TCER Working Paper Series

Output and Welfare Implications of Oligopolistic Third-Degree Price Discrimination

Takanori Adachi
Michal Fabinger

October 2019

Working Paper E-139
http://tcer.or.jp/wp/pdf/e139.pdf

TOKYO CENTER FOR ECONOMIC RESEARCH
1-7-10-703 Iidabashi, Chiyoda-ku, Tokyo 102-0072, Japan

Abstract

Using estimable concepts, this paper provides sufficient conditions for third-degree price discrimination to raise or lower aggregate output, social welfare, and consumer surplus under differentiated oligopoly when all discriminatory markets are open even without price discrimination. Specifically, we permit general demand functions and cost differences across separate markets, and show that our sufficient conditions entail a cross-market comparison of multiplications of two or three of the following key endogenous variables with economic interpretation:pass-through value,market power index, and markup value. Notably, our results based on these“sufficient statistics”can readily be extended to allow heterogeneous firms, suggesting that they would be used as a building block for empirical study of third-degree price discrimination and welfare.

Takanori Adachi
Nagoya University
School of Economics
1 Furo-cho, Chikusa-ku, Nagoya 464-8601, JAPAN
adachi.t@soec.nagoya-u.ac.jp

Michal Fabinger
Tokyo Data Science
NA
NA
fabinger@gmail.com

Output and Welfare Implications of Oligopolistic Third-Degree Price Discrimination*

TAKANORI ADACHI† MICHAL FABINGER‡

October 10, 2019

Abstract

Using estimable concepts, this paper provides sufficient conditions for third-degree price discrimination to raise or lower aggregate output, social welfare, and consumer surplus under differentiated oligopoly when all discriminatory markets are open even without price discrimination. Specifically, we permit general demand functions and cost differences across separate markets, and show that our sufficient conditions entail a cross-market comparison of multiplications of two or three of the following key endogenous variables with economic interpretation: *pass-through value*, *market power index*, and *markup value*. Notably, our results based on these "sufficient statistics" can readily be extended to allow heterogeneous firms, suggesting that they would be used as a building block for empirical study of third-degree price discrimination and welfare.

Keywords: Third-Degree Price Discrimination; Differentiated Oligopoly; Social Welfare; Pass-through; Sufficient Statistics.

JEL classification: D21; D43; D60; L11; L13.

*We are grateful to Iñaki Aguirre, Jacob Burgdorf, Susumu Cato, Yong Chao, Jacob Gramlich, Makoto Hanazono, Hiroaki Ino, Noritaka Kudoh, Toshifumi Kuroda, Jeanine Miklós-Thal, Dan Sasaki, Susumu Sato, Nicolas Schutz, Shohei Tamura, Tsuyoshi Toshimitsu, Yusuke Zennyo, and conference and seminar participants at the 16th Annual International Industrial Organization Conference, the 71st European Summer Meeting of the Econometric Society, Daito Bunka University, Kwansei Gakuin University, Nagoya University, University of Louisville, and University of Tokyo for helpful comments and discussions. Adachi acknowledges a Grant-in-Aid for Scientific Research (C) (15K03425, 18K01567) from the Japan Society of the Promotion of Science. Fabinger acknowledges a Grant-in-Aid for Young Scientists (A) (26705003) from the Society. All remaining errors are our own. The views and opinions expressed in this paper are solely ours and do not necessarily represent those of Tokyo Data Science.

†School of Economics, Nagoya University, Japan. E-mail: adachi.t@soec.nagoya-u.ac.jp

‡Tokyo Data Science and the University of Tokyo, Japan. E-mail: fabinger@gmail.com

1 Introduction

This paper explores welfare effects of third-degree price discrimination in oligopoly as well as its output implications and the effects on consumer surplus. Specifically, we consider a fairly general setting, and present sufficient conditions for oligopolistic third-degree price discrimination to raise or lower aggregate output, consumer surplus, and Marshallian social welfare (i.e., the sum of consumer and producer surplus) when all discriminatory markets are served even without price discrimination. Our analysis is firstly developed under firm symmetry, and then is extended to allow heterogeneous firms in a natural manner. In addition, our analysis permits a moderate amount of cost differences across separate markets. These features, together with the fact that our sufficient conditions are stated in terms of estimable concepts (see below), are appealing because they allow greater flexibility that can accommodate with empirical studies.

Under third-degree price discrimination, consumers are segmented into separate markets and are charged different unit prices according to identifiable characteristics (e.g., age, occupation, and location or time of purchase). In contrast, all consumers face the same price if third-degree price discrimination is not practiced ("uniform pricing"). Without loss of generality, one can consider the case of two markets to understand how price discrimination changes output and welfare in each market. Furthermore, for a simpler exposition, all firms are symmetric and thus the prevailing price is identical. Then, in one market, the discriminatory price will become greater than the uniform price, whereas the unit price in the other market decreases. In Robinson's (1933) terminology, the former market is called a "strong" market (s), and the latter a "weak" market (w). Stated differently, the discriminatory price in the strong market p_s is higher than the equilibrium uniform price, $\overline{p}$, but p_w, the discriminatory price in the weak market, is lower: $p_s > \overline{p} > p_w$.[1]

Given this price change, price discrimination raises output and social welfare in the weak market, whereas they lower in the strong market. Then, what are the overall effects? Our sufficient conditions for oligopolistic price discrimination to raise or lower aggregate output, social welfare, and consumer surplus are provided in a way of a cross-market comparison of multiplications of two or three of the following economic concepts: (i) *pass-through value*, or simply *pass-through* ($\rho > 0$), i.e., how the price responds to a small change in the marginal cost;

[1] In this paper, price discrimination is present when $p_s > p_w$, i.e., when the prices across markets are not uniform. As Clerides (2004, p. 402) states, once cost differentials are allowed, "there is no single, widely accepted definition of price discrimination." To understand this, consider symmetric firms and let mc_s and mc_w be marginal costs at equilibrium output in markets s and w, respectively. Then, there can be two alternative definitions. One is the *margin* definition: price discrimination occurs when $p_s - mc_s > p_w - mc_w$. The other one is the *markup* definition due to Stigler (1987): price discrimination occurs when $p_s/mc_s > p_w/mc_w$. Our simpler definition is in line with the former definition, and is employed for its tractability and its connectivity to the existing literature on third-degree price discrimination with no cost differentials. Moreover, our definition of price discrimination coincides with what Chen and Schwartz (2015) and Chen, Li and Schwartz (2019) call "differential pricing." As long as cost differentials are sufficiently small, these differences will not alter the results significantly because these three definitions are equivalent if $mc_s = mc_w$.

(ii) *market power index*, or simply *market power* ($\theta \in [0, 1]$), which measures the intensity of competition in the market; and (iii) *markup value*, or simply *markup* ($\mu \geq 0$), which is simply the difference between the price and the marginal cost. In particular, in a series of propositions, we show that the product of market power and pass-through, $\theta\rho$, is important for determining the output effects, whereas the product of all the three concepts, $\theta\mu\rho$, consists of the sufficient condition for a welfare change. Intuitively, the product of market power and pass-through measures how output in each separate market changes in response to a marginal change in price. To evaluate a marginal change in welfare, one needs to consider the markup, which measures welfare gain or loss in response to a marginal change in quantity under imperfect competition where the price exceeds the marginal cost. In this way, welfare implications are obtained from a cross-market comparison of the quantity change multiplied by the markup. For the effects on consumer surplus, the product of markup and pass-through, $\mu\rho$, is crucial because it measures the price change multiplied by the amount of output.

We also emphasize that these three concepts consists of the "sufficient statistics" (Chetty 2009) in the sense that no other information is necessary (except for the information for an adjustment; see the explanation in the end of subsection 3.1) and that they are sufficient to determine the output and welfare effects of oligopolistic third-degree price discrimination. Furthermore, pass-through is determined by up to the second-order demand characteristics, and under a game of price setting, market power is expressed by the own- and cross-price elasticities. Obviously, markup is also determined by the own- and cross-price elasticities via the first-order conditions. Thus, once the market demands are specified/estimated, a change in aggregate output, social welfare, or consumer surplus is predicted by computing these "sufficient statistics." For this sake, the researcher does not have to directly compare welfare under price discrimination and that under uniform pricing. In this way, our theoretical predictions, equipped with an estimable framework, would be utilized to understand the mechanism behind an empirical result. For example, in their empirical analysis of within-store brand competition, Hendel and Nevo (2013) show that social welfare is higher under third-degree price discrimination than with the case of no discrimination. However, it is not clear which factor is important in obtaining this empirical result. Although ultimately welfare evaluation is an empirical matter, one still wishes to know more about which force derives the result. Therefore, this paper also aims to fill the gap between the theoretical predictions and the empirical literature on price discrimination where researchers are often agnostic about the mechanism behind the result.[2]

The literature on third-degree price discrimination has a centennial tradition, pioneered by Pigou (1920) and Robinson (1933), with the main focus being on whether price discrimination raises or lowers social welfare. Among others, Schmalensee (1981) and Aguirre, Cowan, and

[2]For other empirical studies of third-degree price discrimination, see, e.g., Leslie (2004); Asplund, Eriksson and Strand (2008); and Boik (2017).

Vickers (2010) study how the demand curvatures are related to the output and welfare effects. In addition to competition being imperfect, third-degree price discrimination entails further inefficiency in production and distribution as consumers with the same marginal valuation may face different prices simply because they belong to different markets. Thus, for third-degree price discrimination to raise social welfare, it must expand aggregate output sufficiently to offset such a misallocation across markets. Schmalensee (1981) shows that an increase in aggregate output is a necessary condition for third-degree price discrimination to raise social welfare—generalized by Varian (1985) and Schwartz (1990)—and Aguirre, Cowan, and Vickers (2010) identify that aggregate output increases with price discrimination if the inverse demand curvature is greater ("more convex") in the weak market than in the strong market, and that with an additional qualification (the discriminatory prices are sufficiently close), social welfare also increases.

However, these studies analyze *monopolistic* third-degree discrimination, and to date, "there are *virtually no predictions* as to how discrimination impacts welfare" (Hendel and Nevo 2013, p. 2723; emphasis added) once *oligopolistic* competition is taken into account. For example, Holmes (1989) employs the same technique used by Schmalensee (1981) and Aguirre, Cowan and Vickers (2010) (see Section 3 for details) to study the output effects of third-degree price discrimination in a symmetric oligopoly. However, Holmes (1989) provides no welfare predictions (see also Dastidar 2006).[3] In this paper, we aim to contribute to this literature by providing fairly general conditions as to whether oligopolistic price discrimination raises or lowers aggregate output, social welfare, and consumer surplus. Our results generalize Aguirre, Cowan, and Vickers' (2010) analysis of monopolistic third-degree price discrimination not only because we study oligopoly but because our results are also stated in terms of the sufficient statistics. Although these sufficient statistics themselves are endogenous variables and thus are less fundamental than demand primitives such as the first- and second-order characteristics (i.e., elasticities and curvatures), our focus on the sufficient statistics enables us to extend our results under firm symmetry to the case of heterogeneous firms (both in demand and production). In addition, we also provide another expression for Holmes' (1989) derivation of the output effects in terms of the key statistics.

Furthermore, our treatment also allows differences in the marginal cost across separate markets. In almost all of the theoretical studies on price discrimination, researchers (somewhat manually) assume that there are no cost differentials across discriminatory markets to focus on the demand side. However, in many real cases of price discrimination, cost differentials are quite

[3] In a similar vein, Armstrong and Vickers (2001) consider a model of symmetric duopoly with product differentiation à la Hotelling (1929), and study the consequences of third-degree price discrimination in the competitive limit around zero transportation cost where the equilibrium prices are almost equal to marginal cost. Under this setting, Armstrong and Vickers (2001) show that price discrimination lowers social welfare if the weak market has a lower value of price elasticity of demand. Adachi and Matsushima (2014) also derive a similar result by assuming linear demand.

often observed, not to mention the typical example of freight charges across regional markets with different costs of transportation and storage (Phlips 1983, pp. 5-7). In the narrow definition of price discrimination, this is not price discrimination because they are regarded as different products. However, airlines are arguably motivated to offer different types of seats mainly because they aim to make use of heterogeneity among consumers. Thus, ideally, a theoretical analysis on price discrimination should also allow a moderate amount of cost differentials across discriminatory markets.

Even if costs differ across markets, sellers, in reality, may have to be engaged in uniform pricing due to the universal service requirement, fairness concerns from consumers, and so on (Okada 2014; Geruso 2017). In the case of monopoly with differentials in the marginal costs across markets, Chen and Schwartz (2015) derive sufficient conditions for consumer surplus to be higher under differential pricing.[4] To ensure that the strong market is indeed strong when cost differentials are allowed, it is sufficient to assume that the marginal cost in the strong market, denoted by mc_s, is higher than that in the weak market: $mc_s > mc_w$ (though mc_s should not be too much higher than mc_w). Then, under uniform pricing, the markup in the strong market $\overline{p} - mc_s$ is smaller than the markup in the weak market $\overline{p} - mc_w$. Differential pricing allows the monopolist to sell more products in the weak market, which is improves efficiency. Chen and Schwartz (2015) find that while differential pricing with no cost differentials (third-degree price discrimination in a traditional manner) tends to increase the average price after differential pricing is allowed, differential pricing with cost differentials does not. In contrast, our analysis provides welfare implications in a more direct manner. As in Chen and Schwartz (2015), this paper does not have to make an explicit assumption on mc_s and mc_w as long as the second-order conditions for profit maximization are satisfied and a large discrepancy between mc_w and mc_w does not change the order of the discriminatory prices from the one with no cost differentials.[5]

The rest of the paper is organized as follows. Section 2 presents our basic model of oligopolistic pricing with symmetric firms and constant marginal costs. Then, we derive output and welfare implications in Section 3. We also provide parametric examples, show the results of the effects on consumer surplus, and discuss the case of non-marginal costs. In Section 4, we argue that our method can be readily extended if firm heterogeneity is introduced. Section 5 concludes the paper.

[4] Chen, Li and Schwartz (2019) extend Chen and Schwartz' (2015) analysis to the case of oligopoly. See also Galera and Zaratiegui (2006) and Bertoletti (2009) as studies of conditions under which price discrimination raises social welfare when cost differentials across markets are allowed.

[5] In the context of reduced-fare parking as a form of third-degree price discrimination with cost differentials, Flores and Kalashnikov (2017) characterize a sufficient condition for free parking (drivers receive a price discount in the form of complementary parking while pedestrians do not) to be welfare improving.

2 The Model of Oligopolistic Pricing

For ease of exposition, we, following Holmes (1989) and Aguirre, Cowan, and Vickers (2010), consider the case of two symmetric firms and two separate markets or consumer groups (simply called markets hereafter). Extending the following analysis to the case of J (≥ 3) symmetric firms and M (≥ 3) separate markets is straightforward. As explained above, we call one market s (strong), where the equilibrium discriminatory price will be higher than the equilibrium uniform price, and the other w (weak), where the opposite is true. Two firms, A and B, have an identical cost structure in each market. Specifically, each firm has an identical cost function, $c_m(q_{jm})$, in market $m = s, w$, where q_{jm} is firm j's output ($j = A$, B). For simplicity of exposition below, we assume, with abuse of notation, that firms have a constant marginal cost in each market m, $c_m \geq 0$; here, in the spirit of Chen and Schwartz (2015), c_s and c_w can be different. However, none of our arguments crucially hinges on this assumption (see Subsection 3.5).

In market $m = s, w$, given firms A and B's prices p_{Am} and p_{Bm}, the representative consumer consumes $q_{Am} > 0$ and $q_{Bm} > 0$, and her (net) utility is quasi-linear and is written as $U_m(q_{Am}, q_{Bm}) - p_{Am}q_{Am} - p_{Bm}q_{Bm}$, where U_m is three-times continuously differentiable, $\partial U_m/\partial q_{jm} > 0$, $\partial^2 U_m/\partial q_{jm}^2 < 0$, $j = A$, B, and $\partial^2 U_m/(\partial q_{Am}\partial q_{Bm}) < 0$. *Direct demands* in market m are derived from the representative consumer's utility maximization: $\partial U_m(q_{jm}, q_{-j,m})/\partial q_{jm} - p_{jm} = 0$, which leads to firm j's demand in market m, $q_{jm} = x_{jm}(p_{jm}, p_{-j,m})$. We assume that x_{jm} is twice continuously differentiable. The corresponding *inverse demand* can be written as $p_{jm} = p_{jm}(q_{jm}, q_{-j,m})$. Because of the assumptions on the utility, firm j's demand in market m falls as its own price increases ($\partial x_{jm}/\partial p_{jm} < 0$ and $\partial p_{jm}/\partial x_{jm} < 0$), and it rises as the rival's price increases ($\partial x_{jm}/\partial p_{-j,m} > 0$ and $\partial p_{jm}/\partial x_{-j,m} < 0$; the firms' products are substitutes). We assume that for a consumer's perspective firms are symmetric: $U_m(q', q'') = U_m(q'', q')$ for any $q' > 0$ and $q'' > 0$. Then, the firms' demands in market m are also symmetric: $x_{Am}(p', p'') = x_{Bm}(p', p'')$ for any $p' > 0$ and $p'' > 0$. Because the firms' technologies are also identical, we, throughout this paper, focus on symmetric Nash equilibrium.

Therefore, we define the demand in symmetric pricing by $q_m(p) \equiv x_{Am}(p, p)$. Another interpretation of $q_m(p)$ is: both firms take $2q_m(p)$ as the joint demand, "cooperatively" choose the same price (behaving as an "industry"), and divide the joint demand equally to obtain $q_m(p)$. Note that:

$$q_m'(p) = \underbrace{\frac{\partial x_{Am}}{\partial p_A}(p_A, p)\bigg|_{p_A=p}}_{<0} + \underbrace{\frac{\partial x_{Am}}{\partial p_B}(p, p_B)\bigg|_{p_B=p}}_{>0 \text{ if substitutes}}.$$

Thus, for $q_m'(p)$ to be negative, we assume that $|\partial x_{Am}(p,p)/\partial p_A| > \partial x_{Am}(p,p)/\partial p_B$. Note also that by symmetry the following relationship also holds (this corresponds to Holmes' (1989) equation 4):

$$\underbrace{\frac{\partial x_{Am}}{\partial p_A}(p,p)}_{\text{own}} = \underbrace{q'_m(p)}_{\text{industry}} - \underbrace{\frac{\partial x_{Bm}}{\partial p_A}(p,p)}_{\text{strategic effects}}.$$

This exchangeability is key in Holmes' (1989) derivation below. Intuitively, under symmetry, each firm treats the industry demand $q_m(p)$ as if it is its own demand. Thus, how a firm's pricing behavior affects its own demand as an industry demand has the following two effects: a small decrease in p_A by firm A by deviating from the "coordinated" price p (i) not only raises its own demand by $\partial x_{Am}/\partial p_A$ as the *residual* monopolist (taking the rival's pricing as fixed; *intrinsic effects*), (ii) firm A can now also obtain some of the consumers originally attached to firm B, and this amount is $\partial x_{Bm}/\partial p_A$ (*strategic effects*).

Under symmetric pricing, we are able to define, following Holmes (1989, p. 245), the *price elasticity of the industry's demand* by $\varepsilon^I_m(p) \equiv -pq'_m(p)/q_m(p)$. This corresponds to ϵ_D in Weyl and Fabinger (2013, p. 542), and it should not "be confused with the elasticity of the residual demand that any of the firms faces." Similarly, the *own-price and the cross-price elasticities of the firm's demand* are defined by $\varepsilon^F_m(p) \equiv -(p/q_m(p))(\partial x_{Am}(p,p)/\partial p_A)$ and by $\varepsilon^C_m(p) \equiv (p/q_m(p))(\partial x_{Bm}(p,p)/\partial p_A)$, respectively. Then, Holmes (1989) shows that under symmetric pricing, $\varepsilon^F_m(p) = \varepsilon^I_m(p) + \varepsilon^C_m(p)$ holds. This implies that the own-price elasticity must be greater than the cross-price elasticity ($\varepsilon^F_m(p) > \varepsilon^C_m(p)$). Here, $\partial^2 x_{jm}(p,p)/\partial p_j^2$ can be positive, zero or negative. Following Dastidar's (2006, p. 234) Assumption 2 (iv), we assume that $\partial^2 x_{jm}(p,p)/\partial p_j^2 + \partial^2 x_{jm}(p,p)/\partial p_j \partial p_{-j} \leq 0$.

Firm j's profit in market m is written as $\pi_{jm}(p_{jm}, p_{-j,m}) = (p_{jm} - c_m)x_{jm}(p_{jm}, p_{-j,m})$. As in Dastidar's (2006, pp. 235-6) Assumptions 3 and 4, for the existence and the global uniqueness of pricing equilibrium under either uniform pricing or price discrimination, we assume that for each firm $j = A, B$, $\partial^2\pi_{jm}/\partial p^2_{jm} < 0$, $\partial^2\pi_{jm}/(\partial p_{jm}\partial p_{-j,m}) > 0$, and $-[\partial^2\pi_{jm}/(\partial p_{jm}\partial p_{-j,m})]$ $/[\partial^2\pi_{jm}/\partial p^2_{jm}] < 1$ (see Dastidar's (2006) Lemmas 1 and 2 for the existence and the uniqueness). We then define the first-order partial derivative of the profit in market m, evaluated at a symmetric price p, by

$$\begin{aligned}\partial_p \pi_m(p) &\equiv \left.\frac{\partial \pi_{jm}(p_{jm}, p_{-j,m})}{\partial p_{jm}}\right|_{p_{jm}=p_{-j,m}=p} \\ &= q_m(p) + (p - c_m)\frac{\partial x_{Am}}{\partial p_A}(p,p). \end{aligned} \tag{1}$$

Then, under symmetric discriminatory pricing, p^*_m satisfies $\partial_p\pi_m(p^*_m) = 0$ for $m = s, w$. Under symmetric uniform pricing, $\overline{p}$ is a (unique) solution of $\partial_p\pi_s(\overline{p}) + \partial_p\pi_w(\overline{p}) = 0$.[6] Throughout

[6]If one considers quantity-setting, rather than price-setting, firms as in Aguirre (2019), then firm j's profit in market m is defined by $\pi_{jm} = p_{jm}(q_{jm}, q_{-j,m})q_{jm} - c_m(q_{jm})$, and thus the first-order partial derivative in symmetric equilibrium is $\partial\pi_{jm}/\partial q_{jm}|_{q_{jm}=q_{-j,m}=q}$, which is equivalent to $p_m(q) - c'_m(q) + q\frac{\partial p_{Am}}{\partial q_A}(q,q) = 0$. Under

this paper, we consider the situation where the weak market is open under uniform pricing: $q_w(\overline{p}) > 0$.[7] Hereafter, functional and parametric restrictions are imposed to assure that $p_s^* > \overline{p} > p_w^*$.[8,9]

The equilibrium discriminatory price in market $m = s, w$, p_m^*, satisfies the following Lerner formula: $\varepsilon_m^F(p_m^*)(p_m^* - c_m)/p_m^* = 1$. This shows that the discriminatory price in market m approaches to the marginal cost as the own-price elasticity for the firm, $\varepsilon_m^F(p_m^*)$, becomes large. Because of Holmes' (1989) elasticity formula explained above, $\varepsilon_m^F(p_m^*)$ can be large (i) when $\varepsilon_m^I(p_m^*)$ is very large even if $\varepsilon_m^C(p_m^*)$ is close to zero, or (ii) when $\varepsilon_m^C(p_m^*)$ is very large even if $\varepsilon_m^I(p_m^*)$ is close to zero. These are two polar cases of a large $\varepsilon_m^F(p_m^*)$: of course if both $\varepsilon_m^I(p_m^*)$ and $\varepsilon_m^C(p_m^*)$ are very large, then $\varepsilon_m^F(p_m^*)$ is also very large. Case (i) is where the industry is under strong pressure from other substitutable industries[10] so that a small price increase in symmetric pricing causes a large number of consumers switching to purchasing a product in other industries instead, although any consumers are very loyal to either firm so that a small price increase by one firm causes a very small number of consumers switching to other rivals' product in the same industry (most of them leave the industry to purchase something outside the industry). For example, in a residential area, consumers (especially young consumers) would have a strong taste for their favorite soda (thus, $\varepsilon_m^C(p_m^*)$ is close to zero), although soda is highly substitutable by mineral water (thus, $\varepsilon_m^I(p_m^*)$ is very large) if consumers just want to relieve their throat (whether soda or water does not matter).

On the other hand, case (ii) is where the competitive pressure from other industries is weak, although inside the industry, firms are fiercely competing for consumers. For example,

uniform pricing, firm j's quantity-setting problem is formulated as $\max_{q_s,q_w} \sum_{m=s,w}[p_{jm}(q_{jm}, q_{-j,m})q_{jm} - c_m(q_{jm})$ subject to $p_{j,s}(q_{j,s}, q_{-j,s}) = p_{j,w}(q_{j,w}, q_{-j,w})$.

[7]Note that $q_w(\overline{p}) > q_w(p_s^*)$ because $q_w(\cdot)$ is strictly decreasing and $p_s^* > \overline{p}$. By Assumption, $q_w(p_s^*) > 0$. Thus, the weak market is open under uniform pricing, i.e., $q_w(\overline{p}) > 0$. Alternatively, we would be able to show that there exist $\underline{c}_s$ and $\overline{c}_s$, $\underline{c}_s < \overline{c}_s$, such that $p_s^* > p_w^*$ and $q_w(\overline{p}) > 0$ for $c_s \in (\underline{c}_s, \overline{c}_s)$ in a similar spirit of Adachi and Matsushima (2014).

[8]However, see Nahata, Ostaszewski, and Sahoo (1990) for an example of all discriminatory prices being lower than the uniform price with a plausible demand structure under monopoly. In the case of oligopoly, Corts (1998) show that best-response asymmetry, by which firms differ in ranking strong and weak markets, is necessary for all discriminatory prices to be lower than the uniform price ("all-out price competition"). As long as symmetric firms are considered, this case never arises. When firm heterogeneity is allowed in Section 4 below, we assume away best-response asymmetry throughout. In addition, if price discrimianation is allowed, each firm may not price discriminate even if it is allowed to do so because it is still able to set an uniform price (i.e., it is not forced to price discriminate). We assume that $\pi_{jm}(\cdot, p_{-j,m}^*)$ is strictly increasing (decreasing) at $p_{jm} = \overline{p}$ in market $m = s$ (m) and thus firm j has an incentive to deviate from the equilbrium uniform price if the other firm chooses $p_{-j,s}^*$ and $p_{-j,w}^*$, and that $\pi_{jm}(\cdot, p_{-j,m}^*)$ attains the global optimum at $p_{jm} = p_{jm}^*$.

[9]Here, the only policy instrument is an enforcement of uniform pricing. Cowan (2018) studies a model of monopoly to consider a more moderate instrument by which a government regulates the monopolist's profit margins or price-marginal cost ratios across different markets.

[10]In the demand construction based on random utility and discrete choice, this would be interpreted as an outside option. Note also that if the industry is defined by the SSNIP ("Small but Significant and Non-transitory Increase in Price") test, then by definition, $\varepsilon_m^I(p_m^*)$ is close to zero, and thus, $\varepsilon_m^F(p_m^*)$ is closely approximated by $\varepsilon_m^C(p_m^*)$.

in a resort, consumers may not care much about the difference between Coca-Cola and Pepsi (thus, $\varepsilon_m^C(p_m^*)$ is very large), though soda would not be easily substitutable by mineral water (thus, $\varepsilon_m^I(p_m^*)$ is close to zero) because consumers want to get perfectly refreshed: having water instead of soda does not relieve their throat. From a firm's perspective, these two polar cases are equivalent with respect to pricing in the sense that if it raises its price by even a small amount, it loses a large number of consumers: whether they leave the industry or switch to rivals' products does not matter to that firm. Thus, in the examples above, the firms' (discriminatory) prices are close to marginal cost both in a residential area and in a resort due to different reasons. Recall again that these are two polar cases: in reality, $\varepsilon_m^F(p_m^*)$ may be large because both $\varepsilon_m^I(p_m^*)$ and $\varepsilon_m^C(p_m^*)$ are large. We can also think of the following alternative possibility: in a resort filled by young visitors, Coca-Cola's and Pepsi's prices are close to the marginal cost because consumers do not care about water or soda as long as they can relieve their throat (i.e., $\varepsilon_m^I(p_m^*)$ is very large), although they are very loyal to either brand once they choose soda ($\varepsilon_m^C(p_m^*)$ is close to zero).

Next, let y_m be per-firm (symmetric) share output in market m,[11] that is, $y_m(p_s, p_w) \equiv q_m(p_m)/[q_s(p_s)+q_w(p_w)]$. Then, the equilibrium uniform price, $\overline{p} \equiv \overline{p}(c_s, c_w)$, satisfies: $\sum_{m=s,w} \overline{y}_m \varepsilon_m^F(\overline{p})(\overline{p} - c_m)/\overline{p} = 1$, where $\overline{y}_m \equiv y_m(\overline{p}(c_s, c_w), \overline{p}(c_s, c_w))$ for $m = s, w$.[12] In the rest of the paper, the dependence of the equilibrium price is often implicit when there are no confusions. In particular, the superscript star denotes price discrimination, whereas the upper bar uniform pricing. For example, we use $(\varepsilon_m^I)^* \equiv \varepsilon_m^I(p_m^*)$ and $\overline{\varepsilon}_m^I \equiv \varepsilon_m^I(\overline{p})$ as the industry's elasticities in equilibrium.

3 Output and Welfare

In the analysis below, we, following Schmalensee (1981), Holmes (1989), and Aguirre, Cowan, and Vickers (2010), add the constraint $p_s - p_w \leq r$, where $r > 0$, to the firms' profit maximization problem (under symmetric pricing).[13] Then, $r = 0$ corresponds to uniform pricing, and $r = r^* \equiv p_s^* - p_w^*$ to price discrimination. We express social welfare (and aggregate output) as a function of r in $[0, r^*]$. Note that under this constrained problem of profit maximization, p_w satisfies $\partial_p \pi_s(p_w + r) + \partial_p \pi_w(p_w) = 0$. Thus, we write the solution by $p_w(r)$. Then, we define $p_s(r) \equiv p_w(r) + r$. Applying the implicit function theorem to this equation yields to

[11] Of course, the total output in market m is $2q_m(p_m)$, aggregated across two symmetric firms.

[12] If there are no cost differentials, i.e., $c_s = c_w$ ($\equiv c$), then the formula is simpler: $\sum_{m=s,w} \overline{x}_m \varepsilon_m^F(\overline{p})(\overline{p} - c)/\overline{p} = 1$ as shown by Holmes (1989, p. 247): the markup rate (common to all markets) is equal to the inverse of the average of own-price elasticities weighted by the output shares.

[13] Alternatively, Vickers (2017) analyzes properties of functions of social welfare or consumer surplus value as a scalar argument to compare social welfare or consumer surplus under price discrimination and uniform pricing in monopoly. Vickers (2017) especially focuses on the cases when the quantity elasticity or the inverse demand curvature is constant for all markets. See also Cowan (2017) for an analysis of the role of the price elasticity and the demand curvature in determining the effects of monopolistic third-degree price discrimination.

$p'_w(r) = -\pi''_s/(\pi''_s + \pi''_w) < 0$ and $p'_s(r) = \pi''_w/(\pi''_s + \pi''_w) > 0$.

Here, note that $\pi''_m(p)$ and $\partial^2_p \pi_m(p)$ are different:

$$\begin{aligned}\pi''_m(p) &= q'_m(p) + \frac{\partial x_{Am}}{\partial p_A}(p,p) + (p - c_m)\frac{d}{dp}\left(\frac{\partial x_{Am}}{\partial p_A}(p,p)\right) \\ &= \partial^2_p \pi_m(p) + \frac{\partial x_{Am}}{\partial p_B}(p,p) + (p - c_m)\frac{\partial^2 x_{Am}}{\partial p_B \partial p_A}(p,p),\end{aligned}$$

where $\partial^2_p \pi_m(p)$ is defined by

$$\partial^2_p \pi_m(p) \equiv \left[2 + (p - c_m)\frac{\partial^2 x_{Am}(p,p)/\partial p_A^2}{\partial x_{Am}(p,p)/\partial p_A}\right]\frac{\partial x_{Am}}{\partial p_A}(p,p),$$

which corresponds to Aguirre, Cowan, and Vickers' (2010, p. 1603) $\pi''_m(p)$. We assume that $\pi''_m(p) < 0$ for all $p \geq 0$.[14]

We define the representative consumer's utility in symmetric pricing by $\widetilde{U}_m(q) = U_m(q,q)$. Aggregate output under symmetric pricing is given by $Q(r) = Q_s(r) + Q_w(r) = 2\{q_s[p_s(r)] + q_w[p_w(r)]\}$. Social welfare under symmetric pricing as a function of r is written as $W(r) \equiv \widetilde{U}_s(q_s[p_s(r)]) + \widetilde{U}_w(q_w[p_w(r)]) - 2c_s \cdot q_s[p_s(r)] - 2c_w \cdot q_w[p_w(r)]$, which implies $W'(r) \equiv (\widetilde{U}'_s - 2c_s) \cdot q'_s \cdot p'_s(r) + (\widetilde{U}'_w - 2c_w) \cdot q'_w \cdot p'_w(r)$. Now, note that $\widetilde{U}'_m = \partial U_m/\partial q_A + \partial U_m/\partial q_B = 2\partial U_m/\partial q_A$ (by symmetry). Thus, $W'(r) = 2(p_s(r) - c_s) \cdot q'_s \cdot p'_s(r) + 2(p_w(r) - c_w) \cdot q'_w \cdot p'_w(r)$.[15]

3.1 Output

Now, we can further proceed:

$$\begin{aligned}\frac{W'(r)}{2} &= (p_s(r) - \overline{p} + \overline{p} - c_s)\, q'_s[p_s(r)]p'_s(r) \\ &\quad + (p_w(r) - \overline{p} + \overline{p} - c_w)\, q'_w[p_w(r)]p'_w(r) \\ &= \underbrace{(p_s(r) - \overline{p})\, q'_s[p_s(r)]p'_s(r)}_{<0} + \underbrace{(p_w(r) - \overline{p})\, q'_w[p_w(r)]p'_w(r)}_{<0} \\ &\quad + \sum_{m=s,w} (\overline{p} - c_m)\, q'_m[p_m(r)]p'_m(r).\end{aligned}$$

This derivation coincides with the case of monopoly as shown in Aguirre, Cowan and Vickers' (2010, p. 1604) equality (3) if there are no cost differentials (i.e., $c_s = c_w \equiv c$), with

[14] Appendix A of Aguirre, Cowan and Vickers (2010) discusses the concavity of the profit function.

[15] As already noted, we consider one strong market and one weak market. More generally, by defining $S \equiv \{m | p^*_m > \overline{p}\}$ and $W \equiv \{m | \overline{p} > p^*_m\}$, $p_s(r)$, $s \in S$, and $p_w(r)$, $w \in S$, consists of the optimal price vector under constraints $|p_m - \overline{p}| \leq r$ for all $m \in S \cup W$, with $r \in [0, \max_m |p^*_m - \overline{p}|]$. Then, for example, social welfare is defined as $W(r; c_s, c_w) \equiv \sum_{s \in S} \widetilde{U}_s(q_s[p_s(r)]) + \sum_{w \in W} \widetilde{U}_w(q_w[p_w(r)]) - 2\sum_{s \in S}(c_s \cdot q_s[p_s(r)]) - 2\sum_{w \in W}(c_w \cdot q_w[p_w(r)])$. Clearly, our two-market analysis does not lose any validity.

two minor modifications: (i) the left hand side is $W'(r)/2$ rather than $W'(r)$ itself, and (ii) the last term of Aguirre, Cowan and Vickers' (2010) equality (3) is replaced by $Q'(r)/2$ rather than $Q'(r)$ because $(1/2)\sum_{m=s,w}(\overline{p}-c_m)\,q'_m[p_m(r)]p'_m(r) = (\overline{p}-c)\,(Q'(r)/2)$. If cost differentials are allowed, it is observed that an increase in the *weighted* aggregate output, $\sum_{m=s,w}(\overline{p}-c_m)\,q'_m[p_m(r)]p'_m(r)$, is *necessary* for price discrimination to raise social welfare, as in the case of monopoly.[16,17]

To proceed further, we define the *curvature of the firm's (direct) demand* in market m by

$$\alpha_m^F(p) \equiv -\frac{p}{\partial x_{Am}(p,p)/\partial p_A}\frac{\partial^2 x_{Am}}{\partial p_A^2}(p,p)$$

(which measures the convexity/concavity of the firm's direct demand, and corresponds to $\alpha_m(p)$ in Aguirre, Cowan and Vickers 2010, p. 1603), and the *elasticity of the cross-price effect* of the firm's direct demand in market m by

$$\begin{aligned}\alpha_m^C(p) &\equiv -\frac{p}{\partial x_{Am}(p,p)/\partial p_B}\frac{\partial^2 x_{Am}}{\partial p_B \partial p_A}(p,p)\\ &= -\underbrace{\frac{p}{\partial x_{Bm}(p,p)/\partial p_A}}_{>0}\underbrace{\frac{\partial^2 x_{Bm}}{\partial p_A^2}(p,p)}_{<0},\end{aligned}$$

which is new to oligopoly.[18] Here, $\alpha_m^F(p)$ is positive (resp. negative) if and only if $\partial^2 x_{Am}(p,p)/\partial p_A^2$ is positive (resp. negative), while $\alpha_m^C(p)$ is always positive (because of our assumption, $\partial^2 x_{jm}(p,p)$ $/(\partial p_A \partial p_B) < 0$). Note that the sign of $\alpha_m^F(p)$ indicates whether the firm's own part of the demand slope under symmetric pricing given the rival's price being p, $\partial x_{Am}(\cdot,p)/\partial p_A$, is *convex* ($\alpha_m^F(p)$ is positive) or *concave* ($\alpha_m^F(p)$ is negative). On the other hand, $\alpha_m^C(p)$ measures how the rival's price level matters to how many of the firm's customers switch to the rival's product when the firm raises its own price ($\partial x_{Bm}/\partial p_A$). Thus, a large $\alpha_m^C(p)$ implies that $\partial x_{Bm}/\partial p_A$ is very responsive to a change in p_B, and vice versa.

Next, we define the *market power index* in market m by $\theta_m(p) \equiv 1 - A_m(p)$,[19,20] where

[16]However, if externalities across consumers, such as network externalities and congestion, exist, then an increase in aggregate output would be no longer a necessary condition, as implied by Adachi (2002, 2005), who studies monopoly with linear demands. See also Czerny and Zhang (2015) as a recent study of price discrimination and congestion.

[17]With a general number of markets (see Footnote 15 above), this condition can be written as $\mathbb{E}[(\overline{p} - c_m)q'_m p'_m] > 0$. For this to hold, $\mathrm{Cov}(\overline{p} - c_m, q'_m p'_m) > 0$, that is, on average, the markup under uniform pricing and $q'_m p'_m > 0$ are positively correlated because $\mathbb{E}[(\overline{p} - c_m)] > 0$ and $\mathbb{E}[q'_m p'_m] > 0$.

[18]This is because $\partial\left(\partial x_{Am}(p,p)/\partial p_B\right)/\partial p_A = \partial\left(\partial x_{Bm}(p,p)/\partial p_A\right)/\partial p_A$.

[19]Conceptually, the market power index defined here corresponds to what is called the conduct parameter when market power itself is a target of estimation without an exact specification of strategic intreracion (see, e.g., Bresnahan 1989; Genesove and Mullin 1998; and Corts 1999).

[20]Alternatively, Weyl and Fabinger (2013, p. 531) and Adachi and Fabinger (2019) define the market power index in a market (which, in our interest in price discrimination, can be indexed by m) by $\theta_m \equiv [(p - c_m)/p]\varepsilon_m^I$

$A_m(p)$ is the *aggregate diversion ratio* (Shapiro 1996) in market m, defined by $A_m(p) \equiv -(\partial x_{Bm}(p,p)/\partial p_A)/(\partial x_{Am}(p,p)/\partial p_A) = \varepsilon_m^C(p)/\varepsilon_m^F(p)$. Here, $A_m(p)$ measures the intensity of *rivalness*: if $A_m(p)$ is close to one, consumers who leave a firm as a response to an increase in its price are nearly all switching to its rival's product. In this way, Aguirre, Cowan and Vickers' (2010) derivation in the case of monopoly (where the method by Schmalensee (1981) is utilized) is connected to Weyl and Fabinger's (2013) condition in the case of symmetric oligopoly. In particular, Aguirre, Cowan and Vickers' (2010, p. 1606) Proposition 2 (a sufficient condition for price discrimination to raise social welfare) is extended to the case of oligopoly in a simpler manner, using the concept of *pass-through* introduced in the next subsection.

As Weyl and Fabinger (2013, p. 544) argue, $\theta_m(p)$ captures the degree of *industry-level brand loyalty or stickiness*[21] in market m: if $\theta_m(p)$ is zero (close to one), market m is captured by perfect competition (almost monopoly): firms' products are perfect substitutes (nearly non-substitutable products).[22] *The markup rate* (the Lerner index), $L_m(p_m) \equiv (p_m - c_m)/p_m$, *alone is not appropriate to measure the rivalness within market* m because it can be the case that p_m is close to c_m (the markup rate is close to zero) simply because the price elasticity of the industry's demand $\varepsilon_m^I(p_m)$ is very large while the brand rivalness is so weak that the cross-price elasticity, $\varepsilon_m^C(p_m)$, remains very small (as a result, in total, $\varepsilon_m^F(p_m)$ is very large, which is actually reason for the low markup rate). However, if $\varepsilon_m^C(p_m)$ is close to $\varepsilon_m^F(p_m)$ (i.e., almost of all consumers who leave a firm as a response to its price increase are switching to other rivals' products), then θ_m becomes close to zero *irrespective of the value of the markup rate*. Thus, $\theta_m(p)$, which ranges between 0 and 1, better captures the brand stickiness than $L_m(p)$ does.

(their mc and ε_D are replaced by our c_m and ε_m^I, respectively) as the Lerner index adjusted by the elasticity of the *industry's* demand. If the first-order condition is given for each market (that is, if full price discrimination is allowed), then $\theta_m(p)$ defined as in Weyl and Fabinger (2013) coincides with $1 - A_m(p)$ because $\varepsilon_m^F[(p_m - c_m)/p_m] = 1$ and thus

$$\begin{aligned}
\varepsilon_m^I(p)\frac{p - c_m}{p} &= \frac{1}{\varepsilon_m^F(p)}\left(-\frac{p}{q_m(p)}\right)q_m'(p) \\
&= -\frac{q_m(p)}{p}\frac{1}{\partial x_{Am}(p,p)/\partial p_A}\left(-\frac{p}{q_m(p)}\right)\left(\frac{\partial x_{Am}}{\partial p_A}(p,p) + \frac{\partial x_{Am}}{\partial p_B}(p,p)\right) \\
&= \frac{\partial x_{Am}(p,p)/\partial p_A + \partial x_{Bm}(p,p)/\partial p_A}{\partial x_{Am}(p,p)/\partial p_A} \text{ (by symmetry)}
\end{aligned}$$

is established.

[21]Even if the firms' products have the same characteristics across different markets (with no product differentiation), the extent of brand loyalty may differ across markets, reflecting differences in market characteristics (summarized in demand functions).

[22]Because $(p_m - c_m)\varepsilon_m^F(p_m)/p_m = 1$ and $\varepsilon_m^F(p_m) = \varepsilon_m^I(p_m) + \varepsilon_m^C(p_m)$, it is verified that $\theta_m(p_m, c_m) + \varepsilon_m^C(p_m)(p_m - c_m)/p_m = 1$. Thus, as long as the products are substitutes ($\varepsilon_m^C(p_m) > 0$), $\theta_m(p_m)$ is less than one.

Now, we consider the effects of price discrimination on aggregate output. First, note that

$$\begin{aligned}\frac{Q'(r)}{2} &= q_w' \cdot p_w' + q_s' \cdot p_s' \\ &= \underbrace{\left(-\frac{q_s' q_w'}{\pi_s'' + \pi_w''}\right)}_{>0} \times \left(\frac{1 - L_s[p_s(r)]\left\{\alpha_s^F[p_s(r)] - (1-\theta_s[p_s(r)])\alpha_s^C[p_s(r)]\right\}}{\theta_s[p_s(r)]}\right. \\ &\qquad \left. - \frac{1 - L_w[p_w(r)]\left\{\alpha_w^F[p_w(r)] - (1-\theta_w[p_w(r)])\alpha_w^C[p_w(r)]\right\}}{\theta_w[p_w(r)]}\right).\end{aligned}$$

Note also that $\varepsilon_m^C(p_m)/\varepsilon_m^I(p_m) = [1 - \theta_m(p_m)]/\theta_m(p_m)$ measures the *substitutability between brands* (adjusted by the elasticity of the industry's demand): if the brand stickiness is very strong (i.e., $\theta_m(p)$ is close to one), $\varepsilon_m^C(p_m)/\varepsilon_m^I(p_m)$ is close to one, while if the brand stickiness is very weak (i.e., $\theta_m(p)$ is close to zero), then $\varepsilon_m^C(p_m)/\varepsilon_m^I(p_m)$ becomes infinitely large. Then, the following lemma holds with cost differentials being allowed.

Lemma 1. $Q'(r) > 0$ *if and only if (suppressing the dependence on* $p_m(r)$*)*

$$L_w \cdot \frac{\alpha_w^F - (1-\theta_w)\alpha_w^C}{\theta_w} - L_s \cdot \frac{\alpha_s^F - (1-\theta_s)\alpha_s^C}{\theta_s} + \frac{1}{\theta_s} - \frac{1}{\theta_w} > 0. \tag{2}$$

Now, suppose that the brand stickiness in the weak market is so weak that $\theta_w(p_w)$ is close to zero ($\theta_w(p_w) \approx 0$), whereas the brand stickiness in the strong market is moderate or strong ($\theta_s(p_s) \gg 0$). Then, the left hand side of inequality (2) above is approximated by

$$\frac{1 - L_s\left[\alpha_s^F - (1-\theta_s)\alpha_s^C\right]}{\theta_s} - \left\{1 - L_w\left[\alpha_w^F - \alpha_w^C\right]\right\}\frac{1}{\theta_w}.$$

Thus, as long as $1 > L_w[\alpha_w^F - \alpha_w^C]$, the left hand side becomes infinitely negative as $\theta_w(p_w)$ approaches to zero (assuming the first term is bounded). Counterintuitively, in the weak market, where price discrimination lowers the price, the brand rivalness has a *negative* effect on an increase in aggregate output by price discrimination. This is because the uniform price is already very low due to the fierce level of competition and thus there is little room for a price reduction by price discrimination to increase the output in the weak market. The opposite argument holds if the strong market is characterized by a low brand stickiness (i.e., $\theta_s(p_s, c_s) \approx 0$). This implies that, as expected, a fierce level of competition in the strong market has a positive effect on an increase in aggregate output by price discrimination. The rivalness in the strong market keeps the price increase by price discrimination small, and thus a reduction in output in the strong market is also kept small.

Following Holmes (1989), we call the first and the second terms in the left hand side of inequality (2) the *adjusted-concavity part*, and the third and the fourth terms the *elasticity-ratio part*:

$$\underbrace{L_w \cdot \frac{\alpha_w^F - (1-\theta_w)\alpha_w^C}{\theta_w} - L_s \cdot \frac{\alpha_s^F - (1-\theta_s)\alpha_s^C}{\theta_s}}_{\text{Adjusted-concavity}} + \underbrace{\frac{1}{\theta_s} - \frac{1}{\theta_w}}_{\text{Elasticity-ratio}}. \tag{2'}$$

Consider the adjusted-concavity part. A larger α_w^F and/or a smaller α_w^C make a positive $Q'(r)$ more likely. A larger α_w^F means that the firm's own part of the demand in the weak market ($\partial x_{A,w}/\partial p_A$) is more convex ("the output expansion effect"). On the other hand, a smaller α_w^C means that how many of the firm's customers switch to the rival's product as response to the firm's price increase is not so much affected by the current price level ("the countervailing effect"). In this sense, the strategic concerns in the firm's pricing are small. Thus, both a larger α_w^F and a smaller α_w^C indicate that the weak market is competitive. Even if $\partial x_{A,w}/\partial p_A$ is not so convex, a smaller α_w^C (i.e., $\partial x_{Bm}/\partial p_A$ is not responsive to the level of p_B) can substitute it. Here, the intensity of competition, $1/\theta_w$, magnifies both effects, resulting in α_w^F/θ_w and $(1/\theta_w - 1)\alpha_w^C$. A similar argument also holds for α_s^F and α_s^C. In part A of the Appendix, we show that Holmes' (1989) expression for $Q'(r)$ (expression (9) in Holmes 1989, p. 247) is equivalent to expression (2') above.

Now, define $h_m(p) \equiv 1/[q_m'(p)/\pi_m''(p)] > 0$ so that

$$\frac{Q'(r)}{2} = \underbrace{\left(-\frac{q_s' q_w'}{\pi_s'' + \pi_w''}\right)}_{>0} \{h_s[p_s(r)] - h_w[p_w(r)]\}. \tag{3}$$

We assume that h_m is *decreasing* (and call it the *Decreasing Inverse Ratio Condition*; DIRC).[23] It is also shown that

$$\frac{Q''(r)}{2} = \left(-\frac{q_s' q_w'}{\pi_s'' + \pi_w''}\right)[h_s' p_s' - h_w' p_w'] + [h_s - h_w]\frac{d}{dr}\left(-\frac{q_s' q_w'}{\pi_s'' + \pi_w''}\right).$$

Then, there exists $\widehat{r}$ such that $Q'(\widehat{r}) = 0$ and $Q''(\widehat{r})/2 = -[q_s' q_w'/(\pi_s'' + \pi_w'')](h_s' p_s' - h_w' p_w') < 0$ because $h_s' p_s' < 0$ and $h_w' p_w' > 0$. Then, $(1/2)Q(r)$ behaves on $[0, r^*]$ in either manner:[24]

1. If $Q'(0) \leq 0$, then $(1/2)Q(r)$ is monotonically decreasing in r, and as a result $\Delta Q/2 =$

[23] Note that $h_m' < 0$ is equivalent to $\pi_m''' > (\pi_m''/q_m')q_m''$, where $(\pi_m''/q_m') > 0$, because $h_m'(p) = [\pi_m''' q_m' - \pi_m'' q_m'']/[q_m']^2$. Thus, the DIRC states that the profit function, starting from the zero price, increases quickly, attaining the optimal price, and the decreases slowly as p becomes larger and larger beyond the optimum. In this way, the optimal price is reached "close" enough to the zero price, rather than "still climbing up" even far away from it. To see this, if $q_m'' > 0$, then it is necessary to assume $\pi_m''' > 0$. This means that π_m'', which is negative, should be larger, that is, the negative slope of π_m'' should be gentler, as p increases. If $q_m'' \leq 0$, then π_m''' should be, whether it is positive or negative, sufficiently large. In either case, as p increases, π_m increases quickly below the optimum, and decreases slowly beyond it.

[24] This is because the modified version of Aguirre, Cowan and Vickers' (2010, p.1605) Lemma also holds in our oligopoly setting.

$[Q(r^*) - Q(0)]/2 < 0$; *price discrimination lowers aggregate output.*

2. If $Q'(0) > 0$, then $(1/2)Q(r)$ either

   (a) is monotonically increasing (if $Q'(r^*) > 0$, this is true), and as a result, $\Delta Q/2 > 0$; *price discrimination raises aggregate output.*

   (b) first increases, and then after the reaching the maximum (where $Q'(r) = 0$), decreases until $r = r^*$. In this case, *price discrimination may raise or lower aggregate output*: it cannot be determined whether $\Delta Q/2 < 0$ or $\Delta Q/2 > 0$ without further functional and/or parametric restrictions.

Now, we determine the sign of $Q'(0)$. It follows that $\text{sign}[Q'(0)] = \text{sign}[\pi_s''(\overline{p})/q_s'(p) - \pi_w''(\overline{p})/q_w'(p)]$, which implies that $Q'(0) \le 0 \Leftrightarrow \pi_w''(\overline{p})/q_w'(\overline{p}) \ge \pi_s''(\overline{p})/q_s'(\overline{p})$. Note also that $\text{sign}[Q'(r^*)] = \text{sign}[h_s(p_s^*) - h_w(p_w^*)]$, which implies that $Q'(r^*) > 0 \Leftrightarrow \pi_s''(p_s^*)/q_s'(p_s^*) > \pi_w''(p_w^*)/q_w'(p_w^*)$. Because

$$\frac{\pi_m''(\overline{p})}{q_m'(\overline{p})} = \frac{\left[2 - L_m(\overline{p})\alpha_m^F(\overline{p})\right] - [1 - \theta_m(\overline{p})]\left[1 - L_m(\overline{p})\alpha_m^C(\overline{p})\right]}{\theta_m(\overline{p})} \tag{4}$$

holds, the following proposition obtains.

Proposition 1. *Given the DIRC, if $\overline{\theta}_s > \overline{\theta}_w$ and*

$$\frac{\overline{\alpha}_s^F - (1 - \overline{\theta}_s)\overline{\alpha}_s^C}{\overline{\theta}_s} > \frac{\overline{\alpha}_w^F - (1 - \overline{\theta}_w)\overline{\alpha}_w^C}{\overline{\theta}_w}, \tag{5}$$

then price discrimination lowers aggregate output. If $\theta_w^* > \theta_s^*$ and

$$\frac{(\alpha_w^F)^* - (1 - \theta_w^*)(\alpha_w^C)^*}{\theta_w^*} > \frac{(\alpha_s^F)^* - (1 - \theta_s^*)(\alpha_s^C)^*}{\theta_s^*}, \tag{6}$$

then price discrimination raises aggregate output.

Proof. See Appendix, part B. □

It is seen that this proposition is a generalization of Aguirre, Cowan, and Vickers' (2010, p. 1609) Proposition 4, (i) and (ii). If θ_s and θ_w are sufficiently close to one (i.e., monopoly) under either uniform pricing or price discrimination, then inequality (5) is approximated by $\alpha_s^F \ge \alpha_w^F$. Thus, the statement in Aguirre, Cowan, and Vickers' (2010, p. 1609) Proposition 4, (ii), "[i]f demand is convex in the strong market and concave, or less convex, in the weak market then output decreases discrimination" should be started with "[i]f the firm's own part of the demand." This is also true for the complementary case where inequality (6) is approximated by $\alpha_w^F \ge \alpha_s^F$.

More interestingly, equality (3) can be rewritten in the following manner. First, we define *pass-through* in market m by $\rho_m \equiv \frac{\partial p_m}{\partial c_m}$. It is a function of $r \in [0, r^*]$ when the constrained problem is considered. In particular, for $r < r^*$, one obtains $\rho_m[p_m(r)] = \frac{\partial x_{Am}/\partial p_A}{\pi_s'' + \pi_w''}$ by applying the implicit function theorem to $\partial_p \pi_s(p_w + r) + \partial_p \pi_w(p_w) = 0$. For $r = r^*$ (i.e., under price discrimination), pass-through is given by $\rho_m^* = \frac{\partial x_{Am}/\partial p_A}{\pi_m''}$ from $\partial_p \pi_m(p_m) = 0$. Then, if the marginal costs are constant, *quantity pass-through* in market m under price discrimination, which is defined by $\frac{dq_m^*}{d\widetilde{q}}$, where $\widetilde{q}$ is an exogenous amount of output with $\pi_{jm}(p_{jm}, p_{-j,m}) = (p_{jm} - c_m)[x_{jm}(p_{jm}, p_{-j,m}) - \widetilde{q}]$, is expressed by

$$\frac{dq_m^*}{d\widetilde{q}} = q_m'(p_m^*) \cdot \frac{dp_m^*}{d\widetilde{q}} = \frac{q_m'}{\frac{\partial x_{Am}}{\partial p_A}} \cdot \frac{\partial x_{Am}}{\partial p_A} \cdot \frac{dp_m^*}{d\widetilde{q}} = \left(\frac{q_m'}{\frac{\partial x_{Am}}{\partial p_A}} \right) \cdot \left(\frac{\frac{\partial x_{Am}}{\partial p_A}}{\pi_m''} \right) = \theta_m^* \cdot \rho_m^*$$

because the first-order condition with $\widetilde{q}$ indicates $\frac{dp_m^*}{d\widetilde{q}} = \frac{1}{\pi_m''}$.[25] Note here that $q_m'/\pi_m'' = \theta_m^* \cdot \rho_m^*$, which implies that from equality (3),

$$\frac{Q'(r^*)}{2} = \left(-\frac{q_s' q_w'}{\pi_s'' + \pi_w''} \right) \left(\frac{1}{\theta_s^* \rho_s^*} - \frac{1}{\theta_w^* \rho_w^*} \right),$$

whereas for $r < r^*$,

$$\begin{aligned} \frac{Q'(r)}{2} &= \left(\frac{q_w'}{\frac{\partial x_{A,w}}{\partial p_A}} \right) \left(\frac{\partial x_{A,w}}{\partial p_A} \right) p_w' + \left(\frac{q_s'}{\frac{\partial x_{A,s}}{\partial p_A}} \right) \left(\frac{\partial x_{A,s}}{\partial p_A} \right) p_s' \\ &= \theta_w \left(\frac{\frac{\partial x_{A,w}}{\partial p_A}}{\pi_s'' + \pi_w''} \right) (\pi_s'' + \pi_w'') p_w' + \theta_s \left(\frac{\frac{\partial x_{A,s}}{\partial p_A}}{\pi_s'' + \pi_w''} \right) (\pi_s'' + \pi_w'') p_s' \\ &= \underbrace{(-\pi_s'' \pi_w'')}_{<0} \left(\frac{\theta_w(r) \rho_w(r)}{\pi_w''} - \frac{\theta_s(r) \rho_s(r)}{\pi_s''} \right). \end{aligned}$$

Summarizing these arguments, we obtain the following proposition.

Proposition 2. *Given the DIRC, if $\theta_w^* \rho_w^* \geq \theta_s^* \rho_s^*$ holds, the price discrimination raises aggregate output. Conversely, if*

$$\frac{\overline{\theta}_w \overline{\rho}_w}{\overline{\theta}_s \overline{\rho}_s} \leq \frac{\pi_w''(\overline{p})}{\pi_s''(\overline{p})}$$

holds, then price discrimination lowers aggregate output.

As stated above, $\theta_m^* \rho_m^*$ is interpreted as quantity pass-through under price discrimination in market m if the marginal cost is constant. Thus, the first part of the proposition simply claims

[25] Note that this is the case where $\frac{dq_m^*}{d\widetilde{q}}$ is evaluated at $\widetilde{q} = 0$: Miklós-Thal and Shaffer (2019b) derive a general formula for $\widetilde{q} > 0$, correcting Weyl and Fabinger's (2013) arguments. If the marginal costs are non-constant, then $\pi_{jm}(p_{jm}, p_{-j,m}) = p_{jm} \cdot [x_{jm}(p_{jm}, p_{-j,m}) - \widetilde{q}] - c_m[x_{jm}(p_{jm}, p_{-j,m}) - \widetilde{q}]$ should be considered, where $c_m(\cdot)$ is the cost function, and thus $\theta_m^* \rho_m^*$ is no longer the quantity pass-through under price discrimination (that is, when $\widetilde{q} = 0$). See Weyl and Fabinger (2013, p. 572) for a precise expression of quantity pass-through with non-constant marginal costs.

that aggregate output is raised by price discrimination if the marginal reduction in quantity caused by a small deviation from price discrimination in the market where price discrimination raises output (i.e., the weak market) is larger than the marginal increase in quantity in the strong market. The second part describes the opposite case, although it does not permit a direct comparison because pass-through is not defined market-wise unless the pricing regime is "perfect" or "full" price discrimination (i.e., $r = r^*$), where the first-order conditions are given market-wise. Note that if $|\pi''_m|$ is small, then π_m is "flat," and thus the price shift $|\Delta p_m|$ in response to some change would be large. Hence, the role of π''_w/π''_s is to adjust measurement units for ρ_w/ρ_s. For example, if $|\pi''_w|$ is very small, then the ρ_w is "over represented," and thus it should be "penalized" so that right hand side of the inequality in the proposition becomes small.

3.2 Social Welfare

Now, we study the effects of allowing third-degree price discrimination on social welfare. To proceed further, note that

$$\begin{aligned}\frac{W'(r)}{2} &= \underbrace{\left(-\frac{\pi''_s\pi''_w}{\pi''_s+\pi''_w}\right)}_{>0} \\ &\quad \times \left(\frac{(p_w(r)-c_w)q'_w[p_w(r)]}{\pi''_w} - \frac{(p_s(r)-c_s)q'_s[p_s(r)]}{\pi''_s}\right).\end{aligned}$$

We follow Aguirre, Cowan, and Vickers (2010, p. 1605), who define $z_m(p) \equiv (p-c_m)q'_m(p)/\pi''_m(p)$, which is "the ratio of the marginal effect of a price increase on social welfare to the second derivative of the profit function." However, our q'_m and π''_m have *strategic effects*. More specifically, our q'_m and π''_m are written as

$$q'_m(p_m) = \underbrace{\frac{\partial x_{Am}}{\partial p_A}(p_m, p_m)}_{<0 \text{ (ACV's } q'_m)} + \underbrace{\frac{\partial x_{Bm}}{\partial p_B}(p_m, p_m)}_{>0 \text{ (strategic)}}$$

and

$$\pi''_m(p_m, c_m) = \underbrace{D^2\pi_m(p_m, c_m)}_{\text{ACV's } \pi''_m} + \underbrace{G_m(p_m, c_m)}_{\text{strategic}},$$

where

$$G_m(p) = \frac{\partial x_{Am}}{\partial p_B}(p,p) + (p-c_m)\left[\frac{d}{dp}\left(\frac{\partial x_{Am}}{\partial p_A}(p,p)\right) - \frac{\partial^2 x_{Am}}{\partial p_A^2}(p,p)\right].$$

As in Aguirre, Cowan, and Vickers (2010, p. 1605), we can write

$$\frac{W'(r)}{2} = \underbrace{\left(-\frac{\pi_s''\pi_w''}{\pi_s''+\pi_w''}\right)}_{>0}\{z_w[p_w(r)] - z_s[p_s(r)]\},$$

and their lemma also holds in our case of oligopoly if we assume z_m is *increasing* (the *Increasing Ratio Condition*; IRC).[26] It is also shown that

$$\frac{W''(r)}{2} = \left(-\frac{\pi_s''\pi_w''}{\pi_s''+\pi_w''}\right)(z_w'p_w' - z_s'p_s') + (h_w - h_s)\frac{d}{dr}\left(-\frac{\pi_s''\pi_w''}{\pi_s''+\pi_w''}\right).$$

Then, as $(1/2)Q(r)$ does, $(1/2)W(r)$ behaves on $[0, r^*]$ in either manner:[27]

1. If $W'(0) \leq 0$, then $(1/2)W(r)$ is monotonically decreasing in r, and as a result $\Delta W/2 = [W(r^*) - W(0)]/2 < 0$; *price discrimination lowers social welfare.*

2. If $W'(0) > 0$, then $(1/2)W(r)$ either

   (a) is monotonically increasing (if $W'(r^*) > 0$, this is true), and as a result, $\Delta W/2 > 0$; *price discrimination raises social welfare.*

   (b) first increases, and then after the reaching the maximum (where $W'(r) = 0$), decreases until $r = r^*$. In this case, *price discrimination may raise or lower social welfare*: it cannot be determined whether $\Delta W/2 < 0$ or $\Delta W/2 > 0$ without further functional and/or parametric restrictions.

Now, we determine the sign of $W'(0)$. First, define the *markup* in market m by $\mu_m(p) \equiv p - c_m$. Then, it follows that $\text{sign}[W'(0)] = \text{sign}[\mu_w(\overline{p})q_w'(\overline{p})/\pi_w''(\overline{p}) - \mu_s(\overline{p})q_s'(\overline{p})/\pi_s''(\overline{p})]$, and thus, the following proposition is obtained.

Proposition 3. *Given the IRC, if the markup in strong market relative to the weak market at the uniform price $\overline{p}$ is sufficiently large, i.e.,*

$$\overline{\mu}_s\frac{q_s'(\overline{p})}{\pi_s''(\overline{p})} \geq \overline{\mu}_w\frac{q_w'(\overline{p})}{\pi_w''(\overline{p})}, \tag{7}$$

then price discrimination lowers social welfare.

[26]Note that $z_m'(p) = \{[(p-c_m)q_m''(p) + q_m'(p)]\pi_m''(p) - (p-c_m)q_m'(p)\pi_m'''(p)\}/[\pi_m''(p)]^2$ and thus, the IRC is equivalent to $[(p-c_m)q_m''(p) + q_m'(p)]\pi_m''(p) > (p-c_m)q_m'(p)\pi_m'''(p)$. Appendix B of Agguire, Cowan and Vickers (2010) discusses sufficient conditions for the IRC in the case of monopoly. If h_m is decreasing, as we assume throughout, then z_m is increasing because $z_m'(p) = [1 - z_m(p)h_m'(p)]/h_m(p)$ so that z_m' is positive if h_m' is negative. That is, the DIRC is a *sufficient* condition for the IRC to hold.

[27]This is because the modified version of Aguirre, Cowan and Vickers' (2010, p. 1605) Lemma also holds in our oligopoly setting.

If there are *no strategic effects* (i.e., $\partial x_{Bm}/\partial p_A = 0$ or $\theta_m(p) = 1$), then $\pi''_m(\overline{p})/q'_m(\overline{p}) = 2 - L_m(\overline{p})\overline{\alpha}^F_m$, and inequality (7) above reduces to $\overline{\mu}_s/\overline{\mu}_w \geq [2 - L_s(\overline{p})\overline{\alpha}^F_s]/[2 - L_w(\overline{p})\overline{\alpha}^F_w]$. On the other hand, if there are *no cost differentials* (i.e., $c_s = c_w \equiv c$), then inequality (7) above reduces to $\pi''_w(\overline{p})/q'_w(\overline{p}) \geq \pi''_s(\overline{p})/q'_s(\overline{p})$ because the markups are the same in the two markets. Thus, if there are no strategic effects and no cost differentials, then inequality (7) coincides with Aguirre, Cowan and Vickers' (2010, p. 1605) Proposition 1 ($\overline{\alpha}^F_s \geq \overline{\alpha}^F_w$ in our notation; in their notation, $\alpha_s(\overline{p}) \geq \alpha_w(\overline{p})$) because $L_s(\overline{p}) = L_w(\overline{p})$. That is, the firm's "direct demand function in the strong market is at least as convex as that in the weak market at the nondiscriminatory price" (Aguirre, Cowan and Vickers 2010, p. 1602).

Recall that in our case of oligopoly, equality (4) holds, which leads to the following corollary, another expression for the sufficient condition for price discrimination to lower social welfare in the case of no cost differentials.

Corollary 1. *Suppose there are no cost differentials across markets ($c_s = c_w$). Given the IRC, if $\overline{\theta}_w \geq \overline{\theta}_s$ and*

$$\frac{\overline{\alpha}^F_s - (1 - \bar{\theta}_s)\bar{\alpha}^C_s}{\bar{\varepsilon}^I_s} \geq \frac{\overline{\alpha}^F_w - (1 - \bar{\theta}_w)\bar{\alpha}^C_w}{\bar{\varepsilon}^I_w},$$

then price discrimination lowers social welfare.

This is because

$$\begin{aligned} & \frac{\pi''_s(\overline{p})}{q'_s(\overline{p})} - \frac{\pi''_w(\overline{p})}{q'_w(\overline{p})} \leq 0 \\ \Leftrightarrow \quad & \overline{L} \cdot \left(\frac{\overline{\alpha}^F_w - (1 - \bar{\theta}_w)\overline{\alpha}^C_w}{\bar{\theta}_w} - \frac{\overline{\alpha}^F_s - (1 - \bar{\theta}_s)\overline{\alpha}^C_s}{\bar{\theta}_s} \right) + \left(\frac{1}{\bar{\theta}_s} - \frac{1}{\bar{\theta}_w} \right) \leq 0, \end{aligned}$$

where $\overline{L} \equiv L_s(\overline{p}) = L_w(\overline{p})$.

Now, using market power, markup, and pass-through, we obtain the following sufficient conditions for price discrimination to raise or lower social welfare.

Proposition 4. *Given the IRC, if $\theta^*_w\mu^*_w\rho^*_w \geq \theta^*_s\mu^*_s\rho^*_s$ holds, then price discrimination raises social welfare. Conversely, if*

$$\frac{\overline{\theta}_w\overline{\mu}_w\overline{\rho}_w}{\overline{\theta}_s\overline{\mu}_s\overline{\rho}_s} \leq \frac{\pi''_w(\overline{p})}{\pi''_s(\overline{p})}$$

holds, then price discrimination lowers social welfare.

Proof. See Appendix, part C. □

Roughly speaking, if either (i) *market power* (θ), (ii) *markup* (μ), or (iii) *pass-through* (ρ) is sufficiently *small* in the *strong* market, then social welfare is likely to be higher under price discrimination. In particular, if these three measures are calculated (or estimated) in each separate market, then it would assist one to judge whether price discrimination is desirable from a society's viewpoint. As explained above, if the marginal costs are constant, $\theta^*_m \rho^*_m$ is interpreted as *quantity* pass-through: $\mu^*_m \times \theta^*_m \rho^*_m$ approximates the trapezoid generated by a small deviation from (perfect) price discrimination that captures the marginal welfare gain in the strong market and the marginal welfare loss in the weak market. If the latter is larger than the former, such a deviation lowers social welfare, and owing to the IRC, this argument extends globally so that the regime switch to uniform pricing definitely lowers social welfare. Note that this comparison will be a little bit more involved when starting at uniform pricing for the same reason as explained after Proposition 2.

Even if there are no cost differentials (i.e., $c_s = c_w$), this expression cannot be further simplified. In other words, this expression is already robust to the inclusion of cost differentials. Now, if we further assume that there are no strategic effects (i.e., $\theta_m = 1$), then the above condition becomes $(p^*_s - c)/(p^*_w - c) \leq (1/\rho^*_s)/(1/\rho^*_w)$, which coincides with $(p^*_w - c)/[2-(\sigma^I_w)^*] \geq (p^*_s - c)/(2 - (\sigma^I_s)^*]$ in Proposition 2 of Aguirre, Cowan and Vickers (2010, p. 1606), where $\sigma^I_m(q) \equiv -qp''/p'$ is the *curvature of the industry's inverse demand function* (in symmetric pricing), because it is shown that $\rho^*_m = 1/(2 - \sigma^*_m)$ in our oligopoly setting as well. Thus, price discrimination raises social welfare "if the discriminatory prices are not far apart and the inverse demand function in the weak market is locally more convex than that in the strong market" (Aguirre, Cowan and Vickers 2010, p. 1602).

This proposition also has the following attractive feature. Suppose that price discrimination is being conducted. Then, to evaluate it from a viewpoint of social welfare, one only needs the local information: first, θ^*_m, μ^*_m and ρ^*_m for each $m = s, w$, are computed, and if the sufficient condition above is satisfied, then the ongoing price discrimination is justified. In addition, to compute θ^*_m, μ^*_m and ρ^*_m in equilibrium, *information on the marginal cost is unnecessary*: once a specific form of demand function in market m for firm j, $q_{jm} = x_{jm}(p_{jm}, p_{-j,m})$ is provided (and if the IRC is satisfied), then the three variables are computed in the following manner: $\theta^*_m = 1 - (\varepsilon^C_m)^*/(\varepsilon^F_m)^*$, $\mu^*_m = p^*_m/(\varepsilon^F_m)^*$, and $\rho^*_m = 1/(2 - \sigma^*_m) = [q'_m(p^*_m)]^2/(2[q'_m(p^*_m)]^2 - q_m(p^*_m)q''_m(p^*_m))$.[28] Thus, if the firm's demand for each market m is estimated and the discriminatory price p^*_m is observed, then one can easily compute θ^*_m, μ^*_m, and ρ^*_m, using up to the second-order demand characteristics.[29]

To the best of our knowledge, this proposition is the most general statement on when allowing oligopolistic firms to price discriminate raises or lowers social welfare, allowing cost differentials as Chen and Schwartz (2015) and Chen, Li, and Schwartz (2019) do for monopoly and oligopoly, respectively. More realistically, different firms would have different values for

[28] An alternative expression for μ^*_m if the cost information is used is $\mu^*_m = c_m/[(\varepsilon^F_m)^* - 1]$.

[29] It should be emphasized that the second-order supply property, i.e., the derivative of the marginal cost, would be necessary if non-constant marginal cost is allowed, as suggested by Adachi and Fabinger (2019) in the context of issues of general "taxation" (pure taxation and other additional costs from external changes).

θ^*_m, μ^*_m and ρ^*_m. This case is analyzed by allowing firm heterogeneity in Section 4.

3.3 Parametric Examples

To consider the following parametric examples, we assume that there are two (strong and weak), and two symmetric firms.

Example 1: *Linear Demands* Linear demands are derived from the quadratic utility of the representative consumer in market m under symmetric product differentiation:

$$U_m(\mathbf{q}_m) = \omega_m \cdot (q_{Am} + q_{Bm}) - (1/2)\left(\beta_m [q_{Am}]^2 + 2\gamma_m q_{Am} q_{Bm} + \beta_m [q_{Bm}]^2\right),$$

which yields linear inverse demands, $p_{jm}(q_{jm}, q_{-j,m}) = \omega_m - \beta_m q_{jm} - \gamma_m q_{-j,m}$, and the corresponding direct demand in market m is

$$x_{jm}(p_{jm}, p_{-j,m}; \omega_m, \beta_m, \gamma_m) = \frac{1}{(1-\delta_m^2)\beta_m}\left[\omega_m(1-\delta_m) - p_{jm} + \delta_m p_{-j,m}\right]$$

for firm j, where $\delta_m \equiv \gamma_m/\beta_m \in [0,1)$ is the *strength of substitutability*: if δ_m is close to one, market m is approximated by perfect competition, while each firm behaves as a monopolist if δ_m is equal to zero. In symmetric equilibrium, the firm's demand in market m is $q_m(p) = (\omega_m - p)/(1+\delta_m)\beta_m$ (and thus, $q'_m(p) = -1/(1+\delta_m)\beta_m$ and $\varepsilon^I_m(p) = p/(\omega_m - p)$) and the firm's own price elasticity in market m is

$$\begin{aligned} \varepsilon^F_m(p) &= -\frac{(1+\delta_m)\beta_m}{\omega_m - p} p \left(-\frac{1}{(1-\delta_m^2)\beta_m}\right) \\ &= \frac{p}{(1-\delta_m)(\omega_m - p)}, \end{aligned}$$

which implies that the discriminatory price in market m satisfies:

$$\begin{aligned} \frac{p^*_m - c_m}{p^*_m} &= \frac{(1-\delta_m)(\omega_m - p^*_m)}{p^*_m} \\ \Leftrightarrow \quad p^*_m = p^*_m(c_m, \omega_m, \delta_m) &\equiv \frac{(1-\delta_m)\omega_m + c_m}{2-\delta_m}. \end{aligned}$$

and thus, $\rho^*_m \equiv (p^*_m)'(c_m) = 1/(2-\delta_m)$. Next, consider the uniform price. In symmetric equilibrium,

$$\bar{y}_m \bar{\varepsilon}^F_m = \frac{\frac{\omega_m - \overline{p}}{(1+\delta_m)\beta_m}}{\sum_{m=s,w} \frac{\omega_m - \overline{p}}{(1+\delta_m)\beta_m}} \frac{\overline{p}}{(1-\delta_m)(\omega_m - \overline{p})}$$

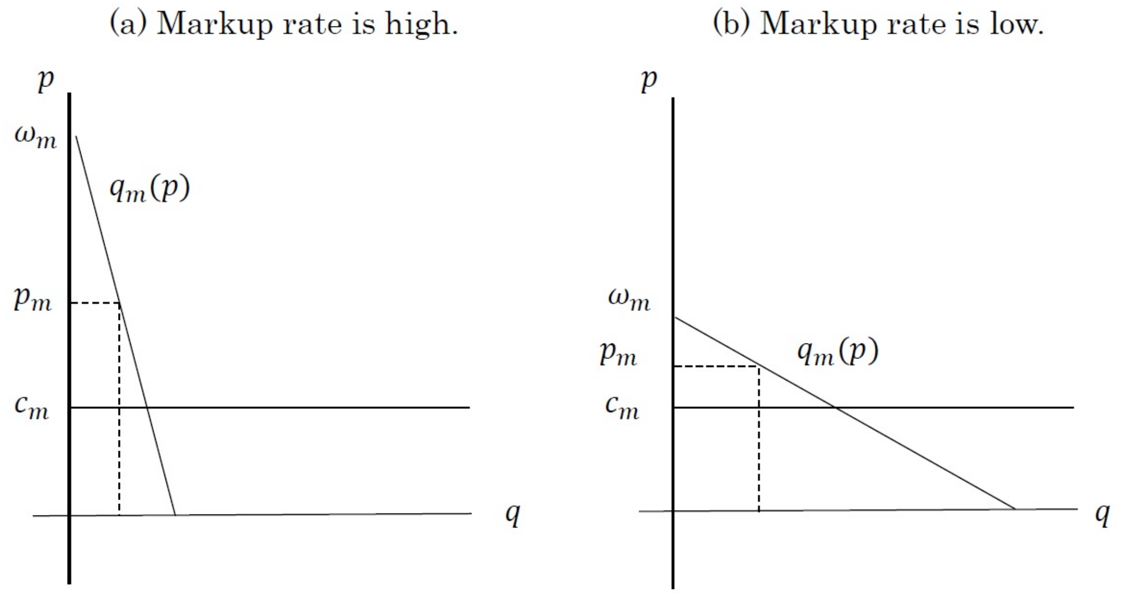

Figure 1: θ_m with linear demands. In both cases, $(p_m - c_m)/(\omega_m - p_m)$ is almost identical.

$$= \frac{\frac{\overline{p}}{(1-\delta_m^2)\beta_m}}{\sum_{m=s,w} \frac{\omega_m - \overline{p}}{(1+\delta_m)\beta_m}}$$

for $m = s$, w, which implies that the equilibrium uniform price $\overline{p}$ satisfies

$$\sum_{m=s,w} \frac{\overline{p} - c_m}{(1-\delta_m^2)\beta_m} = \sum_{m=s,w} \frac{\omega_m - \overline{p}}{(1+\delta_m)\beta_m},$$

leading to the explicit solution:

$$\overline{p} = \overline{p}(\mathbf{c}, \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\delta}) \equiv \frac{\sum_{m=s,w} \frac{(1-\delta_m)\omega_m + c_m}{(1-\delta_m^2)\beta_m}}{\sum_{m=s,w} \frac{2-\delta_m}{(1-\delta_m^2)\beta_m}}.$$

The market power index in the case of linear demands is $\theta_m(p_m) = (p_m - c_m)/(\omega_m - p_m)$. In each panel of Figure 1, the demand curve is depicted so that the ratio of $(p_m - c_m)$ to $(\omega_m - p_m)$ takes almost the same value. However, in the left panel, the markup rate is high. This is mainly due to a low elasticity of the industry's demand rather than a low cross-price elasticity: the rivalness between brands may be sufficiently high. In the right panel, the markup rate is low. However, this does not necessarily mean that brands are in a strong rivalry. Instead, $\varepsilon_m^C(p_m)$ can be low as long as $\varepsilon_m^I(p_m)$ is sufficiently high (that $\varepsilon_m^F(p_m)$ is high, and as a result, the markup rate is low). In both panels, the market is close to monopoly because $\theta_m(p_m)$ is close to one. This graphical example shows that $\theta_m(p_m)$, rather than $L_m(p_m)$, better captures the competitiveness in market m. Because both the curvature of the firm's direct demand and the elasticity of the cross-price effect are zero ($\alpha_m^F(p) = 0$ and $\alpha_m^C(p) = 0$), it is observed that

$$\begin{aligned}
\frac{\pi_m''(p)}{q_m'(p)} &= \frac{\left[2 - L_m(p)\alpha_m^F(p)\right] - (1-\theta_m(p))\left[1 - L_m(p)\alpha_m^C(p)\right]}{\theta_m(p)} \\
&= \frac{1+\theta_m(p)}{\theta_m(p)} \\
&= \frac{\omega_m - c_m}{p - c_m} \\
&= \frac{2-\delta_m}{1-\delta_m},
\end{aligned}$$

which implies that $Q'(r) > 0$ if and only if $\delta_s > \delta_w$, that is, the firms' products are less differentiated in the strong market.

Now, consider the *sufficient* condition for price discrimination to *lower* social welfare in Proposition 3: $G(c_s, c_w) \geq 0$, where

$$\begin{aligned}
G(c_s, c_w) &= \frac{\mu_s[\overline{p}(c_s, c_w)]}{\mu_w[\overline{p}(c_s, c_w)]} - \frac{\pi_s''[\overline{p}(c_s, c_w)]/q_s'[\overline{p}(c_s, c_w)]}{\pi_w''[\overline{p}(c_s, c_w)]/q_w'[\overline{p}(c_s, c_w)]} \\
&= \frac{\mu_s[\overline{p}(c_s, c_w)]}{\mu_w[\overline{p}(c_s, c_w)]} - \frac{(2-\delta_s)(1-\delta_w)}{(1-\delta_s)(2-\delta_w)}.
\end{aligned}$$

Similarly, consider the *sufficient* condition for price discrimination to *raise* social welfare in Proposition 4: $H(c_s, c_w) \leq 0$ where

$$\begin{aligned}
H(c_s, c_w) &= \frac{\mu_s[p_s^*(c_s)]}{\mu_w[p_w^*(c_w)]} - \frac{1/\theta_s[p_s^*(c_s)]\rho_s[p_s^*(c_s)]}{1/\theta_w[p_w^*(c_w)]\rho_w[p_w^*(c_w)]} \\
&= \frac{\mu_s[p_s^*(c_s)]}{\mu_w[p_w^*(c_w)]} - \frac{(2-\delta_s)(1-\delta_w)}{(1-\delta_s)(2-\delta_w)}
\end{aligned}$$

because $\theta_m^* \equiv \theta_m[p_m^*(c_m)] = 1 - \delta_m$. In the case of linear demands with no product differentiation (i.e., monopoly: $\gamma_m = 0$ and hence $\delta_m = 0$),

$$G(c_s, c_w) = \frac{\overline{p} - c_s}{\overline{p} - c_w} - 1$$

so that $G(c_s, c_w) \geq 0 \Leftrightarrow c_w \geq c_s$. Similarly,

$$\begin{aligned}
H(c_s, c_w) &= \frac{p_s^* - c_s}{p_w^* - c_w} - \frac{\theta_w(p_w^*)}{\theta_s(p_s^*)} \\
&= \frac{p_s^* - c_s}{p_w^* - c_w} - \frac{p_w^* - c_w}{p_s^* - c_s} \cdot \frac{\omega_s - p_s^*}{\omega_w - p_w^*}
\end{aligned}$$

so that $H(c_s, c_w) \leq 0 \Leftrightarrow \omega_s - \omega_w \leq c_s - c_w$. In this case of monopoly under linear demands, Chen and Schwartz (2015, p. 454) obtain a *necessary and sufficient* for price discrimination to raise social welfare: in our notation, it is $\omega_s - \omega_w \leq 3(c_s - c_w)$ (with $\gamma_s = 0$ and $\gamma_w = 0$). As Panel (a) in Figure 2 shows, our sufficient condition, $c_s \geq c_w$, for price discrimination to raise social welfare in the case of monopoly is weaker than Chen and Schwartz' (2015, p. 454) necessary and sufficient condition. Under oligopoly, however, the (c_s, c_w) region for sufficiency for $\Delta W \geq 0$, i.e, the region of $H(c_s, c_w) \leq 0$ is now smaller, whereas the region for sufficiency for $\Delta W \leq 0$, i.e, the region of $G(c_s, c_w) \geq 0$ becomes larger (Panel b).

In this numerical example, the line of $c_w = c_s$ is included in the region of $G(c_s, c_w) \geq 0$; price discrimination lowers social welfare when the marginal costs are common across markets. However, it is possible that social welfare is higher under price discrimination in this case, more specifically, if $\delta_s \equiv \gamma_s/\beta_s$ is sufficiently higher than $\delta_w \equiv \gamma_w/\beta_w$, as shown in Figure 3, where $c_s = c_w = 0.2$ and G and H are interpreted as $G(\delta_s, \delta_w)$ and $H(\delta_s, \delta_w)$, respectively. This example is consistent with Adachi and Matsushima's (2014, p.1239) Proposition 1 (and their Figures 4 and 5) with an additional result: here, the sufficient condition for $\Delta W \leq 0$ is also included, whereas their Proposition 1 establishes a necessary and sufficient condition for $\Delta W > 0$ in the case of linear demands.

Example 2: *Logit Demands* In each market $m = 1, 2$, firm j faces the following market share:

$$x_{jm}(p_{jm}, p_{-j,m}) = \frac{\exp(\omega_m - \beta_m p_{jm})}{1 + \sum_{j'=A,B} \exp(\omega_m - \beta_m p_{j'm})} \in (0, 1),$$

where $\omega_m > 0$ is now the product-specific utility and $\beta_m > 0$ is the *responsiveness* of the representative consumer in market m to the price.[30] Then, under symmetric pricing, each firm's share is

$$q_m(p) = \frac{\exp(\omega_m - \beta_m p)}{1 + 2\exp(\omega_m - \beta_m p)}$$

and the symmetric discriminatory equilibrium price $p_m^* = p_m^*(c_m, \omega_m, \beta_m)$ satisfies

$$\underbrace{p_m^* - c_m}_{=\mu_m^*} - \frac{1}{\beta_m(1 - q_m^*)} = 0,$$

where $q_m^* \equiv q_m^*(p_m^*)$:

$$q_m^*(p_m^*) = \frac{\exp(\omega_m - \beta_m p_m^*)}{1 + 2\exp(\omega_m - \beta_m p_m^*)}.$$

Both p_m^* and q_m^* should be jointly solved numerically. Now, it is shown that

$$H(c_s, c_w) = \left(\frac{1 - q_w^*(c_w)}{1 - q_s^*(c_s)}\right)\left(\frac{\beta_w}{\beta_s} - \frac{1 - 2q_w^*(c_w)}{1 - 2q_s^*(c_s)} \cdot \frac{1 - 3q_s^*(c_s) + 3[q_s^*(c_s)]^2}{1 - 3q_w^*(c_w) + 3[q_w^*(c_w)]^2}\right),$$

[30] The indirect utility of the representative consumer in market m is given by $V_m(\mathbf{p}_m) = \ln\left[\exp\left(\omega_m - \beta_m p_{Am}\right) + \exp\left(\omega_m - \beta_m p_{Bm}\right)\right]/\beta_m$ (Anderson, de Palma, and Thisse 1987).

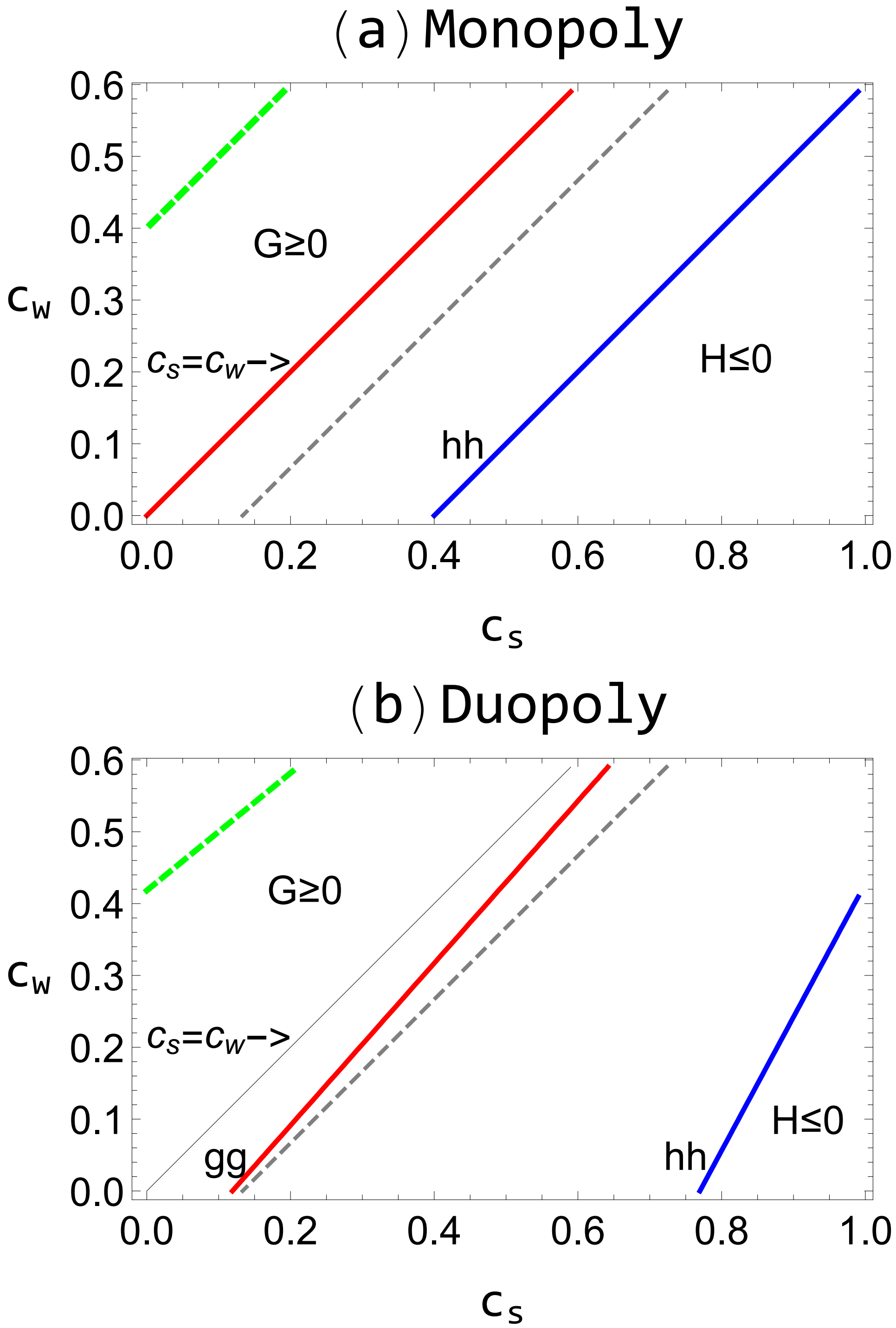

Figure 2: Linear demands with $\omega_s = 1.2$, $\omega_w = 0.8$, $\beta_s = 1.2$, and $\beta_w = 1.4$. For (a), $\gamma_s = \gamma_w = 0$ (monopoly), whereas for (b), $\gamma_s = 0.2$ and $\gamma_w = 0.7$ (duopoly). For p_w^* to be actually lower than p_s^*, c_w, relative to c_s, must be sufficiently small. Specfically, the region of (c_s, c_w) is restricted to the area below the dashed thick line in the upper left. In each panel, the dashed line corrresponds to Chen and Schwarz' (2015) thereshold for the necessity and sufficiency for $\Delta W \geq 0$ in the case of monopoly. The region for $H \leq 0$ is the area below line hh. In panel (a), the region for $G \geq 0$ is the area between the dashed thick line and line $c_s = c_w$, whereas it is the area between the dashed thick line and line gg in panel (b).

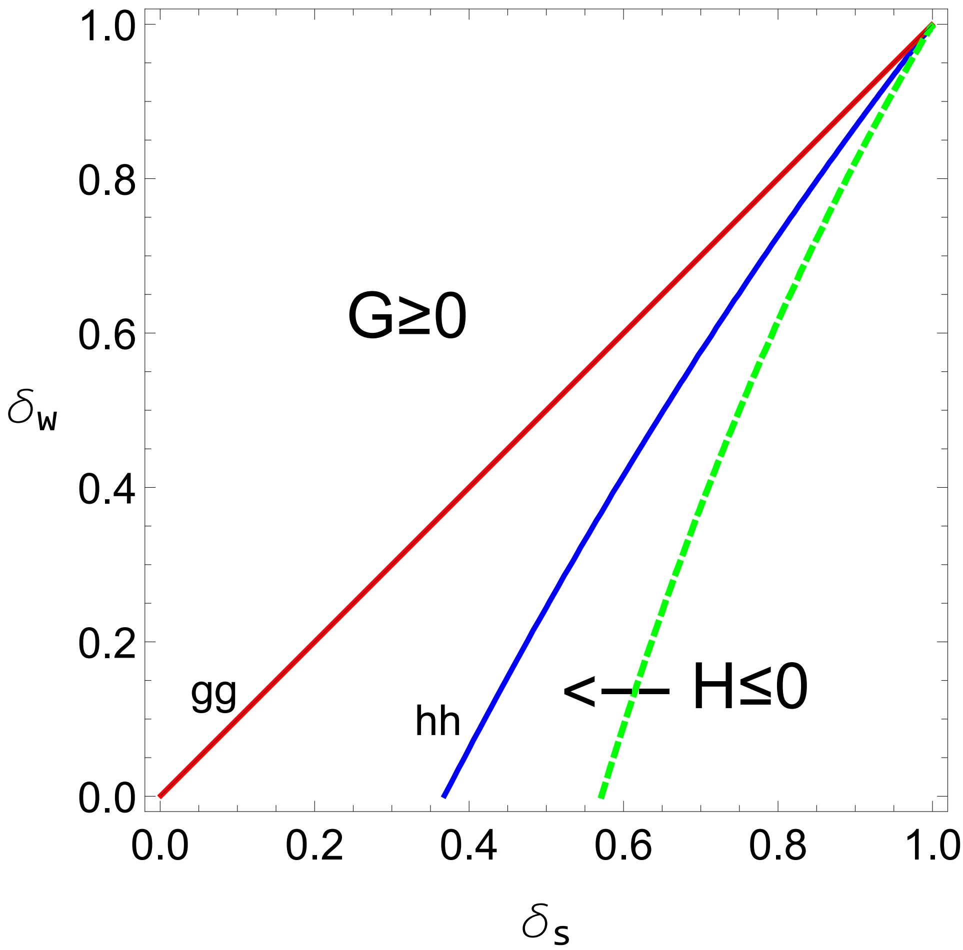

Figure 3: Linear demands with $\omega_s = 1.2$, $\omega_w = 0.8$, $\beta_s = 1.2$, and $\beta_w = 1.4$. It assumed that $c_s = c_w = 0.2$. As in Figure 2, the region of (δ_s, δ_w) is restricted to the area above the dashed thick line in the lower right for p_s^* to be actually higher than p_w^*. The region for $H \leq 0$ is the area between the dashed thick line and line hh, whereas the region for $G \geq 0$ is the area above line gg.

because $(\varepsilon_m^F)^* = \beta_m p_m^*(1 - q_m^*)$, and $(\varepsilon_m^C)^* = \beta_m p_m^* q_m^*$, and thus,

$$\theta_m^* = 1 - \frac{(\varepsilon^C)^*}{(\varepsilon^F)^*} = \frac{1 - 2q_m^*}{1 - q_m^*},$$

and

$$\rho_m^* = \frac{1}{1 - \dfrac{q_m^*(1 - 2q_m^*)}{(1 - q_m^*)^2}}.$$

Note here that in contrast to the linear demand, the two curvatures are positive: $(\alpha_m^F)^* = (\alpha_m^C)^* = \beta_m p_m^*(1 - 2q_m^*) > 0$. That is, the logit demand is *convex* in both own and cross directions.

It is also verified that

$$G(c_s, c_w) = \left(\frac{1 - q_w[\overline{p}(c_s, c_w)]}{1 - q_s[\overline{p}(c_s, c_w)]}\right)\left(\frac{\beta_w}{\beta_s}\right) - \frac{\pi_s''[\overline{p}(c_s, c_w)]/q_s'[\overline{p}(c_s, c_w)]}{\pi_w''[\overline{p}(c_s, c_w)]/q_w'[\overline{p}(c_s, c_w)]},$$

where, with $\alpha_m(p) \equiv \beta_m p[1 - 2q_m(p)]$ being the common value for the curvatures,

$$\begin{aligned}
\frac{\pi_m''(p)}{q_m'(p)} &= \frac{1 + \theta_m(p)}{\theta_m(p)} - L_m(p)\alpha_m(p) \\
&= \frac{2 - 3q_m(p)}{1 - 2q_m(p)} - \beta_m(p - c_m)[1 - 2q_m(p)].
\end{aligned}$$

The equilibrium uniform price $\overline{p} = \overline{p}(\mathbf{c}, \boldsymbol{\omega}, \boldsymbol{\beta})$ satisfies

$$\sum_{m=s,w} q_m(\overline{p})\left\{1 - \beta_m(\overline{p} - c_m)[1 - q_m(\overline{p})]\right\} = 0,$$

which should also be numerically solved.

Here, to see the role of the demand curvatures, we study the region of (β_s, β_w), with a fixed value of $c_s = c_w = 0.2$. Figure 4 shows that a higher value of β_w, relative to β_s, that is, a higher degree of convexity in the weak market, is associated with a *negative* change in social welfare by price discrimination. This result firstly appears not to be consistent with Aguirre, Cowan, and Vickers (2010), who emphasize that as the demand in the weak market becomes more convex, it is more likely that price discrimination raises social welfare because a larger increase in output in the weak market offsets the misallocation effect due to price discrimination. However, Aguirre, Cowan, and Vickers' (2010) result holds if the demand in the strong market is concave. Here, the demand in the strong market is also convex. In this case, the uniform price is kept relatively low; thus an introduction of price discrimination again highlights the misallocation effect.

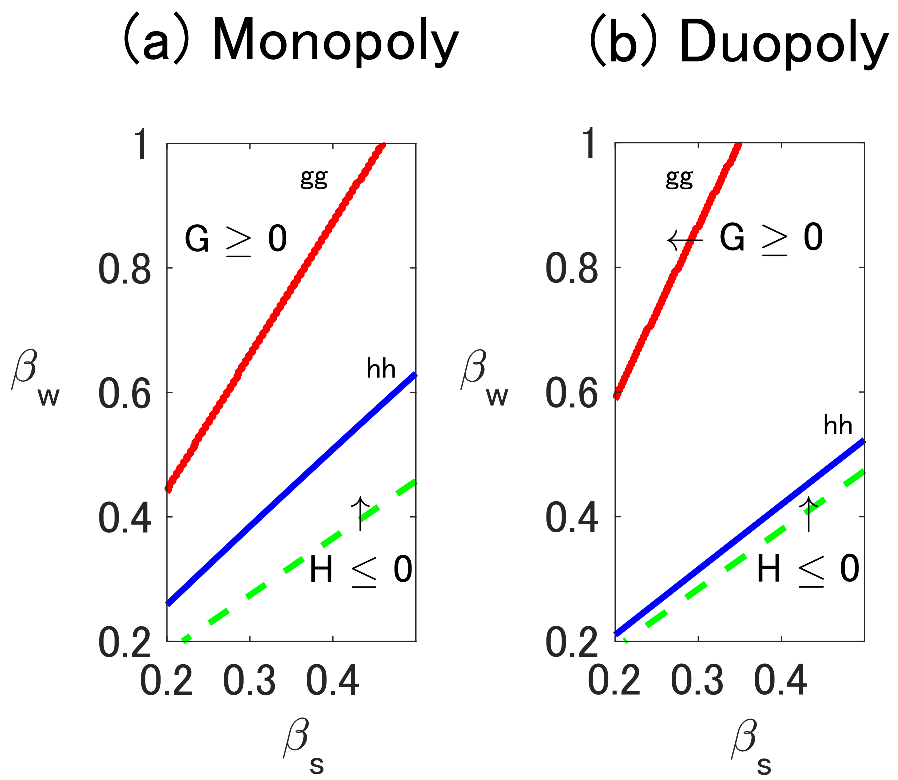

Figure 4: Logit demands with $\omega_s = 1.2$, $\omega_w = 0.8$, and $c_s = c_w = 0.2$. Panels (a) and (b) are monopoly and duopoly, respectively. As in Figures 2 and 3, the region of (β_s, β_w) is restricted to the area above the dashed thick curve in the lower right for p_s^* to be actually higher than p_w^*. The region for $G \geq 0$ is the area above curve gg, whereas the region for $H \leq 0$ is the area between the dashed thick curve and curve hh.

3.4 Consumer Surplus

One can extend the analysis above to study consumer surplus. First, consumer surplus is defined by replacing c_m in $W(r)$ by $p_m(r)$ to define

$$CS(r) = U_s(q_s[p_s(r)]) + U_w(q_w[p_w(r)]) - 2p_s(r)\cdot q_s[p_s(r)] - 2p_w(r)\cdot q_w[p_w(r)],$$

which implies that

$$\begin{aligned}
\frac{CS'(r)}{2} &= p_s(r)\cdot q_s'\cdot p_s'(r) + p_w(r)\cdot q_w'\cdot p_w'(r)\\
&\quad -p_s'(r)(p_s(r)\cdot q_s' + q_s) - p_w'(r)(p_w(r)\cdot q_w' + q_w)\\
&= -(p_s'(r)q_s + p_w'(r)q_w)\\
&= \left(-\frac{\pi_s''\pi_w''}{\pi_s'' + \pi_w''}\right)\left(\frac{q_s}{\pi_s''} - \frac{q_w}{\pi_w''}\right).\\
&= \underbrace{\left(-\frac{\pi_s''\pi_w''}{\pi_s'' + \pi_w''}\right)}_{>0}\{g_s[p_s(r)] - g_w[p_w(r)]\}\,,
\end{aligned}\tag{8}$$

where $g_m(p) \equiv q_m(p)/\pi_m''(p)$. If g_m is assumed to be *decreasing*, then one can use a similar argument. We call this condition the *Decreasing Marginal Consumer Loss Condition* (DMCLC).[31] Then, $(1/2)CS(r)$ behaves on $[0, r^*]$ in either manner:

1. If $CS'(0) \leq 0$, then $(1/2)CS(r)$ is monotonically decreasing in r, and a result $\Delta CS/2 = [CS(r^*) - CS(0)]/2 < 0$; *price discrimination lowers consumer surplus.*
2. If $CS'(0) > 0$, then $(1/2)CS(r)$ either
   (a) is monotonically increasing (if $CS'(r^*) > 0$, this is true), and as a result, $\Delta CS/2 > 0$; *price discrimination raises consumer surplus.*
   (b) first increases, and then after the reaching the maximum (where $CS'(r) = 0$), decreases until $r = r^*$. In this case, *price discrimination may raise or lower consumer surplus*: it cannot be determined whether $\Delta CS/2 < 0$ or $\Delta CS/2 > 0$ without further functional and/or parametric restrictions.

Thus, we determine the sign of $CS'(0)$. it follows that $\text{sign}[CS'(0)] = \text{sign}[q_s(\overline{p})/\pi_s''(\overline{p}) - q_w(\overline{p})/\pi_w''(\overline{p})]$, and thus, the following proposition is obtained.

[31] Now, it is shown that $g_m' < 0$ is equivalent to $\pi_m''' > (q_m'\pi_m'')/q_m > 0$ because $g_m'(p) = [q_m'\pi_m'' - q_m\pi_m''']/[\pi_m'']^2$. Thus, if $(q_m'\pi_m'')/q_m > (q_m''\pi_m'')/q_m' \Leftrightarrow q_m'' < (q_m')^2/q_m$, that is q_m is not "too convex," then the DMCLC is a sufficient condition for the DIRC to hold. Thus, under this "not too convex" assumption, the relationship, "DMCLC $\Rightarrow$ DIRC $\Rightarrow$ IRC", holds.

Proposition 5. *Given the DMCLC, if the output in the weak market at the uniform price $\overline{p}$ is sufficiently large, i.e.,*

$$\frac{q_w(\overline{p})}{\pi''_w(\overline{p})} \geq \frac{q_s(\overline{p})}{\pi''_s(\overline{p})},$$

then price discrimination lowers consumer surplus.

Then, using markup and pass-through, we can rewrite equality (8) as

$$\frac{CS'(r)}{2} = \underbrace{(-\pi''_s\pi''_w)}_{<0}\left(\frac{\mu_w(r)\rho_w(r)}{\pi''_w} - \frac{\mu_s(r)\rho_s(r)}{\pi''_s}\right)$$

for $r < r^*$, and

$$\frac{CS'(r^*)}{2} = \underbrace{\left(-\frac{\pi''_s\pi''_w}{\pi''_s + \pi''_w}\right)}_{>0}(\mu^*_w\rho^*_w - \mu^*_s\rho^*_s)$$

for $r = r^*$ which immediately leads to the following proposition.

Proposition 6. *Given the DMCLC, if $\mu^*_w\rho^*_w \geq \mu^*_s\rho^*_s$ holds, then price discrimination raises consumer surplus. Conversely, if*

$$\frac{\overline{\mu}_w\overline{\rho}_w}{\overline{\mu}_s\overline{\rho}_s} \leq \frac{\pi''_w(\overline{p})}{\pi''_s(\overline{p})}$$

holds, then price discrimination lowers consumer surplus.

Alternatively, it is also possible to directly apply Cowan's (2012) analysis of the effects of monopolistic third-degree price discrimination on consumer surplus to our case of oligopoly. From Cowan's (2012, p. 338) inequality (5), the lower and the upper bounds for a change in consumer surplus, ΔCS, are given by

$$\sum_{m\in S\cup W}[-\pi'_m(\overline{p})]\mu^*_m\rho^*_m\frac{q'_m(p^*_m)}{q'_m(\overline{p})},$$

and

$$\sum_{m\in S\cup W}[-\pi'_m(\overline{p})]\overline{p}\rho_m(\overline{MR}_m)\varepsilon^F_m(\overline{p}),$$

respectively, where $\overline{MR}_m$ is the "virtual" marginal cost with which the uniform price $\overline{p}$ coincides with the equilibrium discriminatory price when no uniform pricing restriction is imposed.

3.5 Non-constant Marginal Costs

Notice that our results so far do not crucially depend on the assumption of constant marginal costs. The only caveat is the definition of pass-through: to properly define pass-through in accommodation with non-constant marginal costs, we introduce a small amount of unit tax $t_m > 0$ in market m: the firm's first-order derivative of the profit with respect to its own price (equality 1) is now replaced by

$$\partial_p \pi_m(p) = q_m(p) + (p - t_m - mc_m[q_m(p)]) \frac{\partial x_{Am}}{\partial p_A}(p, p),$$

where $mc_m = c'_m[q_m(p)]$ is the marginal cost at $q_m(p)$. Then, pass-through is defined by $\rho_m \equiv \frac{\partial p_m}{\partial t_m}$, and no other changes should be made to derive the results above. In fact, the usefulness of pass-through is that it can easily be accommodated with non-constant marginal costs (Weyl and Fabinger 2013; Fabinger and Weyl 2018; and Adachi and Fabinger 2019). An additional caveat is that $\theta_m^* \rho_m^*$ is no longer interpreted as quantity pass-through under price discrimination (Weyl and Fabinger 2013, p. 572): one needs to take into account the "elasticity of the marginal cost" to approximate the trapezoids of the welfare gain and loss by a deviation from (full) price discrimination.

4 Firm Heterogeneity

In this section, we argue that the main thrusts under firm symmetry also hold when heterogeneous firms are introduced. Without loss of generality, we keep considering one strong market and one weak market: as explained in Footnote 15 above, an extension to the case of more than two markets is conceptually straightforward. We assume Corts' (1998, p. 315) *best response symmetry*: all firms agree on which market is strong and which market is weak. The case of best response *asymmetry* is studied by Corts (1998) (see also Footnote 8 above). The number of firms is N (≥ 2). Then, each firm j has the constraint, $p_{js} - p_{jw} \leq r_j$. Then, as above, firm j's equilibrium price in the weak market under all of these constraints is written as $p_{jw}(\mathbf{r})$, where $\mathbf{r} = (r_1, r_2, ..., r_N)$. Accordingly, firm j's equilibrium price in the strong market is $p_{js}(\mathbf{r}) = p_{jw}(\mathbf{r}) + r_j$. Thus, the equilibrium price pair in market $m = w, s$ is written as $\mathbf{p}_m(\mathbf{r}) = (p_{1m}(\mathbf{r}), p_{2m}(\mathbf{r}), ..., p_{Nm}(\mathbf{r}))$. Then, the social welfare is defined as a function of $\mathbf{r}$:

$$W(\mathbf{r}) \equiv U_s(\mathbf{q}_s[\mathbf{p}_s(\mathbf{r})]) + U_w(\mathbf{q}_w[\mathbf{p}_w(\mathbf{r})]) - \mathbf{c}_s^{\mathrm{T}} \cdot \mathbf{q}_s[\mathbf{p}_s(\mathbf{r})] - \mathbf{c}_w^{\mathrm{T}} \cdot \mathbf{q}_w[\mathbf{p}_w(\mathbf{r})],$$

where $\mathbf{q}_m[\mathbf{p}_m(\mathbf{r})] = (q_{1m}[\mathbf{p}_m(\mathbf{r})], q_{2m}[\mathbf{p}_m(\mathbf{r})], ..., q_{Nm}[\mathbf{p}_m(\mathbf{r})])^{\mathrm{T}}$ and $\mathbf{c}_m = (c_{1m}, c_{2m}, ..., c_{Nm})^{\mathrm{T}}$.

Now, let $r_j^* \equiv p_{js}^* - p_{jw}^*$ for each $j = 1, 2, ..., N$. Then, each firm's constraint is written as $0 \leq r_j = \lambda r_j^* \leq r_j^*$, with $\lambda \in [0, 1]$. Using this, we re-define the functions of $\mathbf{r}$ as functions of one-dimensional variable, λ. In particular, the social welfare is written as:

$$W(\lambda) = U_s(\mathbf{q}_s[\mathbf{p}_s(\lambda)]) + U_w(\mathbf{q}_w[\mathbf{p}_w(\lambda)]) - \mathbf{c}_s^{\mathrm{T}} \cdot \mathbf{q}_s[\mathbf{p}_s(\lambda)] - \mathbf{c}_w^{\mathrm{T}} \cdot \mathbf{q}_w[\mathbf{p}_w(\lambda)].$$

Then, we use $\boldsymbol{\partial}_{\mathbf{q}} U_m = \mathbf{p}_m$ from the representative consumer's utility maximization problem in each market m, where $\boldsymbol{\partial}_{\mathbf{q}} U_m \equiv \left(\frac{\partial U_m}{\partial q_{1m}}, \frac{\partial U_m}{\partial q_{2m}}, ..., \frac{\partial U_m}{\partial q_{Nm}}\right)$, to derive

$$\underbrace{W'(\lambda)}_{1\times 1} = \sum_{m=s,w} [\underbrace{(\mathbf{p}_m - \mathbf{c}_m)}_{1\times N}^{\mathrm{T}} \cdot (\underbrace{\boldsymbol{\partial}_{\mathbf{p}_m} \mathbf{q}_m}_{N\times N} \cdot \underbrace{\mathbf{p}'_m}_{N\times 1})],$$

where T denotes transpose, and

$$\boldsymbol{\partial}_{\mathbf{p}_m} \mathbf{q}_m \equiv \left(\underbrace{\begin{pmatrix} \frac{\partial q_{1m}}{\partial p_{1m}} \\ \vdots \\ \frac{\partial q_{Nm}}{\partial p_{1m}} \end{pmatrix}}_{\equiv \boldsymbol{\partial}_{p_{1m}} \mathbf{q}_m} \cdots \underbrace{\begin{pmatrix} \frac{\partial q_{1m}}{\partial p_{Nm}} \\ \vdots \\ \frac{\partial q_{Nm}}{\partial p_{Nm}} \end{pmatrix}}_{\equiv \boldsymbol{\partial}_{p_{Nm}} \mathbf{q}_m} \right)$$

and $\mathbf{p}'_{\mathrm{m}} \equiv (p'_{1m}(\lambda), p'_{2m}(\lambda), ..., p'_{Nm}(\lambda))^{\mathrm{T}}$.

Now, we apply the implicit function theorem to $\mathbf{f}(\mathbf{p_w}, \lambda) = \mathbf{0}$, where

$$\mathbf{f}(\underbrace{\mathbf{p_w}}_{N\times 1}, \underbrace{\lambda}_{1\times 1}) = \begin{pmatrix} \partial_{p_{1s}} \pi_{1s}(\mathbf{p}_w + \lambda \mathbf{r}^*) + \partial_{p_{1w}} \pi_{1w}(\mathbf{p}_w) \\ \vdots \\ \partial_{p_{js}} \pi_{js}(\mathbf{p}_w + \lambda \mathbf{r}^*) + \partial_{p_{jw}} \pi_{jw}(\mathbf{p}_w) \\ \vdots \\ \partial_{p_{Ns}} \pi_{Ns}(\mathbf{p}_w + \lambda \mathbf{r}^*) + \partial_{p_{Nw}} \pi_{Nw}(\mathbf{p}_w) \end{pmatrix},$$

is a collection of all firms' first-order conditions for profit maximization under regime λ, to obtain $\mathbf{p}'_w(\lambda) = -[\mathbf{D}_{\mathbf{p}_w} \mathbf{f}]^{-1} [\mathbf{D}_\lambda \mathbf{f}]$, where

$$\mathbf{D}_{\mathbf{p}_w} \mathbf{f} \equiv \underbrace{\begin{pmatrix} \dfrac{\partial^2 \pi_{1s}}{\partial p_{1s}^2} + \dfrac{\partial^2 \pi_{1w}}{\partial p_{1w}^2} & \cdots & \dfrac{\partial^2 \pi_{1s}}{\partial p_{Ns} \partial p_{1s}} + \dfrac{\partial^2 \pi_{1w}}{\partial p_{Nw} \partial p_{1w}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 \pi_{Ns}}{\partial p_{1s} \partial p_{Ns}} + \dfrac{\partial^2 \pi_{Nw}}{\partial p_{1w} \partial p_{Nw}} & \cdots & \dfrac{\partial^2 \pi_{Ns}}{\partial p_{Ns}^2} + \dfrac{\partial^2 \pi_{Nw}}{\partial p_{Nw}^2} \end{pmatrix}}_{\equiv \mathbf{K}}$$

$$= \underbrace{\begin{pmatrix} \dfrac{\partial^2 \pi_{1s}}{\partial p_{1s}^2} & \cdots & \dfrac{\partial^2 \pi_{1s}}{\partial p_{Ns} \partial p_{1s}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 \pi_{Ns}}{\partial p_{1s} \partial p_{Ns}} & \cdots & \dfrac{\partial^2 \pi_{Ns}}{\partial p_{Ns}^2} \end{pmatrix}}_{\equiv \mathbf{H}_s} + \underbrace{\begin{pmatrix} \dfrac{\partial^2 \pi_{1w}}{\partial p_{1w}^2} & \cdots & \dfrac{\partial^2 \pi_{1w}}{\partial p_{Nw} \partial p_{1w}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 \pi_{Nw}}{\partial p_{1w} \partial p_{Nw}} & \cdots & \dfrac{\partial^2 \pi_{Nw}}{\partial p_{Nw}^2} \end{pmatrix}}_{\equiv \mathbf{H}_w}$$

and

$$\mathbf{D}_\lambda \mathbf{f} = \mathbf{H}_s \underbrace{\begin{pmatrix} r_1^* \\ \vdots \\ r_j^* \\ \vdots \\ r_N^* \end{pmatrix}}_{\equiv \mathbf{r}^*}.$$

Thus,

$$\mathbf{p}_w'(\lambda) = [-\underbrace{\mathbf{K}^{-1}}_{N\times N}\underbrace{\mathbf{H}_s}_{N\times N}]\underbrace{\mathbf{r}^*}_{N\times 1}$$

and

$$\mathbf{p}_s'(\lambda) = -\mathbf{K}^{-1}\mathbf{H}_s\mathbf{r}^* + \mathbf{r}^* = [\underbrace{\mathbf{I} - \mathbf{K}^{-1}\mathbf{H}_s}_{N\times N}]\underbrace{\mathbf{r}^*}_{N\times 1},$$

so that

$$\begin{aligned} W'(\lambda) &= [(\mathbf{p}_s - \mathbf{c}_s)^{\mathrm{T}}\boldsymbol{\partial}_{\mathbf{p}_s}\mathbf{q}_s][\mathbf{I} - \mathbf{K}^{-1}\mathbf{H}_s]\mathbf{r}^* - [(\mathbf{p}_w - \mathbf{c}_w)^{\mathrm{T}}\boldsymbol{\partial}_{\mathbf{p}_w}\mathbf{q}_w][\mathbf{K}^{-1}\mathbf{H}_s]\mathbf{r}^* \\ &= \{[(\mathbf{p}_s - \mathbf{c}_s)^{\mathrm{T}}\boldsymbol{\partial}_{\mathbf{p}_s}\mathbf{q}_s]\mathbf{K}^{-1}[\mathbf{K} - \mathbf{H}_s] - [(\mathbf{p}_w - \mathbf{c}_w)^{\mathrm{T}}\boldsymbol{\partial}_{\mathbf{p}_w}\mathbf{q}_w][\mathbf{K}^{-1}\mathbf{H}_s]\}\mathbf{r}^* \\ &= \big\{\big([(\mathbf{p}_s - \mathbf{c}_s)^{\mathrm{T}}\boldsymbol{\partial}_{\mathbf{p}_s}\mathbf{q}_s]\mathbf{H}_s^{-1}\big)\big(\mathbf{H}_s\mathbf{K}^{-1}[\mathbf{K} - \mathbf{H}_s]\big) \\ &\qquad - \big([(\mathbf{p}_w - \mathbf{c}_w)^{\mathrm{T}}\boldsymbol{\partial}_{\mathbf{p}_w}\mathbf{q}_w]\mathbf{H}_w^{-1}\big)\big(\mathbf{H}_w\mathbf{K}^{-1}\mathbf{H}_s\big)\big\}\,\mathbf{r}^*. \end{aligned}$$

Now, we define

$$\mathbf{Z}_m(\mathbf{p}) \equiv \underbrace{[(\mathbf{p} - \mathbf{c}_m)^{\mathrm{T}}\boldsymbol{\partial}_{\mathbf{p}_m}\mathbf{q}_m(\mathbf{p})]}_{1\times N}\underbrace{\mathbf{H}_m^{-1}(\mathbf{p})}_{N\times N}$$

to proceed:

$$\begin{aligned} W'(\lambda) &= \{\underbrace{\mathbf{Z}_w}_{1\times N} - \underbrace{\mathbf{Z}_s}_{1\times N} \cdot \underbrace{(\mathbf{K} - \mathbf{H}_s)\mathbf{H}_w^{-1}}_{N\times N}\}\underbrace{(-\mathbf{H}_w\mathbf{K}^{-1}\mathbf{H}_s)\mathbf{r}^*}_{N\times 1} \\ &= (\mathbf{Z}_w - \mathbf{Z}_s)(\boldsymbol{\Gamma}\mathbf{r}^*), \end{aligned}$$

where $\boldsymbol{\Gamma} \equiv -\mathbf{H}_w\mathbf{K}^{-1}\mathbf{H}_s \gg \mathbf{0}$ is assumed. We also assume that multi-dimensional version of the IRC: for each market m and each firm j, Z_{jm} is increasing in p_l, $l = 1, 2, ..., N$.

Using this relationship, we can obtain the sufficient condition for price discrimination to improve social welfare with heterogeneous firms, which generalizes Proposition 4 in the previous section.

Proposition 7. *Given the IRC, if* $[[\boldsymbol{\theta}_w^*]^{\mathrm{T}} \circ [\boldsymbol{\mu}_w^*]^{\mathrm{T}}]\boldsymbol{\rho}_w^* > [[\boldsymbol{\theta}_s^*]^{\mathrm{T}} \circ [\boldsymbol{\mu}_s^*]^{\mathrm{T}}]\boldsymbol{\rho}_s^*$ *holds, where* $\circ$ *indicates element-by-element multiplication, then price discrimination raises social welfare. Conversely, if* $[[\overline{\boldsymbol{\theta}}_w]^{\mathrm{T}} \circ [\overline{\boldsymbol{\mu}}_w]^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_w < [[\overline{\boldsymbol{\theta}}_s]^{\mathrm{T}} \circ [\overline{\boldsymbol{\mu}}_s]^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_s\overline{\boldsymbol{\Delta}}$*, where* $\overline{\boldsymbol{\Delta}} \equiv \overline{\mathbf{K}}\,\overline{\mathbf{H}}_s^{-1}\overline{\mathbf{H}}_w\overline{\mathbf{K}}^{-1}$ *is defined for adjustment*

(a) Aggregate Output
If $[\boldsymbol{\theta}_w^*]^{\mathrm{T}}\boldsymbol{\rho}_w^* > [\boldsymbol{\theta}_s^*]^{\mathrm{T}}\boldsymbol{\rho}_s^*$, then $Q^* > \overline{Q}$. If $[\overline{\boldsymbol{\theta}}_w]^{\mathrm{T}}\overline{\boldsymbol{\rho}}_w < [\overline{\boldsymbol{\theta}}_s]^{\mathrm{T}}\overline{\boldsymbol{\rho}}_s\overline{\boldsymbol{\Delta}}$, then $Q^* < \overline{Q}$.
(b) Social Welfare
If $[[\boldsymbol{\theta}_w^*]^{\mathrm{T}} \circ [\boldsymbol{\mu}_w^*]^{\mathrm{T}}]\boldsymbol{\rho}_w^* > [[\boldsymbol{\theta}_s^*]^{\mathrm{T}} \circ [\boldsymbol{\mu}_s^*]^{\mathrm{T}}]\boldsymbol{\rho}_s^*$, then $W^* > \overline{W}$. If $[[\overline{\boldsymbol{\theta}}_w]^{\mathrm{T}} \circ [\overline{\boldsymbol{\mu}}_w]^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_w < [[\overline{\boldsymbol{\theta}}_s]^{\mathrm{T}} \circ [\overline{\boldsymbol{\mu}}_s]^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_s\overline{\boldsymbol{\Delta}}$, then $W^* < \overline{W}$.
(c) Consumer Surplus
If $[\boldsymbol{\mu}_w^*]^{\mathrm{T}}\boldsymbol{\rho}_w^* > [\boldsymbol{\mu}_s^*]^{\mathrm{T}}\boldsymbol{\rho}_s^*$, then $CS^* > \overline{CS}$. If $[\overline{\boldsymbol{\mu}}_w]^{\mathrm{T}}\overline{\boldsymbol{\rho}}_w < [\overline{\boldsymbol{\mu}}_s]^{\mathrm{T}}\overline{\boldsymbol{\rho}}_s\overline{\boldsymbol{\Delta}}$, then $CS^* < \overline{CS}$.

Table 1: Summary of the Sufficient Conditions (with N heterogeneous firms, $\boldsymbol{\theta}_m$, $\boldsymbol{\mu}_m$, and $\boldsymbol{\rho}_m$ are the market power vector ($N \times 1$), the markup vector ($N \times 1$), and the pass-through matrix ($N \times N$), respectively, in market $m = s, w$; asterisks and upper bars indicate price discrimination and uniform pricing, respectively; and $\overline{\boldsymbol{\Delta}}$ is a term for adjustment defined in the text).

($\overline{\mathbf{K}} \equiv \mathbf{K}(\overline{\mathbf{p}})$ and $\overline{\mathbf{H}}_m \equiv \mathbf{H}_m(\overline{\mathbf{p}})$, $m = s, w$, with $\overline{\mathbf{p}}$ being the equilibrium uniform price when $\lambda = 0$), holds, then price discrimination lowers social welfare.

Proof. See Appendix, part D. □

Note that for each j, $Z_{jm}^* = \sum_{k=1}^{N} \theta_{km}^* \mu_{km}^* \rho_{jkm}^*$, which is interpreted as the *weighted sum* of firm j's own pass-through (ρ_{jjm}^*) and the collection of its cross pass-through (ρ_{jkm}^*, $k \neq j$). For aggregate output and consumer surplus, we can readily generalize our previous results to the case of firm heterogeneity in a similar manner. Table 1 summarizes our results for heterogeneous firms.

5 Concluding Remarks

This paper provides theoretical implications of oligopolistic third-degree price discrimination with general nonlinear demands, allowing cost differentials across separate markets. In this sense, this paper, with the help of the methodology proposed by Weyl and Fabinger (2013), synthesizes Aguirre, Cowan, and Vickers' (2010) analysis of monopolistic third-degree price discrimination with general demands and Chen and Schwartz' (2015) analysis of monopolistic differential pricing and extends them to the case of differentiated oligopoly.

Our theoretical analysis, which takes into account firm heterogeneity, can also be utilized to empirically assess the welfare effects of third-degree price discrimination under oligopoly. In particular, in line with the "sufficient statistics" approach (Chetty 2009), our predictions regarding the welfare effects do *not* rely on functional specifications, and are thus considered fairly robust. In practice, one needs to specify demand and supply functions to estimate/calibrate such sufficient statistics as price elasticity, curvature, market power, and pass-through. In this sense, these sufficient statistics can take different values, depending on functional specifications. However, once numerical values of sufficient statistics are obtained, there should be no disagreement as to welfare assessment.

As such, our methodology would also be extended to an analysis of welfare effects of *wholesale/input* third-degree price discrimination (Katz 1987; DeGraba 1990; Yoshida 2000; Inderst and Valletti 2009; Villas-Boas 2009; Arya and Mittenforf 2010; Li 2014; O'Brien 2014; Gaudin and Lestage 2018; and Miklós-Thal and Shaffer 2019a). To do so, one would need to properly define sufficient statistics at each stage of a vertical relationship. Another important issue is to consider *multi-product* oligopolistic firms.[32] What happens if price discrimination is allowed for some products, whereas uniform pricing is enforced for other products? These and other important issues related to third-degree price discrimination await further research.

Appendix

A. Equivalence of Holmes' (1989) and Our Expressions for $Q'(r)$

Holmes (1989, p. 247), who assumes no cost differentials ($c \equiv c_s = c_w$) as in most of the papers on third-degree price discrimination, also derives a necessary and sufficient condition for $Q'(r) > 0$ under symmetric oligopoly. It is (using our notation) written as:

$$\underbrace{\frac{p_s - c}{q_s'(p_s)} \cdot \frac{d}{dp_s}\left(\frac{\partial x_{A,s}(p_s, p_s)}{\partial p_A}\right) - \frac{p_w - c}{q_w'(p_w)} \cdot \frac{d}{dp_w}\left(\frac{\partial x_{A,w}(p_w, p_w)}{\partial p_A}\right)}_{\text{Adjusted-concavity condition (Robinson 1933)}}$$
$$+ \underbrace{\frac{\varepsilon_s^C(p_s)}{\varepsilon_s^I(p_s)} - \frac{\varepsilon_w^C(p_w)}{\varepsilon_w^I(p_w)}}_{\text{Elasticity-ratio condition (Holmes 1989)}} > 0.$$

Recall that $1/\theta_s - 1/\theta_w = \varepsilon_s^C/\varepsilon_s^I - \varepsilon_w^C/\varepsilon_w^I$. The first and the second terms in the left hand side of Holmes' (1989) inequality is rewritten as:

$$\frac{p_s - c}{q_s'(p_s)} \cdot \frac{d}{dp_s}\left(\frac{\partial x_{A,s}(p_s, p_s)}{\partial p_A}\right) - \frac{p_w - c}{q_w'(p_w)} \cdot \frac{d}{dp_w}\left(\frac{\partial x_{A,w}(p_w, p_w)}{\partial p_A}\right)$$

[32] See, e.g., Armstrong and Vickers (2018) and Nocke and Schutz (2018) for recent studies of pricing under multi-product oligopoly.

$$
\begin{aligned}
&= L_w(p_w)\cdot\left[\left(-\frac{p_w}{q_w'(p_w)}\right)\frac{d}{dp_w}\left(\frac{\partial x_{A,w}(p_w,p_w)}{\partial p_A}\right)\right]\\
&\quad -L_s(p_s)\cdot\left[\left(-\frac{p_s}{q_s'(p_s)}\right)\frac{d}{dp_s}\left(\frac{\partial x_{A,s}(p_s,p_s)}{\partial p_A}\right)\right].
\end{aligned}
$$

Now, it is also observed that

$$
\begin{aligned}
\frac{\alpha_m^F-(1-\theta_m)\alpha_m^C}{\theta_m} &= \frac{\alpha_m^F}{\theta_m}-\frac{\partial x_{Bm}/\partial p_A}{q_m'}\frac{p_m}{\partial q_{Bm}/\partial p_A}\frac{\partial^2 x_{Am}}{\partial p_A\partial p_B}\\
&= -\frac{p_m}{q_m'}\left(-\frac{q_m'}{p_m}\frac{\alpha_m^F}{\theta_m}+\frac{\partial^2 x_{Am}}{\partial p_A\partial p_B}\right).
\end{aligned}
$$

This shows that inequality (2) is another expression for Holmes' (1989, p. 247) inequality (9). To see this, note that

$$
\frac{d}{dp_m}\left(\frac{\partial x_{Am}(p_m,p_m)}{\partial p_A}\right)=\frac{\partial^2 x_{Am}}{\partial p_A^2}(p_m,p_m)+\frac{\partial^2 x_{Am}}{\partial p_A\partial p_B}(p_m,p_m)
$$

in Holmes' (1989) expression is equivalent to $-(q_m'/p_m)(\alpha_m^F/\theta_m)+\partial^2 x_{Am}/(\partial p_A\partial p_B)$ because

$$
\begin{aligned}
-\frac{q_m'}{p_m}\frac{\alpha_m^F}{\theta_m} &= -\frac{q_m'}{p_m\theta_m}\left(-\frac{p_m}{\partial x_{Am}/\partial p_A}\right)\frac{\partial^2 x_{Am}}{\partial p_A^2}\\
&= \frac{1}{\theta_m}\frac{\partial x_{Am}/\partial p_A+\partial x_{Bm}/\partial p_A}{\partial x_{Am}/\partial p_A}\frac{\partial^2 x_{Am}}{\partial p_A^2}\\
&= \frac{1}{\theta_m}(1-A_m)\frac{\partial^2 x_{Am}}{\partial p_A^2}\\
&= \frac{\partial^2 x_{Am}}{\partial p_A^2}.
\end{aligned}
$$

B. Proof of Proposition 1

First, note that if

$$
\begin{aligned}
&\frac{\left[2-L_w(\overline{p})\alpha_w^F(\overline{p})\right]-\left[1-\theta_w(\overline{p})\right]\left[1-L_w(\overline{p})\alpha_w^C(\overline{p})\right]}{\theta_w(\overline{p})}\\
&\quad>\frac{\left[2-L_s(\overline{p})\alpha_s^F(\overline{p})\right]-\left[1-\theta_s(\overline{p})\right]\left[1-L_s(\overline{p})\alpha_s^C(\overline{p})\right]}{\theta_s(\overline{p})},
\end{aligned}
$$

then price discrimination lowers aggregate output. The first part is a sufficient condition for this inequality to hold. Next, note that

$$
\frac{\left[2-L_s(p_s^*)\alpha_s^F(p_s^*)\right]-\left[1-\theta_s(p_s^*)\right]\left[1-L_s(p_s^*)\alpha_s^C(p_s^*)\right]}{\theta_s(p_s^*)}
$$

$$> \frac{\left[2 - L_w(p_w^*)\alpha_w^F(p_w^*)\right] - [1 - \theta_w(p_w^*)]\left[1 - L_w(p_w^*)\alpha_w^C(p_w^*)\right]}{\theta_w(p_w^*)},$$

then price discrimination raises aggregate output. Thus, $\theta_w(p_w^*) > \theta_s(p_s^*)$ and

$$L_w(p_w^*)\frac{\alpha_w^F(p_w^*) - [1 - \theta_w(p_w^*)]\alpha_w^C(p_w^*)}{\theta_w(p_w^*)} > L_s(p_s^*)\frac{\alpha_s^F(p_s^*) - [1 - \theta_s(p_s^*)]\alpha_s^C(p_s^*)}{\theta_s(p_s^*)}.$$

C. Proof of Proposition 4

Note first that

$$\begin{aligned} z_m(p_m) &= -\frac{(p_m - c_m)\varepsilon_m^I(p_m)q_m(p_m)}{p_m\pi_m''(p_m)} \\ &= -\theta_m(p_m)\frac{q_m(p_m)}{\pi_m''(p_m)} \end{aligned}$$

holds. Now, define

$$F(p_m, c_m) = \frac{q_m(p_m)}{\frac{\partial x_{Am}}{\partial p_A}(p_m, p_m)} + p_m - c_m$$

so that $F(p_m^*, c_m) = 0$ for $m = s, w$. Then, from the implicit function theorem, it is verified that

$$\begin{aligned} \rho_m &= \frac{1}{1 + \dfrac{q_m'}{\partial x_{Am}/\partial p_A} - q_m\dfrac{d\left(\partial x_{Am}/\partial p_A\right)/dp_m}{\left(\partial x_{Am}/\partial p_A\right)^2}} \\ &= \frac{\partial x_{Am}/\partial p_A}{\dfrac{\partial x_{Am}}{\partial p_A} + q_m' - \dfrac{q_m}{\partial x_{Am}/\partial p_A}\dfrac{d}{dp_m}\left(\dfrac{\partial x_{Am}}{\partial p^A}\right)}, \end{aligned}$$

and under the equilibrium discriminatory prices,

$$\begin{aligned} \rho_m(p_m^*) &= \frac{\partial x_{Am}(p_m^*, p_m^*)/\partial p_A}{q_m'(p_m^*) + \dfrac{\partial x_{Am}}{\partial p_A}(p_m^*, p_m^*) + (p_m^* - c_m)\dfrac{d}{dp_m}\left(\dfrac{\partial x_{Am}}{\partial p_A}(p_m^*, p_m^*)\right)} \\ &= \frac{\partial x_{Am}(p_m^*, p_m^*)/\partial p_A}{\pi_m''(p_m^*, c_m)}, \end{aligned}$$

which implies that

$$\begin{aligned} z_m(p_m^*) &= \theta_m(p_m^*)\left(\frac{-q_m(p_m^*)}{\frac{\partial x_{Am}}{\partial p_A}(p_m^*, p_m^*)}\right)\rho_m(p_m^*) \\ &= \theta_m(p_m^*)\mu_m(p_m^*)\rho_m(p_m^*) \end{aligned}$$

and thus

$$\frac{W'(r^*)}{2} = \left(-\frac{\pi_s''\pi_w''}{\pi_s'' + \pi_w''}\right)(\theta_w^*\mu_w^*\rho_w^* - \theta_s^*\mu_s^*\rho_s^*) \geq 0$$

if $\theta_w^*\mu_w^*\rho_w^* \geq \theta_s^*\mu_s^*\rho_s^*$ holds. Given the IRC, this means that $W(r)$ is strictly increasing in $[0, r^*]$. This completes the proof for the first part.

For $r < r^*$, note that

$$\begin{aligned} z_m(p_m) &= \theta_m \left(\frac{-q_m}{\frac{\partial x_{Am}}{\partial p_A}}\right)\left(\frac{\frac{\partial x_{Am}}{\partial p_A}}{\pi_m''}\right) \\ &= \theta_m\mu_m \underbrace{\left(-\frac{\frac{\partial x_{Am}}{\partial p_A}}{\pi_s'' + \pi_w''}\right)}_{=\rho_m}\left(\frac{\pi_s'' + \pi_w''}{\pi_m''}\right). \end{aligned}$$

Thus, it is verified that

$$\frac{W'(r)}{2} = \underbrace{(-\pi_s''\pi_w'')}_{<0}\left(\frac{\theta_w(r)\mu_w(r)\rho_w(r)}{\pi_w''} - \frac{\theta_s(r)\mu_s(r)\rho_s(r)}{\pi_s''}\right),$$

which implies that given the IRC, $W(r)$ is strictly decreasing in $[0, r^*]$ if $W'(0) \leq 0$. This completes the proof for the second part.

D. Proof of Proposition 7

First, the multi-dimensional version of the market power index is defined by (Weyl and Fabinger 2013, p. 552):

$$\boldsymbol{\theta}_m(\mathbf{p}) \equiv \begin{pmatrix} \dfrac{(\mathbf{p} - \mathbf{c}_m)^{\mathrm{T}}\boldsymbol{\partial}_{p_{1m}}\mathbf{q}_m(\mathbf{p})}{-q_{1m}(\mathbf{p})} \\ \vdots \\ \dfrac{(\mathbf{p} - \mathbf{c}_m)^{\mathrm{T}}\boldsymbol{\partial}_{p_{jm}}\mathbf{q}_m(\mathbf{p})}{-q_{jm}(\mathbf{p})} \\ \vdots \\ \dfrac{(\mathbf{p} - \mathbf{c}_m)^{\mathrm{T}}\boldsymbol{\partial}_{p_{Nm}}\mathbf{q}_m(\mathbf{p})}{-q_{Nm}(\mathbf{p})} \end{pmatrix},$$

which implies that

$$\mathbf{Z}_m[\mathbf{p}_m(\lambda)] = [\underbrace{[\boldsymbol{\theta}_m[\mathbf{p}_m(\lambda)]]^{\mathrm{T}}}_{1\times N} \circ \underbrace{(-q_{1m}[\mathbf{p}_m(\lambda)], ..., -q_{jm}[\mathbf{p}_m(\lambda)], ..., -q_{Nm}[\mathbf{p}_m(\lambda)])}_{1\times N}]\underbrace{\mathbf{H}_m^{-1}[\mathbf{p}_m(\lambda)]}_{N\times N}$$

$$= [[\boldsymbol{\theta}_m[\mathbf{p}_m(\lambda)]]^{\mathrm{T}} \circ [\boldsymbol{\mu}_m[\mathbf{p}_m(\lambda)]]^{\mathrm{T}}] \begin{pmatrix} \frac{1}{\frac{\partial q_{1m}}{\partial p_{1m}}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\frac{\partial q_{2m}}{\partial p_{2m}}} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\frac{\partial q_{Nm}}{\partial p_{Nm}}} \end{pmatrix} \mathbf{H}_m^{-1}[\mathbf{p}_m(\lambda)],$$

where the multi-dimensional version of markup is naturally defined by

$$\boldsymbol{\mu}_m(\mathbf{p}) \equiv \begin{pmatrix} \frac{-q_{1m}(\mathbf{p})}{\frac{\partial q_{1m}}{\partial p_{1m}}(\mathbf{p})} \\ \vdots \\ \frac{-q_{jm}(\mathbf{p})}{\frac{\partial q_{jm}}{\partial p_{jm}}(\mathbf{p})} \\ \vdots \\ \frac{-q_{Nm}(\mathbf{p})}{\frac{\partial q_{jm}}{\partial p_{jm}}(\mathbf{p})} \end{pmatrix}.$$

We then apply the implicit function theorem to $\mathbf{g}(\mathbf{p}_m, \mathbf{c}_m) = \mathbf{0}$, where

$$\mathbf{g}(\underbrace{\mathbf{p}_m}_{N\times 1}, \underbrace{\mathbf{c}_m}_{N\times 1}) = \begin{pmatrix} q_{1m}(\mathbf{p}_m) + (p_{1m} - c_{1m})\dfrac{\partial q_{1m}}{\partial p_{1m}}(\mathbf{p}_m) \\ \vdots \\ q_{jm}(\mathbf{p}_m) + (p_{jm} - c_{jm})\dfrac{\partial q_{jm}}{\partial p_{jm}}(\mathbf{p}_m) \\ \vdots \\ q_{Nm}(\mathbf{p}_m) + (p_{Nm} - c_{Nm})\dfrac{\partial q_{Nm}}{\partial p_{Nm}}(\mathbf{p}_m) \end{pmatrix},$$

to obtain

$$\underbrace{\boldsymbol{\rho}_m(\mathbf{p}_m^*)}_{N\times N} = -[\mathbf{D}_{\mathbf{p}_m}\mathbf{g}]^{-1}[\mathbf{D}_{\mathbf{c}_m}\mathbf{g}] = \begin{pmatrix} \frac{\partial q_{1m}}{\partial p_{1m}} & 0 & \cdots & 0 \\ 0 & \frac{\partial q_{2m}}{\partial p_{2m}} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial q_{Nm}}{\partial p_{Nm}} \end{pmatrix} \mathbf{H}_m^{-1}(\mathbf{p}_m^*),$$

which indicates that

$$\mathbf{Z}_m(\mathbf{p}_m^*) = [[\boldsymbol{\theta}_m(\mathbf{p}_m^*)]^{\mathrm{T}} \circ [\boldsymbol{\mu}_m(\mathbf{p}_m^*)]^{\mathrm{T}}]\boldsymbol{\rho}_m(\mathbf{p}_m^*).$$

Now, note that $W'(1) > 0$ if the inequality in this proposition holds. Thus, given the IRC, $W(\lambda)$ is strictly increasing in $[0, 1]$, meaning that social welfare is higher under price discrimination than under uniform pricing. This complete the proof for the first part of the proposition.

For the second part, we proceed

$$\mathbf{Z}_m(\lambda) = [[\boldsymbol{\theta}_m^{\mathrm{T}} \circ \boldsymbol{\mu}_m^{\mathrm{T}}] \underbrace{\begin{pmatrix} \frac{1}{\frac{\partial q_{1m}}{\partial p_{1m}}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\frac{\partial q_{2m}}{\partial p_{2m}}} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\frac{\partial q_{Nm}}{\partial p_{Nm}}} \end{pmatrix}}_{=\boldsymbol{\rho}_m} \mathbf{K}^{-1}(\mathbf{K}\mathbf{H}_m^{-1}),$$

for $\lambda < 1$, and thus

$$W'(\lambda) = \left\{[\boldsymbol{\theta}_w^{\mathrm{T}} \circ \boldsymbol{\mu}_w^{\mathrm{T}}]\boldsymbol{\rho}_w[\mathbf{K}\mathbf{H}_w^{-1}\mathbf{K}^{-1}] - [\boldsymbol{\theta}_s^{\mathrm{T}} \circ \boldsymbol{\mu}_s^{\mathrm{T}}]\boldsymbol{\rho}_s[\mathbf{K}\mathbf{H}_s^{-1}\mathbf{K}^{-1}]\right\} \underbrace{(\mathbf{K}\boldsymbol{\Gamma}\mathbf{r}^*)}_{<<0}.$$

Then, it is verified that

$$\begin{aligned} W'(0) < 0 &\Leftrightarrow [\overline{\boldsymbol{\theta}}_w^{\mathrm{T}} \circ \overline{\boldsymbol{\mu}}_w^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_w[\overline{\mathbf{K}}\,\overline{\mathbf{H}}_w^{-1}\overline{\mathbf{K}}^{-1}] > [\overline{\boldsymbol{\theta}}_s^{\mathrm{T}} \circ \overline{\boldsymbol{\mu}}_s^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_s[\overline{\mathbf{K}}\,\overline{\mathbf{H}}_s^{-1}\overline{\mathbf{K}}^{-1}] \\ &\Leftrightarrow [\overline{\boldsymbol{\theta}}_w^{\mathrm{T}} \circ \overline{\boldsymbol{\mu}}_w^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_w < [\overline{\boldsymbol{\theta}}_s^{\mathrm{T}} \circ \overline{\boldsymbol{\mu}}_s^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_s[\overline{\mathbf{K}}\,\overline{\mathbf{H}}_s^{-1}\overline{\mathbf{K}}^{-1}][\overline{\mathbf{K}}\,\overline{\mathbf{H}}_w^{-1}\overline{\mathbf{K}}^{-1}]^{-1} \\ &\Leftrightarrow [\overline{\boldsymbol{\theta}}_w^{\mathrm{T}} \circ \overline{\boldsymbol{\mu}}_w^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_w < [\overline{\boldsymbol{\theta}}_s^{\mathrm{T}} \circ \overline{\boldsymbol{\mu}}_s^{\mathrm{T}}]\overline{\boldsymbol{\rho}}_s[\overline{\mathbf{K}}\,\overline{\mathbf{H}}_s^{-1}\overline{\mathbf{H}}_w\overline{\mathbf{K}}^{-1}], \end{aligned}$$

which completes the proof for the second part.

References

Adachi, Takanori. 2002. "A Note on 'Third-Degree Price Discrimination with Interdependent Demands'." *Journal of Industrial Economics*, 50(2), 235.

—. 2005. "Third-Degree Price Discrimination, Consumption Externalities, and Social Welfare." *Economica*, 72(285), 171-178.

—, and Michal Fabinger. 2019. "Multi-Dimensional Pass-Through and Welfare Measures under Imperfect Competition." Unpublished.

—, and Noriaki Matsushima. 2014. "The Welfare Effects of Third-Degree Price Discrimination in a Differentiated Oligopoly." *Economic Inquiry*, 52(3), 1231-1244.

Aguirre, Iñaki. 2019. "Oligopoly Price Discrimination, Competitive Pressure and Total Output." Unpublished.

—, Simon Cowan, and John Vickers. 2010. "Monopoly Price Discrimination and Demand Curvature." *American Economic Review*, 100(4), 1601-1615.

Anderson, Simon P., André de Palma, and Jacques-François Thisse. 1987. "A Representative Consumer Theory of the Logit Model." *International Economic Review*, 29(3), 461-466.

Armstrong, Mark, and John Vickers. 2001. "Welfare-Increasing Third-Degree Price Discrimination." *RAND Journal of Economics*, 32(4), 579-605.

—, and —. 2018. "Multiproduct Pricing Made Simple." *Journal of Political Economy*, 126(4), 1444-1471.

Arya, Anil, and Brian Mittenforf. 2010. "Input Price Discrimination when Buyers Operate in Multiple Markets." *Journal of Industrial Economics*, 58(4), 846-867.

Asplund, Marcus, Rickard Eriksson, and Niklas Strand. 2008. "Price Discrimination in Oligopoly: Evidence from Swedish Newspapers." *Journal of Industrial Economics*, 56(2), 333-346.

Bertoletti, Paolo. 2009. "On the Output Criterion for Price Discrimination." *Economics Bulletin*, 29(4), 2951-2956.

Boik, Andre. 2017. "The Empirical Effects of Competition on Third Degree Price Discrimination in the Presence of Arbitrage." *Canadian Journal of Economics*, 50 (4), 1023-1036.

Bresnahan, Timothy F. 1989. "Empirical Studies of Industries with Market Power." In R. Schmalensee and R. D. Willig (eds.), *Handbook of Industrial Organization*, Vol. 2. Amsterdam, The Netherlands: Elsevier Science Publishers B.V., 1011-1057.

Chen, Yongmin, Jianpei Li, and Marius Schwartz. 2019. "Competitive Differential Pricing." Unpublished.

—, and Marius Schwartz. 2015. "Differential Pricing When Costs Differ: A Welfare Analysis." *RAND Journal of Economics*, 46(2), 442-460.

Chetty, Raj. 2009. "Sufficient Statistics for Welfare Analysis: A Bridge Between Structural and Reduced-Form Methods." *Annual Reviews of Economics*, 1, 451–88.

Clerides, Sofronis K. 2004. "Price Discrimination with Differentiated Products: Definition and Identification." *Economic Inquiry*, 42(3), 402-412.

Corts, Kenneth S. 1998. "Third-Degree Price Discrimination in Oligopoly: All-Out Competition and Strategic Commitment." *RAND Journal of Economics*, 29(2), 306-323.

—. 1999. “Conduct Parameters and the Measurement of Market Power.” *Journal of Econometrics*, 88(2), 227-250.

Cowan, Simon. 2012. “Third-Degree Price Discrimination and Consumer Surplus.” *Journal of Industrial Economics*, 60(2), 333-345.

—. 2016. “Welfare-Increasing Third-Degree Price Discrimination.” *RAND Journal of Economics*, 47(2), 326-340.

—. 2017. “A Model of Third-Degree Price Discrimination with Positive Welfare Effects.” Unpublished.

—. 2018. “Regulating Monopoly Price Discrimination.” *Journal of Regulatory Economics*, 54(1), 1-13.

Czerny, Achim I., and Anming Zhang. 2015. “Third-Degree Price Discrimination in the Presence of Congestion Externality.” *Canadian Journal of Economics*, 48(4), 1430-1455.

Dastidar, Krishnendu Ghosh. 2006. “On Third-Degree Price Discrimination in Oligopoly.” *The Manchester School*, 74(2), 231-250.

DeGraba, Patrick. 1990. “Input Market Price Discrimination and the Choice of Technology.” *American Economic Review*, 80(5), 1246-1253.

Fabinger, Michal, and E. Glen Weyl. 2018. “Functional Forms for Tractable Economic Models and the Cost Structure of International Trade.” Unpublished.

Flores, Daniel, and Vitaliy Kalashnikov. 2017. “Parking Discounts: Price Discrimination with Different Marginal Costs.” *Review of Industrial Organization*, 50(1), 91-103.

Galera, Francisco, and Jesús M. Zaratiegui. 2006. “Welfare and Output in Third-Degree Price Discrimination: A Note.” *International Journal of Industrial Organization*, 24(3), 605-611.

Gaudin, Germain, and Romain Lestage. 2018. “Input Price Discrimination, Demand Forms, and Welfare.” Unpublished.

Genesove, David, and Wallace P. Mullin. 1998. “Testing Static Oligopoly Models: Conduct and Cost in the Sugar Industry, 1890-1914.” *RAND Journal of Economics*, 29(2), 355-377

Geruso, Michael. 2017. “Demand Heterogeneity in Insurance Markets: Implications for Equity and Efficiency.” *Quantitative Economics*, 8(3), 929-975.

Hendel, Igal, and Aviv Nevo. 2013. “Intertemporal Price Discrimination in Storable Goods Markets.” *American Economic Review*, 103(7), 2722-2751.

Holmes, Thomas J. 1989. "The Effects of Third-Degree Price Discrimination in Oligopoly." *American Economic Review*, 79(1), 244-250.

Hotelling, Harold. 1929. "Stability in Competition." *Economic Journal*, 39(153), 41-57.

Inderst, Roman, and Tommaso Valletti. 2009. "Price Discrimination in Input Markets." *RAND Journal of Economics*, 40(1), 1-19.

Katz, Michal L. 1987. "The Welfare Effects of Third-Degree Price Discrimination in Intermediate Goods Markets." *American Economic Review*, 77(1), 154-167.

Leslie, Phillip. 2004. "Price Discrimination in Broadway Theater." *RAND Journal of Economics*, 35(3), 520-541.

Li, Youping. 2014. "A Note on Third Degree Price Discrimination in Intermediate Good Markets." *Journal of Industrial Economics*, 62(3), 554.

Miklós-Thal, Jeanine, and Greg Shaffer. 2019a. "Input Price Discrimination by Resale Market." Unpublished.

—, and —. 2019b. "Pass-Through as an Economic Tool: On Second Derivatives, Social Incidence, and Price Discrimination." Unpublished.

Nahata, Babu, Krzysztof Ostaszewski, and P. K. Sahoo. 1990. "Direction of Price Changes in Third-Degree Price Discrimination." *American Economic Review*, 80(5), 1254-1258.

Nocke, Volker, and Nicolas Schutz. 2018. "Multiproduct-Firm Oligopoly: An Aggregative Games Approach." *Econometrica*, 86(2), 523-557.

O'Brien, Daniel P. 2014. "The Welfare Effects of Third-Degree Price Discrimination in Intermediate Good Markets: The Case of Bargaining." *RAND Journal of Economics*, 45(1), 92-115.

Okada, Tomohisa. 2014. "Third-Degree Price Discrimination with Fairness-Concerned Consumers." *The Manchester School*, 86 (6), 701-715.

Phlips, Louis. 1983. *The Economics of Price Discrimination*. Cambridge, UK: Cambridge University Press.

Pigou, Arthur C. 1920. *The Economics of Welfare*. London, UK: Macmillan.

Robinson, Joan. 1933. *The Economics of Imperfect Competition*. London, UK: Macmillan.

Schmalensee, Richard. 1981. "Output and Welfare Implications of Monopolistic Third-Degree Price Discrimination." *American Economic Review*, 71(1), 242-247.

Schwartz, Marius. 1990. “Third-Degree Price Discrimination and Output: Generalizing a Welfare Result.” *American Economic Review*, 80(5), 1259-1262.

Shapiro, Carl. 1996. “Mergers with Differentiated Products.” *Antitrust*, 10(2), 23-30.

Stigler, George J. 1987. *The Theory of Price, Fourth Edition*. New York, USA: Macmillan.

Varian, Hal R. 1985. “Price Discrimination and Social Welfare.” *American Economic Review*, 75(4), 870-875.

Vickers, John. 2017. “Direct Welfare Analysis of Relative Price Regulation.” Unpublished.

Villas-Boas, Sofia Berto. 2009. “An Empirical Investigation of the Welfare Effects of Banning Wholesale Price Discrimination.” *RAND Journal of Economics*, 40(1), 20-46.

Weyl, E. Glen, and Michal Fabinger. 2013. “Pass-Through as an Economic Tool: Principle of Incidence under Imperfect Competition.” *Journal of Political Economy*, 121(3), 528-583.

Yoshida, Yoshihiro. 2000. “Third-Degree Price Discrimination in Input Markets: Output and Welfare.” *American Economic Review*, 90(1), 240-246.

Omitted Details in Derivations

- The first-order partial derivative of the profit in symmetric pricing is given by

$$\begin{aligned}\Delta\pi_m(p) &\equiv \left.\frac{\partial\pi_{jm}}{\partial p_{jm}}\right|_{p_{jm}=p_{-j,m}=p} \\ &= q_{Am}(p,p)+(p-c_m)\frac{\partial x_{Am}}{\partial p_A}(p,p) \\ &= q_m(p)+(p-c_m)\frac{\partial x_{Am}}{\partial p_A}(p,p).\end{aligned}$$

- The detailed derivation of $\pi''_m(p)$:

$$\begin{aligned}\pi''_m(p) &= q'_m(p)+\frac{\partial x_{Am}}{\partial p_A}(p,p)+(p-c_m)\frac{d}{dp}\left(\frac{\partial x_{Am}}{\partial p_A}(p,p)\right) \\ &= 2\frac{\partial x_{Am}}{\partial p_A}(p,p)+\frac{\partial x_{Am}}{\partial p_B}(p,p)+(p-c_m)\frac{d}{dp}\left(\frac{\partial x_{Am}}{\partial p_A}(p,p)\right) \\ &= D^2\pi_m(p)+\frac{\partial x_{Am}}{\partial p_B}(p,p)+(p-c_m)\left[\frac{d}{dp}\left(\frac{\partial x_{Am}}{\partial p_A}(p,p)\right)-\frac{\partial^2 x_{Am}}{\partial p_A^2}(p,p)\right] \\ &= D^2\pi_m(p)+\frac{\partial x_{Am}}{\partial p_B}(p,p)+(p-c_m)\frac{\partial^2 x_{Am}}{\partial p_B\partial p_A}(p,p),\end{aligned}$$

- $W'(r)$ is computed by

$$\begin{aligned}W'(r) &= 2\left(\frac{\partial U_s}{\partial q_A}-c_s\right)\cdot q'_s\cdot p'_s(r)+2\left(\frac{\partial U_w}{\partial q_A}-c_w\right)\cdot q'_w\cdot p'_w(r) \\ &= 2\left(p_s(r)-c_s\right)\cdot q'_s\cdot p'_s(r)+2\left(p_w(r)-c_w\right)\cdot q'_w\cdot p'_w(r).\end{aligned}$$

- The derivation of $Q'(r)/2$ is given by

$$\begin{aligned}\frac{Q'(r)}{2} &= q'_w\cdot p'_w+q'_s\cdot p'_s \\ &= -\frac{\pi''_s q'_w}{\pi''_s+\pi''_w}+\frac{\pi''_w q'_s}{\pi''_s+\pi''_w} \\ &= \underbrace{\left(-\frac{q'_s q'_w}{\pi''_s+\pi''_w}\right)}_{>0}\left(\frac{\pi''_s}{q'_s}-\frac{\pi''_w}{q'_w}\right),\end{aligned}$$

where

$$\frac{\pi''_m(p)}{q'_m(p)} = \left[2-L_m(p)\alpha^F_m(p)\right]\frac{\partial x_{Am}(p,p)/\partial p_A}{q'_m(p)}$$

$$
\begin{aligned}
& + \left[1 - L_m(p)\alpha_m^C(p)\right] \frac{\partial x_{Bm}(p,p)/\partial p_A}{q_m'(p)} \\
= \quad & \left[2 - L_m(p)\alpha_m^F(p)\right] \frac{\varepsilon_m^F(p)}{\varepsilon_m^I(p)} - \left[1 - L_m(p)\alpha_m^C(p)\right] \frac{\varepsilon_m^C(p)}{\varepsilon_m^I(p)} \\
= \quad & \frac{\left[2 - L_m(p)\alpha_m^F(p)\right] - (1-\theta_m(p))\left[1 - L_m(p)\alpha_m^C(p)\right]}{\theta_m(p)},
\end{aligned}
$$

where the derivation of $\varepsilon_m^C(p_m)/\varepsilon_m^I(p_m)$ is given by

$$
\begin{aligned}
\frac{\varepsilon_m^C(p_m)}{\varepsilon_m^I(p_m)} &= \frac{\varepsilon_m^C(p_m)}{\varepsilon_m^F(p_m) - \varepsilon_m^C(p_m)} \\
&= \frac{1}{1/A_m(p_m) - 1} \\
&= \frac{1-\theta_m(p_m)}{\theta_m(p_m)},
\end{aligned}
$$

which implies that

$$
\frac{\pi_s''}{q_s'} - \frac{\pi_w''}{q_w'} = \frac{1 + \theta_s - L_s[\alpha_s^F - (1-\theta_s)\alpha_s^C]}{\theta_s} - \frac{1 + \theta_w - L_w[\alpha_w^F - (1-\theta_w)\alpha_w^C]}{\theta_w}.
$$

Thus,

$$
\begin{aligned}
\frac{Q'(r)}{2} = \quad & \underbrace{\left(-\frac{\pi_s''\pi_w''}{\pi_s'' + \pi_w''}\right)}_{>0} \\
& \times \Bigg(\frac{1 - L_s[p_s(r)]\left\{\alpha_s^F[p_s(r)] - (1-\theta_s[p_s(r)])\alpha_s^C[p_s(r)]\right\}}{\theta_s[p_s(r)]} \\
& \quad - \frac{1 - L_w[p_w(r)]\left\{\alpha_w^F[p_w(r)] - (1-\theta_w[p_w(r)])\alpha_w^C[p_w(r)]\right\}}{\theta_w[p_w(r)]} \Bigg).
\end{aligned}
$$

- The precise expression for $\boldsymbol{\partial}_{\mathbf{p}_m}\mathbf{q}_m \cdot \mathbf{p}_m'$ is:

$$
\boldsymbol{\partial}_{\mathbf{p}_m}\mathbf{q}_m \cdot \mathbf{p}'_m = \begin{pmatrix} \sum_{j'=1,..,N} \frac{\partial q_{1m}}{\partial p_{j'm}} p'_{j'm} \\ \vdots \\ \sum_{j'=1,..,N} \frac{\partial q_{jm}}{\partial p_{j'm}} p'_{j'm} \\ \vdots \\ \sum_{j'=1,..,N} \frac{\partial q_{jm}}{\partial p_{j'm}} p'_{j'm} \end{pmatrix}.
$$