Output and Welfare Implications of Oligopolistic Third-Degree Price Discrimination

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Keywords: Third-Degree Price Discrimination; Differentiated Oligopoly; Social Welfare; Pass-through; Sufficient Statistics.

JEL classification: D21; D43; D60; L11; L13.

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1 Introduction

This paper explores welfare effects of third-degree price discrimination in oligopoly as well as its output implications and the effects on consumer surplus. Specifically, we consider a fairly general setting, and present sufficient conditions for oligopolistic third-degree price discrimination to raise or lower aggregate output, consumer surplus, and Marshallian social welfare (i.e., the sum of consumer and producer surplus) when all discriminatory markets are served even without price discrimination. Our analysis is firstly developed under firm symmetry, and then is extended to allow heterogeneous firms in a natural manner. In addition, our analysis permits a moderate amount of cost differences across separate markets. These features, together with the fact that our sufficient conditions are stated in terms of estimable concepts (see below), are appealing because they allow greater flexibility that can accommodate with empirical studies.

Under third-degree price discrimination, consumers are segmented into separate markets and are charged different unit prices according to identifiable characteristics (e.g., age, occupation, and location or time of purchase). In contrast, all consumers face the same price if third-degree price discrimination is not practiced (“uniform pricing”). Without loss of generality, one can consider the case of two markets to understand how price discrimination changes output and welfare in each market. Furthermore, for a simpler exposition, all firms are symmetric and thus the prevailing price is identical. Then, in one market, the discriminatory price will become greater than the uniform price, whereas the unit price in the other market decreases. In Robinson’s (1933) terminology, the former market is called a “strong” market (s), and the latter a “weak” market (w). Stated differently, the discriminatory price in the strong market ps is higher than the equilibrium uniform price, $\bar{p}$, but pw, the discriminatory price in the weak market, is lower: $p_s > \bar{p} > p_w$.\(^1\)

Given this price change, price discrimination raises output and social welfare in the weak market, whereas they lower in the strong market. Then, what are the overall effects? Our sufficient conditions for oligopolistic price discrimination to raise or lower aggregate output, social welfare, and consumer surplus are provided in a way of a cross-market comparison of multiplications of two or three of the following economic concepts: (i) pass-through value, or simply pass-through ($\rho > 0$), i.e., how the price responds to a small change in the marginal cost;

\(^1\)In this paper, price discrimination is present when $p_s > p_w$, i.e., when the prices across markets are not uniform. As Clerides (2004, p.402) states, once cost differentials are allowed, “there is no single, widely accepted definition of price discrimination.” To understand this, consider symmetric firms and let $mc_s$ and $mc_w$ be marginal costs at equilibrium output in markets s and w, respectively. Then, there can be two alternative definitions. One is the margin definition: price discrimination occurs when $p_s - mc_s > p_w - mc_w$. The other one is the markup definition due to Stigler (1987): price discrimination occurs when $p_s/mc_s > p_w/mc_w$. Our simpler definition is in line with the former definition, and is employed for its tractability and its connectivity to the existing literature on third-degree price discrimination with no cost differentials. Moreover, our definition of price discrimination coincides with what Chen and Schwartz (2015) and Chen, Li and Schwartz (2019) call “differential pricing.” As long as cost differentials are sufficiently small, these differences will not alter the results significantly because these three definitions are equivalent if $mc_s = mc_w$.\}

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(ii) *market power index*, or simply *market power* ($\theta \in [0,1]$), which measures the intensity of competition in the market; and (iii) *markup value*, or simply *markup* ($\mu \geq 0$), which is simply the difference between the price and the marginal cost. In particular, in a series of propositions, we show that the product of market power and pass-through, $\theta \rho$, is important for determining the output effects, whereas the product of all the three concepts, $\theta \mu \rho$, consists of the sufficient condition for a welfare change. Intuitively, the product of market power and pass-through measures how output in each separate market changes in response to a marginal change in price. To evaluate a marginal change in welfare, one needs to consider the markup, which measures welfare gain or loss in response to a marginal change in quantity under imperfect competition where the price exceeds the marginal cost. In this way, welfare implications are obtained from a cross-market comparison of the quantity change multiplied by the markup. For the effects on consumer surplus, the product of markup and pass-through, $\mu \rho$, is crucial because it measures the price change multiplied by the amount of output.

We also emphasize that these three concepts consists of the “sufficient statistics” (Chetty 2009) in the sense that no other information is necessary (except for the information for an adjustment; see the explanation in the end of subsection 3.1) and that they are sufficient to determine the output and welfare effects of oligopolistic third-degree price discrimination. Furthermore, pass-through is determined by up to the second-order demand characteristics, and under a game of price setting, market power is expressed by the own- and cross-price elasticities. Obviously, markup is also determined by the own- and cross-price elasticities via the first-order conditions. Thus, once the market demands are specified/estimated, a change in aggregate output, social welfare, or consumer surplus is predicted by computing these “sufficient statistics.” For this sake, the researcher does not have to directly compare welfare under price discrimination and that under uniform pricing. In this way, our theoretical predictions, equipped with an estimable framework, would be utilized to understand the mechanism behind an empirical result. For example, in their empirical analysis of within-store brand competition, Hendel and Nevo (2013) show that social welfare is higher under third-degree price discrimination than with the case of no discrimination. However, it is not clear which factor is important in obtaining this empirical result. Although ultimately welfare evaluation is an empirical matter, one still wishes to know more about which force derives the result. Therefore, this paper also aims to fill the gap between the theoretical predictions and the empirical literature on price discrimination where researchers are often agnostic about the mechanism behind the result.²

The literature on third-degree price discrimination has a centennial tradition, pioneered by Pigou (1920) and Robinson (1933), with the main focus being on whether price discrimination raises or lowers social welfare. Among others, Schmalensee (1981) and Aguirre, Cowan, and

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²For other empirical studies of third-degree price discrimination, see, e.g., Leslie (2004); Asplund, Eriksson and Strand (2008); and Boik (2017).
Vickers (2010) study how the demand curvatures are related to the output and welfare effects. In addition to competition being imperfect, third-degree price discrimination entails further inefficiency in production and distribution as consumers with the same marginal valuation may face different prices simply because they belong to different markets. Thus, for third-degree price discrimination to raise social welfare, it must expand aggregate output sufficiently to offset such a misallocation across markets. Schmalensee (1981) shows that an increase in aggregate output is a necessary condition for third-degree price discrimination to raise social welfare—generalized by Varian (1985) and Schwartz (1990)—and Aguirre, Cowan, and Vickers (2010) identify that aggregate output increases with price discrimination if the inverse demand curvature is greater (“more convex”) in the weak market than in the strong market, and that with an additional qualification (the discriminatory prices are sufficiently close), social welfare also increases.

However, these studies analyze monopolistic third-degree discrimination, and to date, “there are virtually no predictions as to how discrimination impacts welfare” (Hendel and Nevo 2013, p. 2723; emphasis added) once oligopolistic competition is taken into account. For example, Holmes (1989) employs the same technique used by Schmalensee (1981) and Aguirre, Cowan and Vickers (2010) (see Section 3 for details) to study the output effects of third-degree price discrimination in a symmetric oligopoly. However, Holmes (1989) provides no welfare predictions (see also Dastidar 2006).

In this paper, we aim to contribute to this literature by providing fairly general conditions as to whether oligopolistic price discrimination raises or lowers aggregate output, social welfare, and consumer surplus. Our results generalize Aguirre, Cowan, and Vickers’ (2010) analysis of monopolistic third-degree price discrimination not only because we study oligopoly but because our results are also stated in terms of the sufficient statistics. Although these sufficient statistics themselves are endogenous variables and thus are less fundamental than demand primitives such as the first- and second-order characteristics (i.e., elasticities and curvatures), our focus on the sufficient statistics enables us to extend our results under firm symmetry to the case of heterogeneous firms (both in demand and production). In addition, we also provide another expression for Holmes’ (1989) derivation of the output effects in terms of the key statistics.

Furthermore, our treatment also allows differences in the marginal cost across separate markets. In almost all of the theoretical studies on price discrimination, researchers (somewhat manually) assume that there are no cost differentials across discriminatory markets to focus on the demand side. However, in many real cases of price discrimination, cost differentials are quite

\(^3\)In a similar vein, Armstrong and Vickers (2001) consider a model of symmetric duopoly with product differentiation à la Hotelling (1929), and study the consequences of third-degree price discrimination in the competitive limit around zero transportation cost where the equilibrium prices are almost equal to marginal cost. Under this setting, Armstrong and Vickers (2001) show that price discrimination lowers social welfare if the weak market has a lower value of price elasticity of demand. Adachi and Matsushima (2014) also derive a similar result by assuming linear demand.
often observed, not to mention the typical example of freight charges across regional markets with different costs of transportation and storage (Phlips 1983, pp. 5-7). In the narrow definition of price discrimination, this is not price discrimination because they are regarded as different products. However, airlines are arguably motivated to offer different types of seats mainly because they aim to make use of heterogeneity among consumers. Thus, ideally, a theoretical analysis on price discrimination should also allow a moderate amount of cost differentials across discriminatory markets.

Even if costs differ across markets, sellers, in reality, may have to be engaged in uniform pricing due to the universal service requirement, fairness concerns from consumers, and so on (Okada 2014; Geruso 2017). In the case of monopoly with differentials in the marginal costs across markets, Chen and Schwartz (2015) derive sufficient conditions for consumer surplus to be higher under differential pricing.\footnote{Chen, Li and Schwartz (2019) extend Chen and Schwartz’ (2015) analysis to the case of oligopoly. See also Galera and Zaratiegui (2006) and Bertoletti (2009) as studies of conditions under which price discrimination raises social welfare when cost differentials across markets are allowed.} To ensure that the strong market is indeed strong when cost differentials are allowed, it is sufficient to assume that the marginal cost in the strong market, denoted by $mc_s$, is higher than that in the weak market: $mc_s > mc_w$ (though $mc_s$ should not be too much higher than $mc_w$). Then, under uniform pricing, the markup in the strong market $p - mc_s$ is smaller than the markup in the weak market $p - mc_w$. Differential pricing allows the monopolist to sell more products in the weak market, which is improves efficiency. Chen and Schwartz (2015) find that while differential pricing with no cost differentials (third-degree price discrimination in a traditional manner) tends to increase the average price after differential pricing is allowed, differential pricing with cost differentials does not. In contrast, our analysis provides welfare implications in a more direct manner. As in Chen and Schwartz (2015), this paper does not have to make an explicit assumption on $mc_s$ and $mc_w$ as long as the second-order conditions for profit maximization are satisfied and a large discrepancy between $mc_w$ and $mc_w$ does not change the order of the discriminatory prices from the one with no cost differentials.\footnote{In the context of reduced-fare parking as a form of third-degree price discrimination with cost differentials, Flores and Kalashnikov (2017) characterize a sufficient condition for free parking (drivers receive a price discount in the form of complementary parking while pedestrians do not) to be welfare improving.}

The rest of the paper is organized as follows. Section 2 presents our basic model of oligopolistic pricing with symmetric firms and constant marginal costs. Then, we derive output and welfare implications in Section 3. We also provide parametric examples, show the results of the effects on consumer surplus, and discuss the case of non-marginal costs. In Section 4, we argue that our method can be readily extended if firm heterogeneity is introduced. Section 5 concludes the paper.
2 The Model of Oligopolistic Pricing

For ease of exposition, we, following Holmes (1989) and Aguirre, Cowan, and Vickers (2010), consider the case of two symmetric firms and two separate markets or consumer groups (simply called markets hereafter). Extending the following analysis to the case of \( J \geq 3 \) symmetric firms and \( M \geq 3 \) separate markets is straightforward. As explained above, we call one market \( s \) (strong), where the equilibrium discriminatory price will be higher than the equilibrium uniform price, and the other \( w \) (weak), where the opposite is true. Two firms, \( A \) and \( B \), have an identical cost structure in each market. Specifically, each firm has an identical cost function, \( c_m(q_{jm}) \), in market \( m = s, w \), where \( q_{jm} \) is firm \( j \)'s output (\( j = A, B \)). For simplicity of exposition below, we assume, with abuse of notation, that firms have a constant marginal cost, \( x_m \), in each market. Specifically, each firm has an identical cost function, \( c_m(x_m) \), in each market. Throughout this paper, focus on symmetric Nash equilibrium.

In market \( m = s, w \), given firms \( A \) and \( B \)'s prices \( p_{Am} \) and \( p_{Bm} \), the representative consumer consumes \( q_{Am} > 0 \) and \( q_{Bm} > 0 \), and her (net) utility is quasi-linear and is written as \( U_m(q_{Am}, q_{Bm}) - p_{Am}q_{Am} - p_{Bm}q_{Bm} \), where \( U_m \) is three-times continuously differentiable, \( \partial U_m/\partial q_{jm} > 0 \), \( \partial^2 U_m/\partial q_{jm}^2 < 0 \), \( j = A, B \), and \( \partial^2 U_m/(\partial q_{Am} \partial q_{Bm}) < 0 \). Direct demands in market \( m \) are derived from the representative consumer’s utility maximization: \( \partial U_m(q_{jm}, q_{-jm})/\partial q_{jm} - p_{jm} = 0 \), which leads to firm \( j \)'s demand in market \( m \), \( q_{jm} = x_{jm}(p_{jm}, p_{-jm}) \). We assume that \( x_{jm} \) is twice continuously differentiable. The corresponding inverse demand can be written as \( p_{jm} = p_{jm}(q_{jm}, q_{-jm}) \). Because of the assumptions on the utility, firm \( j \)'s demand in market \( m \) falls as its own price increases \( (\partial x_{jm}/\partial p_{jm} < 0 \) and \( \partial p_{jm}/\partial x_{jm} < 0 \)), and it rises as the rival’s price increases \( (\partial x_{jm}/\partial p_{-jm} > 0 \) and \( \partial p_{jm}/\partial x_{jm} < 0 \); the firms’ products are substitutes). We assume that for a consumer’s perspective firms are symmetric: \( U_m(q', q'') = U_m(q'', q') \) for any \( q' > 0 \) and \( q'' > 0 \). Then, the firms’ demands in market \( m \) are also symmetric: \( x_{Am}(p', p'') = x_{Bm}(p', p'') \) for any \( p' > 0 \) and \( p'' > 0 \). Because the firms’ technologies are also identical, we, throughout this paper, focus on symmetric Nash equilibrium.

Therefore, we define the demand in symmetric pricing by \( q_m(p) \equiv x_{Am}(p, p) \). Another interpretation of \( q_m(p) \) is: both firms take \( 2q_m(p) \) as the joint demand, “cooperatively” choose the same price (behaving as an “industry”), and divide the joint demand equally to obtain \( q_m(p) \). Note that:

\[
q_m'(p) = \left. \frac{\partial x_{Am}}{\partial p_A} (p_A, p) \right|_{p_A = p} + \left. \frac{\partial x_{Am}}{\partial p_B} (p, p_B) \right|_{p_B = p} .
\]

Thus, for \( q_m'(p) \) to be negative, we assume that \( |\partial x_{Am}(p, p)/\partial p_A| > \partial x_{Am}(p, p)/\partial p_B \). Note also that by symmetry the following relationship also holds (this corresponds to Holmes’ (1989)
In symmetric pricing, we are able to define, following Holmes (1989, p. 245), the **price elasticity of the industry’s demand** by \( \varepsilon^F_m(p) \equiv -(p/q_m(p))(\partial x_{Am}(p,p)/\partial p_A) \) and by \( \varepsilon^C_m(p) \equiv (p/q_m(p))(\partial x_{Bm}(p,p)/\partial p_A) \), respectively. Then, Holmes (1989) shows that under symmetric pricing, \( \varepsilon^F_m(p) = \varepsilon^E_m(p) + \varepsilon^C_m(p) \) holds. This implies that the own-price elasticity must be greater than the cross-price elasticity \( (\varepsilon^F_m(p) > \varepsilon^C_m(p)) \). Here, \( \partial^2 x_{jm}(p,p)/\partial p_j^2 \) can be positive, zero or negative. Following Dastidar’s (2006, p. 234) Assumption 2 (iv), we assume that \( \partial^2 x_{jm}(p,p)/\partial p_j^2 + \partial^2 x_{jm}(p,p)/\partial p_j \partial p_{-j} \leq 0 \).

Firm \( j \)'s profit in market \( m \) is written as \( \pi_{jm}(p_{jm}, p_{-j,m}) = (p_{jm} - c_m)x_{jm}(p_{jm}, p_{-j,m}) \). As in Dastidar’s (2006, pp. 235-6) Assumptions 3 and 4, for the existence and the global uniqueness of pricing equilibrium under either uniform pricing or price discrimination, we assume that for each firm \( j = A, B, \partial^2 \pi_{jm}/\partial p_j^2 < 0, \partial^2 \pi_{jm}/(\partial p_{jm}\partial p_{-j,m}) > 0, \) and \( -(\partial^2 \pi_{jm}/(\partial p_{jm}\partial p_{-j,m})) < 1 \) (see Dastidar’s (2006) Lemmas 1 and 2 for the existence and the uniqueness).

We then define the first-order partial derivative of the profit in market \( m \), evaluated at a symmetric price \( p \), by

\[
\frac{\partial x_{Am}}{\partial p_A}(p, p) = q_m(p) - \frac{\partial x_{Bm}}{\partial p_A}(p, p).
\]

Then, under symmetric discriminatory pricing, \( p_s^* \) satisfies \( \partial p \pi_m(p_s^*) = 0 \) for \( m = s, w \). Under symmetric uniform pricing, \( \bar{p} \) is a (unique) solution of \( \partial_p \pi_s(\bar{p}) + \partial_p \pi_w(\bar{p}) = 0 \).

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6If one considers quantity-setting, rather than price-setting, firms as in Aguierre (2019), then firm \( j \)'s profit in market \( m \) is defined by \( \pi_{jm} = p_{jm}(q_{jm}, q_{-j,m})q_{jm} - c_m(q_{jm}) \), and thus the first-order partial derivative in symmetric equilibrium is \( \partial \pi_{jm}/\partial q_{jm} |_{q_{jm} = q_{-j,m} = q} \), which is equivalent to \( p_{jm}(q) - c_m(q) + q \frac{\partial x_{Am}}{\partial p_A}(q, q) = 0 \). Under
this paper, we consider the situation where the weak market is open under uniform pricing: $q_w(\overline{p}) > 0$.\(^7\) Hereafter, functional and parametric restrictions are imposed to assure that $p^*_m > \overline{p} > p^*_w$.\(^8\)^9.

The equilibrium discriminatory price in market $m = s, w, p^*_m$, satisfies the following Lerner formula: $\varepsilon^F_m(p^*_m)(p^*_m - c_m)/p^*_m = 1$. This shows that the discriminatory price in market $m$ approaches to the marginal cost as the own-price elasticity for the firm, $\varepsilon^F_m(p^*_m)$, becomes large. Because of Holmes’ (1989) elasticity formula explained above, $\varepsilon^F_m(p^*_m)$ can be large (i) when $\varepsilon^E_m(p^*_m)$ is very large even if $\varepsilon^C_m(p^*_m)$ is close to zero, or (ii) when $\varepsilon^C_m(p^*_m)$ is very large even if $\varepsilon^E_m(p^*_m)$ is close to zero. These are two polar cases of a large $\varepsilon^F_m(p^*_m)$: of course if both $\varepsilon^E_m(p^*_m)$ and $\varepsilon^C_m(p^*_m)$ are very large, then $\varepsilon^F_m(p^*_m)$ is also very large. Case (i) is where the industry is under strong pressure from other substitutable industries\(^10\) so that a small price increase in symmetric pricing causes a large number of consumers switching to purchasing a product in other industries instead, although any consumers are very loyal to either firm so that a small price increase by one firm causes a very small number of consumers switching to other rivals’ product in the same industry (most of them leave the industry to purchase something outside the industry). For example, in a residential area, consumers (especially young consumers) would have a strong taste for their favorite soda (thus, $\varepsilon^C_m(p^*_m)$ is close to zero), although soda is highly substitutable by mineral water (thus, $\varepsilon^E_m(p^*_m)$ is very large) if consumers just want to relieve their throat (whether soda or water does not matter).

On the other hand, case (ii) is where the competitive pressure from other industries is weak, although inside the industry, firms are fiercely competing for consumers. For example,
in a resort, consumers may not care much about the difference between Coca-Cola and Pepsi (thus, $\varepsilon^C_m(p^*_m)$ is very large), though soda would not be easily substitutable by mineral water (thus, $\varepsilon^C_m(p^*_m)$ is close to zero) because consumers want to get perfectly refreshed: having water instead of soda does not relieve their throat. From a firm’s perspective, these two polar cases are equivalent with respect to pricing in the sense that if it raises its price by even a small amount, it loses a large number of consumers: whether they leave the industry or switch to rivals’ products does not matter to that firm. Thus, in the examples above, the firms’ (discriminatory) prices are close to marginal cost both in a residential area and in a resort due to different reasons. Recall again that these are two polar cases: in reality, $\varepsilon^F_m(p^*_m)$ may be large because both $\varepsilon^F_m(p^*_m)$ and $\varepsilon^C_m(p^*_m)$ are large. We can also think of the following alternative possibility: in a resort filled by young visitors, Coca-Cola’s and Pepsi’s prices are close to the marginal cost because consumers do not care about water or soda as long as they can relieve their throat (i.e., $\varepsilon^C_m(p^*_m)$ is very large), although they are very loyal to either brand once they choose soda ($\varepsilon^C_m(p^*_m)$ is close to zero).

Next, let $y_m$ be per-firm (symmetric) share output in market $m$,\(^{11}\) that is, $y_m(p_s,p_w) \equiv q_m(p^*_m)/[q_s(p_s)+q_w(p_w)]$. Then, the equilibrium uniform price, $\overline{p} \equiv \overline{p}(c_s,c_w)$, satisfies: $\sum_{m=s,w} \overline{y}_m \varepsilon^F_m(\overline{p})(\overline{p} - c_m)/\overline{p} = 1$, where $\overline{y}_m \equiv y_m(\overline{p}(c_s,c_w),\overline{p}(c_s,c_w))$ for $m = s,w$.\(^{12}\) In the rest of the paper, the dependence of the equilibrium price is often implicit when there are no confusions. In particular, the superscript star denotes price discrimination, whereas the upper bar uniform pricing. For example, we use $(\varepsilon^l_m)^* \equiv \varepsilon^l_m(p^*_m)$ and $\varepsilon^l_m \equiv \varepsilon^l_m(\overline{p})$ as the industry’s elasticities in equilibrium.

### 3 Output and Welfare

In the analysis below, we, following Schmalensee (1981), Holmes (1989), and Aguirre, Cowan, and Vickers (2010), add the constraint $p_s - p_w \leq r$, where $r > 0$, to the firms’ profit maximization problem (under symmetric pricing).\(^{13}\) Then, $r = 0$ corresponds to uniform pricing, and $r = r^* \equiv p^*_s - p^*_w$ to price discrimination. We express social welfare (and aggregate output) as a function of $r$ in $[0,r^*]$. Note that under this constrained problem of profit maximization, $p_w$ satisfies $\partial_p \pi_s(p_w + r) + \partial_p \pi_w(p_w) = 0$. Thus, we write the solution by $p_w(r)$. Then, we define $p_s(r) \equiv p_w(r) + r$. Applying the implicit function theorem to this equation yields to

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\(^{11}\) Of course, the total output in market $m$ is $2q_m(p_m)$, aggregated across two symmetric firms.

\(^{12}\) If there are no cost differentials, i.e., $c_s = c_w \equiv c$, then the formula is simpler: $\sum_{m=s,w} \pi_m^m(\overline{p})(\overline{p} - c)/\overline{p} = 1$ as shown by Holmes (1989, p. 247): the markup rate (common to all markets) is equal to the inverse of the average of own-price elasticities weighted by the output shares.

\(^{13}\) Alternatively, Vickers (2017) analyzes properties of functions of social welfare or consumer surplus value as a scalar argument to compare social welfare or consumer surplus under price discrimination and uniform pricing in monopoly. Vickers (2017) especially focuses on the cases when the quantity elasticity or the inverse demand curvature is constant for all markets. See also Cowan (2017) for an analysis of the role of the price elasticity and the demand curvature in determining the effects of monopolistic third-degree price discrimination.
Clearly, our two-market analysis does not lose any validity. Here, note that \( \pi''_s(p) \) and \( \partial^2 \pi_m(p) \) are different:

\[
\pi''_s(p) = \frac{d}{dp} \left( \pi'_{Ax}(p,p) + \frac{d}{dp} \left( \frac{d}{dp} \pi_{Ax}(p,p) \right) \right)
\]
\[
= \partial^2 \pi_m(p) + \frac{d}{dp} \pi_{Ax}(p,p) + (p - c_m) \frac{d}{dp} \frac{d}{dp} \pi_{Ax}(p,p),
\]

where \( \partial^2 \pi_m(p) \) is defined by

\[
\partial^2 \pi_m(p) = \left[ 2 + (p - c_m) \frac{d^2 \pi_{Ax}(p,p)}{dp^2} \right] \frac{d}{dp} \pi_{Ax}(p,p)
\]

which corresponds to Aguirre, Cowan, and Vickers’ (2010, p. 1603) \( \pi''_s(p) \). We assume that \( \pi''_s(p) < 0 \) for all \( p \geq 0 \).\(^{14}\)

We define the representative consumer’s utility in symmetric pricing by \( \tilde{U}_m(q) = U_m(q,q) \). Aggregate output under symmetric pricing is given by \( Q(r) = Q_s(r) + Q_w(r) = 2 \left\{ q_s[p_s(r)] + q_w[p_w(r)] \right\} \). Social welfare under symmetric pricing as a function of \( r \) is written as \( W(r) = \tilde{U}_s(q_s[p_s(r)]) + \tilde{U}_w(q_w[p_w(r)]) - 2c_s \cdot q_s[p_s(r)] - 2c_w \cdot q_w[p_w(r)] \), which implies \( W'(r) = \left( \tilde{U}_s' - 2c_s \cdot \tilde{q}_s' \cdot p_s'(r) + (\tilde{U}_w - 2c_w) \cdot \tilde{q}_w' \cdot p_w'(r) \right) \). Now, note that \( \tilde{U}_m' = \partial U_m/\partial q_A + \partial U_m/\partial q_B = 20U_m/\partial q_A \) (by symmetry). Thus, \( W'(r) = 2(p_s(r) - c_s) \cdot q_s' \cdot p_s'(r) + 2(p_w(r) - c_w) \cdot q_w' \cdot p_w'(r) \).\(^{15}\)

### 3.1 Output

Now, we can further proceed:

\[
\frac{W'(r)}{2} = \left( p_s(r) - \bar{p} + \bar{p} - c_s \right) q_s'[p_s(r)]p_s'(r) + (p_w(r) - \bar{p} + \bar{p} - c_w) q_w'[p_w(r)]p_w'(r)
\]
\[
= \left( p_s(r) - \bar{p} \right) q_s'[p_s(r)]p_s'(r) + \left( p_w(r) - \bar{p} \right) q_w'[p_w(r)]p_w'(r)
\]
\[
+ \sum_{m=s,w} (p_m - c_m) q_m'[p_m(r)]p_m'(r).
\]

This derivation coincides with the case of monopoly as shown in Aguirre, Cowan and Vickers’ (2010, p. 1604) equality (3) if there are no cost differentials (i.e., \( c_s = c_w = c \), with

\(^{14}\)Appendix A of Aguirre, Cowan and Vickers (2010) discusses the concavity of the profit function.

\(^{15}\)As already noted, we consider one strong market and one weak market. More generally, by defining \( S \equiv \{ m | p_m^* > \bar{p} \} \) and \( W \equiv \{ m | \bar{p} > p_m^* \} \), \( p_s(r), s \in S, \) and \( p_w(r), w \in S \), consists of the optimal price vector under constraints \( [ p_m - \bar{p} ] \leq r \) for all \( m \in S \cup W \), with \( r \in [0, \max_m | p_m^* - \bar{p} ] \). Then, for example, social welfare is defined as \( W(r; c_s, c_w) \equiv \sum_{s \in S} \tilde{U}_s(q_s[p_s(r)]) + \sum_{w \in W} \tilde{U}_w(q_w[p_w(r)]) - 2 \sum_{s \in S} (c_s \cdot q_s[p_s(r)]) - 2 \sum_{w \in W} (c_w \cdot q_w[p_w(r)]) \). Clearly, our two-market analysis does not lose any validity.
two minor modifications: (i) the left hand side is \(W'(r)/2\) rather than \(W'(r)\) itself, and (ii) the last term of Aguirre, Cowan and Vickers’ (2010) equality (3) is replaced by \(Q'(r)/2\) rather than \(Q'(r)\) because \((1/2) \sum_{m=s,w} (\bar{p} - c_m) q_m[p_m(r)]p'_m(r) = (\bar{p} - c) (Q'(r)/2)\). If cost differentials are allowed, it is observed that an increase in the weighted aggregate output, \(\sum_{m=s,w} (\bar{p} - c_m) q_m[p_m(r)]p'_m(r)\), is necessary for price discrimination to raise social welfare, as in the case of monopoly.\(^{16,17}\)

To proceed further, we define the \textit{curvature of the firm’s (direct) demand} in market \(m\) by

\[
\alpha^F_m(p) \equiv - \frac{p}{\partial x_{Am}(p,p)/\partial p_A} \frac{\partial^2 x_{Am}(p,p)}{\partial p_A^2} (p,p)
\]

(which measures the convexity/concavity of the firm’s direct demand, and corresponds to \(\alpha_m(p)\) in Aguirre, Cowan and Vickers 2010, p. 1603), and the \textit{elasticity of the cross-price effect} of the firm’s direct demand in market \(m\) by

\[
\alpha^C_m(p) \equiv - \frac{p}{\partial x_{Am}(p,p)/\partial p_B} \frac{\partial^2 x_{Am}(p,p)}{\partial p_B \partial p_A} (p,p)
\]

\[
= - \frac{p}{\partial x_{Bm}(p,p)/\partial p_A} \frac{\partial^2 x_{Bm}(p,p)}{\partial p_A^2} (p,p),
\]

which is new to oligopoly.\(^{18}\) Here, \(\alpha^F_m(p)\) is positive (resp. negative) if and only if \(\partial^2 x_{Am}(p,p)/\partial p_A^2\) is positive (resp. negative), while \(\alpha^C_m(p)\) is always positive (because of our assumption, \(\partial^2 x_{jm}(p,p)/\partial p_A \partial p_B < 0\)). Note that the sign of \(\alpha^F_m(p)\) indicates whether the firm’s own part of the demand slope under symmetric pricing given the rival’s price being \(p\), \(\partial x_{Am}(\cdot;p)/\partial p_A\), is convex \((\alpha^F_m(p)\) is positive) or concave \((\alpha^F_m(p)\) is negative). On the other hand, \(\alpha^C_m(p)\) measures how the rival’s price level matters to how many of the firm’s customers switch to the rival’s product when the firm raises its own price \((\partial x_{Bm}/\partial p_A)\). Thus, a large \(\alpha^C_m(p)\) implies that \(\partial x_{Bm}/\partial p_A\) is very responsive to a change in \(p_B\), and vice versa.

Next, we define the \textit{market power index} in market \(m\) by \(\theta_m(p) \equiv 1 - A_m(p)\),\(^{19,20}\) where

\(^{16}\)However, if externalities across consumers, such as network externalities and congestion, exist, then an increase in aggregate output would be no longer a necessary condition, as implied by Adachi (2002, 2005), who studies monopoly with linear demands. See also Czerny and Zhang (2015) as a recent study of price discrimination and congestion.

\(^{17}\)With a general number of markets (see Footnote 15 above), this condition can be written as \(E[(\bar{p} - c_m) q'_m p'_m] > 0\). For this to hold, \(\text{Cov}(\bar{p} - c_m, q'_m p'_m) > 0\), that is, on average, the markup under uniform pricing and \(q'_m p'_m > 0\) are positively correlated because \(E[(\bar{p} - c_m)] > 0\) and \(E[q'_m p'_m] > 0\).

\(^{18}\)This is because \(\partial (\partial x_{Am}(p,p)/\partial p_B)/\partial p_A = \partial (\partial x_{Bm}(p,p)/\partial p_A)/\partial p_A\).

\(^{19}\)Conceptually, the market power index defined here corresponds to what is called the conduct parameter when market power itself is a target of estimation without an exact specification of strategic interaction (see, e.g., Bresnahan 1989; Genesove and Mullin 1998; and Corts 1999).

\(^{20}\)Alternatively, Weyl and Fabinger (2013, p. 531) and Adachi and Fabinger (2019) define the market power index in a market (which, in our interest in price discrimination, can be indexed by \(m\)) by \(\theta_m \equiv [(p - c_m)/p] e_m\).
As Weyl and Fabinger (2013, p. 544) argue, $\theta_m(p)$ captures the degree of industry-level brand loyalty or stickiness\(^{21}\) in market $m$: if $\theta_m(p)$ is zero (close to one), market $m$ is captured by perfect competition (almost monopoly): firms’ products are perfect substitutes (nearly non-substitutable products).\(^{22}\) The markup rate (the Lerner index), $L_m(p_m) \equiv (p_m - c_m)/p_m$, alone is not appropriate to measure the rivalry within market $m$ because it can be the case that $p_m$ is close to $c_m$ (the markup rate is close to zero) simply because the price elasticity of the industry’s demand $\epsilon_m^c(p_m)$ is very large while the brand rivalry is so weak that the cross-price elasticity, $\epsilon_m^C(p_m)$, remains very small (as a result, in total, $\epsilon_m^F(p_m)$ is very large, which is actually reason for the low markup rate). However, if $\epsilon_m^C(p_m)$ is close to $\epsilon_m^F(p_m)$ (i.e., almost of all consumers who leave a firm as a response to its price increase are switching to other rivals’ products), then $\theta_m$ becomes close to zero irrespective of the value of the markup rate. Thus, $\theta_m(p)$, which ranges between 0 and 1, better captures the brand stickiness than $L_m(p)$ does. (their $mc$ and $\epsilon_D$ are replaced by our $c_m$ and $\epsilon_m^f$, respectively) as the Lerner index adjusted by the elasticity of the industry’s demand. If the first-order condition is given for each market (that is, if full price discrimination is allowed), then $\theta_m(p)$ defined as in Weyl and Fabinger (2013) coincides with $1 - A_m(p)$ because $\epsilon_m^F(p_m - c_m)/p_m = 1$ and thus

$$
\frac{\epsilon_m^f(p) - c_m}{p} = \frac{1}{\epsilon_m^f(p)} \left( p \right) \frac{q_m(p)}{q_m(p)} q_m(p) = \frac{1}{\epsilon_m^f(p)} \frac{q_m(p)}{p} \frac{\partial x_A (p, p)}{\partial p_A} \left( \frac{\partial x_A (p, p)}{\partial p_B (p, p)} \right) = \frac{\partial x_A (p, p)}{\partial p_A} \left( \frac{\partial x_A (p, p)}{\partial p_A} + \frac{\partial x_B (p, p)}{\partial p_A} \right) \text{(by symmetry)}
$$

is established.

\(^{21}\) Even if the firms’ products have the same characteristics across different markets (with no product differentiation), the extent of brand loyalty may differ across markets, reflecting differences in market characteristics (summarized in demand functions).

\(^{22}\) Because $(p_m - c_m)/\epsilon_m^F(p_m)/p_m = 1$ and $\epsilon_m^F(p_m) = \epsilon_m^c(p_m) + \epsilon_m^C(p_m)$, it is verified that $\theta_m(p_m, c_m) + \epsilon_m^C(p_m)/(p_m - c_m)/p_m = 1$. Thus, as long as the products are substitutes ($\epsilon_m^c(p_m) > 0$), $\theta_m(p_m)$ is less than one.
Now, we consider the effects of price discrimination on aggregate output. First, note that

$$\frac{Q'(r)}{2} = q'_w \cdot p'_w + q'_s \cdot p'_s$$

$$= \left( -\frac{q'_s q'_w}{\pi'_w + \pi'_w} > 0 \right) \times \left( 1 - L_s[p_s(r)] \{ \alpha_s^F[p_s(r)] - (1 - \theta_s[p_s(r)])\alpha_s^C[p_s(r)] \} \right) \left( \frac{\theta_s[p_s(r)]}{\theta_w[p_w(r)]} \right) - \frac{1 - L_w[p_w(r)] \{ \alpha_w^F[p_w(r)] - (1 - \theta_w[p_w(r)])\alpha_w^C[p_w(r)] \} \right) \left( \frac{\theta_w[p_w(r)]}{\theta_s[p_s(r)]} \right).$$

Note also that $\varepsilon^C_m(p_m)/\varepsilon^I_m(p_m) = [1 - \theta_m(p_m)]/\theta_m(p_m)$ measures the substitutability between brands (adjusted by the elasticity of the industry’s demand): if the brand stickiness is very strong (i.e., $\theta_m(p)$ is close to one), $\varepsilon^C_m(p_m)/\varepsilon^I_m(p_m)$ is close to one, while if the brand stickiness is very weak (i.e., $\theta_m(p)$ is close to zero), then $\varepsilon^C_m(p_m)/\varepsilon^I_m(p_m)$ becomes infinitely large. Then, the following lemma holds with cost differentials being allowed.

**Lemma 1.** $Q'(r) > 0$ if and only if (suppressing the dependence on $p_m(r)$)

$$L_w \cdot \frac{\alpha_w^F - (1 - \theta_w)\alpha_w^C}{\theta_w} - L_s \cdot \frac{\alpha_s^F - (1 - \theta_s)\alpha_s^C}{\theta_s} + \frac{1}{\theta_s} - \frac{1}{\theta_w} > 0. \quad (2)$$

Now, suppose that the brand stickiness in the weak market is so weak that $\theta_w(p_w)$ is close to zero ($\theta_w(p_w) \approx 0$), whereas the brand stickiness in the strong market is moderate or strong ($\theta_s(p_s) \gg 0$). Then, the left hand side of inequality (2) above is approximated by

$$\frac{1 - L_s[\alpha_s^F - (1 - \theta_s)\alpha_s^C]}{\theta_s} - \{ 1 - L_w[\alpha_w^F - \alpha_w^C] \} \frac{1}{\theta_w}.$$ 

Thus, as long as $1 > L_w[\alpha_w^F - \alpha_w^C]$, the left hand side becomes infinitely negative as $\theta_w(p_w)$ approaches to zero (assuming the first term is bounded). Counterintuitively, in the weak market, where price discrimination lowers the price, the brand rivalness has a negative effect on an increase in aggregate output by price discrimination. This is because the uniform price is already very low due to the fierce level of competition and thus there is little room for a price reduction by price discrimination to increase the output in the weak market. The opposite argument holds if the strong market is characterized by a low brand stickiness (i.e., $\theta_s(p_s, c_s) \approx 0$). This implies that, as expected, a fierce level of competition in the strong market has a positive effect on an increase in aggregate output by price discrimination. The rivalness in the strong market keeps the price increase by price discrimination small, and thus a reduction in output in the strong market is also kept small.

Following Holmes (1989), we call the first and the second terms in the left hand side of
inequality (2) the adjusted-concavity part, and the third and the fourth terms the elasticity-ratio part:

\[
L_w \cdot \frac{\alpha^F_w - (1 - \theta_w)\alpha^C_w}{\theta_w} - L_s \cdot \frac{\alpha^F_s - (1 - \theta_s)\alpha^C_s}{\theta_s} + \frac{1}{\theta_s} - \frac{1}{\theta_w}.
\]

(2')

Consider the adjusted-concavity part. A larger \( \alpha^F_w \) and/or a smaller \( \alpha^C_w \) make a positive \( Q'(r) \) more likely. A larger \( \alpha^F_w \) means that the firm’s own part of the demand in the weak market \((\partial x_{A,w}/\partial p_A)\) is more convex (“the output expansion effect”). On the other hand, a smaller \( \alpha^C_w \) means that how many of the firm’s customers switch to the rival’s product as response to the firm’s price increase is not so much affected by the current price level (“the countervailing effect”). In this sense, the strategic concerns in the firm’s pricing are small. Thus, both a larger \( \alpha^F_w \) and a smaller \( \alpha^C_w \) indicate that the weak market is competitive. Even if \( \partial x_{A,w}/\partial p_A \) is not so convex, a smaller \( \alpha^C_w \) (i.e., \( \partial x_{B,m}/\partial p_A \) is not responsive to the level of \( p_B \)) can substitute it. Here, the intensity of competition, \( 1/\theta_w \), magnifies both effects, resulting in \( \alpha^F_w/\theta_w \) and \( (1/\theta_w - 1)\alpha^C_w \). A similar argument also holds for \( \alpha^F_s \) and \( \alpha^C_s \). In part A of the Appendix, we show that Holmes’ (1989) expression for \( Q'(r) \) (expression (9) in Holmes 1989, p.247) is equivalent to expression (2') above.

Now, define \( h_m(p) \equiv 1/[q'_m(p)/\pi''_m(p)] > 0 \) so that

\[
Q'(r) = \left(-\frac{q'_s q'_w}{\pi''_s + \pi''_w}\right) \left[ h_s p_s(r) - h_w p_w(r) \right].
\]

(3)

We assume that \( h_m \) is decreasing (and call it the Decreasing Inverse Ratio Condition; DIRC).\(^{23}\)

It is also shown that

\[
\frac{Q''(r)}{2} = \left[-\frac{q'_s q'_w}{\pi''_s + \pi''_w}\right] [h'_s p'_s - h'_w p'_w] + [h_s - h_w] \frac{d}{dr} \left[-\frac{q'_s q'_w}{\pi''_s + \pi''_w}\right].
\]

Then, there exists \( \hat{r} \) such that \( Q'(\hat{r}) = 0 \) and \( Q''(\hat{r})/2 = [q'_s q'_w/(\pi''_s + \pi''_w)](h'_s p'_s - h'_w p'_w) < 0 \) because \( h'_s p'_s < 0 \) and \( h'_w p'_w > 0 \). Then, \( (1/2)Q(r) \) behaves on \([0, \hat{r}]\) in either manner.\(^{24}\)

1. If \( Q'(0) \leq 0 \), then \( (1/2)Q(r) \) is monotonically decreasing in \( r \), and as a result \( \Delta Q/2 = 

\(^{23}\)Note that \( h'_m < 0 \) is equivalent to \( \pi''_m > (\pi''_m/q''_m)q''_m \), where \( (\pi''_m/q''_m) > 0 \), because \( h'_m(p) = [\pi''_m q''_m - \pi''_m q''_m]/[q''_m]^2 \). Thus, the DIRC states that the profit function, starting from the zero price, increases quickly, attaining the optimal price, and the price decreases slowly as \( p \) becomes larger and larger beyond the optimum. In this way, the optimal price is reached “close” enough to the zero price, rather than “still climbing up” even far away from it. To see this, if \( q''_m > 0 \), then it is necessary to assume \( \pi''_m > 0 \). This means that \( \pi''_m \), which is negative, should be larger, that is, the negative slope of \( \pi''_m \) should be gentler, as \( p \) increases. If \( q''_m \leq 0 \), then \( \pi''_m \) should be, whether it is positive or negative, sufficiently large. In either case, as \( p \) increases, \( \pi_m \) increases quickly below the optimum, and decreases slowly beyond it.

\(^{24}\)This is because the modified version of Aguirre, Cowan and Vickers’ (2010, p.1605) Lemma also holds in our oligopoly setting.
\[ (Q(r^*) - Q(0))/2 < 0; \text{ price discrimination lowers aggregate output.} \]

2. If \( Q(0) > 0 \), then \((1/2)Q(r)\) either

(a) is monotonically increasing (if \( Q'(r^*) > 0 \), this is true), and as a result, \( \Delta Q/2 > 0 \); 
\text{ price discrimination raises aggregate output.}

(b) first increases, and then after the reaching the maximum (where \( Q'(r) = 0 \), decreases until \( r = r^* \). In this case, \text{ price discrimination may raise or lower aggregate output};

it cannot be determined whether \( \Delta Q/2 < 0 \) or \( \Delta Q/2 > 0 \) without further functional and/or parametric restrictions.

Now, we determine the sign of \( Q'(0) \). It follows that sign\[Q'(0)] = \text{sign}[\pi''(\bar{p})/q'_w(p) - \pi''(\bar{p})/q'_w(p)]\], which implies that \( Q'(0) \leq 0 \Leftrightarrow \pi''(\bar{p})/q'_w(p) \geq \pi''(\bar{p})/q'_w(p) \). Note also that sign\[Q'(r^*)] = \text{sign}[h_s(p^*_s) - h_w(p^*_w)], which implies that \( Q'(r^*) > 0 \Leftrightarrow \pi''(p^*_s)/q'_w(p^*_w) \geq \pi''(p^*_w)/q'_w(p^*_w) \). Because

\[ \frac{\pi''(\bar{p})}{q'_w(p)} = \frac{[2 - L_m(\bar{p})\alpha_m^F(\bar{p})] - [1 - \theta_m(\bar{p})]}{\theta_m(\bar{p})} \]

holds, the following proposition obtains.

**Proposition 1.** Given the DIRC, if \( \bar{\theta}_s > \bar{\theta}_w \) and

\[ \frac{\pi^F_s - (1 - \bar{\theta}_s)\pi^C_s}{\bar{\theta}_s} > \frac{\pi^F_w - (1 - \bar{\theta}_w)\pi^C_w}{\bar{\theta}_w}, \]

then price discrimination lowers aggregate output. If \( \theta^*_w > \theta^*_s \) and

\[ \frac{(\alpha^F_w)^* - (1 - \theta^*_w)(\alpha^C_w)^*}{\theta^*_w} > \frac{(\alpha^F_s)^* - (1 - \theta^*_s)(\alpha^C_s)^*}{\theta^*_s}, \]

then price discrimination raises aggregate output.

**Proof.** See Appendix, part B.

It is seen that this proposition is a generalization of Aguirre, Cowan, and Vickers’ (2010, p. 1609) Proposition 4, (i) and (ii). If \( \theta_s \) and \( \theta_w \) are sufficiently close to one (i.e., monopoly) under either uniform pricing or price discrimination, then inequality (5) is approximated by \( \alpha^F_s \geq \alpha^F_w \). Thus, the statement in Aguirre, Cowan, and Vickers’ (2010, p. 1609) Proposition 4, (ii), “[i]f demand is convex in the strong market and concave, or less convex, in the weak market then output decreases discrimination” should be started with “[i]f the firm’s own part of the demand.” This is also true for the complementary case where inequality (6) is approximated by \( \alpha^F_w \geq \alpha^F_s \).

More interestingly, equality (3) can be rewritten in the following manner. First, we define \text{pass-through} in market \( m \) by \( \rho_m \equiv \frac{\partial p_m}{\partial c_m} \). It is a function of \( r \in [0, r^*] \) when the constrained
problem is considered. In particular, for \( r < r^* \), one obtains \( \rho_m[p_m(r)] = \frac{\partial x_{Am}/\partial p_A}{\pi''_m + \pi''_w} \) by applying the implicit function theorem to \( \partial_p \pi_s(p_w + r) + \partial_p \pi_m(p_w) = 0 \). For \( r = r^* \) (i.e., under price discrimination), pass-through is given by \( \rho^*_m = \frac{\partial x_{Am}/\partial p_A}{\pi''_m} \) from \( \partial_p \pi_m(p_m) = 0 \). Then, if the marginal costs are constant, quantity pass-through in market \( m \) under price discrimination, which is defined by \( \frac{dq^*_m}{dq} \), where \( \tilde{q} \) is an exogenous amount of output with \( \pi_{jm}(p_{jm}, p_{-j,m}) = (p_{jm} - c_{m})[x_{jm}(p_{jm}, p_{-j,m}) - \tilde{q}] \), is expressed by

\[
\frac{dq^*_m}{dq} = q'_m(p^*_m) \cdot \frac{dp^*_m}{dq} = q'_m \cdot \frac{\partial x_{Am}}{\partial p_A} \cdot \frac{dp^*_m}{dq} = \left( \frac{q'_m}{\frac{\partial x_{Am}}{\partial p_A}} \right) \cdot \left( \frac{\partial x_{Am}}{\partial p_A} \pi''_m \right) = \theta^*_m \cdot \rho^*_m
\]

because the first-order condition with \( \tilde{q} \) indicates \( \frac{dp^*_m}{dq} = \frac{1}{\pi'_m} \). Note here that \( q'_m/\pi''_m = \theta^*_m \cdot \rho^*_m \), which implies that from equality (3),

\[
\frac{Q'(r^*)}{2} = \left( -\frac{q'_m q'_w}{\pi'_s + \pi'_w} \right) \left( \frac{1}{\theta^*_wp^*_w} \right) + \frac{1}{\theta^*_w p^*_w},
\]

whereas for \( r < r^* \),

\[
\frac{Q'(r)}{2} = \left( \frac{q'_w}{\partial x_{Am}/\partial p_A} \right) \pi'_w + \left( \frac{q'_s}{\partial x_{Am}/\partial p_A} \right) \pi'_s = \theta_w \left( \frac{\partial x_{Am}/\partial p_A}{\pi''_s + \pi''_w} \right) \pi''_w + \theta_s \left( \frac{\partial x_{Am}/\partial p_A}{\pi''_s + \pi''_w} \right) \pi''_w,
\]

\[
= \theta_w \left( \frac{\theta_w(p_w)(r) - \theta_s(p_s)(r)}{\pi''_w} \right) - \theta_s \pi''_w,
\]

Summarizing these arguments, we obtain the following proposition.

**Proposition 2.** Given the DIRC, if \( \theta^*_w \rho^*_w \geq \theta^*_s \rho^*_s \) holds, the price discrimination raises aggregate output. Conversely, if

\[
\frac{\bar{q}_w}{\bar{p}_w} \leq \frac{\pi''_w(\bar{p})}{\pi''_w(\bar{p})},
\]

holds, then price discrimination lowers aggregate output.

As stated above, \( \theta^*_m \rho^*_m \) is interpreted as quantity pass-through under price discrimination in market \( m \) if the marginal cost is constant. Thus, the first part of the proposition simply claims

\[25\] Note that this is the case where \( \frac{dq^*_m}{dq} \) is evaluated at \( \tilde{q} = 0 \): Miklós-Thal and Shaffer (2019b) derive a general formula for \( \tilde{q} > 0 \), correcting Weyl and Fabinger’s (2013) arguments. If the marginal costs are non-constant, then \( \pi_{jm}(p_{jm}, p_{-j,m}) = p_{jm} \cdot [x_{jm}(p_{jm}, p_{-j,m}) - \tilde{q}] - c_{m}[x_{jm}(p_{jm}, p_{-j,m}) - \tilde{q}] \) should be considered, where \( c_{m}(\cdot) \) is the cost function, and thus \( \theta^*_m \rho^*_m \) is no longer the quantity pass-through under price discrimination (that is, when \( \tilde{q} = 0 \)). See Weyl and Fabinger (2013, p. 572) for a precise expression of quantity pass-through with non-constant marginal costs.
that aggregate output is raised by price discrimination if the marginal reduction in quantity caused by a small deviation from price discrimination in the market where price discrimination raises output (i.e., the weak market) is larger than the marginal increase in quantity in the strong market. The second part describes the opposite case, although it does not permit a direct comparison because pass-through is not defined market-wise unless the pricing regime is “perfect” or “full” price discrimination (i.e., \( r = r^* \)), where the first-order conditions are given market-wise. Note that if \( |\pi_m''| \) is small, then \( \pi_m \) is “flat,” and thus the price shift \( |\Delta p_m| \) in response to some change would be large. Hence, the role of \( \pi''_w/\pi''_s \) is to adjust measurement units for \( \rho_w/\rho_s \). For example, if \( |\pi''_w| \) is very small, then the \( \rho_w \) is “over represented,” and thus it should be “penalized” so that right hand side of the inequality in the proposition becomes small.

3.2 Social Welfare

Now, we study the effects of allowing third-degree price discrimination on social welfare. To proceed further, note that

\[
\frac{W'(r)}{2} = \left( -\frac{\pi''_w \pi''_w}{\pi''_w + \pi''_w} \right)_{>0} \times \left( (p_w(r) - c_w)q'_w[p_w(r)] - \frac{(p_s(r) - c_s)q'_s[p_s(r)]}{\pi''_s} \right). 
\]

We follow Aguirre, Cowan, and Vickers (2010, p. 1605), who define \( z_m(p) \equiv (p-c_m)q'_m(p)/\pi''_m(p) \), which is “the ratio of the marginal effect of a price increase on social welfare to the second derivative of the profit function.” However, our \( q'_m \) and \( \pi''_m \) have strategic effects. More specifically, our \( q'_m \) and \( \pi''_m \) are written as

\[
q'_m(p_m) = \frac{\partial x_{Am}}{\partial p_A}(p_m, p_m) + \frac{\partial x_{Bm}}{\partial p_B}(p_m, p_m) < 0 \text{ (ACV’s } q'_m \text{)}
\]

and

\[
\pi''_m(p_m, c_m) = D^2 \pi_m(p_m, c_m) + G_m(p_m, c_m),
\]

where

\[
G_m(p) = \frac{\partial x_{Am}}{\partial p_B}(p, p) + (p - c_m) \left[ \frac{d}{dp} \left( \frac{\partial x_{Am}}{\partial p_A}(p, p) \right) - \frac{\partial^2 x_{Am}}{\partial p^2_A}(p, p) \right].
\]
As in Aguirre, Cowan, and Vickers (2010, p. 1605), we can write
\[
\frac{W'(r)}{2} = \left( -\frac{\pi''_w}{\pi'_w + \pi''_w} \right) \{ z_w[p_w(r)] - z_s[p_s(r)] \},
\]
and their lemma also holds in our case of oligopoly if we assume \( z_m \) is increasing (the Increasing Ratio Condition; IRC).\(^{26}\) It is also shown that
\[
\frac{W''(r)}{2} = \left( -\frac{\pi''_w}{\pi'_w + \pi''_w} \right) (z'_w p'_w - z'_s p'_s) + (h_w - h_s) \frac{d}{dr} \left( -\frac{\pi''_w}{\pi'_w + \pi''_w} \right).
\]
Then, as \((1/2)Q(r)\) does, \((1/2)W(r)\) behaves on \([0, r^*]\) in either manner:\(^{27}\)

1. If \(W'(0) \leq 0\), then \((1/2)W(r)\) is monotonically decreasing in \(r\), and as a result \(\Delta W/2 = [W(r^*) - W(0)]/2 < 0\); price discrimination lowers social welfare.

2. If \(W'(0) > 0\), then \((1/2)W(r)\) either
   - (a) is monotonically increasing (if \(W'(r^*) > 0\), this is true), and as a result, \(\Delta W/2 > 0\); price discrimination raises social welfare.
   - (b) first increases, and then after the reaching the maximum (where \(W'(r) = 0\)), decreases until \(r = r^*\). In this case, price discrimination may raise or lower social welfare: it cannot be determined whether \(\Delta W/2 < 0\) or \(\Delta W/2 > 0\) without further functional and/or parametric restrictions.

Now, we determine the sign of \(W'(0)\). First, define the markup in market \(m\) by \(\mu_m(p) \equiv p - c_m\). Then, it follows that \(\text{sign}[W'(0)] = \text{sign}[\mu_w(\bar{p}) q'_w(\bar{p})/\pi''_w(\bar{p}) - \mu_s(\bar{p}) q'_s(\bar{p})/\pi''_s(\bar{p})]\), and thus, the following proposition is obtained.

**Proposition 3.** Given the IRC, if the markup in strong market relative to the weak market at the uniform price \(\bar{p}\) is sufficiently large, i.e.,
\[
\frac{\mu_s(q'_s(\bar{p}))}{\pi''_s(\bar{p})} > \frac{\mu_w(q'_w(\bar{p}))}{\pi''_w(\bar{p})},
\]
then price discrimination lowers social welfare.

\(^{26}\)Note that \(z'_m(p) = \{(p - c_m) q'_m(p) + q'_m(p) \pi''_m(p) - (p - c_m) q'_m(p) \pi''_m(p)\}/(\pi''_m(p))^2\) and thus, the IRC is equivalent to \([(p - c_m) q'_m(p) + q'_m(p) \pi''_m(p) > (p - c_m) q'_m(p) \pi''_m(p)]\) and as a result, the IRC in the case of monopoly. If \(h_m\) is decreasing, as we assume throughout, then \(z_m\) is increasing because \(z'_m(p) = [1 - z_m(p) h_m(p)]/h_m(p)\) so that \(z'_m\) is positive if \(h'_m\) is negative. That is, the DIRC is a sufficient condition for the IRC to hold.

\(^{27}\)This is because the modified version of Aguirre, Cowan and Vickers’ (2010, p. 1605) Lemma also holds in our oligopoly setting.
If there are no strategic effects (i.e., \( \frac{\partial x_{Bm}}{\partial p_A} = 0 \) or \( \theta_m(p) = 1 \)), then \( \frac{\pi''_m(p)}{q'_m(p)} = 2 - L_m(p)\alpha_s^F \), and inequality (7) above reduces to \( \pi''_s(p)/q'_s(p) \geq \pi''_w(p)/q'_w(p) \). On the other hand, if there are no cost differentials (i.e., \( c_s = c_w \equiv c \)), then inequality (7) above reduces to \( \pi''_s(p)/q'_s(p) \geq \pi''_w(p)/q'_w(p) \) because the markups are the same in the two markets. Thus, if there are no strategic effects and no cost differentials, then inequality (7) coincides with Aguirre, Cowan and Vickers’ (2010, p. 1605) Proposition 1 (\( \alpha_s^F \geq \alpha_w^F \) in our notation; in their notation, \( \alpha_s(p) \geq \alpha_w(p) \)) because \( L_s(p) = L_w(p) \). That is, the firm’s “direct demand function in the strong market is at least as convex as that in the weak market at the nondiscriminatory price” (Aguirre, Cowan and Vickers 2010, p. 1602).

Recall that in our case of oligopoly, equality (4) holds, which leads to the following corollary, another expression for the sufficient condition for price discrimination to lower social welfare in the case of no cost differentials.

**Corollary 1.** Suppose there are no cost differentials across markets (\( c_s = c_w \)). Given the IRC, if \( \overline{\theta}_w \geq \overline{\theta}_s \) and

\[
\frac{\alpha_s^F - (1 - \overline{\theta}_s)\alpha_s^C}{\alpha'_s} \geq \frac{\alpha_w^F - (1 - \overline{\theta}_w)\alpha_w^C}{\alpha'_w},
\]

then price discrimination lowers social welfare.

This is because

\[
\frac{\pi''_s(p)}{q'_s(p)} - \frac{\pi''_w(p)}{q'_w(p)} \leq 0 \iff \overline{L} \cdot \left( \frac{\alpha'_w - (1 - \overline{\theta}_w)\alpha_w^C}{\theta_w} - \frac{\alpha'_s - (1 - \overline{\theta}_s)\alpha_s^C}{\overline{\theta}_s} \right) + \left( \frac{1}{\overline{\theta}_s} - \frac{1}{\theta_w} \right) \leq 0,
\]

where \( \overline{L} \equiv L_s(p) = L_w(p) \).

Now, using market power, markup, and pass-through, we obtain the following sufficient conditions for price discrimination to raise or lower social welfare.

**Proposition 4.** Given the IRC, if \( \theta_w^* \mu_w^* \rho_w^* \geq \theta_s^* \mu_s^* \rho_s^* \) holds, then price discrimination raises social welfare. Conversely, if

\[
\frac{\overline{\theta}_w \mu_w \rho_w}{\overline{\theta}_s \mu_s \rho_s} \leq \frac{\pi''_w(p)}{\pi''_s(p)}
\]

holds, then price discrimination lowers social welfare.

**Proof.** See Appendix, part C.

Roughly speaking, if either (i) market power (\( \theta \)), (ii) markup (\( \mu \)), or (iii) pass-through (\( \rho \)) is sufficiently small in the strong market, then social welfare is likely to be higher under price discrimination. In particular, if these three measures are calculated (or estimated) in each
separate market, then it would assist one to judge whether price discrimination is desirable from a society’s viewpoint. As explained above, if the marginal costs are constant, $\theta^*_m \rho^*_m$ is interpreted as quantity pass-through: $\mu^*_m \times \theta^*_m \rho^*_m$ approximates the trapezoid generated by a small deviation from (perfect) price discrimination that captures the marginal welfare gain in the strong market and the marginal welfare loss in the weak market. If the latter is larger than the former, such a deviation lowers social welfare, and owing to the IRC, this argument extends globally so that the regime switch to uniform pricing definitely lowers social welfare. Note that this comparison will be a little bit more involved when starting at uniform pricing for the same reason as explained after Proposition 2.

Even if there are no cost differentials (i.e., $c_s = c_w$), this expression cannot be further simplified. In other words, this expression is already robust to the inclusion of cost differentials. Now, if we further assume that there are no strategic effects (i.e., $\theta_m = 1$), then the above condition becomes $(p^*_s - c)/(p^*_w - c) \leq (1/\rho^*_s)/(1/\rho^*_w)$, which coincides with $(p^*_w - c)/[2 - (\sigma^*_w)^*] \geq (p^*_s - c)/(2 - (\sigma^*_s)^*)$ in Proposition 2 of Aguirre, Cowan and Vickers (2010, p. 1606), where $\sigma^*_m(q) \equiv -q \rho''/\rho'$ is the curvature of the industry’s inverse demand function (in symmetric pricing), because it is shown that $\rho^*_m = 1/(2 - \sigma^*_m)$ in our oligopoly setting as well. Thus, price discrimination raises social welfare “if the discriminatory prices are not far apart and the inverse demand function in the weak market is locally more convex than that in the strong market” (Aguirre, Cowan and Vickers 2010, p. 1602).

This proposition also has the following attractive feature. Suppose that price discrimination is being conducted. Then, to evaluate it from a viewpoint of social welfare, one only needs the local information: first, $\theta^*_m, \mu^*_m$ and $\rho^*_m$ for each $m = s, w$, are computed, and if the sufficient condition above is satisfied, then the ongoing price discrimination is justified. In addition, to compute $\theta^*_m, \mu^*_m$ and $\rho^*_m$ in equilibrium, information on the marginal cost is unnecessary: once a specific form of demand function in market $m$ for firm $j$, $q_{jm} = x_{jm}(p_{jm}, p_{-j,m})$ is provided (and if the IRC is satisfied), then the three variables are computed in the following manner: $\theta^*_m = 1 - (\varepsilon^C_j)/\varepsilon^F_j), \mu^*_m = \rho^*_m = [q_m(p^*_m)]^2/[2[q_m(p^*_m)]^2 - q_m(p^*_m)q_m(p^*_m)].^{28}$

Thus, if the firm’s demand for each market $m$ is estimated and the discriminatory price $p^*_m$ is observed, then one can easily compute $\theta^*_m, \mu^*_m$, and $\rho^*_m$, using up to the second-order demand characteristics.$^{29}$

To the best of our knowledge, this proposition is the most general statement on when allowing oligopolistic firms to price discriminate raises or lowers social welfare, allowing cost differentials as Chen and Schwartz (2015) and Chen, Li, and Schwartz (2019) do for monopoly and oligopoly, respectively. More realistically, different firms would have different values for

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28 An alternative expression for $\mu^*_m$ if the cost information is used is $\mu^*_m = c_m/[(\varepsilon^F_j)^* - 1]$.  
29 It should be emphasized that the second-order supply property, i.e., the derivative of the marginal cost, would be necessary if non-constant marginal cost is allowed, as suggested by Adachi and Fabinger (2019) in the context of issues of general “taxation” (pure taxation and other additional costs from external changes).
3.3 Parametric Examples

To consider the following parametric examples, we assume that there are two (strong and weak), and two symmetric firms.

**Example 1: Linear Demands**  Linear demands are derived from the quadratic utility of the representative consumer in market \( m \) under symmetric product differentiation:

\[
U_m(q_m) = \omega_m \cdot (q_{Am} + q_{Bm}) - (1/2) \left( \beta_m [q_{Am}]^2 + 2\gamma_m q_{Am} q_{Bm} + \beta_m [q_{Bm}]^2 \right),
\]

which yields linear inverse demands, \( p_{jm}(q_{jm}, q_{-j,m}) = \omega_m - \beta_m q_{jm} - \gamma_m q_{j-m} \), and the corresponding direct demand in market \( m \) is

\[
x_{jm}(p_{jm}, p_{-j,m}; \omega_m, \beta_m, \gamma_m) = \frac{1}{(1 - \delta_m^2)\beta_m} [\omega_m (1 - \delta_m) - p_{jm} + \delta_m p_{-j,m}]
\]

for firm \( j \), where \( \delta_m \equiv \gamma_m / \beta_m \in [0, 1) \) is the strength of substitutability: if \( \delta_m \) is close to one, market \( m \) is approximated by perfect competition, while each firm behaves as a monopolist if \( \delta_m \) is equal to zero. In symmetric equilibrium, the firm’s demand in market \( m \) is

\[
q_m(p) = (\omega_m - p)/(1 + \delta_m)\beta_m
\]

and thus, \( \varepsilon_{F_j}^m(p) = -1/(1 + \delta_m)\beta_m \) and \( \epsilon^l_{m}(p) = p/(\omega_m - p) \) and the firm’s own price elasticity in market \( m \) is

\[
\varepsilon_{F_j}^m(p) = -\frac{(1 + \delta_m)\beta_m}{\omega_m - p} \left( -\frac{1}{(1 - \delta_m^2)\beta_m} \right) = \frac{p}{(1 - \delta_m)(\omega_m - p)},
\]

which implies that the discriminatory price in market \( m \) satisfies:

\[
\frac{p_m^* - c_m}{p_m^*} = \frac{(1 - \delta_m)(\omega_m - p_m^*)}{p_m^*} \iff p_m^* = p_m^*(c_m, \omega_m, \delta_m) \equiv \frac{(1 - \delta_m)\omega_m + c_m}{2 - \delta_m},
\]

and thus, \( \rho_m^* \equiv (p_m^*)/(c_m) = 1/(2 - \delta_m) \). Next, consider the uniform price. In symmetric equilibrium,

\[
\bar{y}_m \varepsilon_{F_j}^m = \frac{\omega_m - \bar{p}}{(1 + \delta_m)\beta_m} \left( \frac{\bar{p}}{1 - \delta_m(\omega_m - \bar{p})} \right).
\]
Figure 1: \( \theta_m \) with linear demands. In both cases, \( (p_m - c_m) / (\omega_m - p_m) \) is almost identical.

\[
\theta_m(p_m) = \frac{p_m - c_m}{\omega_m - p_m}
\]

for \( m = s, w \), which implies that the equilibrium uniform price \( \bar{p} \) satisfies

\[
\sum_{m=s,w} \frac{\bar{p} - c_m}{(1 - \delta_m^2)\beta_m} = \sum_{m=s,w} \frac{\omega_m - \bar{p}}{(1 + \delta_m)\beta_m},
\]

leading to the explicit solution:

\[
\bar{p} = \bar{p}(c, \omega, \beta, \delta) \equiv \frac{\sum_{m=s,w} (1 - \delta_m)\omega_m + c_m (1 - \delta_m^2)}{\sum_{m=s,w} (1 - \delta_m^2)^2 \beta_m}.
\]

The market power index in the case of linear demands is \( \theta_m(p_m) = (p_m - c_m) / (\omega_m - p_m) \). In each panel of Figure 1, the demand curve is depicted so that the ratio of \( (p_m - c_m) \) to \( (\omega_m - p_m) \) takes almost the same value. However, in the left panel, the markup rate is high. This is mainly due to a low elasticity of the industry’s demand rather than a low cross-price elasticity: the rivalness between brands may be sufficiently high. In the right panel, the markup rate is low. However, this does not necessarily mean that brands are in a strong rivalry. Instead, \( \varepsilon_m^f(p_m) \) can be low as long as \( \varepsilon_m^l(p_m) \) is sufficiently high (that \( \varepsilon_m^F(p_m) \) is high, and as a result, the markup rate is low). In both panels, the market is close to monopoly because \( \theta_m(p_m) \) is close to one. This graphical example shows that \( \theta_m(p_m) \), rather than \( L_m(p_m) \), better captures the competitiveness in market \( m \). Because both the curvature of the firm’s direct demand and the
elasticity of the cross-price effect are zero ($\alpha^F_m(p) = 0$ and $\alpha^C_m(p) = 0$), it is observed that

$$\frac{\pi''_m(p)}{q''_m(p)} = \frac{[2 - L_m(p)\alpha^F_m(p)] - (1 - \theta_m(p))[1 - L_m(p)\alpha^C_m(p)]}{\theta_m(p)}$$

$$= \frac{1 + \theta_m(p)}{\theta_m(p)}$$

$$= \frac{\omega_m - c_m}{p - c_m}$$

$$= \frac{2 - \delta_m}{1 - \delta_m},$$

which implies that $Q'(r) > 0$ if and only if $\delta_s > \delta_w$, that is, the firms’ products are less differentiated in the strong market.

Now, consider the sufficient condition for price discrimination to lower social welfare in Proposition 3: $G(c_s, c_w) \geq 0$, where

$$G(c_s, c_w) = \frac{\mu_s[p^*_s(c_s)]}{\mu_u[p^*_w(c_w)]} - \frac{1/\theta_s[p^*_w(c_w)]\rho_s[p^*_w(c_w)]}{1/\theta_u[p^*_w(c_w)]\rho_u[p^*_w(c_w)]}$$

$$= \frac{\mu_s[p^*_s(c_s)]}{\mu_u[p^*_w(c_w)]} - \frac{(2 - \delta_s)(1 - \delta_w)}{(1 - \delta_s)(2 - \delta_w)}.$$

Similarly, consider the sufficient condition for price discrimination to raise social welfare in Proposition 4: $H(c_s, c_w) \leq 0$ where

$$H(c_s, c_w) = \frac{p^*_s - c_s}{p^*_w - c_w} - \frac{\theta_w(p^*_w)}{\theta_s(p^*_w)}$$

$$= \frac{p^*_s - c_s}{p^*_w - c_w} - \frac{p^*_w - c_w}{p^*_s - c_s} \frac{\omega_s - p^*_s}{\omega_w - p^*_w}.$$
raise social welfare: in our notation, it is \( \omega_s - \omega_w \leq 3(c_s - c_w) \) (with \( \gamma_s = 0 \) and \( \gamma_w = 0 \)). As Panel (a) in Figure 2 shows, our sufficient condition, \( c_s \geq c_w \), for price discrimination to raise social welfare in the case of monopoly is weaker than Chen and Schwartz’ (2015, p. 454) necessary and sufficient condition. Under oligopoly, however, the \((c_s, c_w)\) region for sufficiency for \( \Delta W \geq 0 \), i.e., the region of \( H(c_s, c_w) \leq 0 \) is now smaller, whereas the region for sufficiency for \( \Delta W \leq 0 \), i.e., the region of \( G(c_s, c_w) \geq 0 \) becomes larger (Panel b).

In this numerical example, the line of \( c_w = c_s \) is included in the region of \( G(c_s, c_w) \geq 0 \); price discrimination lowers social welfare when the marginal costs are common across markets. However, it is possible that social welfare is higher under price discrimination in this case, more specifically, if \( \delta_s \equiv \gamma_s/\beta_s \) is sufficiently higher than \( \delta_w \equiv \gamma_w/\beta_w \), as shown in Figure 3, where \( c_s = c_w = 0.2 \) and \( G \) and \( H \) are interpreted as \( G(\delta_s, \delta_w) \) and \( H(\delta_s, \delta_w) \), respectively. This example is consistent with Adachi and Matsushima’s (2014, p.1239) Proposition 1 (and their Figures 4 and 5) with an additional result: here, the sufficient condition for \( \Delta W \leq 0 \) is also included, whereas their Proposition 1 establishes a necessary and sufficient condition for \( \Delta W > 0 \) in the case of linear demands.

Example 2: Logit Demands In each market \( m = 1, 2 \), firm \( j \) faces the following market share:

\[
x_{jm}(p_{jm}, p_{-j,m}) = \frac{\exp(\omega_m - \beta_m p_{jm})}{1 + \sum_{j'=A,B} \exp(\omega_m - \beta_m p_{j'm})} \in (0, 1),
\]

where \( \omega_m > 0 \) is now the product-specific utility and \( \beta_m > 0 \) is the responsiveness of the representative consumer in market \( m \) to the price. Then, under symmetric pricing, each firm’s share is

\[
q_m(p) = \frac{\exp(\omega_m - \beta_m p)}{1 + 2 \exp(\omega_m - \beta_m p)}
\]

and the symmetric discriminatory equilibrium price \( p^*_m = p^*_m(c_m, \omega_m, \beta_m) \) satisfies

\[
\frac{q^*_m - c_m}{\mu^*_m} = \frac{1}{\beta_m(1 - q^*_m)} = 0,
\]

where \( q^*_m \equiv q^*_m(p^*_m) \):

\[
q^*_m(p^*_m) = \frac{\exp(\omega_m - \beta_m p^*_m)}{1 + 2 \exp(\omega_m - \beta_m p^*_m)}.
\]

Both \( p^*_m \) and \( q^*_m \) should be jointly solved numerically. Now, it is shown that

\[
H(c_s, c_w) = \left( \frac{1 - q^*_w(c_w)}{1 - q^*_s(c_s)} \right) \left( \frac{\beta_w}{\beta_s} \left( \frac{1 - 2q^*_w(c_w)}{1 - 2q^*_s(c_s)} \cdot \frac{1 - 3q^*_s(c_s) + 3[q^*_s(c_s)]^2}{1 - 3q^*_w(c_w) + 3[q^*_w(c_w)]^2} \right) \right).
\]

\[30\]The indirect utility of the representative consumer in market \( m \) is given by \( V_m(p_m) = \ln \{ \exp(\omega_m - \beta_m p_{Am}) + \exp(\omega_m - \beta_m p_{Bm}) \} / \beta_m \) (Anderson, de Palma, and Thissse 1987).
Figure 2: Linear demands with $\omega_s = 1.2$, $\omega_w = 0.8$, $\beta_s = 1.2$, and $\beta_w = 1.4$. For (a), $\gamma_s = \gamma_w = 0$ (monopoly), whereas for (b), $\gamma_s = 0.2$ and $\gamma_w = 0.7$ (duopoly). For $p_w^*$ to be actually lower than $p_s^*$, $c_w$, relative to $c_s$, must be sufficiently small. Specifically, the region of $(c_s, c_w)$ is restricted to the area below the dashed thick line in the upper left. In each panel, the dashed line corresponds to Chen and Schwarz’ (2015) threshold for the necessity and sufficiency for $\Delta W \geq 0$ in the case of monopoly. The region for $H \leq 0$ is the area below line $hh$. In panel (a), the region for $G \geq 0$ is the area between the dashed thick line and line $c_s = c_w$, whereas it is the area between the dashed thick line and line $gg$ in panel (b).
Figure 3: Linear demands with $\omega_s = 1.2$, $\omega_w = 0.8$, $\beta_s = 1.2$, and $\beta_w = 1.4$. It assumed that $c_s = c_w = 0.2$. As in Figure 2, the region of $(\delta_s, \delta_w)$ is restricted to the area above the dashed thick line in the lower right for $p_s^*$ to be actually higher than $p_w^*$. The region for $H \leq 0$ is the area between the dashed thick line and line $hh$, whereas the region for $G \geq 0$ is the area above line $gg$. 

Electronic copy available at: https://ssrn.com/abstract=3006421
because \((\varepsilon_F^*) = \beta_m p_m^*(1 - q_m^*)\), and \((\varepsilon_C^*) = \beta_m p_m^* q_m^*\), and thus,

\[
\theta_m^* = 1 - \frac{(\varepsilon_C^*)}{(\varepsilon_F^*)} = \frac{1 - 2q_m^*}{1 - q_m^*},
\]

and

\[
\rho_m^* = \frac{1}{1 - q_m^*(1 - 2q_m^*)}.
\]

Note here that in contrast to the linear demand, the two curvatures are positive: \((\alpha_F^*) = (\alpha_C^*) = \beta_m p_m^* (1 - 2q_m^*) > 0\). That is, the logit demand is convex in both own and cross directions.

It is also verified that

\[
G(c_s, c_w) = \left( \frac{1 - q_w[p(c_s, c_w)]}{1 - q_s[p(c_s, c_w)]} \right) \left( \frac{\beta_w}{\beta_s} \right) - \frac{\pi_m''[\bar{p}(c_s, c_w)]}{\pi_m''[\bar{p}(c_s, c_w)]} - \frac{\pi_w''[\bar{p}(c_s, c_w)]}{\pi_w''[\bar{p}(c_s, c_w)]} q_m^*[\bar{p}(c_s, c_w)] - \frac{\pi_w''[\bar{p}(c_s, c_w)]}{\pi_w''[\bar{p}(c_s, c_w)]} q_w^*[\bar{p}(c_s, c_w)] = 0
\]

where, with \(\alpha_m(p) \equiv \beta_m p[1 - 2q_m(p)]\) being the common value for the curvatures,

\[
\frac{\pi_m''(p)}{\pi_m'(p)} = \frac{1 + \theta_m(p)}{\theta_m(p)} - L_m(p)\alpha_m(p) = \frac{2 - 3q_m(p)}{1 - 2q_m(p)} - \beta_m(p - c_m)[1 - 2q_m(p)].
\]

The equilibrium uniform price \(\bar{p} = \bar{p}(c, \omega, \beta)\) satisfies

\[
\sum_{m=s,w} q_m(\bar{p}) \{1 - \beta_m(\bar{p} - c_m)[1 - q_m(\bar{p})]\} = 0,
\]

which should also be numerically solved.

Here, to see the role of the demand curvatures, we study the region of \((\beta_s, \beta_w)\), with a fixed value of \(c_s = c_w = 0.2\). Figure 4 shows that a higher value of \(\beta_w\), relative to \(\beta_s\), that is, a higher degree of convexity in the weak market, is associated with a negative change in social welfare by price discrimination. This result firstly appears not to be consistent with Aguirre, Cowan, and Vickers (2010), who emphasize that as the demand in the weak market becomes more convex, it is more likely that price discrimination raises social welfare because a larger increase in output in the weak market offsets the misallocation effect due to price discrimination. However, Aguirre, Cowan, and Vickers’ (2010) result holds if the demand in the strong market is concave. Here, the demand in the strong market is also convex. In this case, the uniform price is kept relatively low; thus an introduction of price discrimination again highlights the misallocation effect.
Figure 4: Logit demands with $\omega_s = 1.2$, $\omega_w = 0.8$, and $c_s = c_w = 0.2$. Panels (a) and (b) are monopoly and duopoly, respectively. As in Figures 2 and 3, the region of $(\beta_s, \beta_w)$ is restricted to the area above the dashed thick curve in the lower right for $p^*_s$ to be actually higher than $p^*_w$. The region for $G \geq 0$ is the area above curve $gg$, whereas the region for $H \leq 0$ is the area between the dashed thick curve and curve $hh$. 
3.4 Consumer Surplus

One can extend the analysis above to study consumer surplus. First, consumer surplus is defined by replacing $c_m$ in $W(r)$ by $p_m(r)$ to define

$$CS(r) = U_s(q_s[p_s(r)]) + U_w(q_w[p_w(r)]) - 2p_s(r) \cdot q_s[p_s(r)] - 2p_w(r) \cdot q_w[p_w(r)],$$

which implies that

$$\frac{CS'(r)}{2} = p_s(r) \cdot q_s'[r] + p_w(r) \cdot q_w'[r] - p_s'(r)(p_s(r) \cdot q_s + q_s) - p_w'(r)(p_w(r) \cdot q_w + q_w)$$

$$= -(p_s'(r)q_s + p_w'(r)q_w)$$

$$= \left( -\frac{\pi''_s p_m''}{\pi'_s + \pi''_w} \right) \left( \frac{q_s}{\pi'_s} - \frac{q_w}{\pi'_w} \right).$$

$$= \left( -\frac{\pi''_s p_m''}{\pi'_s + \pi''_w} \right) \{ q_s[p_s(r)] - g_w(p_w(r)) \},$$

where $g_m(p) \equiv q_m(p)/\pi''_m(p)$. If $g_m$ is assumed to be decreasing, then one can use a similar argument. We call this condition the Decreasing Marginal Consumer Loss Condition (DMCLC).\(^\text{31}\)

Then, $(1/2)CS(r)$ behaves on $[0, r^\star]$ in either manner:

1. If $CS'(0) \leq 0$, then $(1/2)CS(r)$ is monotonically decreasing in $r$, and as a result $\Delta CS/2 = [CS(r^\star) - CS(0)]/2 < 0$; price discrimination lowers consumer surplus.

2. If $CS'(0) > 0$, then $(1/2)CS(r)$ either

   (a) is monotonically increasing (if $CS'(r^\star) > 0$, this is true), and as a result, $\Delta CS/2 > 0$; price discrimination raises consumer surplus.

   (b) first increases, and then after the reaching the maximum (where $CS'(r) = 0$), decreases until $r = r^\star$. In this case, price discrimination may raise or lower consumer surplus: it cannot be determined whether $\Delta CS/2 < 0$ or $\Delta CS/2 > 0$ without further functional and/or parametric restrictions.

Thus, we determine the sign of $CS'(0)$. it follows that $\text{sign}[CS'(0)] = \text{sign}[q_s(\bar{p})/\pi'_s(\bar{p}) - q_w(\bar{p})/\pi'_w(\bar{p})]$, and thus, the following proposition is obtained.

\(^\text{31}\)Now, it is shown that $g_m' < 0$ is equivalent to $\pi''_m > (q_m''p_m)/q_m > 0$ because $g_m(p) = [q_m''p_m - q_m p_m'']/[\pi''_m]$. Thus, if $q_m'' > (q_m''p_m)/q_m$ $\Rightarrow q_m'' < \{ q_m''p_m'/q_m \},$ that is $q_m$ is not "too convex," then the DMCLC is a sufficient condition for the DIRC to hold. Thus, under this "not too convex" assumption, the relationship, "DMCLC $\Rightarrow$ DIRC $\Rightarrow$ IRC", holds.
Proposition 5. Given the DMCLC, if the output in the weak market at the uniform price \( \overline{p} \) is sufficiently large, i.e.,

\[
\frac{q_w(\overline{p})}{\pi''_w(\overline{p})} \geq \frac{q_s(\overline{p})}{\pi''_s(\overline{p})},
\]

then price discrimination lowers consumer surplus.

Then, using markup and pass-through, we can rewrite equality (8) as

\[
\frac{CS'(r)}{2} = \left( -\frac{\pi''_w \pi''_w}{\pi''} \right) \left( \frac{\mu_w(r) \mu_w(r)}{\pi''_w} - \frac{\mu_s(r) \rho_s(r)}{\pi''_s} \right)
\]

for \( r < r^* \), and

\[
\frac{CS'(r^*)}{2} = \left( -\frac{\pi''_w \pi''_w}{\pi''} \right) \left( \frac{\mu_w^* \rho_w^* - \mu_s^* \rho_s^*}{\pi''_w + \pi''_w} \right)
\]

for \( r = r^* \) which immediately leads to the following proposition.

Proposition 6. Given the DMCLC, if \( \mu_w^* \rho_w^* \geq \mu_s^* \rho_s^* \) holds, then price discrimination raises consumer surplus. Conversely, if

\[
\frac{\pi''_w}{\mu_w \rho_w} \leq \frac{\pi''_s(\overline{p})}{\mu_s \rho_s(\overline{p})}
\]

holds, then price discrimination lowers consumer surplus.

Alternatively, it is also possible to directly apply Cowan’s (2012) analysis of the effects of monopolistic third-degree price discrimination on consumer surplus to our case of oligopoly. From Cowan’s (2012, p. 338) inequality (5), the lower and the upper bounds for a change in consumer surplus, \( \Delta CS \), are given by

\[
\sum_{m \in S \cup W} [\pi'_m(\overline{p})] \mu_m^* \rho_m^* q'_m(\overline{p}),
\]

and

\[
\sum_{m \in S \cup W} [\pi'_m(\overline{p})] \rho_m(\overline{MTR}_m) \epsilon_m^F(\overline{p}),
\]

respectively, where \( \overline{MTR}_m \) is the “virtual” marginal cost with which the uniform price \( \overline{p} \) coincides with the equilibrium discriminatory price when no uniform pricing restriction is imposed.

3.5 Non-constant Marginal Costs

Notice that our results so far do not crucially depend on the assumption of constant marginal costs. The only caveat is the definition of pass-through: to properly define pass-through in
accommodation with non-constant marginal costs, we introduce a small amount of unit tax 
$t_m > 0$ in market $m$: the firm’s first-order derivative of the profit with respect to its own price 
(equality 1) is now replaced by 

$$\partial_p \pi_m(p) = q_m(p) + (p - t_m - mc_m[q_m(p)]) \frac{\partial \pi_{Am}}{\partial p_A}(p, p),$$

where $mc_m = c_m[q_m(p)]$ is the marginal cost at $q_m(p)$. Then, pass-through is defined by 
$\rho_m \equiv \frac{\partial p_m}{\partial t_m}$, and no other changes should be made to derive the results above. In fact, the 
usefulness of pass-through is that it can easily be accommodated with non-constant marginal 
costs (Weyl and Fabinger 2013; Fabinger and Weyl 2018; and Adachi and Fabinger 2019). An 
aditional caveat is that $\theta_m^* \rho_m^*$ is no longer interpreted as quantity pass-through under price 
discrimination (Weyl and Fabinger 2013, p. 572): one needs to take into account the “elasticity 
of the marginal cost” to approximate the trapezoids of the welfare gain and loss by a deviation 
from (full) price discrimination.

4 Firm Heterogeneity

In this section, we argue that the main thrusts under firm symmetry also hold when heteroge-
nous firms are introduced. Without loss of generality, we keep considering one strong market 
and one weak market: as explained in Footnote 15 above, an extension to the case of more 
than two markets is conceptually straightforward. We assume Corts’ (1998, p. 315) best re-
sponse symmetry: all firms agree on which market is strong and which market is weak. The 
case of best response asymmetry is studied by Corts (1998) (see also Footnote 8 above). The 
number of firms is $N$ ($\geq 2$). Then, each firm $j$ has the constraint, $p_{js} - p_{jw} \leq r_j$. Then, as 
above, firm $j$’s equilibrium price in the weak market under all of these constraints is written as 
$p_{jw}(r)$, where $r = (r_1, r_2, ..., r_N)$. Accordingly, firm $j$’s equilibrium price in the strong market 
is $p_{js}(r) = p_{jw}(r) + r_j$. Thus, the equilibrium price pair in market $m = w, s$ is written as 
$p_m(r) = (p_{1m}(r), p_{2m}(r), ..., p_{Nm}(r))$. Then, the social welfare is defined as a function of $r$:

$$W(r) \equiv U_s(q_s[p_s(r)]) + U_w(q_w[p_w(r)]) - c_s \cdot q_s[p_s(r)] - c_w \cdot q_w[p_w(r)],$$

where $q_m[p_m(r)] = (q_{1m}[p_m(r)], q_{2m}[p_m(r)], ..., q_{Nm}[p_m(r)])^T$ and $c_m = (c_{1m}, c_{2m}, ..., c_{Nm})^T$.

Now, let $r_j^* \equiv p_{jw}^* - p_{jw}$, for each $j = 1, 2, ..., N$. Then, each firm’s constraint is written as 
$0 \leq r_j = \lambda r_j^* \leq r_j^*$, with $\lambda \in [0, 1]$. Using this, we re-define the functions of $r$ as functions of 
one-dimensional variable, $\lambda$. In particular, the social welfare is written as:

$$W(\lambda) = U_s(q_s[p_s(\lambda)]) + U_w(q_w[p_w(\lambda)]) - c_s^T \cdot q_s[p_s(\lambda)] - c_w^T \cdot q_w[p_w(\lambda)].$$
Then, we use \( \partial_q U_m = p_m \) from the representative consumer's utility maximization problem in each market \( m \), where \( \partial_q U_m \equiv \left( \frac{\partial U_m}{\partial q_{1m}}, \frac{\partial U_m}{\partial q_{2m}}, \ldots, \frac{\partial U_m}{\partial q_{Nm}} \right) \), to derive

\[
W'(\lambda) = \sum_{m=s,w} \left[ (p_m - c_m)^T \cdot (\partial_{p_m} q_m \cdot p'_m) \right],
\]

where \( T \) denotes transpose, and

\[
\partial_{p_m} q_m \equiv \begin{pmatrix}
\frac{\partial q_{1m}}{\partial p_{1m}} \\
\vdots \\
\frac{\partial q_{Nm}}{\partial p_{1m}} \\
\end{pmatrix}
= \partial_{p_1m} q_m
\]

and \( p'_m \equiv (p'_{1m}(\lambda), p'_{2m}(\lambda), \ldots, p'_{Nm}(\lambda))^T \).

Now, we apply the implicit function theorem to \( f(p_w, \lambda) = 0 \), where

\[
f\left( p_w, \frac{\lambda}{N_1} \right) = \begin{pmatrix}
\partial_{p_{1s}} \pi_{1s}(p_w + \lambda r^*) + \partial_{p_{1w}} \pi_{1w}(p_w) \\
\vdots \\
\partial_{p_{js}} \pi_{js}(p_w + \lambda r^*) + \partial_{p_{jw}} \pi_{jw}(p_w) \\
\vdots \\
\partial_{p_{Ns}} \pi_{Ns}(p_w + \lambda r^*) + \partial_{p_{Nw}} \pi_{Nw}(p_w)
\end{pmatrix},
\]

is a collection of all firms' first-order conditions for profit maximization under regime \( \lambda \), to obtain \( p'_w(\lambda) = -[D_{p_w} f]^{-1}[D_\lambda f] \), where

\[
D_{p_w} f \equiv \begin{pmatrix}
\frac{\partial^2 \pi_{1s}}{\partial p_{1s}^2} + \frac{\partial^2 \pi_{1w}}{\partial p_{1w}^2} & \cdots & \frac{\partial^2 \pi_{1s}}{\partial p_{Ns} \partial p_{1s}} + \frac{\partial^2 \pi_{1w}}{\partial p_{Nw} \partial p_{1w}} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 \pi_{Ns}}{\partial p_{1s} \partial p_{Ns}} + \frac{\partial^2 \pi_{Nw}}{\partial p_{1w} \partial p_{Nw}} & \cdots & \frac{\partial^2 \pi_{Ns}}{\partial p_{Ns}^2} + \frac{\partial^2 \pi_{Nw}}{\partial p_{Nw}^2}
\end{pmatrix} = K
\]

\[
= \begin{pmatrix}
\frac{\partial^2 \pi_{1s}}{\partial p_{1s}^2} & \cdots & \frac{\partial^2 \pi_{1s}}{\partial p_{Ns} \partial p_{1s}} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 \pi_{Ns}}{\partial p_{1s} \partial p_{Ns}} & \cdots & \frac{\partial^2 \pi_{Ns}}{\partial p_{Nw}^2}
\end{pmatrix} = H_s
\]

\[
+ \begin{pmatrix}
\frac{\partial^2 \pi_{1w}}{\partial p_{1w}^2} & \cdots & \frac{\partial^2 \pi_{1w}}{\partial p_{Ns} \partial p_{1w}} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 \pi_{Nw}}{\partial p_{1w} \partial p_{Nw}} & \cdots & \frac{\partial^2 \pi_{Nw}}{\partial p_{Nw}^2}
\end{pmatrix} = H_w
\]

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and

\[
D_\lambda f = H_s \begin{pmatrix} r_1^s \\ \vdots \\ r_j^s \\ \vdots \\ r_N^s \end{pmatrix} = r^s
\]

Thus,

\[
p'_w(\lambda) = [-K^{-1}H_s] r^s
\]

and

\[
p'(\lambda) = -K^{-1}H_s r^s + r^s = \left[ I - K^{-1}H_s \right] r^s.
\]

so that

\[
W'(\lambda) = \left( [p - c_s]^T \partial_{p,q_s} [I - K^{-1}H_s] r^s \right) - \left( [p - c_w]^T \partial_{p,q_w} [K^{-1}H_s] r^s \right)
\]

\[
= \left( [p - c_s]^T \partial_{p,q_s} K^{-1} [K - H_s] \right) \left( H_s K^{-1} [K - H_s] \right) r^s
\]

\[
- \left( [p - c_w]^T \partial_{p,q_w} [H_w^{-1}] \left( H_w K^{-1} H_s \right) \right) r^s.
\]

Now, we define

\[
Z_m(p) \equiv \left( [p - c_m]^T \partial_{p,q_m} [p] \right) H_m^{-1}(p)
\]

1 × N

N × N

to proceed:

\[
W'(\lambda) = \left( Z_w - Z_s \right) \cdot \left( K - H_s \right) H_w^{-1}(\lambda) \left( -H_w K^{-1} H_s \right) r^s
\]

\[
= (Z_w - Z_s)(Tr^s),
\]

where \( \Gamma \equiv -H_w K^{-1} H_s \gg 0 \) is assumed. We also assume that multi-dimensional version of the IRC: for each market \( m \) and each firm \( j \), \( Z_{jm} \) is increasing in \( p_l, l = 1, 2, ..., N \).

Using this relationship, we can obtain the sufficient condition for price discrimination to improve social welfare with heterogeneous firms, which generalizes Proposition 4 in the previous section.

**Proposition 7.** Given the IRC, if \( [[\theta_w^*]^T \circ [\mu_w^*]^T] \rho_w^* > [[\theta_s^*]^T \circ [\mu_s^*]^T] \rho_s^* \) holds, where \( \circ \) indicates element-by-element multiplication, then price discrimination raises social welfare. Conversely, if \( [[\theta_w^*]^T \circ [\mu_w^*]^T] \rho_w < [[\theta_s^*]^T \circ [\mu_s]^T] \rho_s \delta \), where \( \delta \equiv K H_s^{-1} H_w K^{-1} \) is defined for adjustment.

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(a) Aggregate Output

If \( \theta^*_w \rho^*_w > \theta^*_s \rho^*_s \), then \( Q^* > Q \).
If \( \theta^*_w \rho^*_w < \theta^*_s \rho^*_s \), then \( Q^* < Q \).

(b) Social Welfare

If \( \theta^*_w \rho^*_w > \theta^*_s \rho^*_s \Delta \), then \( W^* > W \).
If \( \theta^*_w \rho^*_w < \theta^*_s \rho^*_s \Delta \), then \( W^* < W \).

(c) Consumer Surplus

If \( \mu^*_w \rho^*_w > \mu^*_s \rho^*_s \), then \( CS^* > CS \).
If \( \mu^*_w \rho^*_w < \mu^*_s \rho^*_s \Delta \), then \( CS^* < CS \).

Table 1: Summary of the Sufficient Conditions (with \( N \) heterogeneous firms, \( \theta_m, \mu_m, \) and \( \rho_m \) are the market power vector \( (N \times 1) \), the markup vector \( (N \times 1) \), and the pass-through matrix \( (N \times N) \), respectively, in market \( m = s, w \); asterisks and upper bars indicate price discrimination and uniform pricing, respectively; and \( \Delta \) is a term for adjustment defined in the text).

Our theoretical analysis, which takes into account firm heterogeneity, can also be utilized to empirically assess the welfare effects of third-degree price discrimination under oligopoly. In this sense, this paper, with the help of the methodology proposed by Weyl and Fabinger (2013), synthesizes Aguirre, Cowan, and Vickers’ (2010) analysis of monopolistic third-degree price discrimination with general demands and Chen and Schwartz’ (2015) analysis of monopolistic differential pricing and extends them to the case of differentiated oligopoly.

5 Concluding Remarks

This paper provides theoretical implications of oligopolistic third-degree price discrimination with general nonlinear demands, allowing cost differentials across separate markets. In this sense, this paper, with the help of the methodology proposed by Weyl and Fabinger (2013), synthesizes Aguirre, Cowan, and Vickers’ (2010) analysis of monopolistic third-degree price discrimination with general demands and Chen and Schwartz’ (2015) analysis of monopolistic differential pricing and extends them to the case of differentiated oligopoly.

Note that for each \( j \), \( Z_{jm} = \sum_{k=1}^{N} \theta^*_{km} \mu^*_{km} \rho^*_{jkm} \), which is interpreted as the weighted sum of firm \( j \)'s own pass-through \( (\rho^*_{jkm}) \) and the collection of its cross pass-through \( (\rho^*_{jkm}, k \neq j) \). For aggregate output and consumer surplus, we can readily generalize our previous results to the case of firm heterogeneity in a similar manner. Table 1 summarizes our results for heterogeneous firms.

Proof. See Appendix, part D. \( \square \)
particular, in line with the “sufficient statistics” approach (Chetty 2009), our predictions regarding the welfare effects do not rely on functional specifications, and are thus considered fairly robust. In practice, one needs to specify demand and supply functions to estimate/calibrate such sufficient statistics as price elasticity, curvature, market power, and pass-through. In this sense, these sufficient statistics can take different values, depending on functional specifications. However, once numerical values of sufficient statistics are obtained, there should be no disagreement as to welfare assessment.

As such, our methodology would also be extended to an analysis of welfare effects of wholesale/input third-degree price discrimination (Katz 1987; DeGraba 1990; Yoshida 2000; Inderst and Valletti 2009; Villas-Boas 2009; Arya and Mittenforf 2010; Li 2014; O’Brien 2014; Gaudin and Lestage 2018; and Miklós-Thal and Shaffer 2019a). To do so, one would need to properly define sufficient statistics at each stage of a vertical relationship. Another important issue is to consider multi-product oligopolistic firms. What happens if price discrimination is allowed for some products, whereas uniform pricing is enforced for other products? These and other important issues related to third-degree price discrimination await further research.

Appendix

A. Equivalence of Holmes’ (1989) and Our Expressions for $Q'(r)$

Holmes (1989, p.247), who assumes no cost differentials ($c \equiv c_s = c_w$) as in most of the papers on third-degree price discrimination, also derives a necessary and sufficient condition for $Q'(r) > 0$ under symmetric oligopoly. It is (using our notation) written as:

$$
\frac{p_s - c}{q'_s(p_s)} \cdot \frac{d}{dp_s} \left( \frac{\partial x_{A,s}(p_s, p_s)}{\partial p_A} \right) - \frac{p_w - c}{q'_w(p_w)} \cdot \frac{d}{dp_w} \left( \frac{\partial x_{A,w}(p_w, p_w)}{\partial p_A} \right) 
$$

Adjusted-concavity condition (Robinson 1933)

$$
\frac{\varepsilon^C_s(p_s)}{\varepsilon^I_s(p_s)} - \frac{\varepsilon^C_w(p_w)}{\varepsilon^I_w(p_w)} > 0.
$$

Elasticity-ratio condition (Holmes 1989)

Recall that $1/\theta_s - 1/\theta_w = \varepsilon^C_s / \varepsilon^I_s - \varepsilon^C_w / \varepsilon^I_w$. The first and the second terms in the left hand side of Holmes’ (1989) inequality is rewritten as:

$$
\frac{p_s - c}{q'_s(p_s)} \cdot \frac{d}{dp_s} \left( \frac{\partial x_{A,s}(p_s, p_s)}{\partial p_A} \right) - \frac{p_w - c}{q'_w(p_w)} \cdot \frac{d}{dp_w} \left( \frac{\partial x_{A,w}(p_w, p_w)}{\partial p_A} \right)
$$

\begin{align*}
&= L_w(p_w) \cdot \left[ \left( -\frac{p_w}{q'_w(p_w)} \right) \frac{d}{dp_w} \left( \frac{\partial x_{A,w}(p_w, p_w)}{\partial p_A} \right) \right] \\
&- L_s(p_s) \cdot \left[ \left( -\frac{p_s}{q'_s(p_s)} \right) \frac{d}{dp_s} \left( \frac{\partial x_{A,s}(p_s, p_s)}{\partial p_A} \right) \right].
\end{align*}

Now, it is also observed that
\[
\frac{\alpha^F_m - (1 - \theta_m)\alpha^C_m}{\theta_m} = \frac{\alpha^F_m}{\theta_m} - \frac{\partial x_{Bm}/\partial p_A}{p_m} \frac{\partial q'_m}{\partial p_A} \frac{\partial^2 x_{Am}}{\partial p_A \partial p_B}.
\]

This shows that inequality (2) is another expression for Holmes’ (1989, p. 247) inequality (9). To see this, note that
\[
\frac{d}{dp_m} \left( \frac{\partial x_{Am}(p_m, p_m)}{\partial p_A} \right) = \frac{\partial^2 x_{Am}}{\partial p_A^2} (p_m, p_m) + \frac{\partial^2 x_{Am}}{\partial p_A \partial p_B} (p_m, p_m)
\]
in Holmes’ (1989) expression is equivalent to \(-q'_m/p_m)(\alpha^F_m/\theta_m) + \partial^2 x_{Am}/(\partial p_A \partial p_B)\) because
\[
-\frac{q'_m}{p_m} \frac{\alpha^F_m}{\theta_m} = -\frac{q'_m}{p_m} \frac{\partial x_{Am}/\partial p_A}{p_m} \frac{\partial^2 x_{Am}}{\partial p_A^2} = \frac{1}{\theta_m} \frac{\partial x_{Am}/\partial p_A + \partial x_{Bm}/\partial p_A \partial^2 x_{Am}}{\partial p_A^2} = \frac{1}{\theta_m} (1 - A_m) \frac{\partial^2 x_{Am}}{\partial p_A^2} = \frac{\partial^2 x_{Am}}{\partial p_A^2}.
\]

**B. Proof of Proposition 1**

First, note that if
\[
\frac{[2 - L_w(\bar{p})\alpha^C_w(\bar{p})] - [1 - \theta_w(\bar{p})] [1 - L_w(\bar{p})\alpha^C_w(\bar{p})]}{\theta_w(\bar{p})} > \frac{[2 - L_s(\bar{p})\alpha^F_s(\bar{p})] - [1 - \theta_s(\bar{p})] [1 - L_s(\bar{p})\alpha^C_s(\bar{p})]}{\theta_s(\bar{p})},
\]
then price discrimination lowers aggregate output. The first part is a sufficient condition for this inequality to hold. Next, note that
\[
\frac{[2 - L_s(p^*_s)\alpha^F_s(p^*_s)] - [1 - \theta_s(p^*_s)] [1 - L_s(p^*_s)\alpha^C_s(p^*_s)]}{\theta_s(p^*_s)}
\]
\[
\frac{[2 - L_w(p_w^*)\alpha^F_w(p_w^*)] - [1 - \theta_w(p_w^*)] [1 - L_w(p_w^*)\alpha^C_w(p_w^*)]}{\theta_w(p_w^*)},
\]
then price discrimination raises aggregate output. Thus, \(\theta_w(p_w^*) > \theta_s(p_s^*)\) and

\[
L_w(p_w^*)\frac{\alpha^F_w(p_w^*) - [1 - \theta_w(p_w^*)] \alpha^C_w(p_w^*)}{\theta_w(p_w^*)} > L_s(p_s^*)\frac{\alpha^F_s(p_s^*) - [1 - \theta_s(p_s^*)] \alpha^C_s(p_s^*)}{\theta_s(p_s^*)}.
\]

C. Proof of Proposition 4

Note first that

\[
\begin{align*}
z_m(p_m) & = -\frac{(p_m - c_m)\epsilon_m(p_m)q_m(p_m)}{p_m\pi''_m(p_m)} \\
& = -\theta_m(p_m)\frac{q_m(p_m)}{\pi''_m(p_m)}
\end{align*}
\]

holds. Now, define

\[
F(p_m, c_m) = \frac{q_m(p_m)}{\partial x_{Am}/\partial p_A} + p_m - c_m
\]

so that \(F(p_m^*, c_m) = 0\) for \(m = s, w\). Then, from the implicit function theorem, it is verified that

\[
\rho_m = \frac{1}{1 + \left(\frac{q'_m}{\partial x_{Am}/\partial p_A} - q_m \frac{d(\partial x_{Am}/\partial p_A)/dp_m}{(\partial x_{Am}/\partial p_A)^2}\right)}
\]

\[
= \frac{\partial x_{Am}/\partial p_A}{\partial x_{Am}/\partial p_A + q_m - \frac{q_m}{\partial x_{Am}/\partial p_A} \frac{d}{dp_m} \left(\frac{\partial x_{Am}}{\partial p_A}\right)},
\]

and under the equilibrium discriminatory prices,

\[
\rho_m(p_m^*) = \frac{\partial x_{Am}(p_m^*, p_m^*)/\partial p_A}{q'_m(p_m^*) + \frac{\partial x_{Am}(p_m^*, p_m^*)}{\partial p_A} + (p_m^* - c_m) \frac{d}{dp_m} \left(\frac{\partial x_{Am}(p_m^*, p_m^*)}{\partial p_A}\right)}
\]

\[
= \frac{\partial x_{Am}(p_m^*, p_m^*)/\partial p_A}{\pi''_m(p_m^*, c_m)},
\]

which implies that

\[
\begin{align*}
z_m(p_m^*) & = \theta_m(p_m^*) \left(\frac{-q_m(p_m^*)}{\partial x_{Am}(p_m^*, p_m^*)}\right) \rho_m(p_m^*) \\
& = \theta_m(p_m^*) \mu_m(p_m^*) \rho_m(p_m^*)
\end{align*}
\]

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and thus

\[
\frac{W'(r^*)}{2} = \left(-\frac{\pi''_w\pi''_w}{\pi''_s + \pi''_w}\right) (\theta_w^*\mu_w^*\rho_w^* - \theta_s^*\mu_s^*\rho_s^*) \geq 0
\]

if \(\theta_w^*\mu_w^*\rho_w^* \geq \theta_s^*\mu_s^*\rho_s^*\) holds. Given the IRC, this means that \(W(r)\) is strictly increasing in \([0, r^*)\).

This completes the proof for the first part.

For \(r < r^*\), note that

\[
z_m(p_m) = \theta_m \left(\frac{\partial x_{Am}}{\partial p_m} \frac{\partial x_{Am}}{\partial p_m} \pi''_m \right) \left(\frac{-q_{1m}(p_m)}{\pi''_w} - \frac{-q_{jm}(p_m)}{\pi''_w} \pi''_s \right) = \rho_m \left(\frac{-q_{1m}(p_m)}{\pi''_w} - \frac{-q_{jm}(p_m)}{\pi''_w} \pi''_s \right).
\]

Thus, it is verified that

\[
\frac{W'(r)}{2} = \left(-\frac{\pi''_w\pi''_w}{\pi''_s + \pi''_w}\right) \left(\frac{\theta_w(r)\mu_w(r)\rho_w(r)}{\pi''_w} - \frac{\theta_s(r)\mu_s(r)\rho_s(r)}{\pi''_w} \pi''_s \right),
\]

which implies that given the IRC, \(W(r)\) is strictly decreasing in \([0, r^*)\) if \(W'(0) \leq 0\). This completes the proof for the second part.

**D. Proof of Proposition 7**

First, the multi-dimensional version of the market power index is defined by (Weyl and Fabinger 2013, p. 552):

\[
\theta_m(p) \equiv \begin{pmatrix}
(p - c_m)^T \theta_{p1m} q_m(p) \\
-q_{1m}(p) \\
& \vdots \\
(p - c_m)^T \theta_{pNm} q_m(p) \\
-q_{Nm}(p)
\end{pmatrix},
\]

which implies that

\[
Z_m[p_m(\lambda)] = \left[\theta_m[p_m(\lambda)]\right]^T \circ \left(-q_{1m}[p_m(\lambda)], ..., -q_{jm}[p_m(\lambda)], ..., -q_{Nm}[p_m(\lambda)]\right) \left[H^{-1}_m[p_m(\lambda)]\right]
\]

Electronic copy available at: https://ssrn.com/abstract=3006421
\[
[\theta_m(p_m(\lambda))]^T \circ [\mu_m(p_m(\lambda))]^T \quad \begin{pmatrix}
\frac{1}{\sigma_{q1m}} & 0 & \cdots & 0 \\
0 & \frac{1}{\sigma_{p1m}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{\sigma_{Nm}} \\
\end{pmatrix}
\quad H_m^{-1}[p_m(\lambda)],
\]

where the multi-dimensional version of markup is naturally defined by

\[
\mu_m(p) \equiv \begin{pmatrix}
-q_{1m}(p) \\
\frac{\partial q_{1m}}{\partial p_{1m}}(p) \\
\vdots \\
-q_{jm}(p) \\
\frac{\partial q_{jm}}{\partial p_{jm}}(p) \\
\vdots \\
-q_{Nm}(p) \\
\frac{\partial q_{Nm}}{\partial p_{Nm}}(p) \\
\end{pmatrix}.
\]

We then apply the implicit function theorem to \(g(p_m, c_m) = 0\), where

\[
g(p_m, c_m) = \begin{pmatrix}
q_{1m}(p_m) + (p_{1m} - c_{1m})\frac{\partial q_{1m}}{\partial p_{1m}}(p_m) \\
\vdots \\
q_{jm}(p_m) + (p_{jm} - c_{jm})\frac{\partial q_{jm}}{\partial p_{jm}}(p_m) \\
\vdots \\
q_{Nm}(p_m) + (p_{Nm} - c_{Nm})\frac{\partial q_{Nm}}{\partial p_{Nm}}(p_m)
\end{pmatrix},
\]

to obtain

\[
\rho_m(p_m^*) = -[D_{p_m}g]^{-1}[D_{c_m}g] = \begin{pmatrix}
\frac{\partial q_{1m}}{\partial p_{1m}} & 0 & \cdots & 0 \\
0 & \frac{\partial q_{2m}}{\partial p_{2m}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{\partial q_{Nm}}{\partial p_{Nm}} \\
\end{pmatrix} \quad H_m^{-1}(p_m^*),
\]

which indicates that

\[
Z_m(p_m) = [[\theta_m(p_m^*)]^T \circ [\mu_m(p_m^*)]^T] \rho_m(p_m^*).
\]

Now, note that \(W'(1) > 0\) if the inequality in this proposition holds. Thus, given the IRC, \(W(\lambda)\) is strictly increasing in \([0, 1]\), meaning that social welfare is higher under price discrimination.
than under uniform pricing. This complete the proof for the first part of the proposition.

For the second part, we proceed

\[
Z_m(\lambda) = \left[ \begin{array}{c}
\frac{1}{\partial q_1^{\mu}} \\
\frac{1}{\partial q_2^{\mu}} \\
\vdots \\
\frac{1}{\partial q_N^{\mu}} \\
\end{array} \right] \begin{pmatrix}
0 & \cdots & 0 \\
0 & 1 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & \cdots \\
\end{pmatrix} \begin{pmatrix}
\theta^{T} \circ \mu^{T}_{m} \\
\theta^{T} \circ \mu^{T}_{s} \\
\vdots \\
\theta^{T} \circ \mu^{T}_{N} \\
\end{pmatrix}
\]

\[ KH^{-1} - \left( \begin{array}{c}
\partial q_1^{\mu} \\
\partial q_2^{\mu} \\
\vdots \\
\partial q_N^{\mu} \\
\end{array} \right) = \rho_{m}, \]

for \( \lambda < 1 \), and thus

\[
W'(\lambda) = \left\{ \left[ \theta^{T} \circ \mu^{T}_{w} \right] \rho_{w} \left[ KH^{-1} \right] - \left[ \theta^{T} \circ \mu^{T}_{s} \right] \rho_{s} \left[ KH^{-1} \right] \right\} \left( KH^{-1} \right).
\]

Then, it is verified that

\[
W'(0) < 0 \iff \left[ \theta^{T} \circ \mu^{T}_{w} \right] \rho_{w} \left[ KH^{-1} \right] > \left[ \theta^{T} \circ \mu^{T}_{s} \right] \rho_{s} \left[ KH^{-1} \right]
\]

\[
\iff \left[ \theta^{T} \circ \mu^{T}_{w} \right] \rho_{w} < \left[ \theta^{T} \circ \mu^{T}_{s} \right] \rho_{s} \left[ KH^{-1} \right]^{-1}
\]

which completes the proof for the second part.

References


Omitted Details in Derivations

- The first-order partial derivative of the profit in symmetric pricing is given by

\[
\Delta \pi_m(p) \equiv \frac{\partial \pi_{jm}}{\partial p_{jm}} \bigg|_{p_{jm}=p_{-j,m}=p}
= q_{Am}(p,p) + (p - c_m) \frac{\partial x_{Am}}{\partial p_A}(p,p)
= q_m(p) + (p - c_m) \frac{\partial x_{Am}}{\partial p_A}(p,p).
\]

- The detailed derivation of \(\pi''_m(p)\):

\[
\pi''_m(p) = q'_m(p) + \frac{\partial x_{Am}}{\partial p_A}(p,p) + (p - c_m) \frac{d}{dp} \left( \frac{\partial x_{Am}}{\partial p_A}(p,p) \right)
= 2 \frac{\partial x_{Am}}{\partial p_A}(p,p) + \frac{\partial x_{Am}}{\partial p_B}(p,p) + (p - c_m) \left[ \frac{d}{dp} \left( \frac{\partial x_{Am}}{\partial p_A}(p,p) \right) - \frac{\partial^2 x_{Am}}{\partial p_A^2}(p,p) \right]
= D^2 \pi_m(p) + \frac{\partial x_{Am}}{\partial p_B}(p,p) + (p - c_m) \frac{\partial^2 x_{Am}}{\partial p_B \partial p_A}(p,p).
\]

- \(W'(r)\) is computed by

\[
W'(r) = 2 \left( \frac{\partial U_s}{\partial q_A} - c_s \right) \cdot q'_s \cdot p'_s(r) + 2 \left( \frac{\partial U_w}{\partial q_A} - c_w \right) \cdot q'_w \cdot p'_w(r)
= 2 \left( p_s(r) - c_s \right) \cdot q'_s \cdot p'_s(r) + 2 \left( p_w(r) - c_w \right) \cdot q'_w \cdot p'_w(r).
\]

- The derivation of \(Q'(r)/2\) is given by

\[
\frac{Q'(r)}{2} = q'_w \cdot p'_w + q'_s \cdot p'_s
= -\frac{\pi''_w q'_w}{\pi''_s + \pi''_w} + \frac{\pi''_w q'_s}{\pi''_s + \pi''_w}
= -\frac{q'_s q'_w}{\pi''_s + \pi''_w} \left( \frac{\pi''_w}{q'_w} - \frac{\pi''_s}{q'_s} \right).
\]

where

\[
\frac{\pi''_m(p)}{q'_m(p)} = \left[ 2 - L_m(p) \alpha_m^F(p) \right] \frac{\partial x_{Am}(p,p)/\partial p_A}{q'_m(p)}
\]
\[
+ \left[ 1 - L_m(p)\alpha_m^C(p) \right] \frac{\partial x_{Bm}(p,p)/\partial p_A}{q_m'(p)}
\]
\[
= \left[ 2 - L_m(p)\alpha_m^F(p) \right] \frac{\varepsilon_m^F(p)}{\varepsilon_m^I(p)} - \left[ 1 - L_m(p)\alpha_m^C(p) \right] \frac{\varepsilon_m^C(p)}{\varepsilon_m^I(p)}
\]
\[
= \frac{\left[ 2 - L_m(p)\alpha_m^F(p) \right] - (1 - \theta_m(p)) \left[ 1 - L_m(p)\alpha_m^C(p) \right]}{\theta_m(p)}
\]

where the derivation of \(\varepsilon_m^C(p_m)/\varepsilon_m^I(p_m)\) is given by
\[
\frac{\varepsilon_m^C(p_m)}{\varepsilon_m^I(p_m)} = \frac{\varepsilon_m^C(p_m)}{\varepsilon_m^I(p_m) - \varepsilon_m^C(p_m)} = \frac{1}{1/A_m(p_m) - 1} = \frac{1 - \theta_m(p_m)}{\theta_m(p_m)},
\]

which implies that
\[
\frac{\pi_s''}{q_s'} - \frac{\pi_w''}{q_w'} = \frac{1 + \theta_s - L_s[\alpha_s^F - (1 - \theta_s)\alpha_s^C]}{\theta_s} - \frac{1 + \theta_w - L_w[\alpha_w^F - (1 - \theta_w)\alpha_w^C]}{\theta_w}.
\]

Thus,
\[
\frac{Q'(r)}{2} = \left( -\frac{\pi_s''\pi_w''}{\pi_s'' + \pi_w''} \right)_{>0}
\]
\[
\times \left[ 1 - L_s[p_s(r)] \left\{ \alpha_s^F[p_s(r)] - (1 - \theta_s[p_s(r)])\alpha_s^C[p_s(r)] \right\} \theta_s[p_s(r)]
\]
\[
- 1 - L_w[p_w(r)] \left\{ \alpha_w^F[p_w(r)] - (1 - \theta_w[p_w(r)])\alpha_w^C[p_w(r)] \right\} \theta_w[p_w(r)] \right)
\]

- The precise expression for \(\partial_{p_m} q_m \cdot p_m'\) is:
\[
\partial_{p_m} q_m \cdot p_m' = \left( \sum_{j'=1,..,N} \frac{\partial q_{1m}}{\partial p_{j'm}} p_{j'm}' \right)
\]
\[
\vdots
\]
\[
\sum_{j'=1,..,N} \frac{\partial q_{jm}}{\partial p_{j'm}} p_{j'm}'
\]
\[
\vdots
\]
\[
\sum_{j'=1,..,N} \frac{\partial q_{jm}}{\partial p_{j'm}} p_{j'm}'
\]