - (m-n)\varepsilon_2]\,[1-(n-1)\delta_1] - \frac{1}{n}(m-n)\varepsilon_2\right\} + \varepsilon(1-m\varepsilon_2) = 0, \\ \delta_1 \left\{\varepsilon_1 - (n-1)\delta_1\varepsilon_1 - [1-(n-1)\delta_1]\,[1-(n-1)\varepsilon_1 - (m-n)\varepsilon_2]\right\} + \delta(1-n\delta_1) = 0. \end{cases}$$

Let Df denote the Jacobian matrix of f_1, f_2, f_3 with respect to $\varepsilon_1, \varepsilon_2, \delta_1$, that is,

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial \varepsilon_1} & \frac{\partial f_1}{\partial \varepsilon_2} & \frac{\partial f_1}{\partial \delta_1} \\ \frac{\partial f_2}{\partial \varepsilon_1} & \frac{\partial f_2}{\partial \varepsilon_2} & \frac{\partial f_2}{\partial \delta_1} \\ \frac{\partial f_3}{\partial \varepsilon_1} & \frac{\partial f_3}{\partial \varepsilon_2} & \frac{\partial f_3}{\partial \delta_1} \end{pmatrix}.$$

Let $\det(Df(\mathbf{x}))$ denote the determinant of $Df(\mathbf{x})$ at point $\mathbf{x} = (\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta)$. Since

$$Df(\mathbf{0}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1-n}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

at the point $(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = (0,0,0;0,0)$, we have $\det\big(Df(\mathbf{0})\big) \neq 0$ with $n \geq 2$.

By the implicit function theorem, our reduced system $F = 0$ defines $\varepsilon_1, \varepsilon_2$, and δ_1 as continuously differentiable functions of ε and δ in some neighborhood of $(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = (0,0,0;0,0)$.[7] We denote these functions by $\varepsilon_1(\varepsilon,\delta), \varepsilon_2(\varepsilon,\delta)$, and $\delta_1(\varepsilon,\delta)$.

[7] That is, we obtain unique candidates of values of $\varepsilon_1, \varepsilon_2$, and δ_1, for each pair of mutation rates, (ε, δ).

Inserting $q_{ji} = \frac{1}{n}$ and $p_{ij} = \varepsilon_2$ for each $(i,j) \in I_3$, into

$$\dot{q}_{ji} = q_{ji}(p_{ij} - \sum_{t=1}^{n} q_{jt}p_{tj}) + \delta(1 - nq_{ji}) \quad \text{with } (i,j) \in I_3,$$

we obtain

$$\dot{q}_{ji} = \tilde{q}_{ji}(\tilde{p}_{ij} - \sum_{t=1}^{n} \tilde{q}_{jt}\tilde{p}_{tj}) + \delta(1 - n\tilde{q}_{ji}) = \frac{1}{n}(\varepsilon_2 - n \times \frac{1}{n} \times \varepsilon_2) + \delta(1 - n \times \frac{1}{n}) = 0.$$

The constant values, $\tilde{q}_{ji} = \frac{1}{n}$ and $\tilde{p}_{ij} = \varepsilon_2$ for each $(i,j) \in I_3$, of the symmetric rest point are consistent with the condition required of the rest point, $\dot{q}_{ji} = 0$.

It remains to be proven that the functions $\varepsilon_1(\varepsilon,\delta), \varepsilon_2(\varepsilon,\delta)$, and $\delta_1(\varepsilon,\delta)$ satisfy the equations $\dot{p}_{ij} = 0$ and $\dot{q}_{ji} = 0$ for each $(i,j) \in I_1$ which are removed. From our fixed form of the rest point, we can assert that, for each $(i,j) \in I_2$, $\dot{p}_{ij} = \dot{\varepsilon_1}(\varepsilon,\delta) = 0$, $\dot{q}_{ji} = \dot{\delta}_1(\varepsilon,\delta) = 0$, and for each $(i,j) \in I_3$, $\dot{p}_{ij} = \dot{\varepsilon_2}(\varepsilon,\delta) = 0$. Since we also fix $p_{ij} = 1 - (n-1)\varepsilon_1 - (m-n)\varepsilon_2$ and $q_{ji} = 1 - (n-1)\delta_1$ for each $(i,j) \in I_1$, we obtain $\dot{p}_{ij} = -(n-1)\dot{\varepsilon_1}(\varepsilon,\delta) - (m-n)\dot{\varepsilon_2}(\varepsilon,\delta) = 0$ and $\dot{q}_{ji} = -(n-1)\dot{\delta}_1(\varepsilon,\delta) = 0$.

Therefore, we conclude that, for all $(i,j) \in M \times N$, $\dot{p}_{ij} = \dot{q}_{ji} = 0$ with $(\varepsilon_1, \varepsilon_2, \delta_1) = (\varepsilon_1(\varepsilon,\delta), \varepsilon_2(\varepsilon,\delta), \delta_1(\varepsilon,\delta))$, which proves the theorem. $\square$

Proof of Corollary 1

Tayler's formula for the function $(\varepsilon_1(\varepsilon,\delta), \varepsilon_2(\varepsilon,\delta), \delta_1(\varepsilon,\delta))$ about $(\varepsilon,\delta) =$

$(0,0)$ is given by

$$\begin{pmatrix} \varepsilon_1(\varepsilon,\delta) \\ \varepsilon_2(\varepsilon,\delta) \\ \delta_1(\varepsilon,\delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(0,0) \\ \varepsilon_2(0,0) \\ \delta_1(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial\varepsilon_1}{\partial\varepsilon}(0,0) & \frac{\partial\varepsilon_1}{\partial\delta}(0,0) \\ \frac{\partial\varepsilon_2}{\partial\varepsilon}(0,0) & \frac{\partial\varepsilon_2}{\partial\delta}(0,0) \\ \frac{\partial\delta_1}{\partial\varepsilon}(0,0) & \frac{\partial\delta_1}{\partial\delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \end{pmatrix}.$$

Because $(\varepsilon_1(0,0), \varepsilon_2(0,0), \delta_1(0,0))$ is a solution of the system

$f_I(\varepsilon_1(0,0), \varepsilon_2(0,0), \delta_1(0,0); 0,0) = 0, I = 1,2,3,$

we obtain $(\varepsilon_1(0,0), \varepsilon_2(0,0), \delta_1(0,0)) = (0,0,0)$.

By the implicit function theorem and the fact that

$$Df(\mathbf{0}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1-n}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \frac{\partial\varepsilon_1}{\partial\varepsilon}(0,0) & \frac{\partial\varepsilon_1}{\partial\delta}(0,0) \\ \frac{\partial\varepsilon_2}{\partial\varepsilon}(0,0) & \frac{\partial\varepsilon_2}{\partial\delta}(0,0) \\ \frac{\partial\delta_1}{\partial\varepsilon}(0,0) & \frac{\partial\delta_1}{\partial\delta}(0,0) \end{pmatrix} = -(Df(\mathbf{0})^{-1}) \begin{pmatrix} \frac{\partial f_1}{\partial\varepsilon}(\mathbf{0}) & \frac{\partial f_1}{\partial\delta}(\mathbf{0}) \\ \frac{\partial f_2}{\partial\varepsilon}(\mathbf{0}) & \frac{\partial f_2}{\partial\delta}(\mathbf{0}) \\ \frac{\partial f_3}{\partial\varepsilon}(\mathbf{0}) & \frac{\partial f_3}{\partial\delta}(\mathbf{0}) \end{pmatrix}$$

$$= - \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1-n}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n}{n-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\mathbf{0} = (0,0,0;0,0)$. Thus, Tayler's formula described above becomes

$$\begin{pmatrix} \varepsilon_1(\varepsilon,\delta) \\ \varepsilon_2(\varepsilon,\delta) \\ \delta_1(\varepsilon,\delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(0,0) \\ \varepsilon_2(0,0) \\ \delta_1(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial\varepsilon_1}{\partial\varepsilon}(0,0) & \frac{\partial\varepsilon_1}{\partial\delta}(0,0) \\ \frac{\partial\varepsilon_2}{\partial\varepsilon}(0,0) & \frac{\partial\varepsilon_2}{\partial\delta}(0,0) \\ \frac{\partial\delta_1}{\partial\varepsilon}(0,0) & \frac{\partial\delta_1}{\partial\delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{n}{n-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \end{pmatrix},$$

where $o_I(\varepsilon,\delta)$,$I = 1,2,3$, stands for the second- or higher-order terms of ε and δ. Thus, we obtain the first-order approximated values of $\varepsilon_1, \varepsilon_2$, and δ_1, respectively, as follows:

$$\begin{aligned} \varepsilon_1 &= \varepsilon + o_1(\varepsilon,\delta), \\ \varepsilon_2 &= \tfrac{n}{n-1}\varepsilon + o_2(\varepsilon,\delta), \\ \delta_1 &= \delta + o_3(\varepsilon,\delta). \end{aligned}$$

Replacing ε_1, ε_2, and δ_1 in Definition 3 by these values above, we find the first-order approximated rest point. $\square$

Proof of Theorem 2

The characteristic equation of the first-order approximated Jacobian matrix evaluated at the symmetric rest point close to *any* extended-signaling system is given by

$$
\begin{aligned}
&[\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1]^{n-1}\\
&\times[\lambda + \varepsilon - n\delta + 1]^{n(n-1)}\\
&\times[\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}]^{n(m-n)-1}\\
&\times[\lambda - \tfrac{-2n+mn+1}{n-1}\varepsilon - (n-2)\delta + 1]^{n}\\
&\times[\lambda - \tfrac{mn-n}{n-1}\varepsilon + \delta + 1]^{n(n-1)}\\
&\times[\lambda + \tfrac{1}{n-1}\varepsilon + n\delta]^{n(m-n)-1}\\
&\times[\lambda - \tfrac{mn-n^2-1}{n-1}\varepsilon + n\delta]\\
&\times[\lambda + \tfrac{1}{n-1}(n^3 - n^2 + n - mn^2 + mn)\varepsilon + (1-n)\delta + 1 - \tfrac{1}{n}]\\
&\times[\lambda + (mn - n^2 - n + 2)\varepsilon + (1-n)\delta + 1] = 0
\end{aligned}
$$

where λ is the eigenvalue.

We briefly explain the procedure for obtaining the above equation. Let $A = J\Phi - \lambda I$, that is, $\det A$, denote the characteristic polynomial. Let $a_{ij} \in A$ be the entries of the matrix A, and A_{ij} the corresponding (i,j)th cofactor. Then, we disregard any term that is a second- or higher-order term in ε, δ because of the continuity of the characteristic polynomial with respect to ε and δ.[8] Consequently, we may regard most of the entries of the Jacobian matrix, except its diagonal factors, as 0, or linear forms of ε and δ. Referring to Table2.A and noting that $|I_1| = n, |I_2| = (n-1)$, and $|I_3| = n(m-n)$, we expand the matrix A along any ith row. We thus obtain the following polynomial.

[8]This is a normal procedure for determining the stability of a rest point, which Hofbauer and Hutteger (2015) also follow.

$$
\begin{aligned}
&\det A\\
=\ & \textstyle\sum_{k=1}^{2nm}(-1)^{i+k}a_{ik}A_{ik}\\
=\ & [-\lambda+\tfrac{mn-3n+2}{n-1}\varepsilon+(n-1)\delta-1]^n[-\lambda-\varepsilon+n\delta-1]^{n(n-1)}[\lambda-\tfrac{n}{n-1}\varepsilon+(n-1)\delta-1+\tfrac{1}{n}]^{n(m-n)}\\
&\times[-\lambda+\tfrac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta-1]^n[-\lambda+\tfrac{mn-n}{n-1}\varepsilon-\delta-1]^{n(n-1)}[-\lambda-\tfrac{1}{n-1}\varepsilon-n\delta]^{n(m-n)}\\
&-(-\tfrac{n}{n-1}\varepsilon)(-\tfrac{1}{n}-\tfrac{2n-mn-1}{n(n-1)}\varepsilon)[-\lambda+\tfrac{mn-3n+2}{n-1}\varepsilon+(n-1)\delta+1]^{n-1}[-\lambda-\varepsilon+n\delta+1]^{n(n-1)}[-\lambda-\tfrac{n}{n-1}\varepsilon+(n-1)\delta+1+\tfrac{1}{n}]^{n(m-n)-1}\\
&\times[-\lambda+\tfrac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta-1]^n[-\lambda+\tfrac{mn-n}{n-1}\varepsilon-\delta+1]^{n(n-1)}[-\lambda-\tfrac{1}{n-1}\varepsilon-n\delta]^{n(m-n)}\times n(m-n)\\
&-(\tfrac{1}{n}-\tfrac{1}{n^2})\tfrac{n}{n-1}\varepsilon[-\lambda+\tfrac{mn-3n+2}{n-1}\varepsilon+(n-1)\delta-1]^{n-1}[-\lambda-\varepsilon+n\delta-1]^{n(n-1)}[-\lambda-\tfrac{n}{n-1}\varepsilon+(n-1)\delta-1+\tfrac{1}{n}]^{n(m-n)-1}\\
&\times[-\lambda+\tfrac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta-1]^n[-\lambda+\tfrac{mn-n}{n-1}\varepsilon-\delta-1]^{n(n-1)}[-\lambda-\tfrac{1}{n-1}\varepsilon-n\delta]^{n(m-n)-1}\times n(m-n).
\end{aligned}
$$

This polynomial is the sum of three parts. The first part comprises all the diagonal factors of the characteristic polynomial $\det A$, that is, $[-\lambda+\frac{mn-3n+2}{n-1}\varepsilon+(n-1)\delta-1]^n[-\lambda-\varepsilon+n\delta-1]^{n(n-1)}[-\lambda-\frac{n}{n-1}\varepsilon+(n-1)\delta-1+\frac{1}{n}]^{n(m-n)}[-\lambda+\frac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta-1]^n[-\lambda+\frac{mn-n}{n-1}\varepsilon-\delta-1]^{n(n-1)}[-\lambda-\frac{1}{n-1}\varepsilon-n\delta]^{n(m-n)}$. In Table2.A, $\frac{\partial\dot{p}_{ij}}{\partial p_{ij}}$ or $\frac{\partial\dot{q_{ji}}}{\partial q_{ji}}$ for $s=i,t=j$ of each column correspond to each diagonal factor.

The second part with the negative sign comprises all the diagonal factors except an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_1,(s,t)\in I_3,s=i,t=j$, and an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_1,(s,t)\in I_3,s=i$. The value of an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_1,(s,t)\in I_3,s=i,t=j$ is $-\frac{n}{n-1}\varepsilon$. The value of an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_1,(s,t)\in I_3,s=i$ is $-\frac{1}{n}-\frac{2n-mn-1}{n(n-1)}\varepsilon$. The number of such terms is $n(m-n)$.

The third with the negative sign is composed of all diagonal factors except an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_3,(s,t)\in I_3,t=j,s=i$ and an entry $\frac{\partial\dot{p}_{ij}}{\partial q_{ts}}$ for $(i,j)\in I_3,(s,t)\in I_3,s=i,t=j$. The value of an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_3,(s,t)\in I_3,t=j,s=i$ is $\frac{1}{n}-\frac{1}{n^2}$. The value of an entry $\frac{\partial\dot{p}_{ij}}{\partial q_{ts}}$ for $(i,j)\in I_3,(s,t)\in I_3,s=i,t=j$ is $\frac{n}{n-1}\varepsilon$. The number of such terms is $n(m-n)$.

By factoring and arranging these parts, we obtain the characteristic polynomial as follows.

$$
\begin{aligned}
& [\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1]^{n-1}[\lambda + \varepsilon - n\delta + 1]^{n(n-1)}[\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}]^{n(m-n)-1} \\
& \times[\lambda - \tfrac{-2n+mn+1}{n-1}\varepsilon - (n-2)\delta + 1]^{n}[\lambda - \tfrac{mn-n}{n-1}\varepsilon + \delta + 1]^{n(n-1)}[\lambda + \tfrac{1}{n-1}\varepsilon + n\delta]^{n(m-n)-1} \\
& \times\big([\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1][\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}][\lambda + \tfrac{1}{n-1}\varepsilon + n\delta] \\
& +(-\tfrac{n}{n-1}\varepsilon)(-\tfrac{1}{n} - \tfrac{2n-mn-1}{n(n-1)}\varepsilon)[\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}] \times n(m-n) \\
& +(\tfrac{1}{n} - \tfrac{1}{n^2})\tfrac{n}{n-1}\varepsilon[\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1] \times n(m-n)\big) \\
= \; & [\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1]^{n-1} \\
& \times[\lambda + \varepsilon - n\delta + 1]^{n(n-1)} \\
& \times[\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}]^{n(m-n)-1} \\
& \times[\lambda - \tfrac{-2n+mn+1}{n-1}\varepsilon - (n-2)\delta + 1]^{n} \\
& \times[\lambda - \tfrac{mn-n}{n-1}\varepsilon + \delta + 1]^{n(n-1)} \\
& \times[\lambda + \tfrac{1}{n-1}\varepsilon + n\delta]^{n(m-n)-1} \\
& \times[\lambda - \tfrac{mn-n^2-1}{n-1}\varepsilon + n\delta] \\
& \times[\lambda + \tfrac{1}{n-1}(n^3 - n^2 + n - mn^2 + mn)\varepsilon + (1-n)\delta + 1 - \tfrac{1}{n}] \\
& \times[\lambda + (mn - n^2 - n + 2)\varepsilon + (1-n)\delta + 1].
\end{aligned}
$$

□

Proof of Theorem 3

We now find the values of the entries of the symmetric rest points, $\varepsilon_{1_{ij}}$, $\delta_{1_{ji}}$, for each $i \in N$ and each $j \in M \backslash K_i$. These entries are consistent with the conditions required for the rest points, $\dot{p}_{ij} = 0$ and $\dot{q}_{ji} = 0$ for each $i \in N$ and $j \in M$.

Our dynamical system $S' = \Phi(S)$ of the selection–mutation dynamics consists of $2mn$ differential equations. They are divided into four equations

$$
\begin{cases}
\dot{p}_{ij} = 0 & \text{for each } i \in N \text{ and each } j \in K_i, \\
\dot{p}_{ij} = 0 & \text{for each } i \in N \text{ and each } j \in M \backslash K_i, \\
\dot{q}_{ji} = 0 & \text{for each } i \in N \text{ and each } j \in K_i, \\
\dot{q}_{ji} = 0 & \text{for each } i \in N \text{ and each } j \in M \backslash K_i.
\end{cases}
$$

We remove redundant equations, $\dot{p}_{ij} = 0$ and $\dot{q}_{ji} = 0$ for each $i \in$

N and each $j \in K_i$. Thus, we obtain $\dot{p}_{ij} = 0$ and $\dot{q}_{ji} = 0$ for each $i \in N$ and each $j \in M \backslash K_i$.

By substituting the entries $(\tilde{p}_{ij}, \tilde{q}_{ji})$ of Definition 5 into the above equations, we obtain the following reduced system $F = 0$ that consists of $f_I(\varepsilon_{1_{ij}}, \delta_{1_{ji}}; \varepsilon, \delta) = \mathbf{0}, I = 1, 2$, for each $i \in N$ and each $j \in M \backslash K_i$:

$$F = \mathbf{0}$$

$$\Leftrightarrow \begin{cases} f_1(\varepsilon_{1_{ij}}, \delta_{1_{ji}}; \varepsilon, \delta) = 0, \text{for each } i \in N \text{and each } j \in M \backslash K_i \\ f_2(\varepsilon_{1_{ij}}, \delta_{1_{ji}}; \varepsilon, \delta) = 0, \text{for each } i \in N \text{and each } j \in M \backslash K_i. \end{cases}$$

$$\Leftrightarrow \begin{cases} \varepsilon_{1_{ij}} \left\{ \delta_{1_{ji}} - |K_i| \left(\frac{1}{|K_i|} - \frac{1}{|K_i|} \left(\sum_l^{N \backslash \{i\}} \sum_{k_l}^{K_l} \varepsilon_{1_{ik_l}} \right) \left(1 - \sum_l^{N \backslash \{i\}} \delta_{1_{k_l i}} \right) - \left(\sum_l^{N \backslash \{i\}} \sum_{k_l}^{K_l} \right) \varepsilon_{1_{ik_l}} \delta_{1_{k_l i}} \right\} + \varepsilon (1 - m \varepsilon_{1_{ji}}) = 0, \\ \text{for each } i \in N \text{ and each } j \in M \backslash K_i, \\ \delta_{1_{ji}} \left\{ \varepsilon_{1_{ij}} - \sum_l^{N \backslash \{i\}} \delta_{1_{jl}} \varepsilon_{1_{lj}} - \left(\frac{1}{|K_{i'}|} - \frac{1}{|K_{i'}|} \left(\sum_l^{N \backslash \{i'\}} \sum_{k_l}^{K_l} \varepsilon_{1_{ik_l}} \right) \right) \left(1 - \sum_l^{N \backslash \{i'\}} \delta_{1_{li}} \right) \right\} + \delta (1 - n \delta_{1_{ji}}) = 0. \\ \text{for each } i \in N \text{and each } j \in M \backslash K_i (j \in K_{i'}, i \neq i'). \end{cases}$$

From this, we see that $(\varepsilon_{1_{ij}}, \delta_{1_{ji}}; \varepsilon, \delta) = (0, 0; 0, 0)$ for each $i \in N$ and each $j \in M \backslash K_i$ is a solution to the reduced system, that is, $f_I(0, 0; 0, 0) = 0$, $I = 1, 2$, for each $i \in N$ and each $j \in M \backslash K_i$. Let Df be the Jacobian matrix of f_1, f_2 with respect to $\varepsilon_{1_{ij}}, \delta_{1_{ji}}$. At the point $(\varepsilon_{1_{ij}}, \delta_{1_{ji}}; \varepsilon, \delta) = (0, 0; 0, 0)$ for each $i \in N$ and each $j \in M \backslash K_i$, we have $\det(Df(\mathbf{0})) \neq 0$. $\square$

Proof of Corollary 2

Tayler's formula for the function $(\varepsilon_{1_{ij}}(\varepsilon, \delta), \delta_{1_{ji}}(\varepsilon, \delta))$ with each $i \in N$ and each $j \in M \backslash K_i$ about $(\varepsilon, \delta) = (0, 0)$ is:

for each $i, ..., i' \in N$ and each $j, ..., j' \in M \backslash K_i$,

$$\begin{pmatrix} \varepsilon_{1_{ij}}(\varepsilon,\delta) \\ \vdots \\ \varepsilon_{1_{i'j'}}(\varepsilon,\delta) \\ \delta_{1_{ji}}(\varepsilon,\delta) \\ \vdots \\ \delta_{1_{j'i'}}(\varepsilon,\delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_{1_{ij}}(0,0) \\ \vdots \\ \varepsilon_{1_{i'j'}}(0,0) \\ \delta_{1_{ji}}(0,0) \\ \vdots \\ \delta_{1_{j'i'}}(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial \varepsilon_{1_{ij}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{1_{ij}}}{\partial \delta}(0,0) \\ \vdots & \\ \frac{\partial \varepsilon_{1_{i'j'}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{1_{i'j'}}}{\partial \delta}(0,0) \\ \frac{\partial \delta_{1_{ji}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{1_{ji}}}{\partial \delta}(0,0) \\ \vdots & \\ \frac{\partial \delta_{1_{j'i'}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{1_{j'i'}}}{\partial \delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_{1_{ij}}(\varepsilon,\delta) \\ \vdots \\ o_{1_{i'j'}}(\varepsilon,\delta) \\ o_{2_{ji}}(\varepsilon,\delta) \\ \vdots \\ o_{2_{j'i'}}(\varepsilon,\delta) \end{pmatrix}.$$

Because $(\varepsilon_{1_{ij}}(0,0), \delta_{1_{ji}}(0,0))$ for each $i \in N$ and each $j \in M \backslash K_i$ is a solution of the system

$f_I(\varepsilon_{1_{ij}}(0,0), \delta_{1_{ji}}(0,0); 0,0) = 0, I = 1,2,$ for each $i \in N$ and each $j \in M \backslash K_i$, we have $(\varepsilon_{1_{ij}}(0,0), \delta_{1_{ji}}(0,0)) = (0,0)$ for each $i \in N$ and each $j \in M \backslash K_i$.

By the implicit function theorem and the fact that

$$Df(\mathbf{0}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{|K_l|} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{|K_{l'}|} \end{pmatrix},$$

for each $i, ..., i' \in N,$ each $l, ..., l' \in N(i \neq l, ..., i' \neq l'),$ and for each $j \in K_l, ..., j' \in K_{l'}$.

we obtain for each $i, ..., i' \in N,$ each $l, ..., l' \in N(i \neq l, ..., i' \neq l'),$ and for each $j \in$

$K_l, ..., j' \in K_{l'}$,

$$\begin{pmatrix} \frac{\partial \varepsilon_{1_{ij}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{1_{ij}}}{\partial \delta}(0,0) \\ \vdots & \vdots \\ \frac{\partial \varepsilon_{1_{i'j'}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{1_{i'j'}}}{\partial \delta}(0,0) \\ \frac{\partial \delta_{1_{ji}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{1_{ji}}}{\partial \delta}(0,0) \\ \vdots & \vdots \\ \frac{\partial \delta_{1_{j'i'}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{1_{j'i'}}}{\partial \delta}(0,0) \end{pmatrix} = -(Df(\mathbf{0})^{-1}) \begin{pmatrix} \frac{\partial f_1}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_1}{\partial \delta}(\mathbf{0}) \\ \vdots & \vdots \\ \frac{\partial f_1}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_1}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_2}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_3}{\partial \delta}(\mathbf{0}) \\ \vdots & \vdots \\ \frac{\partial f_2}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_3}{\partial \delta}(\mathbf{0}) \end{pmatrix}$$

$$= - \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{|K_l|} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{|K_{l'}|} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -|K_l| & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -|K_{l'}| \end{pmatrix},$$

where $\mathbf{0} = (0, 0; 0, 0)$. From the above equation, we obtain for each $i, ..., i' \in$

$N,$ each $l,...,l' \in N(i \neq l,...,i' \neq l'),$ and each $j \in K_l,...,j' \in K_{l'}$

$$\begin{pmatrix} \varepsilon_{1_{ij}}(\varepsilon,\delta) \\ \vdots \\ \varepsilon_{1_{ij}}(\varepsilon,\delta) \\ \delta_{1_{ji}}(\varepsilon,\delta) \\ \vdots \\ \delta_{1_{ji}}(\varepsilon,\delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_{1_{ij}}(0,0) \\ \vdots \\ \varepsilon_{1_{i'j'}}(0,0) \\ \delta_{1_{ji}}(0,0) \\ \vdots \\ \delta_{1_{j'i'}}(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial \varepsilon_{1_{ij}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{1_{ij}}}{\partial \delta}(0,0) \\ \vdots & \\ \frac{\partial \varepsilon_{1_{i'j'}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{1_{i'j'}}}{\partial \delta}(0,0) \\ \frac{\partial \delta_{1_{ji}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{1_{ji}}}{\partial \delta}(0,0) \\ \vdots & \\ \frac{\partial \delta_{1_{j'i'}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{1_{j'i'}}}{\partial \delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_{1_{ij}}(\varepsilon,\delta) \\ \vdots \\ o_{1_{i'j'}}(\varepsilon,\delta) \\ o_{2_{ji}}(\varepsilon,\delta) \\ \vdots \\ o_{2_{j'i'}}(\varepsilon,\delta) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & |K_l| & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & |K_{l'}| \end{pmatrix}, \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon \\ \delta \\ \vdots \\ \delta \end{pmatrix} + \begin{pmatrix} o_{1_{ij}}(\varepsilon,\delta) \\ \vdots \\ o_{1_{i'j'}}(\varepsilon,\delta) \\ o_{2_{ji}}(\varepsilon,\delta) \\ \vdots \\ o_{2_{j'i'}}(\varepsilon,\delta) \end{pmatrix},$$

where $o_I(\varepsilon,\delta), I = 1,2,$ stands for the second- or higher-order terms of ε and δ. Thus, we obtain the first-order approximated values of ε_1 and δ_1, respectively, as follows: for each $i,...,i' \in N,$ each $l,...,l' \in N(i \neq l,...,i' \neq l'),$ and each $j \in K_l,...,j' \in K_{l'},$

$$\begin{aligned} \varepsilon_{1_{ij}} &= \varepsilon + o_{1_{ij}}(\varepsilon,\delta), \\ &\vdots \\ \varepsilon_{1_{i'j'}} &= \varepsilon + o_{1_{i'j'}}(\varepsilon,\delta), \\ \delta_{1_{ji}} &= |K_l|\delta + o_{2_{ji}}(\varepsilon,\delta). \\ &\vdots \\ \delta_{1_{j'i'}} &= |K_{l'}|\delta + o_{2_{j'i'}}(\varepsilon,\delta). \end{aligned}$$

Replacing $\varepsilon_{1_{ij}}$ and δ_{1ji} for each $i \in N$ and each $j \in M \backslash K_i$ in Definition 3 by these values above, we find the first-order approximated rest point. $\square$

Proof of Theorem 4

We follow similiar procedure in the proof of Theorem 2. The characteristic equation of the first-order approximated Jacobian matrix evaluated at the symmetric rest point close to *any* unilaterally mixed strategy is given by

$$
\begin{aligned}
&[\lambda + 2\varepsilon]^{\frac{m-2}{2}} \\
&\times[\lambda - (m-4)\varepsilon - 2(m-2)\delta + 1]^{\frac{m-2}{2}} \\
&\times[\lambda + 2\varepsilon - 2n\delta + 1]^{(n-1)m} \\
&\times[\lambda - \tfrac{m-2}{2}\varepsilon - (n-2)\delta + \tfrac{1}{2}]^{m-1} \\
&\times[\lambda - \tfrac{m}{2}\varepsilon - (n-2)\delta + \tfrac{1}{2}]^{m(n-1)} \\
&\times[\lambda + (4-m)\varepsilon + (2 - 2n - \tfrac{1}{2}m + \tfrac{mn}{2})\delta + 1] \\
&\times[\lambda + (1 - \tfrac{1}{2}m)\varepsilon + (2-n)\delta + \tfrac{1}{2}] \\
&\times[\lambda + 2\varepsilon - \tfrac{m(n-1)}{2}\delta] = 0
\end{aligned}
$$

where λ is the eigenvalue. $\square$

Proof of Theorem 5

We find the values of the entries of the symmetric rest point, $\varepsilon_1, \varepsilon_{2_{k_l}}, \varepsilon_{3_l}, \varepsilon_{4_{lk_{l'}}}$ and $\delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \delta_{4_{k_{l'}l}}$ for each $l \in L$, each $l' \in L \backslash \{l\}$, each $k_l \in K_l$ and each $k_{l'} \in K_{l'}$. These entries are consistent with the conditions required for the rest point, $\dot{p}_{ij} = 0$ and $\dot{q}_{ji} = 0$ for each $(i,j) \in I_1^{**}, I_2^{**}, I_3^{**}, I_4^{**}, I_5^{**}, I_6^{**}$.

Our dynamical system $S' = \Phi(S)$ of the selection–mutation dynamics consists of $2mn$ differential equations. They are divided into 12 equations:

$$
F = \mathbf{0} \Leftrightarrow \begin{cases} \dot{p}_{ij} = 0 & \text{for each } (i,j) \in I_1^{**}, I_2^{**}, I_3^{**}, I_4^{**}, I_5^{**}, I_6^{**}, \\ \dot{q}_{ji} = 0 & \text{for each } (i,j) \in I_1^{**}, I_2^{**}, I_3^{**}, I_4^{**}, I_5^{**}, I_6^{**}. \end{cases}
$$

We remove redundant equations, that is, $\dot{p}_{ij} = 0$ and $\dot{q}_{ji} = 0$ for each $(i, j) \in I_1^{**}, I_6^{**}$. Thus, we obtain the following eight equations, $\dot{p}_{ij} = 0$ and $\dot{q}_{ji} = 0$, for each $(i, j) \in I_2^{**}, I_3^{**}, I_4^{**}, I_5^{**}$.

By substituting the entries $(\tilde{p}_{ij}, \tilde{q}_{ji})$ of Definition 7 into the above equations, we obtain the following reduced system, $F = \mathbf{0}$, which consists of $f_I(\varepsilon_1, \varepsilon_{2_{k_l}}, \varepsilon_{3_l}, \varepsilon_{4_{lk_{l'}}}, \delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \delta_{4_{k_{l'}l}}; \varepsilon, \delta) = \mathbf{0}, I = 1, 2, 3, 4, 5, 6, 7, 8,$ for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$ and each $k_{l'} \in K_{l'}$:

$F(\varepsilon_1, \varepsilon_{2_{k_l}}, \varepsilon_{3_l}, \varepsilon_{4_{lk_{l'}}}, \delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \delta_{4_{k_{l'}l}}; \varepsilon, \delta) = \mathbf{0}$, for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$ and each $k_{l'} \in K_{l'}$

$$\Leftrightarrow \begin{cases} f_1(\varepsilon_1, \varepsilon_{2_{k_l}}, \delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \varepsilon, \delta) = 0, \text{for each } l \in L, \text{each } l' \in L\backslash\{l\}, \text{each } k_l \in K_l \text{ and each } k_{l'} \in K_{l'}, \\ f_2(\varepsilon_1, \varepsilon_{2_{k_l}}, \varepsilon_{3_l}, \delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \varepsilon, \delta) = 0, \text{for each } l \in L \text{ and each } k_l \in K_l, \\ f_3(\varepsilon_{3_l}, \varepsilon_{4_{lk_{l'}}}, \delta_1, \delta_{2_l}, \delta_{4_{k_{l'}l}}; \varepsilon, \delta) = 0, \text{for each } l \in L, \text{each } l' \in L\backslash\{l\}, \text{and each } k_{l'} \in K_{l'}, \\ f_4(\varepsilon_{3_l}, \varepsilon_{4_{lk_{l'}}}, \delta_{2_l}, \delta_{4_{k_{l'}l}}; \varepsilon, \delta) = 0, \text{for each } l \in L, \text{each } l' \in L\backslash\{l\}, \text{and each } k_{l'} \in K_{l'}, \\ f_5(\varepsilon_1, \varepsilon_{3_l}, \delta_1, \delta_{2_l}, \varepsilon, \delta) = 0, \text{for each } l \in L, \\ f_6(\varepsilon_1, \varepsilon_{3_l}, \delta_1, \delta_{2_l}, \varepsilon, \delta) = 0, \text{for each } l \in L, \\ f_7(\varepsilon_{3_l}, \varepsilon_{4_{lk_{l'}}}, \delta_{3_{k_{l'}}}, \delta_{4_{k_{l'}l}}; \varepsilon, \delta) = 0, \text{for each } l \in L, \text{each } l' \in L\backslash\{l\}, \text{and each } k_{l'} \in K_{l'}, \\ f_8(\varepsilon_{3_l}, \varepsilon_{4_{lk_{l'}}}, \delta_{3_{k_{l'}}}, \delta_{4_{k_{l'}l}}; \varepsilon, \delta) = 0 \text{for each } l \in L, \text{each } l' \in L\backslash\{l\}, \text{and each } k_{l'} \in K_{l'}, \end{cases}$$

$$\Leftrightarrow \begin{cases} \varepsilon_1 \left\{ \delta_1 - (m-|K|-1)\varepsilon_1\delta_1 - \left[1-(m-|K|-1)\varepsilon_1 - \sum_l^L \sum_{k_l}^{K_l} \varepsilon_{2_{k_l}}\right] \left[1-(n-|L|-1)\delta_1 - \sum_l^L \delta_{2_l}\right] - \sum_l^L \sum_{k_l}^{K_l} \varepsilon_{2_{k_l}}\delta_{3_{k_l}} \right\} + \varepsilon(1-m\varepsilon_1) = 0, \\ \varepsilon_{2_{k_l}} \left\{ \delta_{3_{k_l}} - (m-|K|-1)\varepsilon_1\delta_1 - \left[1-(m-|K|-1)\varepsilon_1 - \sum_l^L \sum_{k_l}^{K_l} \varepsilon_{2_{k_l}}\right] \left[1-(n-|L|-1)\delta_1 - \sum_l^L \delta_{2_l}\right] - \sum_l^L \sum_{k_l}^{K_l} \varepsilon_{2_{k_l}}\delta_{3_{k_l}} \right\} + \varepsilon(1-m\varepsilon_{2_{k_l}}) = 0, \\ \text{, for each } l \in L, \text{ each } k_l \in K_l, \text{ and each } k_l \in K_l, \\ \varepsilon_{3_l} \left\{ \delta_{2_l} - |K_l|(\frac{1}{|K_l|} - \frac{1}{|K_l|}((m-|K|-1)\varepsilon_{3_l} + \sum_{l'}^{L\backslash\{l\}} \sum_{k_{l'}}^{K_{l'}} \varepsilon_{4_{lk_{l'}}}))(1-(n-|L|-1)\delta_{3_{k_{l'}}} - \sum_{l'}^{L\backslash\{l\}} \delta_{4_{k_l l'}}) - (m-|K|)\varepsilon_{3_l}\delta_{2_l} - \sum_{l'}^{L\backslash\{l\}} \sum_{k_{l'}}^{K_{l'}} \varepsilon_{4_{lk_{l'}}}\delta_{4_{k_{l'}l}} \right\} + \varepsilon(1-m\varepsilon_{3_l}) = 0, \\ \text{for each } l \in L, \text{each } l' \in L\backslash\{l\}, \text{ each } k_l \in K_l, \text{ and each } k_{l'} \in K_{l'}, \\ \varepsilon_{4_{lk_{l'}}} \left\{ \delta_{4_{k_{l'}l}} - |K_l|(\frac{1}{|K_l|} - \frac{1}{|K_l|}((m-|K|-1)\varepsilon_{3_l} + \sum_{l'}^{L\backslash\{l\}} \sum_{k_{l'}}^{K_{l'}} \varepsilon_{4_{lk_{l'}}}))(1-(n-|L|-1)\delta_{3_{k_l}} - \sum_{l'}^{L\backslash\{l\}} \delta_{4_{k_l l'}}) - (m-|K|)\varepsilon_{3_l}\delta_{2_l} - \sum_{l'}^{L\backslash\{l\}} \sum_{k_{l'}}^{K_{l'}} \varepsilon_{4_{lk_{l'}}}\delta_{4_{k_{l'}l}} \right\} + \varepsilon(1-m\varepsilon_{4_{lk_{l'}}}) = 0, \\ \text{for each } l \in L, \text{each } l' \in L\backslash\{l\}, \text{ each } k_l \in K_l, \text{ and each } k_{l'} \in K_{l'}, \\ \delta_1 \left\{ \varepsilon_1 - (n-|L|-1)\delta_1\varepsilon_1 - (1-(n-|L|-1)\delta_1 - \sum_l^L \delta_{2_l})(1-(m-|K|-1)\varepsilon_1 - \sum_l^L \sum_{k_l}^{K_l} \varepsilon_{2_{k_l}}) - \sum_l^L \varepsilon_{3_l}\delta_{2_l} \right\} + \delta(1-n\delta_1) = 0, \\ \delta_{2_l} \left\{ \varepsilon_{3_l} - (n-|L|-1)\delta_1\varepsilon_1 - (1-(n-|L|-1)\delta_1 - \sum_l^L \delta_{2_l})(1-(m-|K|-1)\varepsilon_1 - \sum_l^L \sum_{k_l}^{K_l} \varepsilon_{2_{k_l}}) - \sum_l^L \varepsilon_{3_l}\delta_{2_l} \right\} + \delta(1-n\delta_{2_l}) = 0, \\ \text{for each } l \in L, \\ \delta_{3_{k_{l'}}} \left\{ \varepsilon_{2_{k_{l'}}} - (n-|L|)\delta_{3_{k_{l'}}}\varepsilon_{2_{k_{l'}}} - \sum_l^{L\backslash\{l'\}} \delta_{4_{k_{l'}l}}\varepsilon_{4_{lk_{l'}}} - \left[1-(n-|L|-1)\delta_{3_{k_{l'}}} - \sum_l^{L\backslash\{l'\}} \delta_{4_{k_{l'}l}}\right] \left[\frac{1}{|K_{l'}|} - \frac{1}{|K_{l'}|}((m-|K_{l'}|)\varepsilon_{3_{k_{l'}}} + \sum_{l'}^{L\backslash l} \sum_{k_l}^{K_l} \varepsilon_{4_{lk_l}})\right] \right\} + \delta(1-n\delta_{3_{k_{l'}}}) = 0, \\ \text{for each } l \in L, \text{each } l' \in L\backslash\{l\}, \text{each } k_l \in K_l, \text{and each } k_{l'} \in K_{l'}, \\ \delta_{4_{k_{l'}l}} \left\{ \varepsilon_{4_{lk_{l'}}} - (n-|L|)\delta_{3_{k_{l'}}}\varepsilon_{2_{k_{l'}}} - \sum_{l'}^{L\backslash l} \delta_{4_{k_{l'}l}}\varepsilon_{4_{lk_{l'}}} - \left[1-(n-|L|-1)\delta_{3_{k_{l'}}} - \sum_{l'}^{L\backslash l} \delta_{4_{k_{l'}l}}\right] \left[\frac{1}{|K_{l'}|} - \frac{1}{|K_{l'}|}((m-k)\varepsilon_{3_{k_{l'}}} + \sum_l^{L\backslash l} \sum_l^{K_l} \varepsilon_{4_{l'k_l}})\right] \right\} + \delta(1-n\delta_{4_{k_{l'}l}}) = 0, \\ \text{for each } l \in L, \text{each } l' \in L\backslash\{l\}, \text{each } k_l \in K_l, \text{and each } k_{l'} \in K_{l'}. \end{cases}$$

From this, we see that $(\varepsilon_1, \varepsilon_{2_{k_l}}, \varepsilon_{3_l}, \varepsilon_{4_{lk_{l'}}}, \delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \delta_{4_{k_{l'}l}}; \varepsilon, \delta) = (0, 0, 0, 0, 0, 0, 0, 0; 0, 0)$

for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$, and each $k_{l'} \in K_{l'}$, is a solution to the reduced system, that is, $f_I(0,0,0,0,0,0,0,0;0,0) = 0$, $I = 1,2,3,4,5,6,7,8$. Let Df be the Jacobian matrix of $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$ with respect to $\varepsilon_1, \varepsilon_{2_{k_l}}, \varepsilon_{3_l}, \varepsilon_{4_{lk_{l'}}}, \delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \delta_{4_{k_{l'}l}}$ for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$, and each $k_{l'} \in K_{l'}$.

At the point $(\varepsilon_1, \varepsilon_{2_{k_l}}, \varepsilon_{3_{k_l}}, \varepsilon_{4_{lk_{l'}}}, \delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \delta_{4_{k_{l'}l}}; \varepsilon, \delta) = (0,0,0,0,0,0;0,0)$ for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$, and each $k_{l'} \in K_{l'}$, we obtain $\det(Df(\mathbf{0})) \neq 0$. $\square$

Proof of Corollary 3

We prove the case in which the value of $\frac{1}{|K_l|}$ for each $l \in L$ and each $k_l \in K_l$ is a constant for simplicity. The Tayler's formula for the function

$(\varepsilon_1(\varepsilon,\delta), \varepsilon_{2_{lk_l}}(\varepsilon,\delta), \varepsilon_{3_l}(\varepsilon,\delta), \varepsilon_{4lk_l}(\varepsilon,\delta), \delta_1(\varepsilon,\delta), \delta_{2_l}(\varepsilon,\delta), \delta_{3_{k_{l'}}}(\varepsilon,\delta), \delta_{4_{lk_{l'}}}(\varepsilon,\delta))$
about $(\varepsilon,\delta) = (0,0)$ is given by,
for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$, and each $k_{l'} \in K_{l'}$,

$$\begin{pmatrix} \varepsilon_1(\varepsilon,\delta) \\ \varepsilon_{2_{k_l}}(\varepsilon,\delta) \\ \varepsilon_{3_l}(\varepsilon,\delta) \\ \varepsilon_{4_{lk_l}}(\varepsilon,\delta) \\ \delta_1(\varepsilon,\delta) \\ \delta_{2_l}(\varepsilon,\delta) \\ \delta_{3_{k_{l'}}}(\varepsilon,\delta) \\ \delta_{4_{lk_{l'}}}(\varepsilon,\delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(0,0) \\ \varepsilon_{2_{k_l}}(0,0) \\ \varepsilon_{3_l}(0,0) \\ \varepsilon_{4_{lk_l}}(0,0) \\ \delta_1(0,0) \\ \delta_{2_l}(0,0) \\ \delta_{3_{k_{l'}}}(0,0) \\ \delta_{4_{lk_{l'}}}(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_1}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_{2_{k_l}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{2_{k_l}}}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_{3_l}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{3_l}}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_{4_{lk_l}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{4_{lk_l}}}{\partial \delta}(0,0) \\ \frac{\partial \delta_1}{\partial \varepsilon}(0,0) & \frac{\partial \delta_1}{\partial \delta}(0,0) \\ \frac{\partial \delta_{2_l}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{2_l}}{\partial \delta}(0,0) \\ \frac{\partial \delta_{3_{k_{l'}}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{3_{k_{l'}}}}{\partial \delta}(0,0) \\ \frac{\partial \delta_{4_{lk_{l'}}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{4_{lk_{l'}}}}{\partial \delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \\ o_4(\varepsilon,\delta) \\ o_5(\varepsilon,\delta) \\ o_6(\varepsilon,\delta) \\ o_7(\varepsilon,\delta) \\ o_8(\varepsilon,\delta) \end{pmatrix}.$$

Because $(\varepsilon_1(0,0), \varepsilon_{2_{k_l}}(0,0), \varepsilon_{3_l}(0,0), \varepsilon_{4_{lk_l}}(0,0), \delta_1(0,0), \delta_{2_l}(0,0), \delta_{3_{k_{l'}}}(0,0), \delta_{4_{lk_{l'}}}(0,0))$

for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$, and each $k_{l'} \in K_{l'}$ is a solution of the system

$f_I(\varepsilon_1(0,0), \varepsilon_{2_{k_l}}(0,0), \varepsilon_{3_l}(0,0), \varepsilon_{4_{lk_l}}(0,0), \delta_1(0,0), \delta_{2_l}(0,0), \delta_{3_{k_{l'}}}(0,0), \delta_{4_{lk_{l'}}}(0,0); 0,0) = 0, I = 1,2,3,4,5,6,7,8$, for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$ and each $k_{l'} \in K_{l'}$. we obtain, for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$ and each $k_{l'} \in K_{l'}$, $(\varepsilon_1(0,0), \varepsilon_{2_{k_l}}(0,0), \varepsilon_{3_l}(0,0), \varepsilon_{4_{lk_l}}(0,0), \delta_1(0,0), \delta_{2_l}(0,0), \delta_{3_{k_l}}(0,0), \delta_{4_{lk_{l'}}}(0,0))$

.

By the implicit function theorem and the fact that

$$Df(\mathbf{0}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{|K_{l'}|} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{|K_{l'}|} \end{pmatrix}.$$

we obtain, for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$, and each $k_{l'} \in K_{l'}$,

$$\begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_1}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_{2_{k_l}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{2_{k_l}}}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_{3_l}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{3_l}}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_{4_{lk_l}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{4_{lk_l}}}{\partial \delta}(0,0) \\ \frac{\partial \delta_1}{\partial \varepsilon}(0,0) & \frac{\partial \delta_1}{\partial \delta}(0,0) \\ \frac{\partial \delta_{2_l}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{2_l}}{\partial \delta}(0,0) \\ \frac{\partial \delta_{3_{k_{l'}}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{3_{k_{l'}}}}{\partial \delta}(0,0) \\ \frac{\partial \delta_{4_{lk_{l'}}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{4_{lk_{l'}}}}{\partial \delta}(0,0) \end{pmatrix} = -(Df(\mathbf{0})^{-1}) \begin{pmatrix} \frac{\partial f_1}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_1}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_2}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_2}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_3}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_3}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_4}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_4}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_5}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_5}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_6}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_6}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_7}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_7}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_8}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_8}{\partial \delta}(\mathbf{0}) \end{pmatrix}$$

$$= -\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{|K_{l'}|} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{|K_{l'}|} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},$$

where $\mathbf{0} = (0,0,0;0,0)$. The above equation can be rewritten for each $l \in L$, each $l' \in L\backslash\{l\}$, each $k_l \in K_l$, and each $k_{l'} \in K_{l'}$ as

$$\begin{pmatrix} \varepsilon_1(\varepsilon,\delta) \\ \varepsilon_{2_{k_l}}(\varepsilon,\delta) \\ \varepsilon_{3_l}(\varepsilon,\delta) \\ \varepsilon_{4_{lk_l}}(\varepsilon,\delta) \\ \delta_1(\varepsilon,\delta) \\ \delta_{2_l}(\varepsilon,\delta) \\ \delta_{3_{k_{l'}}}(\varepsilon,\delta) \\ \delta_{4_{lk_{l'}}}(\varepsilon,\delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(0,0) \\ \varepsilon_{2_{k_l}}(0,0) \\ \varepsilon_{3_l}(0,0) \\ \varepsilon_{4_{lk_l}}(0,0) \\ \delta_1(0,0) \\ \delta_{2_l}(0,0) \\ \delta_{3_{k_{l'}}}(0,0) \\ \delta_{4_{lk_{l'}}}(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_1}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_{2_{k_l}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{2_{k_l}}}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_{3_l}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{3_l}}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_{4_{lk_l}}}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_{4_{lk_l}}}{\partial \delta}(0,0) \\ \frac{\partial \delta_1}{\partial \varepsilon}(0,0) & \frac{\partial \delta_1}{\partial \delta}(0,0) \\ \frac{\partial \delta_{2_l}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{2_l}}{\partial \delta}(0,0) \\ \frac{\partial \delta_{3_{k_{l'}}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{3_{k_{l'}}}}{\partial \delta}(0,0) \\ \frac{\partial \delta_{4_{lk_{l'}}}}{\partial \varepsilon}(0,0) & \frac{\partial \delta_{4_{lk_{l'}}}}{\partial \delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \\ o_4(\varepsilon,\delta) \\ o_5(\varepsilon,\delta) \\ o_6(\varepsilon,\delta) \\ o_7(\varepsilon,\delta) \\ o_8(\varepsilon,\delta) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & |K_{l'}| \\ 0 & |K_{l'}| \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \\ o_4(\varepsilon,\delta) \\ o_5(\varepsilon,\delta) \\ o_6(\varepsilon,\delta) \\ o_7(\varepsilon,\delta) \\ o_8(\varepsilon,\delta) \end{pmatrix},$$

where $o_I(\varepsilon,\delta)$,$I = 1,2,3,4,5,6,7,8$, stands for the second- or higher-order terms of ε and δ. Thus, we obtain the first-order approximated values of $\varepsilon_1, \varepsilon_{2_{k_l}}, \varepsilon_{3_l}, \varepsilon_{4_{lk_l}}$, and $\delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \delta_{4_{lk_{l'}}}$ for each $l \in L$, each $l' \in L\backslash\{l\}$, each

$k_l \in K_l$, and each $k_{l'} \in K_{l'}$, respectively, as follows:

$$\begin{aligned}
\varepsilon_1 &= \varepsilon + o_1(\varepsilon, \delta), \\
\varepsilon_{2_{k_l}} &= \varepsilon + o_2(\varepsilon, \delta), \\
\varepsilon_{3_l} &= \varepsilon + o_3(\varepsilon, \delta), \\
\varepsilon_{4_{lk_l}} &= \varepsilon + o_4(\varepsilon, \delta), \\
\delta_1 &= \delta + o_5(\varepsilon, \delta), \\
\delta_{2_l} &= \delta + o_6(\varepsilon, \delta), \\
\delta_{3_{k_{l'}}} &= |K_{l'}|\delta + o_7(\varepsilon, \delta), \\
\delta_{4_{lk_{l'}}} &= |K_{l'}|\delta + o_8(\varepsilon, \delta).
\end{aligned}$$

Replacing $\varepsilon_1, \varepsilon_{2_{k_l}}, \varepsilon_{3_l}, \varepsilon_{4lk_l}$, and $\delta_1, \delta_{2_l}, \delta_{3_{k_{l'}}}, \delta_{4_{lk_{l'}}}$ for each $l \in L$, each $l' \in L \backslash \{l\}$, each $k_l \in K_l$ and each $k_{l'} \in K_{l'}$ in Definition 7 by these values above, we find the first-order approximated rest point. In the case in which the value of $\frac{1}{|K_{l'}|}$ for each $l' \in L$ is not a constant, we can prove the theorem through the same procedure as corollary 2. $\square$

Proof of Theorem 6

We follow a similiar procedure in the proof of Theorem 2. The characteristic equation of the first-order approximated Jacobian matrix evaluated at the symmetric rest point close to *any* extended-signaling system is given by

$$
\begin{aligned}
&[\lambda-(m-2)\varepsilon-(n-1)\delta+1]^{m-|K|}\\
&\times[\lambda+\varepsilon-n\delta+1]^{(m-|K|-1)(m-|K|)}\\
&\times[\lambda+\varepsilon-(n+1)\delta+1]^{(m-|K|)|K|}\\
&\times[\lambda+2\varepsilon-(2n-1)\delta+1]^{|L|(m-|K|)}\\
&\times[\lambda+2\varepsilon-2n\delta+1]^{(|L|-1)|K|}\\
&\times[\lambda-(m-1)\varepsilon-(n-2)\delta+1]^{(m-|K|)}\\
&\times[\lambda-m\varepsilon+\delta+1]^{(m-|K|)(n-1)}\\
&\times[\lambda-\tfrac{m}{2}\varepsilon+\delta+\tfrac{1}{2}]^{(m-|K|)|K|}\\
&\times[\lambda-\tfrac{m}{2}\varepsilon+\delta+\tfrac{1}{2}]^{(|L|-1)|K|}\\
&\times[\lambda+2\varepsilon]^{\frac{|K|-2}{2}}\\
&\times[\lambda-(m-4)\varepsilon-2(n-1)\delta+1]^{\frac{|K|-2}{2}}\\
&\times[\lambda-\tfrac{m-2}{2}\varepsilon-(n-2)\delta+\tfrac{1}{2}]^{|K|-1}\\
&\times[\lambda+(4-m)\varepsilon+(2-2n+\tfrac{kn}{2}-\tfrac{1}{2}k)\delta+1]\\
&\times[\lambda+(1-\tfrac{1}{2}m)\varepsilon+(2-n)\delta+\tfrac{1}{2}]\\
&\times[\lambda+2\varepsilon-\tfrac{k}{2}(n-1)\delta]=0
\end{aligned}
$$

where λ is the eigenvalue. $\square$

Table 1: The first-order approximated Jacobian matrix of the selection-mutation dynamics evaluated at the rest point close to an extended-signaling system (P_4^*, Q_4^*)

$$P_4^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, Q_4^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \text{ where } Z_{P_4^*} = \{3\}.$$

	∂p_{11}	∂p_{12}	∂p_{13}	∂p_{21}	∂p_{22}	∂p_{23}	∂q_{11}	∂q_{12}	∂q_{21}	∂q_{22}	∂q_{31}	∂p_{32}
$\partial \dot{p}_{11}$	$-1+2\varepsilon+\delta$	$-\delta$	$-\frac{1}{2}+\frac{3}{2}\varepsilon$	0	0	0	3ε	0	$-\varepsilon$	0	-2ε	0
$\partial \dot{p}_{12}$	$-\varepsilon$	$-1-\varepsilon+2\delta$	$-\frac{1}{2}\varepsilon$	0	0	0	$-\varepsilon$	0	ε	0	0	0
$\partial \dot{p}_{13}$	-2ε	0	$-\frac{1}{2}-2\varepsilon+\delta$	0	0	0	-2ε	0	0	0	2ε	0
$\partial \dot{p}_{21}$	0	0	0	$-1-\varepsilon+2\delta$	$-\varepsilon$	$-\frac{1}{2}\varepsilon$	0	ε	0	$-\varepsilon$	0	0
$\partial \dot{p}_{22}$	0	0	0	$-\delta$	$-1+2\varepsilon+\delta$	$-\frac{1}{2}+\frac{3}{2}\varepsilon$	0	$-\varepsilon$	0	3ε	0	-2ε
$\partial \dot{p}_{23}$	0	0	0	0	-2ε	$-\frac{1}{2}-2\varepsilon+\delta$	0	0	0	-2ε	0	2ε
$\partial \dot{q}_{11}$	δ	0	0	$-\delta$	0	0	$-1+3\varepsilon$	$-\varepsilon$	0	0	0	0
$\partial \dot{q}_{12}$	$-\delta$	0	0	δ	0	0	$-\delta$	$-1+4\varepsilon-\delta$	0	0	0	0
$\partial \dot{q}_{21}$	0	δ	0	0	$-\delta$	0	0	0	$-1+4\varepsilon-\delta$	$-\delta$	0	0
$\partial \dot{q}_{22}$	0	$-\delta$	0	0	δ	0	0	0	$-\varepsilon$	$-1+3\varepsilon$	0	0
$\partial \dot{q}_{31}$	0	0	$\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	0	0	0	$-\varepsilon-2\delta$	$-\varepsilon$
$\partial \dot{q}_{32}$	0	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	$-\varepsilon$	$-\varepsilon-2\delta$

Table 2: The list of all entries of the first-order approximated Jacobian matrix of the selection-mutation dynamics evaluated at the rest point close to each extended-signaling sysytem (for $n \leq m$)

	$s=i, t=j$	$s=i,(s,t)\in I_2$	$s=i,(s,t)\in I_3$	$s\neq i$		$t=j, s=i,$	$t=j, s\neq i$	$t\neq j, s=i, (s,t)\in I_2$	$t=j, s\neq i, (i,j)\in I_2$	$t\neq j, s=i, (s,t)\in I_3$	$t=j, s\neq i, (i,j)\in I_3$
$\frac{\partial \dot{p}_{ij}}{\partial p_{st}}$; $(i,j)\in I_1$	$-1+\frac{-3n+mn+2}{n-1}\varepsilon+(n-1)\delta$	$-\delta$	$-\frac{1}{n}+\frac{mn-2n+1}{n(n-1)}\varepsilon$	0	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}}$; $(i,j)\in I_1$	$-\frac{2n-mn-1}{n-1}\varepsilon$	0	$-\varepsilon$	0	$-\frac{n}{n-1}\varepsilon$	0
the number of entries	$n\cdot 1$	$n\cdot(n-1)$	$n\cdot(m-n)$	$n\cdot m(n-1)$		$n\cdot 1$	$n\cdot(n-1)$	$n\cdot(n-1)$	$n\cdot(n-1)^2$	$n\cdot(m-n)$	$n\cdot(n-1)(m-n)$

	$s=i, (s,t)\in I_1$	$s=i, t=j$	$s=i, (s,t)\in I_2$	$s=i, (s,t)\in I_3$	$s\neq i$		$t=j, s=i$	$t=j, s\neq i, (i,j)\in I_1$	$t\neq j, s=i, (s,t)\in I_1$	$t=j, s\neq i, (s,t)\notin I_1$	$t\neq j, s\neq i, (s,t)\notin I_1$
$\frac{\partial \dot{p}_{ij}}{\partial p_{st}}$; $(i,j)\in I_2$	$-\varepsilon$	$-1-\varepsilon+n\delta$	0	$-\frac{1}{n}\varepsilon$	0	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}}$; $(i,j)\in I_2$	ε	0	$-\varepsilon$	0	0
the number of entries	$n(n-1)\cdot 1$	$n(n-1)\cdot 1$	$n(n-1)\cdot(n-2)$	$n(n-1)\cdot(m-n)$	$n(n-1)\cdot m(n-1)$		$n(n-1)\cdot 1$	$n(n-1)\cdot(n-1)$	$n(n-1)\cdot 1$	$n(n-1)\cdot 1$	$n(n-1)\cdot n(m-2)$

	$s=i, (s,t)\in I_1$	$s=i, (s,t)\in I_2$	$s=i, t=j$	$s=i, (s,t)\in I_3$	$s\neq i$		$s=i, (s,t)\in I_1$	$s=i, (s,t)\in I_2$	$t=j, s=i$	$s\neq i$
$\frac{\partial \dot{p}_{ij}}{\partial p_{st}}$; $(i,j)\in I_3$	$-\frac{n}{n-1}\varepsilon$	0	$\frac{1}{n}-1-\frac{n}{n-1}\varepsilon+(n-1)\delta$	$-\frac{1}{n-1}\varepsilon$	0	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}}$; $(i,j)\in I_3$	$-\frac{n}{n-1}\varepsilon$	0	$\frac{n}{n-1}\varepsilon$	0
the number of entries	$n(m-n)\cdot 1$	$n(m-n)\cdot(n-1)$	$n(m-n)\cdot 1$	$n(m-n)\cdot(m-n-1)$	$n(m-n)\cdot m(n-1)$		$n(m-n)\cdot 1$	$n(m-n)\cdot(n-2)$	$n(m-n)\cdot 1$	$n(m-n)\cdot m(n-1)$

	$s=i, t=j$	$s=i, (s,t)\in I_2$	$s=i, (s,t)\in I_3$	$t=j, s\neq i, (s,t)\in I_2, I_3$	$t\neq j$		$t=j, s=i$	$t=j, s\neq i$	$t\neq j$
$\frac{\partial \dot{q}_{ji}}{\partial p_{st}}$; $(i,j)\in I_1$	δ	0	0	$-\delta$	0	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}}$; $(i,j)\in I_1$	$-1+\frac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta$	$-\varepsilon$	0
the number of entries	$n\cdot 1$	$n\cdot(n-1)$	$n\cdot(m-n)$	$n\cdot(n-1)$	$n\cdot(n-1)(m-1)$		$n\cdot 1$	$n\cdot(n-1)$	$n\cdot n(m-1)$

	$s\neq i, t=j, (s,t)\in I_1$	$s=i, t\neq j, (i,j)\in I_1$	$s=i, t=j$	$s=i, t\neq j$	$t\neq j$		$t=j, s\neq i, (s,t)\in I_1$	$t=j, s=i$	$t=j, s\neq i, (s,t)\notin I_1$	$t\neq j$
$\frac{\partial \dot{q}_{ji}}{\partial p_{st}}$; $(i,j)\in I_2$	$-\delta$	0	δ	0	0	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}}$; $(i,j)\in I_2$	$-\delta$	$-1+\frac{mn-n}{n-1}\varepsilon-\delta$	0	0
the number of entries	$n(n-1)\cdot 1$	$n(n-1)\cdot(m-1)$	$n(n-1)\cdot 1$	$n(n-1)\cdot(m-1)$	$n(n-1)\cdot m(m-2)$		$n(n-1)\dot{1}$	$n(n-1)\dot{1}$	$n(n-1)\dot{(}n-2)$	$n(n-1)\dot{n}(m-1)$

	$s=i, t=j$	$s=i, t\neq j$	$s\neq i, t=j$	$s\neq i, t\neq j$		$t=j, s\neq i$	$t=j, s=i$	$t\neq j$
$\frac{\partial \dot{q}_{ji}}{\partial p_{st}}$; $(i,j)\in I_3$	$\frac{1}{n}-\frac{1}{n^2}$	0	$-\frac{1}{n^2}$	0	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}}$; $(i,j)\in I_3$	$-\frac{1}{n-1}\varepsilon$	$-\frac{1}{n-1}\varepsilon-n\delta$	0
the number of entries	$n(m-n)\cdot 1$	$n(m-n)\cdot(m-1)$	$n(m-n)\cdot(n-1)$	$n(m-n)\cdot(n-1)(m-1)$		$n(m-n)\cdot(n-1)$	$n(m-n)\cdot 1$	$n(m-n)\cdot n(m-1)$

References

[1] Hofbauer, J., The selection mutation equation. *Journal of Mathematical Biology*, 23(1):41-53, 1985.

[2] Hofbauer, J., Huttegger, S., Feasibility of communication in binary signaling. *Journal of Theoretical Biology*, 254:843-849, 2008.

[3] Hofbauer, J., Huttegger, S., Selection-mutation dynamics of signaling games. *Games*, 6(1):2-31, 2015.

[4] Huttegger, S, Evolution and the explanation of meaning. *Philosophy*, 74:127, 2007.

[5] Kuhn, H.W. Lectures on the Theory of Games; Princeton University Press: Princeton, NJ, USA, 2003.

[6] Lewis, D., Convention: A philosophical study. Harvard University Press, Cambridge, MA. 1969.

[7] Nowak, M.A., Krakauer, D., The evolution of language. *Proceedings of National Academy of Science*. USA 96, 8028-8033, 1999.

[8] Nowak, M.A., Plotkin J, Krakauer, D., The evolution language game. *Journal of Theoretical Biology*, 147-162;200, 1999.

[9] Pawlowitsch, C., Finite populations choose an optimal language. *Journal of Theoretical Biology*, 249, 606616, 2007.

[10] Pawlowitsch, C., Why evolution does not always lead to an optimal signaling system." *Games and Economic Behavior*, 63:203-226, 2008.

[11] Trapa, P., Nowak, M.A., Nash equilibria for an evolutionary language game. *Mathematical Biology*, 41:172-188, 2000.

[12] Wärneryd, K., Cheap talk, coordination, and evolutionary stability. *Games and Economic Behavior*, 5:532-546, 1993.

[13] Hofbauer, J.; Sigmund, K. Evolutionary Games and Population Dynamics; Cambridge University Press: Cambridge, UK, 1998.