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Abstract

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Optimality in an OLG model with nonsmooth preferences

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Abstract: It is a well-known observation that, in the overlapping generations (OLG) model with the complete market, we can judge optimality of an equilibrium allocation by examining the associated equilibrium price. Motivated by recent remarkable development in decision theory under ambiguity, this study reexamines the above observation in a stochastic OLG model with convex but not necessarily smooth preferences. It is shown that, under such preferences, optimality of an equilibrium allocation depends on the set of possible *supporting* prices, not necessarily on the associated equilibrium price itself. Therefore, observations of an equilibrium price do not necessarily tell us precise information on optimality of the equilibrium allocation.

Keywords: Nonsmooth preference; Conditional Pareto optimality; Conditional golden rule optimality; Dominant root criterion; Stochastic overlapping generations model.

JEL Classification Numbers: D60; D81; E40.

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1 Introduction

In the overlapping generations (OLG) model, competitive equilibrium might not achieve an optimal allocation, even when markets operate perfectly, as in the Arrow-Debreu abstraction. It is now understood that this sort of inefficiency is caused by the lack of market clearing at infinity (Geanakoplos, 1987). In order to design active policies (such as social security) which remedy this type of inefficiency, it is important to identify optimality with easily verifiable conditions. A cornerstone of the literature about characterizations of optimality in the OLG model is a pair of works by Cass (1972) and Balasko and Shell (1980), the latter of which sophisticated Cass's argument to be more suitable for a deterministic pure-endowment OLG environment. It contributed to the literature by demonstrating that optimality of an equilibrium allocation is characterized by conditions on the equilibrium price corresponding to the allocation. One of implications of this result is that, in a deterministic OLG model (with complete markets), we can examine whether the equilibrium allocation is optimal by *observing* the associated equilibrium price and therefore the policymaker no longer needs to examine the allocation itself (nor to know preferences).

Thanks to previous studies, we now know that the Cass-Balasko-Shell type of criteria of optimality in a deterministic environment, more precisely celebrated as the *Cass criterion*, can be naturally extended to a stochastic environment. For stationary feasible allocations, Peled (1984), Aiyagari and Peled (1991), Manuelli (1990), Chattopadhyay (2001), and Ohtaki (2013) found that optimality can be characterized by the *dominant root criterion*, which is a special case of the Cass criterion, *i.e.*: optimality is characterized by a certain condition on the dominant root of the contingent price matrix related to a stationary equilibrium. For general feasible allocations, on the other hand, Chattopadhyay and Gottardi (1999), Chattopadhyay (2006), and Bloise and Calciano (2008) founded in a various level of generality that the Cass criterion is still applicable to equilibrium contingent price processes.[1] Therefore, even in a stochastic OLG environment, we might be able to examine optimality of equilibrium allocations by observing the associated price (contingent upon date-events).

Although one of important restrictions to obtain these results is a pair of convexity and smoothness of preferences,[2] we aim to reexamine characterizations of optimality in a stochastic OLG model with convex but *not necessarily* smooth preferences. This reexamination is motivated by recent remarkable development in decision theory under uncertainty, such as the maxmin expected utility (MEU) preference axiomatized by Gilboa and Schmeidler (1989) and Casadesus-Masanell, Klibanoff, and Ozdenoren (2000). Differently from the standard expected utility hypothesis, a decision maker endowed with an MEU preference assigns a *set* of probability measures, not a *single* probability measure, to uncertainty and behaves as if she maximizes the minimum of expected utilities over the set of measures. This multiplicity of priors is often called *ambiguity*, which is a case of *true* uncertainty in the sense of Knight (1921).[3] The MEU preference is known as one of reasonable ways to explain several anomalies such as the Ellsberg paradox (1961). It can be convex when its von Neumann-Morgenstern utility index function is concave but might not be differentiable at some points as a result of the minimization of expected utilities. In the last three decades, lots of classes of such convex but not necessarily smooth preferences have been axiomatized. In order to import such various development in decision theory to the study on the OLG model, this study considers a more general class of convex preferences, which can also include the class of preferences with multiple discount rates à la Wakai (2008), for example.

[1] The Cass Criterion can be also extended to the production economy. Interested readers might be found, for example, Demange and Laroque (1999, 2000), Barbie, Hagedorn, and Kaul (2007), and Gottardi and Kübler (2011).

[2] For importance of smoothness of preferences, see Balasko and Shell (1980, Footnote 5) for example.

[3] In the tradition of Knight (1921), "uncertainty" is *risk* if it is reducible to a single probability measure and otherwise true *uncertainty*.

In order to capture the role of convex but not necessarily smooth preferences in an OLG environment, we consider a simple stationary pure-endowment stochastic OLG model consisting of infinite horizon with discrete time periods running from the initial period to $+\infty$, finite Markov states, one perishable commodity by period, and two-period-lived agents by generation. Although such models are canonical, their stationary equilibria with sequentially complete markets are often of interest as benchmarks of analysis of a variety of macroeconomic issues such as social security systems, financial mechanisms, and so on.[4] In this study, therefore, we restrict our attention on optimality of stationary allocations and adopt conditional Pareto optimality (CPO) and conditional golden rule optimality (CGRO) as optimality criteria.[5] While CPO considers welfare of both initial olds and newly born agents, CGRO takes care of welfare only of newly born agents. According to these criteria, agents' welfare is evaluated by conditioning their utility on the state at the date of their birth.

In such a framework with convex but not necessarily smooth preferences, four observations are mainly provided. First, CPO and CGRO of stationary *feasible* allocations are characterized by conditions on the *set* of dominant roots of matrices of marginal rates of substitution at the allocation. While CPO requires that the set of dominant roots contains some number less than or equal to unity, CGRO requires that the set contains unity. Second, CPO and CGRO of a stationary *equilibrium* allocation are characterized by conditions on the set of dominant roots of matrices of *supporting prices*, not the realized equilibrium price matrix itself. Third, similar to the existing results, the introduction of money in constant supply achieves CPO, especially CGRO. Fourth, the introduction of equity also achieves CPO.

This study contributes to the literature by providing characterizations of optimality criteria in a stochastic OLG model with convex but not necessarily smooth preferences. Under smooth preferences, CPO [resp. CGRO] is characterized by the dominant root of the matrix of marginal rates of substitution, being less than or equal to one [resp. being equal to one] (Ohtaki, 2013). The first observation is therefore a natural extension of the existing result. The third and the forth observations also emphasize the robustness of optimality of introducing money in constant supply and equity founded in the literature by indicating that intergenerational trade through such assets ensures optimality of equilibrium allocations. On the other hand, the second observation has a remarkable implication, *i.e.*: observing the equilibrium price *does not necessarily* tell us whether the associated allocation is optimal. This is because, in order to examine optimality, we should now consider the set of *supporting* price matrices, not the *observed* equilibrium price matrix itself. This study also provides several examples illustrating situations, wherein equilibrium price matrices do not necessarily reveal precise information on optimality of equilibrium allocations. For example, this study presents an example, wherein a stationary equilibrium can achieve CPO even when lump-sum money taxes are introduced. This example makes a sharp contrast to a well-known result, that is, when preferences are smooth, an introduction of lump-sum money taxes results inefficiency.[6]

Finally, we should mention that this study also contributes to the literature about the application of decision making under ambiguity to dynamic economics and finance. Decision making under ambiguity is already applied to a wide range of intertemporal macroeconomic models: asset pricing as in Epstein and Wang (1994, 1995), search theory as in Nishimura and Ozaki (2004), real option as in Nishimura and Ozaki (2007), learning as in Epstein and Schneider (2007), and growth theory as in Fukuda (2008) are such examples but these does not necessarily addressed to the issue about optimality of allocations. Actually, there seems few work characterizing optimality of allocations

[4]See also the work of Labadie (2004), which studied optimality in a simple stochastic OLG model and examine several financial mechanisms to achieve optimal allocations.

[5]The concept of CPO is first introduced by Muench (1977).

[6]See, for example, McCandless and Wallace (1992, Ch.10).

in a dynamic general equilibrium setting with ambiguity. One of exception is the work by Dana and Riedel (2013).[7] However, differently from ours, their results are obtained in a finite-horizontal economy with the incomplete preference à la Bewley (2002) and without overlapping of generations. To our best knowledge, therefore, this study is the first of characterizing optimality under ambiguity in an infinite-horizontal general equilibrium setting with overlapping of generations.

The organization of this paper is as follows: Section 2 presents details of the model. Section 3 introduces concepts of CPO and CGRO and characterizes them for stationary feasible allocations. Section 4 applies results given in the previous section to stationary equilibrium allocations. There also exist three appendixes. The Appendix A introduces, according to Ohtaki (2014), a graphical device for analyzing two-state, one-agent, pure-endowment stochastic OLG model. Proofs of main results are provided in the Appendix B. The Appendix C provides some of mathematical tools using this study.

2 The Economy

This study considers a stationary, one-good, finite-state, pure-endowment stochastic overlapping generations model with two-period-lived agents, as studied by Aiyagari and Peled (1991) and Chattopadhyay (2001). The crucial difference from the previous studies is that agents are endowed with convex but *not necessarily* smooth preferences. This section provides a formal description of the model.

2.1 Ingredients

Time is discrete and runs from $t = 1$ to ∞. The stochastic environment is modeled by a stationary Markov process with its finite state space $\mathcal{S} = \{1, \ldots, S\}$. For each $t \geq 0$, we denote by s_t the state realized in period t, where $s_0 \in \mathcal{S}$ is the state in (implicitly defined) period 0 and is treated as given.[8] The set of all probability measures on $\mathcal{S}$ is denoted by $\Delta_{\mathcal{S}}$.

After the realization of state $s_t \in \mathcal{S}$ in each period $t \geq 1$, a new generation, the members of which are called *agents*, is born, lives for two periods, and dies. The set of agents of each generation is denoted by $\mathcal{H} := \{1, \ldots, H\}$. We will assume that the economy is stationary, *i.e.*: the endowments and preference structures of each agent $h \in \mathcal{H}$ depend only on the realizations of states $s, s' \in \mathcal{S}$ during his/her lifetime, not on time nor on past realizations. Thus, an agent $h \in \mathcal{H}$ born at state $s \in \mathcal{S}$ is endowed with (i) $\omega_s^h = (\omega_s^{hy}, (\omega_{ss'}^{ho})_{s' \in \mathcal{S}}) \in \Re_{++} \times \Re_+^{\mathcal{S}}$ as the initial endowment stream and (ii) $\succsim^{hs} \subset (\Re_+ \times \Re_+^{\mathcal{S}})^2$ as his/her preference relation, where ω_s^{hy} and $\omega^{ho} = (\omega_{ss'}^{ho})_{s' \in \mathcal{S}}$ are endowments when young and old, respectively.

In addition, a one-period lived generation, the member of which are called *initial old agents* or more simply *initial olds*, is born after the realization of the state s_1 in period 1. The set of initial olds is given by $\mathcal{H}$ as defined above. Each initial old $h \in \mathcal{H}$ born at state $s_1 \in \mathcal{H}$ is assumed to be endowed with $\omega_{s_0 s_1}^{ho}$ units of the consumption good in his/her lifetime and his/her consumption

[7] We can find a lot of studies characterizing optimality of allocations in a static, not dynamic, general equilibrium environment: Billot, Chateauneuf, Gilboa, and Tallon (2000), Chateauneuf, Dana, and Tallon (2000), Dana (2004), Kajii and Ui (2009), Rigotti and Shannon (2005, 2012), Rigotti, Shannon, and Strzalecki (2008), Dana and Le Van (2010), Strzalecki and Werner (2011), and Carlier and Dana (2013) are such examples. Interested readers might be able to find other applications of ambiguity to the static economic environment including Dow and Werlang (1992), Mukerji and Tallon (1998, 2001, 2004a,b), Kajii and Ui (2005), Karni (2009), Rinaldi (2009), de Castro and Chateauneuf (2011), Condie and Ganguli (2011), Gollier (2011), Lopomo, Rigotti, and Shannon (2011), Mandler (2013), and de Castro, Liu, and Yannelis (2017).

[8] This study implicitly considers a standard date-event tree as seen in, for example, Chattopadhyay (2001). Therefore, the initial state s_0 can be interpreted as the root of the date-event tree.

stream $c^{ho}_{0s_1} \in \Re_+$ is ranked according to a utility function $u_{h0}(c^{ho}_{0s_1}) := c^{ho}_{0s_1}$, where $\omega^{ho}_{s_0 s_1}$ is defined as above.

Let $\bar{\omega}_{ss'} := \sum_{h \in \mathcal{H}}(\omega^{hy}_{s'} + \omega^{ho}_{ss'})$ for each $(s, s') \in \mathcal{S} \times \mathcal{S}$, which is the total endowment when the current and preceding states are s' and s, respectively. The economy faces *aggregate uncertainty* given the preceding state $s \in \mathcal{S}$ if there are some $s', s'' \in \mathcal{S}$ such that $\bar{\omega}_{ss'} \neq \bar{\omega}_{ss''}$ and *idiosyncratic uncertainty* given the preceding state $s \in \mathcal{S}$ if $\bar{\omega}_{ss'}$ is constant with respect to s'. One should note that, even when $\omega^{ho}_{ss'} = \omega^{ho}_{ss''}$ for every $h \in \mathcal{H}$ and every $s, s', s'' \in \mathcal{S}$, the economy might face aggregate uncertainty.[9]

2.2 Utility Representations

Throughout this study, it is assumed that the preference relation $\succsim^{hs}$ of agent h born at state s can be represented by a lifetime utility function $U^{hs} : \Re_+ \times \Re^{\mathcal{S}}_+ \to \Re$, *i.e.*: $c^h_s \succsim^{hs} b^h_s \Leftrightarrow U^{hs}(c^h_s) \geq U^{hs}(b^h_s)$ for each $c^h_s = (c^{hy}_s, (c^{ho}_{ss'})_{s' \in \mathcal{S}})$ and each $b^h_s = (b^{hy}_s, (b^{ho}_{ss'})_{s' \in \mathcal{S}})$. Furthermore, we impose the following assumption on preferences.

Assumption. *For each $(h, s) \in \mathcal{H} \times \mathcal{S}$, U^{hs} is strongly monotone increasing,[10] strictly concave, and continuous.*

We do *not necessarily* assume differentiability of U^{hs}. This type of utility functions includes many classes of preferences, in particular:

- the class of *expected utility* (EU) preferences

$$U^{hs}(c^h_s) = \sum_{s' \in \mathcal{S}} u^{hs}(c^{hy}_s, c^{ho}_{ss'})\pi^h_{ss'},$$

  where $u^{hs} : \Re_+ \times \Re_+ \to \Re$ and $\pi^h_s \in \Delta_{\mathcal{S}}$,
- the class of *maxmin expected utility* (MEU) preferences of Gilboa and Schmeidler (1989)

$$U^{hs}(c^h_s) = \min_{\pi^h_s \in \Pi^h_s} \sum_{s' \in \mathcal{S}} u^{hs}(c^{hy}_s, c^{ho}_{ss'})\pi^h_{ss'}, \tag{1}$$

  where $u^{hs} : \Re_+ \times \Re_+ \to \Re$ and $\Pi^h_s \subset \Delta_{\mathcal{S}}$ is nonempty, compact, and convex,[11]
- the class of variational preferences of Maccheroni, Marinacci, and Rustihici (2006)

$$U^{hs}(c^h_s) = \min_{\pi^h_s \in \Delta_{\mathcal{S}}} \left[\sum_{s' \in \mathcal{S}} u^{hs}(c^{hy}_s, c^{ho}_{ss'})\pi^h_{ss'} + \theta^h_s(\pi^h_s)\right],$$

  where $u^{hs} : \Re_+ \times \Re_+ \to \Re$ and $\theta^h_s : \Delta_{\mathcal{S}} \to \Re$ is convex,

[9] This is because it might hold that $\omega^{hy}_{s'} \neq \omega^{hy}_{s''}$ for some $h \in \mathcal{H}$ and some $s', s'' \in \mathcal{S}$.

[10] A function $f : \Re^n \to \Re$ is *strongly monotone increasing* if $f(x) > f(y)$ for each $x \in \Re^n$ and each $y \in \Re^n$ such that $x_i \geq y_i$ for each $i = 1, \ldots, n$ and $x_j > y_j$ for some $j = 1, \ldots, n$.

[11] The class of MEU preferences also includes the class of Choquet EU preferences with convex capacities (Schmeidler, 1989), the class of ε-contaminated EU preferences (Nishimura and Ozaki, 2006), the class of Cobb-Douglas preferences under uncertainty (Faro, 2013), and so on. The axiomatization of MEU preferences over general acts, not Anscombe-Aumann acts, is given, for example, by Casadesus-Masanell, Klibanoff, and Ozdenoren (2000).

- the class of multiplier preferences of Hansen and Sargent (2001)

$$U^{hs}(c^h_s) = \min_{\pi^h_s \in \Delta_{\mathcal{S}}} \left[\sum_{s' \in \mathcal{S}} u^{hs}(c^{hy}_s, c^{ho}_{ss'}) \pi^h_{ss'} + \theta^h_s(\pi^h_s \| \ \mu^h_s) \right]$$

where u^{hs} is a real-valued function on $\Re_+ \times \Re_+$ and

$$\theta^h_s(\pi^h_s \| \ \mu^h_s) = \begin{cases} \theta \sum_{s' \in \mathcal{S}} \pi^h_{ss'} \log_e \dfrac{\pi^h_{ss'}}{\mu^h_{ss'}} & \text{if} \quad \pi^h_s \ll \ \mu^h_s \\ \infty & \text{otherwise,} \end{cases}$$

for some $\mu^h_s \in \Delta_{\mathcal{S}}$,[12]

- the class of preferences with multiple discount rates à la Wakai (2008)

$$U^{hs}(c^h_s) = \min_{\delta \in \hat{D}^h_s} \left[(1-\delta) u^{hs}(c^{hy}_s) + \delta \sum_{s' \in \mathcal{S}} u^{hs}(c^{ho}_{ss'}) \pi^h_{ss'} \right],$$

where $u^{hs} : \Re_+ \to \Re$, $\pi^h_s \in \Delta_{\mathcal{S}}$, and $\hat{D}^h_s \subset \,]0,1[$ is nonempty, closed, and convex,[13]

and so on.[14] As well known, preferences belonging to the last four classes might be nondifferentiable at some points on their domain even when the utility index function u^{hs} is differentiable. We will often invoke the class of MEU preferences to present examples illustrating cases, wherein preferences are nondifferentiable. In such situations, the utility index function u^{hs} is assumed to be strongly monotone increasing, strictly concave, and continuously differentiable on the interior of its domain.

3 Optimality Criteria

The goal of this section is to define and characterize optimality criteria. Subsection 3.1 gives definitions of optimality criteria and Subsection 3.3 provides their characterizations. On the other hand, Subsection 3.2 introduces a convenient notion on the marginal rate of substitution.

3.1 Definitions of Optimality Criteria

Recall that, at each state s' in each period t with preceding state s, there are $\bar{\omega}_{ss'} := \sum_{h \in \mathcal{H}} (\omega^{hy}_{s'} + \omega^{ho}_{ss'})$ units of the consumption good. As usual, a stationary allocation is a profile of amounts of the consumption good allocated to H young agents and H old agents at that state. A *stationary feasible allocation* is then a family $c = \{c^{ho}_0, c^{hy}, c^{ho}\}_{h \in \mathcal{H}}$, or simply $c = \{c^{hy}, c^{ho}\}$, of functions $c^{ho}_0 : \mathcal{S} \to \Re_+$, $c^{hy} : \mathcal{S} \to \Re_+$, and $c^{ho} : \mathcal{S} \times \mathcal{S} \to \Re_+$ such that

$$\begin{aligned} \sum_{h \in \mathcal{H}} c^{hy}_{s'} + \sum_{h \in \mathcal{H}} c^{ho}_{0s'} &= \bar{\omega}_{s_0 s'} \\ \sum_{h \in \mathcal{H}} c^{hy}_{s'} + \sum_{h \in \mathcal{H}} c^{ho}_{ss'} &= \bar{\omega}_{ss'} \end{aligned} \tag{2}$$

[12]The axiomatization of multiplier preferences can be found in Strzalecki (2011).

[13]To be more precise, Wakai (2008) axiomatized a preference for infinite horizontal choices, not a preference under uncertainty. See also Chambers and Echenique (2018) for preferences with multiple discount rates. Drugeon, Ha-Huy, and Nguyen (2018) applies such preferences to dynamic programming.

[14]Although our preference includes lots of utility representations under ambiguity, there exist some exceptions. For example, the α-maxmin expected utility preference (Ghirardate, Mccheroni, and Marinacci, 2004) is not necessarily convex. See also Ghirardate and Marinacci (2002) and Xue (2018) for the α-maxmin expected utility preference.

for each $s, s' \in \mathcal{S}$, where $c^{ho}_{0s'} \in \Re_+$ is the consumption of the initial old h born at state s' in period 1 and $c^h_s = (c^{hy}_s, (c^{ho}_{ss'})_{s' \in \mathcal{S}})$ is a consumption stream (contingent upon realizations of states in the second period of the life) of each agent $h \in \mathcal{H}$ born at state $s \in \mathcal{S}$.[15] We denote by $\mathcal{A}$ the set of stationary feasible allocations. We also denote by $\mathcal{A}'$ the set of *steady state allocations*, *i.e.*: the set of stationary feasible allocations c satisfying that $c^{ho}_{0s'} = c^{ho}_{s_0 s'}$ for each $s' \in \mathcal{S}$.

A stationary feasible allocation $c = \{c^{ho}_0, c^{hy}, c^{ho}\}_{h \in \mathcal{H}}$ is said to be *interior* if $c^{hy}_s > 0$ and $c^{ho}_{ss'} > 0$ for each $h \in \mathcal{H}$ and each $s, s' \in \mathcal{S}$. It is also *fully-insured* if $c^{ho}_{s\tau} = c^{ho}_{s\kappa}$ for each $h \in \mathcal{H}$ and each $s, \tau, \kappa \in \mathcal{S}$.

For any two stationary feasible allocation b and c, we say that b *CPO-dominates* c if

$$(\forall (h,s) \in \mathcal{H} \times \mathcal{S}) \quad b^{ho}_{0s} \geq c^{ho}_{0s} \quad \text{and} \quad U^{hs}(b^h_s) \geq U^{hs}(c^h_s)$$

with strict inequality somewhere. We can then define "conditional Pareto optimality" as follows:

Definition 1 A stationary feasible allocation c is said to be *conditionally Pareto optimal* (CPO) if there exists no other stationary feasible allocation b that CPO-dominates c.

CPO considers welfare of both initial olds and newly born agents. On the other hand, we can also consider another optimality criterion, which takes care of welfare only of newly born agents. For any two stationary feasible allocation b and c, we say that b *CGRO-dominates* c if

$$(\forall (h,s) \in \mathcal{H} \times \mathcal{S}) \quad U^{hs}(b^h_s) \geq U^{hs}(c^h_s)$$

with strict inequality somewhere. The concept of "conditional golden rule optimality" is then defined as follows:

Definition 1′ A steady state allocation c is said to be *conditionally golden rule optimal* (CGRO) if there exists no other steady state allocation b that CGRO-dominates c.

In the above definitions, "conditionally" means the fact that agents' welfare is evaluated by *conditioning* their lifetime utilities on the state at the date of birth.

We close this subsection with a remark on the relationship between CPO and CGRO.

Remark 1 One might have an intuition that CGRO always implies CPO. However, as presented in Example 1 of Ohtaki (2013), we can construct several examples in which CGRO does not necessarily imply CPO.[16] As shown in Proposition 1 of Ohtaki (2013), such anomalous situations are avoidable by imposing, for example, strict quasi-concavity on lifetime utility functions. Because this study has assumed strict concavity of lifetime utility functions, we can consider that all CGRO allocations are also CPO.

3.2 The Set of Matrices of Marginal Rates of Substitution

In order to provide characterizations of CPO and CGROallocations, we introduce several notations. In the current setting, each utility function U^{hs} is not necessarily differentiable. However, we can define the "superdifferential" of U^{hs}. The *superdifferential* of U^{hs} at c^h_s is defined by

$$\partial U^{hs}(c^h_s) := \{v^h_s \in \Re^{1+S} : \ (\forall b^h_s \in \text{dom } U^{hs}) \quad U^{hs}(b^h_s) \leq U^{hs}(c^h_s) + \langle v^h_s, b^h_s - c^h_s \rangle\}$$

[15] The equal sign in the feasibility condition (2) can be relaxed to the (weak) one of ineqality.

[16] Such an anomalous example is given, for example, when lifetime utility functions are linear. Under such preferences, Ohtaki (2013, Example 1) illustrates an anomalous situation wherein the set of CPO allocations becomes a proper subset of the set of CGRO allocations.

and each of its elements, denoted by $v_s^h = (v_s^{hy}, (v_{ss'}^{ho})_{s' \in \mathcal{S}}) \in \Re^{1+S}$ for example, is called a *super-gradient* of U^{hs} at c_s^h.[17] Because U^{hs} is concave, one can immediately show that $\partial U^{hs}(c_s^h)$ is closed and convex. It also follows from Theorem C.1 in the Appendix that $\partial U^{hs}(c_s^h)$ is nonempty and bounded, provided that $c_s^h \gg 0$. Therefore, $\partial U^{hs}(c_s^h)$ is nonempty, compact, and convex for each $c_s^h \gg 0$. Furthermore, one can easily observe that all of coordinates of each $v_s^h \in \partial U^{hs}(c_s^h)$ are positive because of strong monotonicity of U^{hs}.[18] We can then define the set of $S \times S$ positive matrices

$$\mathcal{M}^h(c^h) := \left\{ \left[\frac{v_{ss'}^{ho}}{v_s^{hy}} \right]_{s,s' \in \mathcal{S}} : \ (\forall s \in \mathcal{S}) \quad (v_s^{hy}, (v_{ss'}^{ho})_{s' \in \mathcal{S}}) \in \partial U^{hs}(c_s^h) \right\}$$

for each $h \in \mathcal{H}$ and each $c^h = (c_s^h)_{s \in \mathcal{S}}$. Because $\partial U^{hs}(c_s^h)$ is nonempty and each coordinates of $v_s^h \in \partial U^{hs}(c_s^h)$ is positive, $\mathcal{M}^h(c^h)$ is well-defined and nonempty, provided that $c_s^h \gg 0$. As shown in following two examples, this can be interpreted as the set of matrices of *marginal rates of substitution*.

Example 1 (Differentiable Case) Assume that, for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, U^{hs} is differentiable on the interior on its domain. This example illustrates that, under such an assumption, $\mathcal{M}^h(c^h)$ is actually a singleton, the element of which is the matrix of marginal rates of substitution. In fact, it follows from Theorem C.2 in the Appendix that

$$\partial U^{hs}(c_s^h) = \{\nabla U^{hs}(c_s^h)\} = \{(U_y^{hs}(c_s^h), (U_{s'}^{hs}(c_s^h))_{s' \in \mathcal{S}})\},$$

where $U_y^{hs}(c_s^h) = \partial U^{hs}(c_s^h)/\partial c_s^{hy}$ and $U_{s'}^{hs}(c_s^h) = \partial U^{hs}(c_s^h)/\partial c_{s'}^{ho}$ are standard marginal utilities with respect to the first- and second-period consumptions. Therefore,

$$\mathcal{M}^h(c^h) = \left\{ \left[\frac{U_{s'}^{hs}(c_s^h)}{U_y^{hs}(c_s^h)} \right]_{s,s' \in \mathcal{S}} \right\},$$

which is a singleton and its unique element is the matrix of marginal rates of substitution. ■

Unlike Example 1, the set $\mathcal{M}^h(c^h)$ might have multiple elements when U^{hs} is not differentiable. The following example illustrates this fact.

Example 2 (Nondifferentiable Case with MEU) Here, we consider the class of MEU preferences as in Eq.(1). As well known, the MEU preference is not differentiable at some points. As shown in Theorem C.7 in the Appendix, we can verify that, under the current assumption,

$$\partial U^{hs}(c_s^h) = \left\{ \left(\sum_{s' \in \mathcal{S}} u_y^{hs}(c_s^{hy}, c_{ss'}^{ho}) \pi_{ss'}^h, (u_o^{hs}(c_s^{hy}, c_{ss'}^{ho}) \pi_{ss'}^h)_{s' \in \mathcal{S}} \right) : \ \pi_s^h \in \mathcal{B}_s^h(c_s^h) \right\},$$

where $\mathcal{B}_s^h(c_s^h)$ is the set of active beliefs given the consumption stream c_s^h, *i.e.*:

$$\mathcal{B}_s^h(c_s^h) := \arg\min_{\pi_s^h \in \Pi_s^h} \sum_{s' \in \mathcal{S}} u^{hs}(c_s^{hy}, c_{ss'}^{ho}) \pi_{ss'}^h;$$

and $u_y^{hs} = \partial u^{hs}/\partial c_s^{hy}$ and $u_o^{hs} = \partial u^{hs}/\partial c_{ss'}^{ho}$. The set of matrices of marginal rates of substitution, $\mathcal{M}^h(c^h)$, is then given by

$$\mathcal{M}^h(c^h) = \left\{ M_{\pi^h}^h(c^h) : \ (\forall s \in \mathcal{S}) \quad \pi_s^h \in \mathcal{B}_s^h(c_s^h) \right\}, \tag{3}$$

[17] For each $x, y \in \Re^n$, $\langle x, y \rangle := x_1 y_1 + \cdots + x_n y_n$ represents their inner product, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

[18] See Theorem C.3 in the Appendix.

where

$$M^h_{\pi^h}(c^h) = \left[\frac{u^{hs}_o(c^{hy}_s, c^{ho}_{ss'})\pi^h_{ss'}}{\sum_{\tau\in\mathcal{S}} u^{hs}_y(c^{hy}_s, c^{ho}_{s\tau})\pi^h_{s\tau}} \right]_{s,s'\in\mathcal{S}}. \tag{4}$$

We can observe that this set can actually have multiple elements when c^h_s is fully-insured, *i.e.*: $c^{ho}_{s\tau} = c^{ho}_{s\kappa}$ for each $\tau, \kappa \in \mathcal{S}$. This is because $\mathcal{B}^h_s(c^h_s) = \Pi^h_s$ for each c^h_s such that $c^{ho}_{s\tau} = c^{ho}_{s\kappa}$ for each $\tau, \kappa \in \mathcal{S}$. ■

One should also note that, even when we consider a deterministic environment, $\mathcal{M}^h(c^h)$ might have multiple elements.

Example 3 (Nondifferentiable Case in Deterministic Environment) Suppose that $\mathcal{S}$ is a singleton and we omit the script s at U^{hs}, c^h_s, and so on. Here, we consider a lifetime utility function in the from given by

$$U^h(c^{hy}, c^{ho}) = \min_{\delta\in D^h}[(1-\delta)u(c^y) + \delta u(c^o)], \tag{5}$$

where $D^h := [\underline{\delta}^h, \overline{\delta}^h]$ for some $\underline{\delta}^h$ and some $\overline{\delta}^h$ satisfying that $0 < \underline{\delta}^h \le \overline{\delta}^h < 1$. This is a special case of Wakai (2008). Then, the set of superdifferential of U^h is calculated as

$$\partial U^h(c^h) = \left\{ ((1-\delta)u'(c^{hy}), \delta u'(c^{ho})) : \ \delta \in AD^h(c^h) \right\},$$

where $AD^h(c^h)$ is the set of active discount factor given c^h, *i.e.*:

$$AD^h(c^h) := \arg\min_{\delta\in D^h}[(1-\delta)u(c^y) + \delta u(c^o)].$$

Therefore, the set of 1×1 matrices of marginal rates of substitution is given by

$$\mathcal{M}^h(c^h) = \left\{ \left[\frac{\delta}{1-\delta} \frac{u'(c^{ho})}{u'(c^{hy})} \right] : \ \delta \in AD^h(c^h) \right\}.$$

One can easily observe that this set has multiple elements when $c^{hy} = c^{ho}$. This is because $AD^h(c^h) = D^h$ for each $c^h = (c^{hy}, c^{ho})$ such that $c^{hy} = c^{ho}$.

Finally, we also introduce several notations. The current restrictions on preferences implies that $\mathcal{M}^h(c^h)$ turns out to be the set of $S \times S$ "positive" square matrices. A matrix is *positive* if all of its components are strictly positive. By the Perron-Frobenius theorem,[19] any positive square matrix M has a unique dominant root, denoted by $\lambda^f(M)$, and it holds that $My(M) = \lambda^f(M)y(M)$ for some positive vector, $y(M)$, unique up to positive scalar multiple. The rest of this paper will repeatedly invoke this fact. Also let, for each stationary feasible allocation $c = (c^h)_{h\in\mathcal{H}}$,

$$\mathcal{M}(c) = \bigcap_{h\in\mathcal{H}} \mathcal{M}^h(c^h),$$

which is the intersection of sets of marginal rates of substitution over agents $h \in \mathcal{H}$.

[19]See, for example, Debreu and Herstein (1953) and Takayama (1974, Theorems 4.B.1 and 4.B.2) for more details on the Perron-Frobenius theorem.

3.3 Characterization

We are now ready to characterize CPO and CGRO allocations. As shown in the next two theorems, CPO and CGRO allocations are characterized by conditions on the dominant roots of matrices of marginal rates of substitution.

Theorem 1 *An interior stationary feasible allocation c is conditionally Pareto optimal if and only if the set $(\lambda^f \circ \mathcal{M})(c)$ contains some number less than or equal to one.*[20]

Theorem 1′ *An interior steady state allocation c is conditionally golden rule optimal if and only if the set $(\lambda^f \circ \mathcal{M})(c)$ contains unity.*

In other words, an interior stationary feasible allocation c is CPO if and only if there exists at least one element M of $\mathcal{M}(c)$ such that $\lambda^f(M) \leq 1$ and an interior steady state allocation c is CGRO if and only if there exists at least one element M of $\mathcal{M}(c)$ such that $\lambda^f(M) = 1$. Although we might not be able to determine uniquely the matrix of marginal rates of substitution under nonsmoothness of preferences, we can say that an interior c is CPO [resp. CGRO] if there is at least one matrix of marginal rates of substitution, which is common to all newly born agents and of which dominant root is less than or equal to one [resp. is exactly equal to one]. As shown in Corollaries 1 and 1′ presented below, these characterizations of optimality criteria are natural extensions of those under smoothness of preferences.[21]

Note that the set of marginal rates of substitution is derived from $\partial U^{hs}(c^h_s)$. Therefore, even when U^{hs} has some nondifferentiable points, the condition in Theorem 1 degenerate into the standard one if $\partial U^{hs}(c^h_s)$ is singleton, *i.e.*: an interior stationary feasible allocation c is CPO if and only if the dominant root of a unique elements of $\mathcal{M}(c)$ is less than or equal to one.[22] Of course, a similar statement can be given to Theorem 1′ and CGRO.

When preferences are differentiable as in some previous literature or in Example 1, the set of matrices of the marginal rates of substitution degenerates into a singleton. Therefore, we can obtain the following corollaries.

Corollary 1 *Suppose that, for each $h \in \mathcal{H}$ and $s \in \mathcal{S}$, U^{hs} is differentiable on the interior of its domain. Then, an interior stationary feasible allocation c is conditionally Pareto optimal if and only if there exists some $S \times S$ positive matrix $M = [m_{ss'}]_{s,s' \in \mathcal{S}}$ such that*

$$(\forall h \in \mathcal{H})(\forall s, s' \in \mathcal{S}) \quad m_{ss'} = \frac{U^{hs}_{s'}(c^h_s)}{U^{hs}_y(c^h_s)} \tag{6}$$

and $\lambda^f(M) \leq 1$.

Corollary 1′ *Suppose that, for each $h \in \mathcal{H}$ and $s \in \mathcal{S}$, U^{hs} is differentiable on the interior of its domain. Then, an interior steady state allocation c is conditionally golden rule optimal if and only if there exists some $S \times S$ positive matrix $M = [m_{ss'}]_{s,s' \in \mathcal{S}}$ and $\lambda^f(M) = 1$, where $m_{ss'}$ satisfies Eq.(6).*

These corollaries are consistent with Theorems 1 and 2 of Ohtaki (2013), which characterize CPO and CGRO when lifetime utility functions are differentiable. By simple comparison, we can know that Theorems 1 and 1′ are natural extensions of these corollaries.

[20]For each stationary feasible allocation c, $(\lambda^f \circ \mathcal{M})(c) = \lambda^f(\mathcal{M}(c)) = \{\lambda^f(M) : M \in \mathcal{M}(c)\}$.

[21]See, for example, Ohtaki (2013, Theorems 1 and 2)

[22]See also Corollary 1 below.

When preferences belong to the MEU class, the characterization becomes a bit complex. Suppose that preferences are represented as in Eq.(1). In such a case, $\mathcal{M}(c)$ is rewritten as in Eq.(3), so that we can obtain the next corollaries.

Corollary 2 *Suppose that preferences belong to the class of MEU preferences as in Eq.(1). An interior stationary feasible allocation c is then conditionally Pareto optimal if and only if there exists some $S \times S$ positive matrix M and some $\pi_s^h \in \mathcal{B}_s^h(c_s^h)$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$ such that $M = M_{\pi^h}^h(c^h)$ for each $h \in \mathcal{H}$ and $\lambda^f(M) \leq 1$, where $M_{\pi^h}^h(c^h)$ is defined as in Eq.(4).*

Corollary 2′ *Suppose that preferences belong to the class of MEU preferences as in Eq.(1). An interior steady state allocation c is then conditionally golden rule optimal if and only if there exists some $S \times S$ positive matrix M and some $\pi_s^h \in \mathcal{B}_s^h(c_s^h)$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$ such that $M = M_{\pi^h}^h(c^h)$ for each $h \in \mathcal{H}$ and $\lambda^f(M) = 1$, where $M_{\pi^h}^h(c^h)$ is defined as in Eq.(4).*

Note that the set of matrices of marginal rates of substitution, $\mathcal{M}^h(c^h)$, is calculated for $\mathcal{B}_s^h(c_s^h)$, not necessarily for Π_s^h. Therefore, even when Π_s^h has multiple elements for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, if $\mathcal{B}_s^h(c_s^h)$ is a singleton for each h and each s, the condition in Corollary 2 degenerates into the standard one, *i.e.*: c is CPO if and only if the dominant root of a unique matrix of marginal rates of substitution is less than or equal to one. However, when we consider a fully-insured stationary feasible allocation, the set of matrices of marginal rates of substitution is calculated for Π_s^h and $\mathcal{M}(c)$ might have multiple elements. A similar statement can be given to CGRO.

We give an example for Corollaries 2 and 2′.

Example 4 Suppose that $\mathcal{H}$ is a singleton (and therefore we omit sub/superscript h, which represents agents' types) and $\mathcal{S} = \{\alpha, \beta\}$. Furthermore, specify the economy by setting as $(\bar{\omega}_\alpha, \bar{\omega}_\beta) = (6.5, 5)$, $\Pi_s = \{\pi_s \in \Delta_{\mathcal{S}} : \ 0.25 \leq \pi_{s\alpha} \leq 0.75\}$, and

$$U^s(c_s^y, c_{s\alpha}^o, c_{s\beta}^o) = \min_{\pi_s \in \Pi_s} \sum_{s' \in \mathcal{S}} (\ln c_s^y + \ln c_{ss'}^o)\pi_{ss'} = \ln c_s^y + \min_{\pi_s \in \Pi_s} \sum_{s' \in \mathcal{S}} \ln c_{ss'}^o \pi_{ss'},$$

which belongs to the class of MEU preferences.[23] Also let $\mu_s = (\mu_{s\alpha}, \mu_{s\beta}) = (0.25, 0.75)$ and $\nu_s = (\nu_{s\alpha}, \nu_{s\beta}) = (0.75, 0.25)$ for each $s \in \mathcal{S}$. Now, as an example, we examine optimality of an interior stationary feasible allocation $(c_0^o, c) = (c_0^o, (c_s^y, c_{ss'}^o)_{s,s' \in \mathcal{S}})$ such that $c_\alpha^y = 3.65$, $c_\beta^y = 2.15$, and $c_0^o = c_{ss'}^o = 2.85$ for each $s, s' \in \mathcal{S}$. At this allocation, we have $\mathcal{B}_s(c_s) = \Pi_s$ for each $s \in \mathcal{S}$ and $\mathcal{M}(c) = \{M_\pi(c) : \ (\forall s \in \mathcal{S}) \ \pi_s \in \Delta_{\mathcal{S}} \text{ and } 0.25 \leq \pi_{s\alpha} \leq 0.75\}$ because c_s^o is fully-insured. Then, we can obtain that $\lambda^f(M_\mu(c)) \approx 0.89$ and $\lambda^f(M_\nu(c)) \approx 1.15$. Because $\mu \in \mathcal{B}_s(c_s)$ for each s and the dominant root of $M_\mu(c)$ is less than one, we can conclude that the allocation (c_0^o, c) here is CPO. Furthermore, one can easily verify that the steady state allocation c is also CGRO, *i.e.*: there exists some $\pi = (\pi_\alpha, \pi_\beta) \in \Pi_\alpha \times \Pi_\beta$ such that $\lambda^f(M_\pi(c)) = 1$. ■

Remark 2 Ohtaki (2014) found that, for a two-state and one-agent stochastic OLG model, a stationary feasible allocation can be identified with a point in an appropriate box diagram and each agent's preference defined over such points can be represented by indifference curves restricted on the box. Moreover, the dominant root characterization of CGRO gives us a graphical intuition, *i.e.*: at a CGRO allocation, indifference curves of agent born at state α and agent born at state β are tangent to each other (and the slope of them must be positive), where $\mathcal{S} = \{\alpha, \beta\}$. According to

[23] Interested readers can find in the work by Faro (2013) an axiomatization of the maxmin expected utility preference with logarithmic index functions.

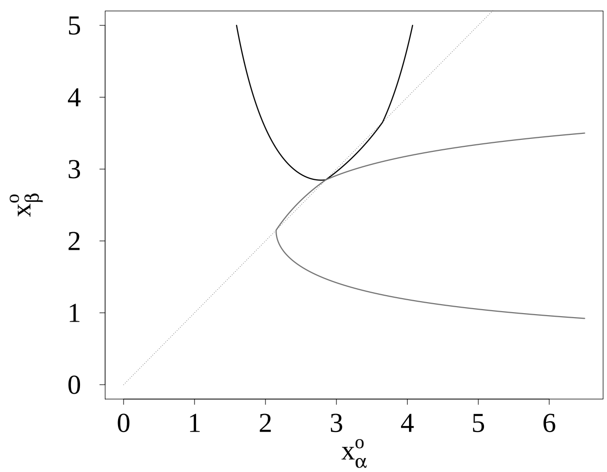


Figure 3.1: Conditional Pareto Optimality in the Edgeworth-Ohtaki Box

Ohtaki (2014), we can give Example 3 the box diagram as in Figure 3.1.[24] In the figure, indifference curves of agents born at state α and β through the stationary feasible allocation considered in Example 3, which is identified with the point $(x^o_\alpha, x^o_\beta) = (2.85, 2.85)$ in the box, are depicted as the black solid line and the gray solid line, respectively. One can find that those indifference curves are tangent to each other.

Finally, we present results in a deterministic environment. In order to tailor the presented results to meet the deterministic situation, assume throughout the rest of this section that $\mathcal{S}$ is a singleton and we omit the script s at U^{hs}, c^h_s, and so on. Then, for each consumption stream for agent h, $c^h = (c^{hy}, c^{ho})$, the supperdifferential of U^h at c^h is defined by

$$\partial U^h(c^h) := \{v^h \in \Re^2 : \ (\forall b^h \in \text{dom } U^h) \ U^h(b^h) \leq U^h(c^h) + \langle v^h, b^h - c^h \rangle\}$$

and the set of 1×1 matrices of marginal rates of substitution for agent h is given by

$$\mathcal{M}^h(c^h) := \left\{ \lambda^f \left(\left[\frac{v^{ho}}{v^{hy}} \right] \right) : \ (\forall h \in \mathcal{H}) \ v^h = (v^{hy}, v^{ho}) \in \partial U^h(c^h) \right\}.$$

Because the dominant root of 1×1 matrix $[m]$ is equal to m for each $m > 0$, each dominant root $\lambda^f([m])$ of $\mathcal{M}(c^h)$ is also the marginal rate of substitution m. Also, we often identify the number m with the 1×1 matrix $[m]$. We can now obtain the following results.

Corollary 3 *An interior stationary feasible allocation c is (conditionally) Pareto optimal if and only if there exists some positive number $m \leq 1$ such that*

$$(\forall h \in \mathcal{H})(\exists \bar{v}^h \in \partial U^h(c^h)) \quad m = \frac{\bar{v}^{ho}}{\bar{v}^{hy}}.$$

Corollary 3′ *An interior steady state allocation c is (conditionally) golden rule optimal if and only if*

$$(\forall h \in \mathcal{H})(\exists \bar{v}^h \in \partial U^h(c^h)) \quad \frac{\bar{v}^{ho}}{\bar{v}^{hy}} = 1.$$

[24]For more detail, see the Appendix A, Ohtaki (2014), or Ohtaki and Ozaki (2015). Figures in this study are drawn by using a well-known free software environment for statistical computing and graphics, R.

Of course, when U^h is differentiable, $\bar{v}^{ho}/\bar{v}^{hy}$ is equal to $U_o^h(c^h)/U_y^h(c^h)$ and therefore is actually the marginal rate of substitution for agent h. In such a case, the necessary and sufficient condition for CPO in Corollary 3 degenerates into a well-known one that claims the marginal rates of substitution for every agent is less than or equal to one. Of course, a similar argument is given to CGRO and Corollary $3'$.

4 Optimality of Stationary Equilibrium Allocations

The previous section characterized optimality criteria of stationary "feasible" allocations. The results also correspond to the welfare analyses of stationary "equilibrium" allocations. This section examines the relationship between optimality criteria and stationary equilibrium allocations.

4.1 Supporting Price Matrix

We first define the concept of supporting price matrices, which can be interpreted as candidates of the equilibrium prices given an allocation:

Definition 2 Let c be a stationary feasible allocation. A positive matrix $P = [p_{ss'}]_{s,s' \in \mathcal{S}}$ is a *supporting price matrix* of c if, for each $b = \{b^{hy}, b^{ho}\}$, each $s \in \mathcal{S}$, and each $h \in \mathcal{H}$,

$$U^{hs}(b_s^h) > U^{hs}(c_s^h) \quad \text{implies that} \quad b_s^{hy} + \sum_{s' \in \mathcal{S}} b_{ss'}^{ho} p_{ss'} > c_s^{hy} + \sum_{s' \in \mathcal{S}} c_{ss'}^{ho} p_{ss'}.$$

Also, we denote by $\mathcal{P}(c)$ the set of all supporting price matrices of c.

The set of supporting price matrices has a closed representation:

Proposition 1 *For each interior stationary feasible allocation c, $\mathcal{P}(c) = \mathcal{M}(c)$.*

Furthermore, combining Theorems 1 and $1'$ with Proposition 1, we can obtain the following characterizations for CPO and CGRO of stationary equilibrium allocations, respectively:

Theorem 2 *An interior stationary feasible allocation c is conditionally Pareto optimal if and only if there exists at least one supporting price matrix $P \in \mathcal{P}(c)$, of which the dominant root is less than or equal to unity.*

Theorem $2'$ *An interior steady state allocation c is conditionally golden rule optimal if and only if there exists at least one supporting price matrix $P \in \mathcal{P}(c)$, of which the dominant root is equal to unity.*

According to these theorems, we can find that optimality of stationary feasible allocations are characterized by conditions on the set of supporting price matrices: CPO requires that the set of dominant roots of supporting price matrices contains number less than or equal to unity and CGRO requires the set contains unity.

Remark 3 When preferences are smooth, a supporting price matrix of a stationary feasible allocation is uniquely determined and it completely retains information about marginal rates of substitutions and, so, about welfare improving redistributions. In such a situation, by following Chattopadhyay and Gottardi (1999) or Ohtaki (2013) for example, one can conclude an allocation is suboptimal in the sense of CPO [resp. CGRO] if the dominant root of a unique supproting price matrix is greater than [rep. not equal to] unity. However, in our model with not necessarily smooth preferences, a stationary feasible allocations might have multiple supporting price matrices because of nondifferentiability of lifetime utility functions. As a result, some of supporting price matrices might not completely retain sufficient information about welfare improving redistributions. So, one can find that an interior stationary feasible allocation might be CPO [resp. CGRO] even when the dominant root of its given supporting price matrix is strictly greater than [resp. not equal to] unity. This implies that, in an OLG structure with not necessarily smooth preferences, prices do not necessarily tell us precise information on optimality of the equilibrium allocation. This is a remarkable difference from the standard argument with smooth preferences and we present several examples later.

4.2 Complete Market

We now define a stationary equilibrium with a complete market, *i.e.*: a stationary equilibrium at which agents can buy and sell all contingent commodities in a centralized market.

Definition 3 A pair (P^*, c^*) of a positive price matrix $P^* = \left[p^*_{ss'}\right]_{s,s' \in \mathcal{S}}$ of contingent commodities and a stationary feasible allocation c^* is called a *stationary equilibrium* (*with respect to the initial endowments* ω) if

- for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, c^*_s belongs to the set

$$\underset{(c_s^{hy}, c_s^{ho}) \in \Re_+ \times \Re_+^{\mathcal{S}}}{\arg\max} \left\{ U^{hs}(c_s^h) : \, c_s^{hy} + \sum_{s' \in \mathcal{S}} c_{ss'}^{ho} p^*_{ss'} \leq \omega_s^{hy} + \sum_{s' \in \mathcal{S}} \omega_{ss'}^{ho} p^*_{ss'} \right\}$$

  given $p^*_s = (p^*_{ss'})_{s' \in \mathcal{S}}$; and

- for each $s' \in \mathcal{S}$, $\sum_{h \in \mathcal{H}} c_{s'}^{*hy} + \sum_{h \in \mathcal{H}} c_{ss'}^{*ho} = \bar{\omega}_{ss'}$.

In this definition, the former condition is the optimization condition of each agent $s \in \mathcal{S}$ subject to a lifetime budget constraint, and the latter is the market clearing conditions. Moreover, for each stationary feasible allocation c, we denote by $\mathcal{P}^*(c)$ the set of all positive price matrices $P = [p_{ss'}]_{s,s' \in \mathcal{S}}$ such that (P, c) is a stationary equilibrium. Note that $\mathcal{P}^*(c)$ might be empty. Because it can be easily verified that $\mathcal{P}^*(c) \subset \mathcal{P}(c)$, we can obtain the following propositions.

Proposition 2 *An interior stationary feasible allocation c is conditionally Pareto optimal* ***if*** *there exists at least one element P of* $\mathcal{P}^*(c)$ *such that* $\lambda^f(P) \leq 1$.

Proposition 2′ *An interior steady state allocation c is conditionally golden rule optimal* ***if*** *there exists at least one element P of* $\mathcal{P}^*(c)$ *such that* $\lambda^f(P) = 1$.

In the existing literature with smooth preferences, one of advantages of the dominant root criterion for optimality of equilibrium allocations is that we can examine optimality of the allocation by examining the dominant roots of the *observed* equilibrium price and the policy maker does not need information about the allocation nor preferences. On the other hand, when preferences are nonsmooth, we should remark the fact that we may not say anything about optimality of an observed stationary equilibrium (P, c) because optimality of its allocation is examined by the *set* of supporting prices, $\mathcal{P}(c)$, not an observed equilibrium contingent price matrix, P. Exceptionally, however, when the observed price matrix has the dominant root being less than or equal to one, we can say that the corresponding equilibrium allocation is optimal.

Corollary 4 *For each stationary equilibrium* (P, c) *with* $c_s^h \gg 0$ *for each* $h \in \mathcal{H}$ *and each* $s \in \mathcal{S}$*,* c *is conditionally Pareto optimal* ***if*** $\lambda^f(P) \leq 1$*.*

Corollary 4′ *For each stationary equilibrium* (P, c) *with* $c_s^h \gg 0$ *for each* $h \in \mathcal{H}$ *and each* $s \in \mathcal{S}$*,* c *is conditionally golden rule optimal* ***if*** $\lambda^f(P) = 1$*.*

These follow immediately from the previous propositions.

We can also provide results in the case of smooth preferences. When U^{hs} is differentiable for each $h \in \mathcal{H}$ and $s \in \mathcal{S}$, the model degenerates into the standard one and we can obtain the well-known characterizations of CPO and CGRO for stationary equilibrium allocations.

Corollary 5 *Suppose that* U^{hs} *is differentiable for each* $h \in \mathcal{H}$ *and each* $s \in \mathcal{S}$*. Then, for each stationary equilibrium* (P, c) *with* $c \gg 0$*,* c *is conditionally Pareto optimal if and only if* $\lambda^f(P) \leq 1$*.*

Corollary 5′ *Suppose that* U^{hs} *is differentiable for each* $h \in \mathcal{H}$ *and each* $s \in \mathcal{S}$*. Then, for each stationary equilibrium* (P, c) *with* $c \gg 0$*,* c *is conditionally golden rule optimal if and only if* $\lambda^f(P) = 1$*.*

These are consistent with Propositions 2 and 3 of Ohtaki (2013), which characterize CPO and CGRO of stationary equilibrium allocations when lifetime utility functions are differentiable.

We close this subsection with an important result on existence of optimal equilibrium.

Proposition 3 *There exists a stationary equilibrium, of which allocation is conditionally Pareto optimal.*

The proof of this proposition follows immediately from that of Theorem 2 of Aiyagari and Peled (1991). More precisely, Aiyagari and Peled provided the above result in a model with the class of differentiable expected utility preferences. However, their proof strategy does not depend on differentiability of lifetime utility functions and therefore it is applicable to our model. Following Proposition 3, we can say that a stationary equilibrium with money and social securities, defined in Subsection 4.4, always exists when every nonmonetary equilibrium, an equilibrium wherein money has no value, is not CPO.

4.3 Efficient No-trade Equilibria and Price Indeterminacy

In the current setting, agents are conditioned on the state at the date of their birth. Therefore, one might have an interpretation that agents have some prior, which might be common to all agents, but also have some posterior conditioned on the state realized in their first period. The current setting allows us to have such an interpretation. For example, if agents' preferences are given in the standard expected utility form such as

$$U^{hs}(c_s^h) = \sum_{s' \in \mathcal{S}} u^h(c_s^{hy}, c_{ss'}^{ho}) \pi_{ss'}^h$$

for some $\pi_s^h \in \Delta_{\mathcal{S}}$, then π_s^h can be interpreted as agent h's posterior conditioned on the first-period state s. Of course, agents' posteriors might be different from each other. We might be able to interpret such a situation as a result of some private information. According to these interpretations, it is natural to investigate whether new information provided by the state realized in agents' first period leads to trade.

Here, we argue about "no-trade equilibrium." A *stationary no-trade equilibrium* is a stationary equilibrium (P^*, c^*) such that $c^* = \omega$, *i.e.* : in that equilibrium, each agent does not buy nor sell at all and, as a result, what each agent consumes is exactly his/her initial endowment. Under strict concavity of utility functions, we can show that all of stationary equilibria are stationary no-trade equilibria if and only if the allocation corresponding to initial endowments is conditionally Pareto optimal.

Proposition 4 *The set of all stationary equilibria is given by* $\{(P,\omega) : P \in \mathcal{P}(\omega)\}$ *if and only if* ω *is conditionally Pareto optimal.*

Therefore, whenever the allocation corresponding to initial endowments is CPO, agents never buy nor sell contingent commodities at all. With the rearrangement of the setting so as to be able to buy and sell financial asset such as contingent claims, not contingent commodities, we can also say that there is no trade of financial assets whenever the profile of initial endowments is CPO.

Remark 4 One might find a relation between the previous proposition and the work of Milgrom and Stokey (1982), which observed in the literature of financial economics that if (1) agents are rational and strictly risk averse, (2) initial endowments are Pareto optimal, (3) agents' prior beliefs are concordant, and (4) the structure for agents to acquire information is itself common knowledge, then "no-trade equilibrium" can occur. The observation in Milgrom and Stokey (1982) says that, even when agents receive new inside information, no trade occurs under (1)–(4). Kajii and Ui (2009) and Martins-da-Rocha (2010) reexamined their results in models with preferences under ambiguity. Although our statement is similar to them, our setting does not explicitly belong to Bayesian models.

Proposition 4 also has a relation to Epstein and Wang (1994), which showed possibilities of indeterminacy of security prices in an infinitely-lived-agent model with the MEU preference. In their model, if there is no aggregate risk, price indeterminacy arises but the equilibrium allocation is equal to that correspond to the initial endowment, *i.e.*: they have observed nominal indeterminacy, not real indeterminacy. Even in our setting, a similar result can be obtained, *i.e.*: if ω is CPO and the set $\mathcal{M}(\omega)$, which is equal to the set of supporting price matrices $\mathcal{P}(\omega)$, has multiple elements, there are multiple price matrices P such that (P,ω) becomes stationary equilibrium. The following example illustrates nominal indeterminacy, not real indeterminacy, under the class of MEU preferences.

Example 5 Consider the same economy and specifications as in Example 4 and assume that $(\omega^o_\alpha, \omega^o_\beta) = (2.85, 2.85)$. In such an environment, as argued in the paragraph following Example 3, the stationary feasible allocation corresponding to $(\omega^o_\alpha, \omega^o_\beta)$, denoted by ω, is conditionally Pareto optimal. Then, it follows from the previous proposition that $\mathcal{P}(\omega) = \mathcal{P}^*(\omega)$, which implies that a pair (P,ω) can be a stationary equilibrium for each $P \in \mathcal{P}(\omega)$. Furthermore, it follows from Proposition 1 that $\mathcal{P}(\omega) = \{M_\pi(\omega) : \pi \in \Pi\}$ because ω is fully insured. Therefore, equilibrium price matrix is indeterminate because $M_\pi(\omega)$ for each $\pi \in \Pi$ is in fact a stationary equilibrium price matrix. ■

Note that this example also illustrates a situation wherein prices of a stationary equilibrium do not necessarily reveal optimality of the corresponding allocation. In fact, both $(M_\mu(\omega),\omega)$ and $(M_\nu(\omega),\omega)$ are, for example, stationary equilibria, wherein the observable price matrices are $M_\mu(\omega)$ and $M_\nu(\omega)$, respectively. However, both $\lambda^f(M_\mu(\omega))$ and $\lambda^f(M_\nu(\omega))$ are not equal to one. Actually, it is calculated that $\lambda^f(M_\mu(\omega)) \approx 0.89$ and $\lambda^f(M_\nu(\omega)) \approx 1.15$, which are not equal to one. Therefore, if the realized price matrix is $M_\nu(\omega)$ and one follows the classical dominant root criterion, which requires that the dominant root of the price matrix is less than or equal to one, the allocation ω is judged as a suboptimal one, whereas ω is CPO, especially CGRO, as argued in Example 3. This implies that observed equilibrium price matrices do not necessarily reveal optimality of corresponding allocations. We should note that, even when $\lambda^f(P) > 1$ [resp. $\lambda^f(P) \neq 1$] for some equilibrium price matrix P, the corresponding equilibrium allocation might be CPO [resp. CGRO].

4.4 Welfare Theorems in Sequentially Complete Markets with Constant Money Supply

As mentioned in Subsections 4.2 and 4.3, a stationary equilibrium itself might not be CPO even when markets operate perfectly. However, it is well-known in the literature that we can construct a market mechanism which generates a CPO allocation by introducing an infinitely-lived outside asset, which yields no dividend, money. In this section, we reexamine this well-known observation and show not only the first but also the second theorems of welfare economics in financial economy under constant money supply. In order to introduce the possibility of transfers, needed for the second welfare theorem, we introduce in addition to fiat money a mandatory unfunded social security system.

Mandatory Unfunded Social Security. We consider lump-sum transfers as a social security system. Each agent $h \in \mathcal{H}$ born at state $s \in \mathcal{H}$ pays τ_s^{hy} when she is young and receives $\tau_{ss'}^{ho}$ at state $s' \in \mathcal{S}$ when she is old. It is assumed that the transfer, $\tau = \{\tau^{hy}, \tau^{ho}\}_{h\in\mathcal{H}}$, satisfies that $\tau_s^{hy} < \omega_s^{hy}$ and $\tau_{ss'}^{ho} > -\omega_{ss'}^{ho}$ for each $h \in \mathcal{H}$ and each $s, s' \in \mathcal{S}$. It is also assumed that the authority's policy is balanced, *i.e.*: $\sum_{h\in\mathcal{H}} \tau_{ss'}^{ho} = \sum_{h\in\mathcal{H}} \tau_{s'}^{hy}$ for each $s, s' \in \mathcal{S}$.

Financial Assets. Suppose in this subsection that there exists one unit of money. Also suppose that spot markets of one-period contingent claims exist and are complete.

Definition 4 A triplet (q^*, P^*, c^*) of a positive money price vector $q^* \in \Re_{++}^{\mathcal{S}}$, a positive price matrix $P^* = \left[p_{ss'}^*\right]_{s,s'\in\mathcal{S}}$ of contingent claims, and a stationary feasible allocation c^* is called a *stationary equilibrium with money and social securities* if there exists some money holdings $m^* = (m^{*h}) \in (\Re^{\mathcal{S}})^{\mathcal{H}}$ and some contingent claim portfolio matrices $\theta^* \in (\Re^{\mathcal{S}\times\mathcal{S}})^{\mathcal{H}}$ such that

- for all $s \in \mathcal{S}$, $(c_s^{*h}, m_s^{*h}, \theta_s^{*h})$ belongs to the set

$$\underset{(c_s^{hy}, c_s^{ho}, m_s^h, \theta_s^h)}{\arg\max} \left\{ U^{hs}(c_s^h) : \begin{array}{l} c_s^{hy} = \omega_s^{hy} - \tau_s^{hy} - q_s^* m_s^h - \sum_{s'\in\mathcal{S}} \theta_{ss'}^h p_{ss'} \\ (\forall s' \in \mathcal{S})\, c_{ss'}^{ho} = \omega_{ss'}^{ho} + \tau_{ss'}^{ho} + q_{s'}^* m_s^h + \theta_{ss'}^h \end{array} \right\}$$

  given $q^* = (q_s^*)_{s\in\mathcal{S}}$ and $p_s^* = (p_{ss'}^*)_{s'\in\mathcal{S}}$; and

- for all $s, s' \in \mathcal{S}$, $\sum_{h\in\mathcal{H}} m_s^{*h} = 1$ and $\sum_{h\in\mathcal{H}} \theta_{ss'}^{*h} = 0$.

In this definition, the former is the optimization condition of each agent $h \in \mathcal{H}$ born at state $s \in \mathcal{S}$ subject to sequential budget constraints, and the latter is the pair of market clearing conditions for money and contingent claims. One can easily verify that the good market equilibrium condition also holds at a stationary equilibrium with money and social securities.

We can then find that an introduction of money can generate a CGRO allocation:

Theorem 3 *An interior stationary feasible allocation of a stationary equilibrium with money and social securities, if any, is always conditionally golden rule optimal.*

In other words, when a stationary equilibrium with money exists, it always generates a CGRO allocation. This theorem is an analog of the first fundamental theorem of welfare economics. The financial intermediate role of money for remedying inefficiency in the OLG model is a well-known result in the literature and the last theorem showed that the result still holds even when preferences are nonsmooth.

Remark 5 Gottardi (1996) considered a stochastic OLG model, wherein each generation consists of heterogeneous agents with differentiable lifetime utility functions and several securities exist, and showed that a stationary monetary equilibrium generically exists and is locally isolated.[25] Applying his result to our model, we can show generic existence of stationary equilibrium with money. This is because his proof of generic existence itself is independent of differentiability of lifetime utility functions. We should remark, however, that stationary equilibrium with money might not be locally isolated because lifetime utility functions in our model are not necessarily differentiable. Indeterminacy and its robustness in a stochastic OLG model under ambiguity has been studied by Ohtaki and Ozaki (2015).

Because a stationary equilibrium with money and social securities always achieves CGRO, we should note that there might exists a CPO allocation that cannot be implemented as a stationary equilibrium with money and social securities. Therefore, the second welfare theorem might not hold in stochastic OLG models if we adopt CPO as an optimality criterion. However, by adopting CGRO, not CPO, as an optimality criterion, we can also obtain the second welfare theorem.

Theorem 4 *Each interior conditionally golden rule optimal allocation, if any, can be achieved by some stationary equilibrium with money and appropriate social securities.*

This theorem is an analog of the second welfare theorem. As noted above, we might not be able to obtain this theorem when we adopt CPO instead of CGRO as an optimality criterion. Theorem 4 has an important implication, *i.e.*: any interior CGRO allocation can be implemented as a stationary equilibrium with money in constant supply under an appropriate social security system.

4.5 Sequentially Complete Markets with Lump-sum Money Transfers

As shown in the previous subsection, an introduction of money in constant supply can achieve CPO, especially CGRO. However, it is also well-known in the deterministic OLG model with smooth preferences that an introduction of lump-sum money tax causes inefficiency (McCandless and Wallace, 1992, Ch.10). In this subsection, we reexamine this well-known result under nonsmooth preferences.

In order to consider an economy with lump-sum money taxes/subsidies, we first introduce a policymaker who issues an outside asset yielding no dividend, money. The stock of money in each period $t \geq 1$ is denoted by M_t and satisfies that $M_t = \mu M_{t-1}$ for each $t \geq 1$, where $\mu > 0$ is the "gross" rate of growth of money and independent of realizations of states and M_0 is the initial stock distributed to the initial olds. The newly issued money in period $t \geq 1$, $M_t - M_{t-1} = (\mu - 1)M_{t-1}$, is equally distributed as lump-sum money transfer among old agents in that period.[26]

Now, denote the real price of money by $q(\sigma_t)$ for each $\sigma_t = (s_0, s_1, ..., s_t)$, which is the history of states realized from (implicitly defined) period 0 until period t. Although the real money balance $q(\sigma_t)M_t$ might also depend on histories of states σ_t, we wish to explore a stationary situation wherein the real money balance at the history σ_t depends only on the current state s_t, not on time nor on past realizations of states, and can be denoted by $\rho^*_{s_t}$. In such a situation, then, we can obtain that

$$q(\sigma_t) = \frac{\rho^*_{s_t}}{M_t}$$

[25] By imposing not only smoothness but also additive separability and some elasticity condition on the lifetime utility function, one can observe uniqueness of stationary equilibrium with money. See for example Ohtaki (2015).

[26] One can assume that the newly issued money in each period is distributed among all agents, not only old agents, in that period.

and define a stationary equilibrium with lump-sum money transfers by ρ^* instead of q.

Definition 5 A triplet (ρ^*, P^*, c^*) of a positive real money balance vector $\rho^* \in \Re^{\mathcal{S}}_{++}$, a positive price matrix $P^* = \left[p^*_{ss'}\right]_{s,s'\in\mathcal{S}}$ of contingent claims, and a stationary feasible allocation c^* is called a *stationary equilibrium with lump-sum money transfers* if there exists some money holdings $m^*_t = (m^{*h}_t)_{h\in\mathcal{H}} \in (\Re^{\mathcal{S}})^{\mathcal{H}}$ and some contingent claim portfolio matrices $\theta^* \in (\Re^{\mathcal{S}\times\mathcal{S}})^{\mathcal{H}}$ such that

- for all $s \in \mathcal{S}$, $(c^{*h}_s, m^{*h}_s, \theta^{*h}_s)$ belongs to the set

$$\underset{(c^{hy}_s, c^{ho}_s, m^h_{t,s}, \theta^h_s)}{\arg\max} \left\{ U^{hs}(c^h_s) : \begin{array}{c} c^{hy}_s = \omega^{hy}_s - \dfrac{\rho^*_s}{M_t} m^h_{t,s} - \displaystyle\sum_{s'\in\mathcal{S}} \theta^h_{ss'} p_{ss'} \\ (\forall s' \in \mathcal{S})\, c^{ho}_{ss'} = \omega^{ho}_{ss'} + \dfrac{\rho^*_{s'}}{M_{t+1}} \left(m^h_{t,s} + (\mu - 1)\dfrac{M_t}{H} \right) + \theta^h_{ss'} \end{array} \right\}$$

  given $\rho^* = (\rho^*_s)_{s\in\mathcal{S}}$ and $p^*_s = (p^*_{ss'})_{s'\in\mathcal{S}}$; and

- for each period t and arbitrary states $s, s' \in \mathcal{S}$, $\sum_{h\in\mathcal{H}} m^{*h}_{t,s} = M_t$ and $\sum_{h\in\mathcal{H}} \theta^{*h}_{ss'} = 0$.

In this definition, the former is the optimization condition of each agent $h \in \mathcal{H}$ born at state $s \in \mathcal{S}$ subject to sequential budget constraints, and the latter is the asset market clearing conditions. One can easily verify that the good market equilibrium condition also holds at a stationary equilibrium with lump-sum money transfers.

Proposition 5 *For each interior stationary feasible allocation of a stationary equilibrium with lump-sum money transfers, denoted by (ρ^*, P^*, c^*) if any, it holds that $\lambda^f(P^*) \leq 1$ if $\mu \leq 1$ and otherwise $\lambda^f(P^*) > 1$.*

As an immediate corollary of this proposition, we can say that a stationary monetary equilibrium with lump-sum money transfers can achieve CPO if $\mu \leq 1$. Furthermore, when preferences are smooth, we can obtain the following corollary.

Corollary 6 *When U^{hs} is differentiable for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, the allocation of a stationary equilibrium with lump-sum money transfers is conditionally Pareto optimal if $\mu \leq 1$ and otherwise conditionally Pareto suboptimal.*

As mentioned above, in the deterministic OLG model with smooth preferences that an introduction of lump-sum money tax causes inefficiency. Corollary 6 is an immediate extension of this well-known observation to a stochastic environment with smooth preferences.

Contrary to a case with smooth preferences, we can find in a case with nonsmooth preferences that the condition that $\mu > 1$ does not necessarily mean suboptimality of an equilibrium allocation. This is because its CPO depends on the set of supporting prices, not on the realized equilibrium prices of contingent claims. In fact, we can construct examples, which illustrate that the equilibrium allocation is still optimal even when lump-sum money tax is introduced.

Example 6 Consider the same economy as Example 4. Moreover, specify the initial endowments as $\omega^y_\alpha = 5.5$, $\omega^y_\beta = 3.05$, $\omega^o_{s\alpha} = 1$, and $\omega^o_{s\beta} = 1.95$ for each $s \in \mathcal{S}$. In this economy, one can immediately verify that the vector of real money balances, $\rho = (\rho_s)_{s\in\mathcal{S}}$, can be characterized as a solution of the following inclusion:

$$\mu \begin{bmatrix} \rho_\alpha \\ \rho_\beta \end{bmatrix} \in \left\{ \begin{bmatrix} \dfrac{v_o'(\omega^o_{\alpha\alpha}+\rho_\alpha)\pi_{\alpha\alpha}}{v_y'(\omega^y_\alpha-\rho_\alpha)} & \dfrac{v_o'(\omega^o_{\alpha\beta}+\rho_\beta)\pi_{\alpha\beta}}{v_y'(\omega^y_\alpha-\rho_\alpha)} \\ \dfrac{v_o'(\omega^o_{\beta\alpha}+\rho_\alpha)\pi_{\beta\alpha}}{v_y'(\omega^y_\beta-\rho_\beta)} & \dfrac{v_o'(\omega^o_{\beta\beta}+\rho_\beta)\pi_{\beta\beta}}{v_y'(\omega^y_\beta-\rho_\beta)} \end{bmatrix} \begin{bmatrix} \rho_\alpha \\ \rho_\beta \end{bmatrix} : (\forall s \in \mathcal{S})\ \pi_s \in \mathcal{B}_s(c) \right\}$$

$$= \left\{ \begin{bmatrix} \dfrac{(5.5-\rho_\alpha)\pi_{\alpha\alpha}}{1+\rho_\alpha} & \dfrac{(5.5-\rho_\alpha)\pi_{\alpha\beta}}{1.95+\rho_\beta} \\ \dfrac{(3.05-\rho_\beta)\pi_{\beta\alpha}}{1+\rho_\alpha} & \dfrac{(3.05-\rho_\beta)\pi_{\beta\beta}}{1.95+\rho_\beta} \end{bmatrix} \begin{bmatrix} \rho_\alpha \\ \rho_\beta \end{bmatrix} : (\forall s \in \mathcal{S})\ \pi_s \in \mathcal{B}_s(c) \right\},$$

where $v_y(x) = v_o(x) = \ln x$ and c is an associated equilibrium allocation. Here, let $1 < \mu < 1.14$ and consider $(\rho_A, \rho_B) = (1.85, 0.9)$ as a candidate of an equilibrium vector of real money balances. Then, $\mathcal{B}_s(c_s)$ in the previous equation is replaced by Π_s because $c^o_{ss'} = 2.85$ for each $s, s' \in \mathcal{S}$. Moreover, because $\lambda^f(M_\mu(c)) \approx 0.89 < \mu < 1.149 \approx \lambda^f(M_\nu(c))$,[27] the vector of real money balances, ρ, is satisfied the last inclusion and therefore it is in fact the equilibrium vector of real money balances. Now consider the equilibrium allocation $(c^o_0, c) = (c^o_0, (c^y_s, c^o_{ss'})_{s,s'\in\mathcal{S}})$ such that $c^y_\alpha = 3.65$, $c^y_\beta = 2.15$, and $c^o_0 = c^o_{ss'} = 2.85$ for each $s, s' \in \mathcal{S}$. As investigated in Example 3, this equilibrium allocation is CPO, especially CGRO, even though the associated equilibrium price matrix of contingent claims has the dominant root being greater than one. Therefore, this example illustrates a situation wherein equilibrium prices do not necessarily reveal optimality of the corresponding allocation. ■

Even in a deterministic environment, we can also construct an example similar to Example 6.

Example 7 Suppose that $\mathcal{S}$ and $\mathcal{H}$ are singletons. Also assume that $\omega^y > \omega^o$ and that the lifetime utility function is given as in Eq.(5), *i.e.*:

$$U(c^y, c^o) = \min_{\delta \in D}[(1-\delta)u(c^y) + \delta u(c^o)],$$

where $D := [\underline{\delta}, \overline{\delta}]$ for some $\underline{\delta}$ and some $\overline{\delta}$ satisfying that $0 < \underline{\delta} \le \overline{\delta} < 1$. Then, a real money balance of a stationary equilibrium with lump-sum money transfers, ρ, is characterized by a solution of the inequality

$$\min_{\delta \in AD(c^y,c^o)}[-(1-\delta)\rho u'(c^y) + \delta\frac{\rho}{\mu}u(c^o)] \le 0 \le \max_{\delta \in AD(c^y,c^o)}[-(1-\delta)\rho u'(c^y) + \delta\frac{\rho}{\mu}u(c^o)]$$

where $(c^y, c^o) = (\omega^y - \rho, \omega^o + \rho)$ and

$$AD(c^y, c^o) := \arg\min_{\delta \in D}[(1-\delta)u(c^y) + \delta u(c^o)],$$

which is the set of active discount rates. Here, as a candidate of an equilibrium real money balance, let $\rho^* = 0.5(\omega^y - \omega^o)$. We can immediately find that ρ^* satisfies the above inequality if and only if

$$\frac{\underline{\delta}}{1-\underline{\delta}} \le \mu \le \frac{\overline{\delta}}{1-\overline{\delta}}.$$

So, ρ^* is actually an equilibrium real money balance, provided that the last inequality holds. Furthermore, one can find that the associated allocation $(c^o_0, c^y, c^o) = (0.5\bar{\omega}, 0.5\bar{\omega}, 0.5\bar{\omega})$ is Pareto optimal, especially golden rule optimal, if

$$\frac{\underline{\delta}}{1-\underline{\delta}} \le 1 \le \frac{\overline{\delta}}{1-\overline{\delta}}.$$

[27]To be more precise, one can find some $\pi = (\pi_\alpha, \pi_\beta) \in \Pi_\alpha \times \Pi_\beta$ such that $\lambda^f(M_\pi(c)) = \mu$.

which is equivalent to the condition that $\underline{\delta} \leq 0.5 \leq \bar{\delta}$. Therefore, we can conclude that, even when $\mu > 1$, an equilibrium allocation can be Pareto optimal if $\underline{\delta} < 0.5$ and $\mu/(1+\mu) \leq \bar{\delta}$. ■

4.6 Sequentially Complete Markets with Equity

We finally consider an economy with "equity" instead of "money." In an OLG environment with equity, there are seemingly different views on efficiency of equilibrium allocations. Dow and Gorton (1993) claimed that, in a deterministic environment, equilibrium allocations are never optimal. On the other hand, Sakai (1988) claimed that, in a stochastic environment, equilibrium allocations are conditionally Pareto optimal. As shown in Corollary 7 below, the gap between these two claims comes from the gap between definitions of optimality criteria, *i.e.*: Dow and Gorton considered golden rule optimality, whereas Sakai did (conditional) Pareto optimality. In this subsection, we reexamine optimality of allocations of equilibrium with equity, especially paying attention to the difference between optimality criteria.

Suppose in this subsection that there exists one unit of an infinitely-lived asset yielding a dividend of $d_s \geq 0$ units of the consumption good at state $s \in \mathcal{S}$, where $d \in \Re^{\mathcal{S}}_{+} \setminus \{0\}$.[28] Also suppose that spot markets of one-period contingent claims exist and are complete.

Definition 6 A triplet (q^*, P^*, c^*) of a positive equity price vector $q^* \in \Re^{\mathcal{S}}_{++}$, a positive price matrix $P^* = \left[p^*_{ss'}\right]_{s,s' \in \mathcal{S}}$ of contingent claims, and a stationary feasible allocation c^* is called a *stationary equilibrium with equity* if there exists some profile of equity holding vectors $z^* \in (\Re^{\mathcal{S}})^{\mathcal{H}}$ and some profile of contingent claim portfolio matricies $\theta^* \in (\Re^{\mathcal{S}\times\mathcal{S}})^{\mathcal{H}}$ such that

- for all $s \in \mathcal{S}$, $(c^{*h}_s, z^{*h}_s, \theta^{*h}_s)$ belongs to the set
$$\underset{(c^{hy}_s, c^{ho}_s, z^h_s, \theta^h_s)}{\arg\max} \left\{ U^{hs}(c^h_s) : \begin{array}{l} c^{hy}_s = \omega^{hy}_s - q^*_s z^h_s - \sum_{s' \in \mathcal{S}} \theta^h_{ss'} p_{ss'} \\ (\forall s' \in \mathcal{S})\, c^{ho}_{ss'} = \omega^{ho}_{ss'} + (q^*_{s'} + d_{s'}) z^h_s + \theta_{ss'} \end{array} \right\}$$
given q^* and p^*_s; and
- for all $s, s' \in \mathcal{S}$, $\sum_{h\in\mathcal{H}} z^{*h}_s = 1$ and $\sum_{h\in\mathcal{H}} \theta^{*h}_{ss'} = 0$.

In this definition, the former is the optimization condition of each agent $h \in \mathcal{H}$ born at state $s \in \mathcal{S}$ subject to sequential budget constraints, and the latter is the asset market clearing conditions. One can easily verify that the good market equilibrium condition also holds at a stationary equilibrium with equity.

We should note that, even in the economy with equity, CPO [resp. CGRO] of an interior stationary *feasible* allocation can be characterized as Theorem 1 [resp. Theorem 1′] by redefining the total endowment as $\bar{\omega}_{ss'} = \sum_{h\in\mathcal{H}} \omega^{hy}_{s'} + \sum_{h\in\mathcal{H}} \omega^{ho}_{ss'} + d_{s'}$ for each $s, s' \in \mathcal{S}$. Therefore, our task is to examine optimality of stationary equilibrium allocations with equity. The following statement is the last theorem of this paper:

Proposition 6 *For any stationary equilibrium with equity, denoted by (q, P, c) if any, it holds that $\lambda^f(P) < 1$.*

In other words, even when preferences are nonsmooth, a stationary equilibrium with equity generates a CPO allocation. This is a natural extension of the result of Sakai (1988) to the case with nonsmooth preferences.

Under smooth preferences, we can obtain a bit stronger result.

[28] Therefore, the asset can be the equity of any productive asset like "land" or a "Lucas tree." It can be also identified with money if $d_s = 0$ for each $s \in \mathcal{S}$.

Corollary 7 *When U^{hs} is differentiable for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, the allocation of a stationary equilibrium with equity is conditionally Pareto optimal but is never conditionally golden rule optimal.*

This result bridges the gap between those of Sakai (1988) and Dow and Gorton (1993). An introduction of equity causes too much second-period consumption and newly born agents wish to transfer some of it to the first-period consumption for consumption smoothing, whereas the initial olds welcome such second-period consumption. This is the reason why the equilibrium allocation is CPO but not CGRO.

In order to explore the role of nonsmoothness of preferences, we should investigate whether a result similar to Corollary 7 still holds when lifetime utility functions might not be differentiable. However, as shown in the next example, the answer is negative.[29]

Example 8 Consider the same economy as Example 4, except for specifications on initial endowments. Instead, specify dividends and initial endowments by $d_\alpha = 0.01$, $d_\beta = 0.02$, $\omega^y_\alpha = 5.49$, $\omega^y_\beta = 3.03$, $\omega^o_{s\alpha} = 1$, $\omega^o_{s\beta} = 1.95$ for each $s \in \mathcal{S}$.[30] Then, one can easily verify that the price vector for equity, $q = (q_\alpha, q_\beta)$, is characterized as a solution of the system of inclusions: for each $s \in \mathcal{S}$,

$$0 \in \left\{ -q_s v'_y(\omega^y_s - q_s) + \sum_{s' \in \mathcal{S}} (q_{s'} + d_{s'}) v'_o(\omega^o_{ss'} + d_{s'} + q_{s'}) \pi_{ss'} : (\forall s \in \mathcal{S})\ \pi_s \in \mathcal{B}_s(c_s) \right\} \tag{7}$$

where $v_y(x) = v_o(x) = \ln x$ and c is an associated equilibrium allocation. Here, we consider $(q_\alpha, q_\beta) = (1.84, 0.88)$ as a candidate of an equilibrium price vector for equity. Because $c^o_{ss'} = \omega^o_{ss'} + q_{s'} = 2.85$ for each $s, s' \in \mathcal{S}$, $\mathcal{B}_s(c_s)$ in the previous equation can be replaced by Π_s. Moreover, by numerical calculations, we can confirm that

$$\begin{aligned}
-q_\alpha v'_y(\omega^y_\alpha - q_\alpha) + \sum_{s' \in \mathcal{S}} (q_{s'} + d_{s'}) v'_o(\omega^o_{ss'} + d_{s'} + q_{s'}) \mu_{\alpha s'} &\approx -0.10, \\
-q_\alpha v'_y(\omega^y_\alpha - q_\alpha) + \sum_{s' \in \mathcal{S}} (q_{s'} + d_{s'}) v'_o(\omega^o_{ss'} + d_{s'} + q_{s'}) \nu_{\alpha s'} &\approx 0.06, \\
-q_\beta v'_y(\omega^y_\beta - q_\beta) + \sum_{s' \in \mathcal{S}} (q_{s'} + d_{s'}) v'_o(\omega^o_{ss'} + d_{s'} + q_{s'}) \mu_{\beta s'} &\approx -0.01, \\
-q_\beta v'_y(\omega^y_\beta - q_\beta) + \sum_{s' \in \mathcal{S}} (q_{s'} + d_{s'}) v'_o(\omega^o_{ss'} + d_{s'} + q_{s'}) \nu_{\beta s'} &\approx 0.16,
\end{aligned}$$

so that the price vector for equity, q, is satisfied the equilibrium inclusions (7) and therefore it is in fact the equilibrium price vector. Now consider the equilibrium allocation $(c^o_0, c) = (c^o_0, (c^y_s, c^o_{ss'})_{s,s' \in \mathcal{S}})$ such that $c^y_\alpha = 3.65$, $c^y_\beta = 2.15$, and $c^o_0 = c^o_{ss'} = 2.85$ for each $s, s' \in \mathcal{S}$. As investigated in Example 3, this equilibrium allocation is not only CPO but also CGRO, even though the associated equilibrium price matrix of contingent claims has the dominant root being less than one. Therefore, this example illustrates a situation wherein equilibrium prices do not necessarily reveal precise information on optimality of the corresponding allocation. ■

Even in a deterministic environment, we can also construct an example similar to Example 8.

[29] Even in a deterministic environment, one can also construct an example similar to Example 6 by using preferences with multiple discount rates as in Eq.(2).

[30] Note that, under these specifications, $\bar{\omega}_\alpha = 6.5$ and $\bar{\omega}_\beta = 5$.

Example 9 Suppose that $\mathcal{S}$ and $\mathcal{H}$ are singletons. Also assume that $\omega^y > \omega^o + d$ and that the lifetime utility function is given as in Example 7. Then, a equilibrium price of equity, q, is characterized by a solution of the inequality

$$\min_{\delta \in AD(c^y, c^o)} [-(1-\delta)qu'(c^y) + \delta(q+d)u(c^o)] \leq 0 \leq \max_{\delta \in AD(c^y, c^o)} [-(1-\delta)qu'(c^y) + \delta(q+d)u(c^o)]$$

where $(c^y, c^o) = (\omega^y - q, \omega^o + d + q)$ and

$$AD(c^y, c^o) := \arg\min_{\delta \in D} [(1-\delta)u(c^y) + \delta u(c^o)],$$

which is the set of active discount rates. Here, as a candidate of an equilibrium price of equity, define q^* by $q^* = 0.5(\omega^y - \omega^o - d)$. Also let $R = (q^* + d)/q^* > 1$, which is the rate of return of equity. One can immediately find that q^* satisfies the above inequality if and only if

$$\frac{\underline{\delta}}{1-\underline{\delta}} \leq \frac{1}{R} \leq \frac{\overline{\delta}}{1-\overline{\delta}}.$$

Furthermore, one can find that the associated allocation $(c_0^o, c^y, c^o) = (0.5\bar{\omega}, 0.5\bar{\omega}, 0.5\bar{\omega})$ is Pareto optimal, especially golden rule optimal, if $\underline{\delta} \leq 0.5 \leq \overline{\delta}$, where $\bar{\omega} = \omega^y + \omega^o + d$. Therefore, we can conclude that, an equilibrium allocation is not only Pareto optimal but also golden rule optimal if $\underline{\delta} \leq 1/(1+R)$ and $0.5 < \overline{\delta}$. ■

5 Concluding Remarks

This study has reexamined optimality of stationary feasible allocations in an OLG model with convex but not necessarily smooth. It has been shown that optimality criteria of stationary *feasible* allocations are characterized by the condition on the set of dominant roots of matrices of marginal rates of substitution: conditional Pareto optimality requires that the set contains some number being less than or equal to unity but conditional golden rule optimality requires that the set contains unity. It has been also shown that optimality criteria of stationary *equilibrium* allocations are characterized by the condition on the set of supporting price matrices, not necessarily on the equilibrium price matrix itself. In contrast to the existing results under smooth preferences, the latter result indicates that the realized equilibrium prices do not necessarily tell us whether the associated equilibrium allocations are optimal. We have also provided several trivial and nontrivial examples illustrating this fact.

The above contrast between the existing and presented results appears at allocations, at which lifetime utility functions are not differentiable. When we consider the class of the MEU preferences, for example, such allocations should be fully insured with respect to the second-period consumptions and the set of such allocations might have Lebesgue measure zero with respect to an appropriate universal set of allocations.[31] So, one might seem that our results fall under the scope of results of Rigotti and Shannon (2012) and are less important. However, as shown in Ohtaki and Ozaki (2015), an OLG model with not necessarily smooth preferences can often generate fully-insured equilibria for a nonnegligible set of initial endowments (and associated economies exhibit aggregate uncertainty). Examples 6 and 8 in this study partially illustrate such equilibria. Therefore, our results will give beneficial suggestions to future studies which examine optimality of various economic mechanisms in OLG models with not necessarily smooth preferences.

[31] Even when we consider a more general class of convex preferences, the set of nondifferentiable points will have Lebesgue measure zero because we assume strict concavity of lifetime utility functions.

Appendix A: Box Diagram for 2-state and 1-agent Economy

In this appendix, we introduce a graphical device, which is developed by Ohtaki (2014), to analyze a 2-state and 1-agent pure-endowment stochastic OLG model. Assume that $\mathcal{H}$ is a singleton (and therefore we omit sub/superscript h, which represents agents' types) and $\mathcal{S} = \{\alpha, \beta\}$. Furthermore, we assume that the second-period endowment vector $(\omega^o_{ss'})_{s' \in \mathcal{S}}$ of agent born at state s is independent of the state at which the agent is born, *i.e.*: $\omega^o_{ss'} = \omega^o_{ts'}$ for each $s, t, s' \in \mathcal{S}$. By the latter assumption, it follows that the total endowment depends only on the current state, *i.e.*: for each $s, t, s' \in \mathcal{S}$, $\bar{\omega}_{ss'} = \bar{\omega}_{ts'}$, so that we denote by $\bar{\omega}_{s'}$ the total endowment when the current state is s'.

Under assumptions that $\mathcal{H}$ is a singleton and that the total endowment depends only on the current state, the set of stationary feasible allocation is identifiable with the set of pairs (x^y, x^o) of functions from $\mathcal{S}$ to $\Re_+$ such that $x^y_s + x^o_s = \bar{\omega}_s$ for each $s \in \mathcal{S}$ or, more simply, with $X := [0, \bar{\omega}_\alpha] \times [0, \bar{\omega}_\beta] = \{x^o \in \Re^{\mathcal{S}} : (\forall s' \in \mathcal{S})\ 0 \leq x^o_{s'} \leq \bar{\omega}_{s'}\}$.[32] Similarly, an interior stationary feasible allocation is related to an element of int.X, the interior of X.

Because the set X, elements of which are second-period consumptions, can be identified with the set of stationary feasible allocations, we can derive another utility function on X, denoted by $\hat{U}^s$, from U^s. To be more precise, for each $s \in \mathcal{S}$, define the function $\hat{U}^s : X \to \Re$ by

$$(\forall x^o \in X) \quad \hat{U}^s(x^o) := U^s(\bar{\omega}_s - x^o_s, (x^o_{s'})_{s' \in \mathcal{S}}).$$

One should note that, by this derived utility function $\hat{U}^s$, we can draw indifference curves in X.

Example A.1 Consider the same economy with that appeared in Example 3. Then, the Panel (b) of Figure A.1 depicts, in the box X, indifference curves derived from $\hat{U}^s$ through $(c^o_\alpha, c^o_\beta) = (1, 1)$ and $(c^o_\alpha, c^o_\beta) = (2, 2)$ for each s. The U-shaped and $\subset$-shaped curves are related to indifference curves for agents born at α and β, respectively. Note that $\hat{U}^s(1, 1) < \hat{U}^s(2, 2)$. Also note that indifference curves have kinks on the line satisfying that $c^o_\alpha = c^o_\beta$. This comes from nondifferentiability of MEU functions. In fact, when $\Pi_s = \{(0.75, 0.25)\}$, the model degenerates to one with smooth preferences and we can obtain smooth indifference curves as in the Panel (a) of Figure A.1. ■

In order to consider kinks of indifference curves appeared in Example A.1, we examine the slope of indifference curves. When U^s is differentiable for each s, the slope of the agent s's indifference curve at $x^o \in X$, denoted by $\widehat{MRS}_s(x^o)$ if any, can be calculated by $\widehat{MRS}_s(x^o) = dx^o_\beta / dx^o_\alpha = -\hat{U}^s_1(x^o)/\hat{U}^s_2(x^o)$ because $0 = \hat{U}^s_1(x^o) dx^o_\alpha + \hat{U}^s_2(x^o) dx^o_\beta$ on $\{x' \in X : \hat{U}^s(x') = \hat{U}^s(x^o)\}$.[33]

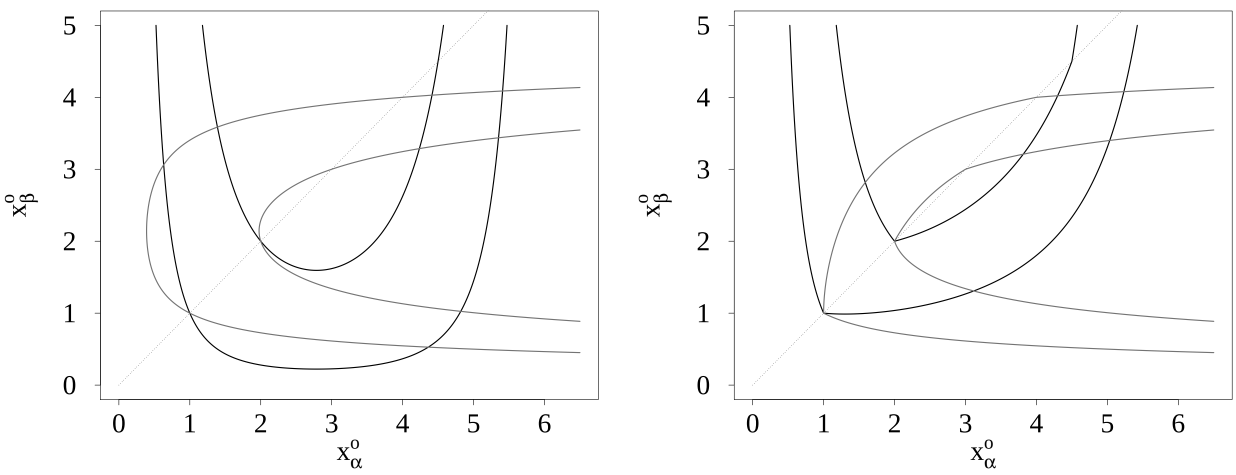
On the other hand, suppose that U^s belongs to the class of MEU preferences and is assumed to be given by

$$U^s(c_s) = \min_{\pi_s \in \Pi_s} \sum_{s' \in \mathcal{S}} [v_y(c^y_s) + v_o(c^o_{ss'})]\, \pi_{ss'}, \tag{A.1}$$

where $\Pi_s = \{\pi_s \in \Delta_{\mathcal{S}} : \mu_{s\alpha} \leq \pi_{s\alpha} \leq \nu_{s\alpha}\}$ for some $\mu_s, \nu_s \in \Delta_{\mathcal{S}}$ such that $0 < \mu_{s\alpha} \leq \nu_{s\alpha} < 1$ and v_y and v_o are real-valued functions on $\Re_+$, which are strictly monotone, strictly concave, and continuously differentiable on their domain, respectively. Then, it holds that, for each $s \in \mathcal{S}$ and each $x^o \in X$,

[32] Note that, for each $s, t, s' \in \mathcal{S}$, $c^y_{s'} + c^o_{ss'} = \bar{\omega}_{s'} = c^y_{s'} + c^o_{ts'}$, which implies that $c^o_{ss'} = c^o_{ts'}$. Therefore, we can identify a stationary feasible allocation c with an element x^o of X by setting as $c^o_{ss'} = x^o_{s'}$ and $c^y_{s'} = \bar{\omega}_{s'} - x^o_{s'}$ for each $s, s' \in \mathcal{S}$.

[33] Do not confuse $\widehat{MRS}_s(x^o)$ with $M_\pi(c)$ defined as in Eq.(5).



(a) $(\mu_{s\alpha}, \nu_{s\alpha}) = (0.75, 0.75)$ for each $s \in \mathcal{S}$ (b) $(\mu_{s\alpha}, \nu_{s\alpha}) = (0.25, 0.75)$ for each $s \in \mathcal{S}$

Figure A.1: Indifference Curves

$$
\begin{aligned}
\hat{U}^s(x^o) &= \min_{\pi_s \in \Pi_s} \sum_{\tau \in \mathcal{S}} \left[v_y(\bar{\omega}_s - x^o_s) + v_o(x^o_\tau) \right] \pi_{s\tau} \\
&= v_y(\bar{\omega}_s - x^o_s) + v_o(x^o_\beta) + \min_{\pi_s \in \Pi_s} \left[\left(v_o(x^o_\alpha) - v_o(x^o_\beta) \right) \pi_{s\alpha} \right] \\
&= \begin{cases} v_y(\bar{\omega}_s - x^o_s) + \mu_{s\alpha} v_o(x^o_\alpha) + (1 - \mu_{s\alpha}) v_o(x^o_\beta) & \text{if} \quad x^o_\alpha > x^o_\beta, \\ v_y(\bar{\omega}_s - x^o_s) + v_o(x^o_s) & \text{if} \quad x^o_\alpha = x^o_\beta, \quad \text{and} \\ v_y(\bar{\omega}_s - x^o_s) + \nu_{s\alpha} v_o(x^o_\alpha) + (1 - \nu_{s\alpha}) v_o(x^o_\beta) & \text{if} \quad x^o_\alpha < x^o_\beta. \end{cases}
\end{aligned}
$$

Therefore, the slope of the agent s's indifference curve at $x^o \in X$, if any, can be calculated as:

$$
\widehat{MRS}_\alpha(x^o) = -\frac{\hat{U}^\alpha_1(x^o)}{\hat{U}^\alpha_2(x^o)} = \begin{cases} \dfrac{v'_y(\bar{\omega}_\alpha - x^o_\alpha) - \mu_{\alpha\alpha} v'_o(x^o_\alpha)}{(1 - \mu_{\alpha\alpha}) v'_o(x^o_\beta)} & \text{if} \quad x^o_\alpha > x^o_\beta \\[2ex] \dfrac{v'_y(\bar{\omega}_\alpha - x^o_\alpha) - \nu_{\alpha\alpha} v'_o(x^o_\alpha)}{(1 - \nu_{\alpha\alpha}) v'_o(x^o_\beta)} & \text{if} \quad x^o_\alpha > x^o_\beta \end{cases}
$$

and

$$
\widehat{MRS}_\beta(x^o) = -\frac{\hat{U}^\beta_1(x^o)}{\hat{U}^\beta_2(x^o)} = \begin{cases} \dfrac{\mu_{\beta\alpha} v'_o(x^o_\alpha)}{v'_y(\bar{\omega}_\beta - x^o_\beta) - \mu_{\beta\beta} v'_o(x^o_\beta)} & \text{if} \quad x^o_\alpha > x^o_\beta, \\[2ex] \dfrac{\nu_{\beta\alpha} v'_o(x^o_\alpha)}{v'_y(\bar{\omega}_\beta - x^o_\beta) - \nu_{\beta\beta} v'_o(x^o_\beta)} & \text{if} \quad x^o_\alpha > x^o_\beta, \end{cases}
$$

if $x^o_\alpha \neq x^o_\beta$, but it might not be calculated if $x^o_\alpha = x^o_\beta$ (This is true when $\mu_{s\alpha} < \nu_{s\alpha}$).

Using the box diagram, we can obtain graphical intuitions on optimality criteria, especially on conditional golden rule optimality (CGRO). Consider an interior allocation $x = (x_\alpha, x_\beta)$ in the box, where $x_s = (x^y_s, x^o_\alpha, x^o_\beta)$ for each $s \in \mathcal{S}$. First, assume that U^s is differentiable. Then, $\mathcal{M}(x)$ is given by

$$
\mathcal{M}(x) = \begin{bmatrix} \dfrac{U^\alpha_\alpha(x_\alpha)}{U^\alpha_y(x_\alpha)} & \dfrac{U^\alpha_\beta(x_\alpha)}{U^\alpha_y(x_\alpha)} \\[2ex] \dfrac{U^\beta_\alpha(x_\beta)}{U^\beta_y(x_\beta)} & \dfrac{U^\beta_\beta(x_\beta)}{U^\beta_y(x_\beta)} \end{bmatrix}.
$$

The dominant root criterion for CGRO,[34] $\lambda^f(\mathcal{M}(x)) = 1$, is equivalent to the following condition:[35]

$$\mathcal{M}(x)y(\mathcal{M}(x)) = \lambda^f(\mathcal{M}(x))y(\mathcal{M}(x)) = y(\mathcal{M}(x)) \tag{A.2}$$

or equivalently

$$U^\alpha_\alpha(x_\alpha)y_\alpha(\mathcal{M}(x)) + U^\alpha_\beta(x_\alpha)y_\beta(\mathcal{M}(x)) = U^\alpha_y(x_\alpha)y_\alpha(\mathcal{M}(x))$$

and

$$U^\beta_\alpha(x_\beta)y_\alpha(\mathcal{M}(x)) + U^\beta_\beta(x_\beta)y_\beta(\mathcal{M}(x)) = U^\beta_y(x_\alpha)y_\beta(\mathcal{M}_c(\pi)).$$

These equations can be rewritten as

$$\frac{U^\alpha_y(x_\alpha) - U^\alpha_\alpha(x_\alpha)}{U^\alpha_\beta(x_\alpha)} = \frac{y_\beta(\mathcal{M}(x))}{y_\alpha(\mathcal{M}(x))} = \frac{U^\beta_\alpha(x_\beta)}{U^\beta_y(x_\alpha) - U^\beta_\beta(x_\beta)}, \tag{A.3}$$

which is equivalent to

$$\widehat{MRS}_\alpha(x^o_\alpha, x^o_\beta) = \widehat{MRS}_\beta(x^o_\alpha, x^o_\beta) > 0, \tag{A.4}$$

where $\widehat{MRS}_s$ has defined above and that (A.3) implies (A.4) follows from the definition of $\widehat{MRS}_s$ and the fact that $y(\mathcal{M}(x))$ is a positive vector. In order to show that (A.4) implies (A.3), let $a > 0$ be such that

$$a = \widehat{MRS}_\alpha(x^o_\alpha, x^o_\beta) = \widehat{MRS}_\beta(x^o_\alpha, x^o_\beta)\,.$$

Then an easy calculation shows that

$$\mathcal{M}(x)\begin{bmatrix} 1 \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix}.$$

This implies that the dominant root of $\mathcal{M}_c(\pi)$ is equal to one,[36] which is the dominant root criterion. Therefore, we can say that, at a CPO allocation, indifference curves of agents α and β are tangent to each other (and the slope of them must be positive).

By a similar calculation, we can say that, when preferences are represented as in Eq.(A.1), the condition that $\lambda^f(\mathcal{M}(x)) \ni 1$ is equivalent to the pair of inequalities:

$$\frac{v'_y(c^{hy}_\alpha) - v'_o(c^{ho}_\alpha)\mu_{\alpha\alpha}}{v'_o(c^{ho}_\beta)\mu_{\alpha\beta}} \leq a \leq \frac{v'_o(c^{ho}_\alpha)\mu_{\beta\alpha}}{v'_y(c^{hy}_\beta) - v'_o(c^{ho}_\beta)\mu_{\beta\beta}}$$

and

$$\frac{v'_o(c^{ho}_\alpha)\nu_{\beta\alpha}}{v'_y(c^{hy}_\beta) - v'_o(c^{ho}_\beta)\nu_{\beta\beta}} \leq a \leq \frac{v'_y(c^{hy}_\alpha) - v'_o(c^{ho}_\alpha)\nu_{\alpha\alpha}}{v'_o(c^{ho}_\beta)\nu_{\alpha\beta}}.$$

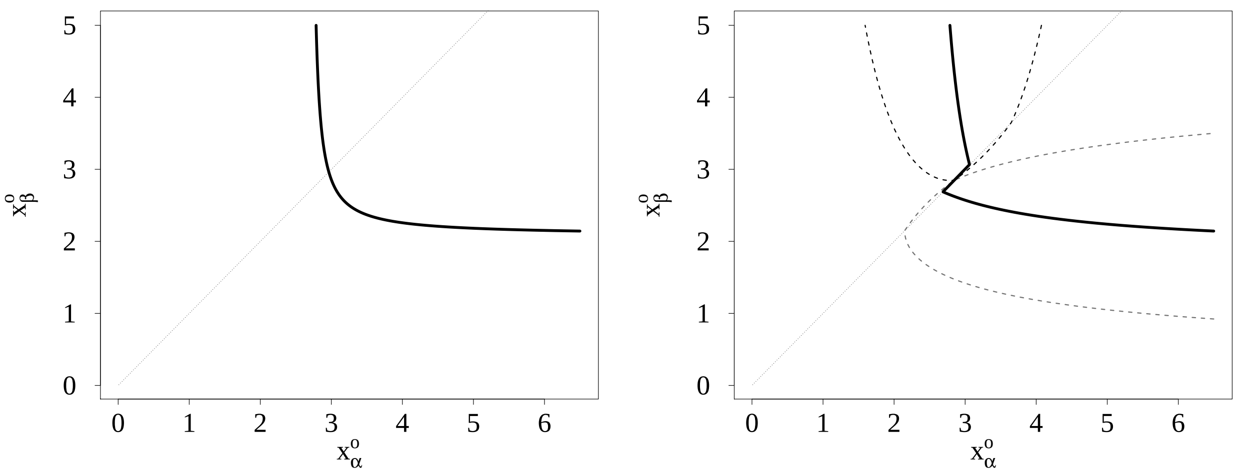
Although preferences belong to the class of MEU preferences, one can find that these inequalities still imply the tangency condition on indifference curves.

---

[34] See Corollary 1′ in Section 3.

[35] This equivalence holds because of the fact that for any positive matrix A, if there exist a nonnegative real number λ and a nonnegative nonzero vector x such that $Ax = \lambda x$, then λ is the dominant root of A. See Takayama (1974, p.372, Theorem 4.B.1(iv)).

[36] See the previous footnote.



(a) $(\mu_{s\alpha}, \nu_{s\alpha}) = (0.75, 0.75)$ for each $s \in \mathcal{S}$ (b) $(\mu_{s\alpha}, \nu_{s\alpha}) = (0.25, 0.75)$ for each $s \in \mathcal{S}$

Figure A.2: Contract Curve (The Set of CGRO Allocations)

Example A.2 Consider the same economy with that appeared in Example 3, except for the specification on Π_s. When $\Pi_s = \{(0.75, 0.25)\}$, preferences are smooth and the set of CGRO allocations can be depicted by the solid line as in Panel (a) of Figure A.2. On the other hand, when $\Pi_s = \{\pi_s \in \Delta_{\mathcal{S}} : \ 0.25 \leq \pi_{s\alpha} \leq 0.75\}$, preferences are smooth and the set of CGRO allocations can be depicted by the solid line as in Panel (b) of Figure A.2. In the panel, one might find that, at $x^o = (2.85, 2.85)$, indifference curves of agents born at state α and β are tangent to each other. ■

One can also find that the set of conditional Pareto optimality allocations is depicted by the area right-upper of the CGRO curve. See Ohtaki (2014) for more details.

Appendix B: Proofs

B.1 Proofs of Theorems 1 and 1′

Proof of Theorem 1. Let $c = \{c^{hy}, c^{ho}\}_{h \in \mathcal{H}}$ be an interior stationary feasible allocation. It is easy to verify that c is a CPO allocation if and only if there exists some Pareto weights, $\gamma_0 : \mathcal{H} \times \mathcal{S} \to \Re_+$ and $\gamma : \mathcal{H} \times \mathcal{S} \to \Re_{++}$, such that

$$c \in \arg\max_{b \in \mathcal{A}} \left[\sum_{(h,s') \in \mathcal{H} \times \mathcal{S}} \gamma_0^{hs'} b_{0s'}^{ho} + \sum_{(h,s) \in \mathcal{H} \times \mathcal{S}} \gamma^{hs} U^{hs}(b_s^h) \right]$$

where $\mathcal{A}$ is the set of stationary feasible allocations. Then, by applying Theorems C.4, C.6, and C.9 in the Appendix C, the interior stationary feasible allocation c is CPO if and only if there exists some Pareto weights, $\gamma_0 : \mathcal{H} \times \mathcal{S} \to \Re_+$ and $\gamma : \mathcal{H} \times \mathcal{S} \to \Re_{++}$, and some Lagrange multipliers, $\lambda_0 : \mathcal{S} \to \Re_+$ and $\lambda : \mathcal{S} \times \mathcal{S} \to \Re_+$, such that

$$\gamma_0^{hs} - \lambda_{0s} \leq 0 \quad \text{with equality if} \quad c_{0s}^{ho} > 0 \tag{B.5}$$

and, for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, there exists some $\bar{v}_s^h \in \partial U^{hs}(c_s^h)$ satisfying that

$$\gamma^{hs} \bar{v}_s^{hy} = \lambda_{0s} + \sum_{\tau \in \mathcal{S}} \lambda_{\tau s}, \tag{B.6}$$

$$\gamma^{hs} \bar{v}_{ss'}^{ho} = \lambda_{ss'} \tag{B.7}$$

for each $s' \in \mathcal{S}$. Note that it follows from strong monotonicity of U^{hs} and Theorem C.3 that $\lambda \gg 0$.

We should now claim the equivalence between the existence of γ_0, γ, λ_0, λ satisfying Eqs.(B.5), (B.6), and (B.7) and the condition that $(\lambda^f \circ \mathcal{M})(c)$ contains the number less than or equal to unity. Assume first the existence of γ_0, γ, λ_0, λ satisfying Eqs.(B.5), (B.6), and (B.7) in order to show that $(\lambda^f \circ \mathcal{M})(c)$ contains the number less than or equal to unity. By Eqs.(B.6) and (B.7), we can obtain that

$$\frac{\bar{v}^{ho}_{ss'}}{\bar{v}^{hy}_s} = \frac{\lambda_{ss'}}{\lambda_{0s} + \sum_{\tau \in \mathcal{S}} \lambda_{\tau s}} =: m_{ss'}$$

for each $h \in \mathcal{H}$ and each $s, s' \in \mathcal{S}$. By its definition, the positive $S \times S$ matrix $M := [m_{ss'}]_{s,s' \in \mathcal{S}}$ belongs to $\mathcal{M}^h(c^h_s)$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$ and therefore $M \in \mathcal{M}(c)$. It also follows that

$$\lambda_{ss'} = \left(\lambda_{0s} + \sum_{\tau \in \mathcal{S}} \lambda_{\tau s} \right) m_{ss'}$$

for each $s, s' \in \mathcal{S}$. Summing up this equation over $s \in \mathcal{S}$, we have

$$\alpha = (\lambda_0 + \alpha) M,$$

where $\alpha_s := \sum_{\tau \in \mathcal{S}} \lambda_{\tau s}$ for each $s \in \mathcal{S}$. Then, it is straightforward to show that $\lambda^f(M) \leq 1$.[37] With the fact that $M \in \mathcal{M}$, this implies that $(\lambda^f \circ \mathcal{M})(c)$ contains the number less than or equal to unity.

Assume now that there exists at least one $S \times S$ matrix $M = [m_{ss'}]_{s,s' \in \mathcal{S}}$ of $\mathcal{M}(c)$ such that $\lambda^f(M) \leq 1$. By the definition of $\mathcal{M}(c)$, for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, there exists some $\bar{v}^h_s \in \partial U^{hs}(c^h_s)$ such that

$$m_{ss'} = \frac{\bar{v}^{ho}_{ss'}}{\bar{v}^{hy}_s} > 0.$$

Then, it follows from the Perron-Frobenius theorem that $(I - M)^{-1} \gg 0$, where I is the $S \times S$ identity matrix. Let λ_0 be an arbitrary element of $\Re^{\mathcal{S}}_{++}$ if $\lambda^f(M) < 1$ and otherwise let $\lambda_0 = 0 \in \Re^{\mathcal{S}}_{+}$. Also define $\alpha \in \Re^{\mathcal{S}}_{++}$ by

$$\alpha := \lambda_o M (I - M)^{-1} \gg 0$$

if $\lambda^f(M) < 1$ and otherwise by the positive eigenvector α of M, which satisfies that $M\alpha = \lambda^f(M)\alpha = \alpha$.[38] Now, for each $h \in \mathcal{H}$ and each $s, s' \in \mathcal{S}$, let

$$\begin{aligned} \gamma^{hs} &:= \frac{\lambda_{0s} + \alpha_s}{\bar{v}^{hy}_s}, \\ \lambda_{ss'} &:= \gamma^{hs} \bar{v}^{ho}_{ss'} = (\lambda_{0s} + \alpha_s) m_{ss'}, \\ \gamma^{hs}_0 &:= \lambda_{0s}. \end{aligned}$$

It is now easy to verify that γ, λ, γ_0, and λ_0 satisfy Eqs.(B.5), (B.6), and (B.7). This completes the proof of this theorem. Q.E.D.

[37] See, for example, Aiyagari and Peled (1991, p.76).

[38] Existence of such a positive eigenvector follows from the Perron-Frobenius theorem.

Proof of Theorem 1′. The proof strategy is almost same with the previous theorem. Let $c = \{c^{hy}, c^{ho}\}_{h\in\mathcal{H}}$ be an interior steady state allocation. It is easy to verify that c is a CGRO allocation if and only if there exists some Pareto weight $\gamma : \mathcal{H} \times \mathcal{S} \to \Re_{++}$ such that

$$c \in \arg\max_{b\in\mathcal{A}'} \sum_{(h,s)\in\mathcal{H}\times\mathcal{S}} \gamma^{hs} U^{hs}(b^h_s)$$

where $\mathcal{A}'$ is the set of steady state allocations. Then, by applying Theorems C.4, C.6, and C.9 in the Appendix C, the interior stationary feasible allocation c is CGRO if and only if there exists some Pareto weight, $\gamma : \mathcal{H} \times \mathcal{S} \to \Re_{++}$, and some Lagrange multipliers, $\lambda : \mathcal{S} \times \mathcal{S} \to \Re_{+}$, such that, for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, there exists some $\bar{v}^h_s \in \partial U^{hs}(c^h_s)$ satisfying that

$$\gamma^{hs}\bar{v}^{hy}_s = \sum_{\tau\in\mathcal{S}} \lambda_{\tau s}, \tag{B.8}$$

$$\gamma^{hs}\bar{v}^{ho}_{ss'} = \lambda_{ss'} \tag{B.9}$$

for each $s' \in \mathcal{S}$. Note that it follows from strong monotonicity of U^{hs} and Theorem C.3 that $\lambda \gg 0$.

We should now claim the equivalence between the existence of γ and λ satisfying Eqs.(B.8) and (B.9) and the condition that $(\lambda^f \circ \mathcal{M})(c)$ contains unity. Assume first the existence of γ_0, γ, λ_0, λ satisfying Eqs.(B.8) and (B.9) in order to show that $(\lambda^f \circ \mathcal{M})(c)$ contains unity. By Eqs.(B.8) and (B.9), we can obtain that

$$\frac{\bar{v}^{ho}_{ss'}}{\bar{v}^{hy}_s} = \frac{\lambda_{ss'}}{\sum_{\tau\in\mathcal{S}} \lambda_{\tau s}} =: m_{ss'}$$

for each $h \in \mathcal{H}$ and each $s, s' \in \mathcal{S}$. By its definition, the positive $S \times S$ matrix $M := [m_{ss'}]_{s,s'\in\mathcal{S}}$ belongs to $\mathcal{M}^h(c^h_s)$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$ and therefore $M \in \mathcal{M}(c)$. It also follows that

$$\lambda_{ss'} = \sum_{\tau\in\mathcal{S}} \lambda_{\tau s} m_{ss'}$$

for each $s, s' \in \mathcal{S}$. Summing up this equation over $s \in \mathcal{S}$, we have

$$\alpha = \alpha M,$$

where $\alpha_s := \sum_{\tau\in\mathcal{S}} \lambda_{\tau s}$ for each $s \in \mathcal{S}$. Then, it follows from the Perron-Frobenius theorem that $\lambda^f(M) = 1$. With the fact that $M \in \mathcal{M}$, this implies that $(\lambda^f \circ \mathcal{M})(c)$ contains unity.

Assume now that there exists at least one $S \times S$ matrix $M = [m_{ss'}]_{s,s'\in\mathcal{S}}$ of $\mathcal{M}(c)$ such that $\lambda^f(M) = 1$. By the definition of $\mathcal{M}(c)$, for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, there exists some $\bar{v}^h_s \in \partial U^{hs}(c^h_s)$ such that

$$m_{ss'} = \frac{\bar{v}^{ho}_{ss'}}{\bar{v}^{hy}_s}.$$

By the Perron-Frobenius theorem, there also exists some eigenvector $\alpha \gg 0$, which satisfies that $\alpha \cdot (I - M) = 0$, where I is the $S \times S$ identity matrix. For each $h \in \mathcal{H}$ and each $s, s' \in \mathcal{S}$, define $\gamma : \mathcal{H} \times \mathcal{S} \to \Re_{++}$ and $\lambda : \mathcal{S} \times \mathcal{S} \to \Re$ by

$$\gamma^{hs} := \frac{\alpha_s}{\bar{v}^{hy}_s} \quad \text{and} \quad \lambda_{ss'} := \gamma^{hs}\bar{v}^{ho}_{ss'}.$$

By their definitions, we can obtain that, for each $s, s' \in \mathcal{S}$,

$$\lambda_{ss'} = \alpha_s \frac{\bar{v}^{ho}_{ss'}}{\bar{v}^{hy}_s} = \alpha_s m_{ss'},$$

so that $\alpha_{s'} = \sum_{\tau\in\mathcal{S}} \lambda_{\tau s'}$. It is now easy to verify that the pair of γ and λ satisfies Eqs.(B.8) and (B.9). This completes the proof of this theorem. Q.E.D.

B.2 Proofs of Proposition 1 and Theorems 2 and 2′

Proof of Proposition 1. Let c be an interior stationary feasible allocation. We first show $\mathcal{P}(c) \subset \mathcal{M}(c)$. Let $P = [p_{ss'}]_{s,s' \in \mathcal{S}} \in \mathcal{P}(c)$ be a supporting price matrix. By their definitions, (P, c) satisfies that $U^{hs}(c_s^h) \geq U^{hs}(b_s^h)$ for each $h \in \mathcal{H}$, each $s \in \mathcal{S}$, and each stationary feasible allocation b satisfying that $b_s^{hy} + \sum_{s' \in S} b_{ss'}^{ho} p_{ss'} \leq c_s^{hy} + \sum_{s' \in S} c_{ss'}^{ho} p_{ss'}$. So, c_s^h belongs to the set

$$\arg\max_{b_s^h} \left\{ U^{hs}(b_s^h) : b_s^{hy} + \sum_{s' \in S} b_{ss'}^{ho} p_{ss'} \leq c_s^{hy} + \sum_{s' \in S} c_{ss'}^{ho} p_{ss'} \right\}.$$

Then, it follows from Theorem C.9 that, for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, c_s^h must be characterized by the existence of $\lambda_s^h \in \Re_+$ such that

$$0 \in \left\{ \left(v_s^{hy} - \lambda_s^h, (v_{ss'}^{ho} - \lambda_s^h p_{ss'})_{s' \in \mathcal{S}} \right) : v_s^h \in \partial U^{hs}(c_s^h) \right\}, \tag{B.10}$$

which implies the existence of $\bar{v}_s^h \in \partial U^{hs}(c_s^h)$ such that

$$\frac{\bar{v}_{ss'}^{ho}}{v_s^{hy}} = p_{ss'}$$

for each $s' \in \mathcal{S}$. Therefore, it follows that $P \in \mathcal{M}(c)$, which implies that $\mathcal{P}(c) \subset \mathcal{M}(c)$.

Conversely, let $M = [m_{ss'}]_{s,s' \in \mathcal{S}} \in \mathcal{M}(c)$. For each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, it follows from its definition that there exists some $\bar{v}_s^h \in \partial U^{hs}(c)$ such that, for each $s, s' \in \mathcal{S}$,

$$m_{ss'} = \frac{\bar{v}_{ss'}^{ho}}{\bar{v}_s^{hy}},$$

or equivalently,

$$\bar{v}_s^{hy} - \lambda_s^h = 0 \quad \text{and} \quad \bar{v}_{ss'}^{ho} - \lambda_s^h m_{ss'} = 0,$$

where $\lambda_s^h := \bar{v}_s^{hy}$. Then, it follows from Eq.(B.10) that M must be a supporting price matrix of c, which implies $M \in \mathcal{P}(c)$. This completes the proof. Q.E.D.

Proof of Theorem 2. It follows from Theorem 1 and Proposition 1. Q.E.D.

Proof of Theorem 2′. It follows from Theorem 1′ and Proposition 1′. Q.E.D.

B.3 Proofs of Proposition 2 and 2′

Lemma 1 *For each interior stationary feasible allocation c, $\mathcal{P}^*(c) \subset \mathcal{P}(c)$.*

Proof of Lemma 1. Let c be an interior stationary feasible allocation (such that $\mathcal{P}^*(c) \neq \emptyset$). Let $P = [p_{ss'}]_{s,s' \in \mathcal{S}} \in \mathcal{P}^*(c)$ and b be an arbitrary stationary feasible allocation satisfying that $U^{hs}(b_s^h) > U^{hs}(c_s^h)$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$. Then, it follows that $b_s^{hy} + \sum_{s' \in \mathcal{S}} b_{ss'}^{ho} p_{ss'} > \omega_s^{hy} + \sum_{s' \in \mathcal{S}} \omega_{ss'}^{ho} p_{ss'} \geq c_s^{hy} + \sum_{s' \in \mathcal{S}} c_{ss'}^{ho} p_{ss'}$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$ because (P, c) is a stationary equilibrium. Therefore, $P \in \mathcal{P}(c)$, which implies that $\mathcal{P}^*(c) \subset \mathcal{P}(c)$. Q.E.D.

Proof of Proposition 2. It follows from Theorem 2 and Lemma 1. Q.E.D.

Proof of Proposition 2′. It follows from Theorem 2′ and Lemma 1. Q.E.D.

B.4 Proof of Proposition 3

We omit the proof of Proposition 3 because it is almost same with that of Aiyagari and Peled (1991, Theorem 2).

B.5 Proof of Proposition 4

We prepare three lemmas to obtain Proposition 3.

Lemma 2 *If ω is conditionally Pareto optimal, then $\mathcal{P}^*(\omega) = \mathcal{P}(\omega)$.*

Proof of Lemma 2. Suppose that ω is conditionally Pareto optimal in order to observe that $\mathcal{P}^*(\omega) = \mathcal{P}(\omega)$. Because $\mathcal{P}^*(\omega) \subset \mathcal{P}(\omega)$, we should show that $\mathcal{P}(\omega) \subset \mathcal{P}^*(\omega)$. Let $P = [p_{ss'}]_{s,s' \in \mathcal{S}}$ be an arbitrary element of $\mathcal{P}(\omega)$. Then, for each $h \in \mathcal{H}$, each $s \in \mathcal{S}$, and each element $c_s^h = (c_s^{hy}, c_s^{ho}) \in \Re_+ \times \Re_+^{\mathcal{S}}$ satisfying that

$$c_s^{hy} + \sum_{s' \in \mathcal{S}} c_{ss'}^{ho} p_{ss'} \leq \omega_s^{hy} + \sum_{s' \in \mathcal{S}} \omega_{s'}^{ho} p_{ss'},$$

we can obtain that $U^{hs}(\omega_s^h) \geq U^{hs}(c_s^h)$. Furthermore, it holds that $\sum_{h \in \mathcal{H}}(\omega_s^{hy} + \omega_{\tau s}^{ho}) = \bar{\omega}_{\tau s}$ for each $\tau, s \in \mathcal{S}$. Therefore, (P, ω) is a stationary equilibrium. This argument implies that $\mathcal{P}(\omega) \subset \mathcal{P}^*(\omega)$ and completes the proof of this lemma. Q.E.D.

Note that the proof of Lemma 2 does not require strict concavity of lifetime utility functions.

Lemma 3 *If ω is conditionally Pareto optimal, there is no stationary equilibrium (P, c) with $c \neq \omega$.*

Proof of Lemma 3. Suppose that ω is conditionally Pareto optimal but there is a stationary equilibrium $(P, c) = ([p_{ss'}]_{s,s' \in \mathcal{S}}, c)$ such that $c \neq \omega$, which implies that $c_\tau^k \neq \omega_\tau^k$ for some $k \in \mathcal{H}$ and some $\tau \in \mathcal{S}$. Because (P, c) is a stationary equilibrium, it follows that $\sum_{h \in \mathcal{H}}(c_{s'}^{hy} + c_{ss'}^{ho}) = \bar{\omega}_{ss'}$ for each $s, s' \in \mathcal{S}$ and $U^{hs}(c_s^h) \geq U^{hs}(\omega_s^h)$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$. Now define $d = \alpha c + (1-\alpha)\omega$, where $0 < \alpha < 1$. Then, we can observe that

- $\sum_{h \in \mathcal{H}}(d_{s'}^{hy} + d_{ss'}^{ho}) = \sum_{h \in \mathcal{H}}[\alpha(c_{s'}^{hy} + c_{ss'}^{ho}) + (1-\alpha)(\omega_{s'}^{hy} + \omega_{ss'}^{ho})] = \bar{\omega}_{ss'}$ for each $s, s' \in \mathcal{S}$
- $U^{hs}(d_s^h) \geq \alpha U^{hs}(c_s^h) + (1-\alpha)U^{hs}(\omega^{hs}) \geq U^{hs}(\omega_s^h)$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$, and
- $U^{k\tau}(d_\tau^k) > \alpha U^{k\tau}(c_\tau^k) + (1-\alpha)U^{k\tau}(\omega_\tau^k) \geq U^{k\tau}(\omega_\tau^k)$ for the pair (k, τ) defined above.

These imply that d CPO-dominates ω, which contradicts the fact that ω is conditionally Pareto optimal. Therefore, there is no stationary equilibrium (P, c) such that $c \neq \omega$ if ω is conditionally Pareto optimal. Q.E.D.

Lemma 4 *Then, ω is conditionally Pareto optimal if the set of stationary equilibrium is given by $\{(P, \omega) : P \in \mathcal{P}(\omega)\}$.*

Proof of Lemma 4. Suppose that the set of stationary equilibrium is given by $\{(P, \omega) : P \in \mathcal{P}(\omega)\}$. Because it follows from Proposition 3 that there must at least one optimal equilibrium, the unique equilibrium allocation ω must be conditionally Pareto optimal. Q.E.D.

Proof of Proposition 4. This follows from Lemmas 2, 3, and 4. Q.E.D.

B.6 Proofs of Theorem 3 and 4

Proof of Theorem 3. By the sequential budget constraints of an agent, we can obtain the agent's lifetime budget constraint such that: for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$,

$$c_s^{hy} + \sum_{s' \in \mathcal{S}} c_{ss'}^{ho} p_{ss'} = (\omega_s^{hy} - \tau_s^{hy}) + \sum_{s' \in \mathcal{S}} (\omega_{ss'}^{ho} + \tau_{ss'}^{ho}) p_{ss'} + \left(\sum_{s' \in \mathcal{S}} q_{s'} p_{ss'} - q_s \right) m^h.$$

By this equation, we can obtain the no arbitrage condition when the money price is positive, *i.e.*: $q = P \cdot q$ for any stationary equilibrium with circulating money, (q, P, c), with $c_s^h \gg 0$ for each $s \in \mathcal{S}$. In order to verify this, we should show that

$$(\forall s \in \mathcal{S}) \quad q_s = \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}.$$

Suppose the contrary that $q_s \neq \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}$ for some $s \in \mathcal{S}$. If $q_s < \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}$, then agent h born at state s will choose ∞ as m^m and his/her optimization problem has no solution. On the other hand, if $q_s > \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}$, then agent born h at state s will choose $-\infty$ as m^h and his/her optimization problem has no solution.[39] In any cases, we obtain a contradiction, so that $q_s = \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}$ for all $s \in \mathcal{S}$.

Suppose now that there exists at least one stationary equilibrium with money and social securities, (q, P, c), satisfying that $c_s^h \gg 0$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$. We have obtained that $Pq = q$, at which the lifetime budget constraint coincides with that in the complete market. Because q_s is now positive for all $s \in \mathcal{S}$, it follows from the Perron-Frobenius theorem that the $S \times S$ matrix P with positive coefficients has the dominant root equal to unity. Now it follows from Corollary 4′ that the equilibrium allocation c is CGRO. This completes the proof of Theorem 3.[40] Q.E.D.

Before proving Theorem 4, we should investigate first order conditions of each agent's optimization problem. By substituting sequential budget constraints with consumptions in the lifetime utility and applying Theorem C.8 with respect to choices of money holdings, we can verify that a solution of each agent's optimization problem, denoted here by $(c_s^h, m_s^h, \theta_s^h)$, is characterized by

$$\begin{aligned}
c_s^{hy} &= \omega_s^{hy} - \tau_s^{hy} - q_s m_s^h - \sum_{s' \in \mathcal{S}} \theta_{ss'}^h p_{ss'}, \\
c_{ss'}^{ho} &= \omega_{ss'}^{ho} + \tau_{ss'}^{ho} + q_{s'} m_{s'} + \theta_{ss'}^h, \\
0 &\in \left\{ -q_s v_s^{hy} + \sum_{s' \in \mathcal{S}} q_{s'} v_{ss'}^{ho} : \ v_s^h \in \partial U^{hs}(c_s^h) \right\}.
\end{aligned}$$

We will use these characterizations in the following proof.

[39] When one wished to impose the lower bound for possible m^h, $m^h \geq 0$ for example, the agent h born at state s chooses 0 as the amount of money holding. However, this contradicts the fact that $\sum_h m^h$ should be equal to 1 at a stationary equilibrium with circulating money.

[40] To prove Theorem 3, we have adopted an indirect way of applying the dominant root criterion. Applying the technique provided by Sakai (1988), one can provide a more direct proof of Theorem 3.

Proof of Theorem 4. Let $c = \{c^{hy}, c^{ho}\}$ be an interior CGRO allocation. Because it is CGRO, there exists some positive matrix $P = [p_{ss'}]_{s,s' \in \mathcal{S}}$ such that $P \in \mathcal{M}^h(c^h)$ for all $h \in \mathcal{H}$, which implies the existence of $\bar{v}^h_s \in \partial U^{hs}(c^h_s)$ satisfying that $p_{ss'} = \bar{v}^{ho}_{ss'}/\bar{v}^{hy}_s$ for each $s, s' \in \mathcal{S}$, and its dominant root is equal to one, *i.e.*: $\lambda^f(P) = 1$. By the Perron-Frobenius theorem, there exists a unique $q \in \Re^{\mathcal{S}}_{++}$ (up to normalization) such that $P \cdot q = \lambda^f(P) q = q$, which implies that $-q_s \bar{v}^{hy}_s + \sum_{s' \in \mathcal{S}} q_{s'} \bar{v}^{ho}_{ss'} = 0$ for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$. Choose the Euclidean norm of q to be small enough and take any $m = \{m^h_s\}_{(h,s) \in \mathcal{H} \times \mathcal{S}} \in \Re^{\mathcal{H} \times \mathcal{S}}_{++}$, any $\tau = \{\tau^{hy}, \tau^{ho}\}_{h \in \mathcal{H}}$, and any $\theta = [\theta^h_{ss'}]_{s,s' \in \mathcal{S}, h \in \mathcal{H}}$ to satisfy (a) the budget constraint in the first period of each agent $(h, s) \in \mathcal{H} \times \mathcal{S}$:

$$c^{hy}_s = \omega^{hy}_s - \tau^{hy}_s - q_s m^h_s - \sum_{s' \in \mathcal{S}} \theta^h_{ss'} p_{ss'}; \tag{B.11}$$

(b) $\omega^{hy}_s > \tau^{hy}_s$ and $\tau^{ho}_{ss'} > -\omega^{ho}_{ss'}$ for each $h \in \mathcal{H}$ and each $s, s' \in \mathcal{S}$; (c) $\sum_{h \in \mathcal{H}} m^h_s = 1$ and $\sum_{h \in \mathcal{H}} \theta^h_s = 0$ for each $s \in \mathcal{S}$; and (d) $\sum_{h \in \mathcal{H}} \tau^{hy}_s = \sum_{h \in \mathcal{H}} \tau^{ho}_{s's}$ for each $s', s \in \mathcal{S}$.[41] By their constructions, the first order conditions of all agents' optimization problems at the stationary equilibrium with money and social securities are satisfied. Because other market clearing conditions immediately also hold, (q, P, c) is a stationary equilibrium with money and social securities. Q.E.D.

B.7 Proof of Proposition 5

Proof of Proposition 5. By the sequential budget constraints of an agent, we can obtain the agent's lifetime budget constraint such that: for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$,

$$c^{hy}_s + \sum_{s' \in \mathcal{S}} c^{ho}_{ss'} p_{ss'} = \omega^{hy}_s + \sum_{s' \in \mathcal{S}} \omega^{ho}_{ss'} p_{ss'} + \left(\sum_{s' \in \mathcal{S}} \rho^*_{s'} p_{ss'} - \mu \rho^*_s \right) \frac{m^h_{t,s}}{M_{t+1}} + \frac{1}{H} \frac{\mu - 1}{\mu} \sum_{s' \in \mathcal{S}} \rho^*_{s'} p_{ss'}.$$

By this equation, we can obtain the no arbitrage condition such that $\mu \rho^* = P \rho^*$ for any stationary equilibrium with lump-sum money transfers, (ρ, P, c), with $c_s \gg 0$ for each $s \in \mathcal{S}$. In order to verify this, we should show that

$$(\forall s \in \mathcal{S}) \quad \mu \rho^*_s = \sum_{s' \in \mathcal{S}} p_{ss'} \rho^*_{s'}.$$

Suppose the contrary that $\mu \rho^*_s \neq \sum_{s' \in \mathcal{S}} p_{ss'} \rho^*_{s'}$ for some $s \in \mathcal{S}$. If $\mu \rho^*_s < \sum_{s' \in \mathcal{S}} p_{ss'} \rho^*_{s'}$, then the agent h born at state s will choose ∞ as $m^h_{t,s}$ and his/her optimization problem has no solution. On the other hand, if $\mu \rho^*_s > \sum_{s' \in \mathcal{S}} p_{ss'} \rho^*_{s'} >$, then agent h born at state s will choose $-\infty$ as $m^h_{t,s}$ and his/her optimization problem has also no solution. In any cases, we obtain a contradiction, so that $\mu \rho^*_s = \sum_{s' \in \mathcal{S}} p_{ss'} \rho^*_{s'}$ for all $s \in \mathcal{S}$.

Suppose now that there exists at least one stationary equilibrium with lump-sum money transfers, (ρ, P, c), satisfying that $c_s \gg 0$ for all $s \in \mathcal{S}$. Because $\mu > 0$, $P \gg 0$, and $P \rho^* = \mu \rho^*$, it follows from the Perron-Frobenius theorem that $\lambda^f(P) = \mu$. Then, the statement in Proposition 5 immediately follows. Q.E.D.

B.8 Proof of Proposition 6

[41] Actually, let $m^h_s := 1/H$, $\tau^{hy}_s := \omega^{hy}_s - c^{hy}_s - q_s/H$, and $\theta^h_{ss'} := 0$ for each $h \in \mathcal{H}$ and each $s, s' \in \mathcal{S}$. By choosing the Euclidean norm of q to be sufficiently small, m, τ^{hy}, and θ can satisfy Eq.(B.11) and conditions (a)–(d) with some $\{\tau^{ho}\}_{h \in \mathcal{H}}$ such that $\tau^{ho}_{ss'} > -\omega^{ho}_{ss'}$ and $\sum_{h \in \mathcal{H}} \tau^{hy}_s = \sum_{h \in \mathcal{H}} \tau^{ho}_{s's}$ for $s, s' \in \mathcal{S}$.

Proof of Proposition 6. By the sequential budget constraints of an agent, we can obtain the agent's lifetime budget constraint such that: for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$,

$$c_s^{hy} + \sum_{s' \in \mathcal{S}} c_{ss'}^{ho} p_{ss'} = \omega_s^{hy} + \sum_{s' \in \mathcal{S}} \omega_{ss'}^{ho} p_{ss'} + \left(\sum_{s' \in \mathcal{S}} (q_{s'} + d_{s'}) p_{ss'} - q_s \right) z_s^h.$$

By this equation, we can obtain the no arbitrage condition such that $q = P \cdot (q + d)$ for any stationary equilibrium with equity, (q, P, c), with $c_s \gg 0$ for each $s \in \mathcal{S}$, where $q + d = (q_s + d_s)_{s \in \mathcal{S}}$. In order to verify this, we should show that

$$(\forall s \in \mathcal{S}) \quad q_s = \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'}).$$

Suppose the contrary that $q_s \neq \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'})$ for some $s \in \mathcal{S}$. If $q_s < \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'})$, then the agent h born at state s will choose ∞ as z_s^h and his/her optimization problem has no solution. On the other hand, if $q_s > \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'})$, then agent h born at state s will choose $-\infty$ as z_s^h and his/her optimization problem has also no solution. In any cases, we obtain a contradiction, so that $q_s = \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'})$ for all $s \in \mathcal{S}$.

Suppose now that there exists at least one stationary equilibrium with equity, (q, P, c), satisfying that $c_s \gg 0$ for all $s \in \mathcal{S}$. We have obtained that $P(q + d) = q$, at which the lifetime budget constraint coincides with that in the complete market. By the fact that $d \in \Re_+^S \setminus \{0\}$, it holds that $Pq < P(q + d) = q$. Therefore, it follows from the Perron-Frobenius theorem that the $\mathcal{S} \times \mathcal{S}$ matrix P with positive coefficients has the dominant root less than unity. Now it follows from Corollary 4 that the equilibrium allocation c is CPO. This completes the proof of Proposition 6. Q.E.D.

Appendix C: Superdifferential and its Calculus

This appendix aims to introduce the definition and calculus rules of superdifferential. We first define the concept of superdifferential following Rockafellar (1970, pp.214–215) and Hiriart-Urruty and Lemaréchal (2004, Definition D.1.2.1, p.167).[42]

Definition C.1 For each real-valued function f on $\mathrm{dom} f \subset \Re^n$ and each $x \in \mathrm{dom} f$, the set

$$\partial f(x) := \{ s \in \Re^n : (\forall y \in \Re^n) \quad f(y) \leq f(x) + \langle s, y - x \rangle \}$$

and each of its elements are called the *superdifferantial* and a *supergradient of f at x*, respectively.

Obviously, $\partial f(x)$ is closed and convex for each concave real-valued function f on a nonempty and convex set $\mathrm{dom} f \subset \Re^n$ and each $x \in \mathrm{dom} f$. Furthermore, we can obtain a necessary and sufficient condition for nonemptiness and boundedness of the superdifferential.

Theorem C.1 *For each concave real-valued function f on a nonempty convex set $\mathrm{dom} f \subset \Re^n$ and each $x \in \mathrm{dom} f$, $\partial f(x)$ is nonempty and bounded if and only if $x \in \mathrm{int.dom} f$, where $\mathrm{int.}X$ is the interior of the set X.*

Proof of Theorem C.1. See Rockafellar (1970, Theorem 23.4, p.217). Q.E.D.

Furthermore, if f is differentiable, its unique supergradient coincides the (standard) gradient.

[42] To be more precise, they define the *subdifferential*, which is defined by

$$\partial f(x) := \{ s \in \Re^n : (\forall y \in \Re^n) \quad f(y) \geq f(x) + \langle s, y - x \rangle \}$$

for each real-valued function f on $\Re^n$ and each $x \in \Re^n$

Theorem C.2 *Let f be a concave real-valued function on $\Re^n$ and $x \in \Re^n$. Then, if f is differentiable at x, $\partial f(x) = \{\nabla f(x)\}$. Conversely, if $\partial f(x)$ is a singleton, a unique element of which is denoted by s, then f is differentiable at x and $s = \nabla f(x)$.*

Proof of Theorem C.2. See Rockafellar (1970, Theorem 25.1, p.242) and Hiriart-Urruty and Lemaréchal (2004, Corollary D.2.1.4, p.175). Q.E.D.

As shown in the following theorem, monotonicity determines the signs of supergradients.

Theorem C.3 *For each strongly monotone and concave real-valued function f on $\Re^n$, and each $x \in \Re^n$, if $s \in \partial f(x)$, then $s \gg 0$.*

Proof of Theorem C.3. To observe this fact, for each $i = 1, \ldots, n$, let $e^i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \Re^n$, where the 1 is in i-th position. Then, for each $s \in \partial f(x)$ and each $i = 1, \ldots, n$,

$$f(x) < f(x + e^i) \leq f(x) + \langle s, e^i \rangle = f(x) + s_i,$$

where the first strict inequality comes from the strong monotonicity of f and the second inequality comes from the definition of the superdifferential. Therefore, for each $s \in \partial f(x)$, $s = (s_1, \cdots, s_i, \ldots, s_n) \gg 0$. Q.E.D.

The superdifferential is linear in the sense of the following theorem.

Theorem C.4 *For any concave real-valued functions f_1 and f_2 on $\Re^n$, any positive numbers a_1 and a_2, and each $x \in \Re^n$, $\partial(a_1 f_1 + a_2 f_2)(x) = a_1 \partial f_1(x) + a_2 \partial f_2(x)$.*

Proof of Theorem C.4. See Hiriart-Urruty and Lemaréchal (2004, Theorem C.4.4.1.1, p.183).

We should note that this observation does not necessarily hold for more general concave functions (Rockafellar, 1970, Theorem 23.8, p.223).

The following theorem provides the result on the superdifferential of composite functions.

Theorem C.5 *Let $f : \Re^m \to \Re$ be concave and increasing componentwise and $g : \Re^n \to \Re^m$ be concave. Then, for each $x \in \Re^n$,*

$$\partial(f \circ g)(x) = \left\{ \sum_{i=1}^{m} \rho_i s_i : \ (\rho_1, \ldots, \rho_m) \in \partial f(g(x)) \quad \text{and} \quad s_i \in \partial g_i(x) \quad \forall i = 1, \ldots, m \right\},$$

where $g_i(x)$ is the i-th component of $g(x)$, i.e.: $g(x) = (g_1(x), \ldots, g_m(x))$.

Proof of Theorem C.5. See Hiriart-Urruty and Lemaréchal (2004, Theorem C.4.4.3.1, p.186).

We can also obtain the result on the superdifferential of partially constant functions.

Theorem C.6 *Let f be a real-valued function on $\Re^m$. Also define the real-valued function g on $\Re^m \times \Re^n$ by $g(x, y) = f(x)$ for each $(x, y) \in \Re^m \times \Re^n$. Then, for each $(x, y) \in \Re^m \times \Re^n$,*

$$\partial g(x, y) = \{(s, 0) \in \Re^m \times \Re^n : s \in \partial f(x)\}.$$

Proof of Theorem C.6. Let $A := \{(s, 0) \in \Re^m \times \Re^n : s \in \partial f(x)\}$ and $(x, y) \in \Re^m \times \Re^n$. We first show that each $(s, 0) \in A$ belongs to $\partial g(x, y)$. For this aim, let $(x', y') \in \Re^m \times \Re^n$. Then, it follows that

$$g(x', y') - g(x, y) - \langle (s, 0), (x' - x, y' - y) \rangle = f(x') - f(x) - \langle s, x' - x \rangle \geq 0,$$

where the last inequality follows from the fact that $s \in \partial f(x)$. This implies that $(s, 0) \in \partial g(x, y)$.

We then show that each $(s, t) \in \partial g(x, y)$ belongs to A by two steps. Let $(s, t) \in \partial g(x, y)$. As the first step, we show that $s \in \partial f(x)$. By the definition of $\partial g(x, y)$, it follows that, for each $x' \in \Re^m$,

$$0 \leq g(x', y) - g(x, y) - \langle (s, t), (x' - x, y - y) \rangle = f(x') - f(x) - \langle s, x' - x \rangle,$$

which implies that $s \in \partial f(x)$. As the second step, we show that $t = 0$. For this aim let $e^i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \Re^n$, where the 1 is in i-th position for each $i = 1, \ldots, n$. Then, it follows from the definition of $\partial g(x, y)$ that

$$\begin{aligned} 0 &\leq g(x, y + e^i) - g(x, y) - \langle (s, t), (x - x, y + e^i - y) \rangle = f(x) - f(x) - t_i = -t_i, \\ 0 &\leq g(x, y - e^i) - g(x, y) - \langle (s, t), (x - x, y - e^i - y) \rangle = f(x) - f(x) + t_i = t_i, \end{aligned}$$

which implies that $t_i = 0$. Therefore, $t = (t_1, \ldots, t_n) = 0$. The above two steps impliy that $(s, t) \in A$. This completes the proof of Theorem C.6. Q.E.D.

The next theorem provides a result on the supperdifferential of the infimum of concave functions.

Theorem C.7 *Let J be a compact set in some metric space and $\{f_j\}_{j \in J}$ be a family of differentiable concave real-valued functions on $\Re^n$, where $j \mapsto f_j(x)$ is upper semi-continuous for each x. Define the real-valued function f on $\Re^n$ by*

$$f(x) := \inf_{j \in J} f_j(x)$$

and let $J(x) := \{j \in J : f_j(x) = f(x)\}$ for each $x \in \Re^n$. Then, it follows that

$$\partial f(x) = \operatorname{co} \{\nabla f_j(x) : j \in J(x)\}.$$

Proof of Theorem C.7. See Hiriart-Urruty and Lemaréchal (2004, Corollary D.4.4.4, p.191). Q.E.D.

We also provide useful results for solving optimization problems.

Theorem C.8 *For each concave real-valued function f on $\Re^n$ and each $x \in \Re^n$, $f(x) \geq f(y)$ for each $y \in \Re^n$ if and only if $0 \in \partial f(x)$.*

Proof of Theorem C.8. See Hiriart-Urruty and Lemaréchal (2004, Theorem C.4.2.2.1, p.177). Q.E.D.

Finally, we provide the Karash-Kuhn-Tucker theorem for concave but nondifferentiable cases. Consider the constrained optimization problem (P) such that

$$\begin{aligned} \max_{x \in C} \quad & f_0(x) \\ \text{subject to} \quad & f_1(x) \geq 0, \ldots, f_r(x) \geq 0, \\ & f_{r+1}(x) = 0, \ldots, f_m(x) = 0, \end{aligned}$$

where C is a nonempty convex set in $\Re^n$, f_i is concave on C for $i = 0, 1, \ldots, r$, and f_i is affine on C for $i = r + 1, \ldots, m$.

Theorem C.9 *An element $x^* \in C$ is an optimal solution to (P) if and only if there exists $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \in \Re^m$ satisfying that*

(a) $\lambda_i^* \geq 0$, $f_i(x^*) \geq 0$, *and* $\lambda_i^* f_i(x^*) = 0$ *for* $i = 1, \ldots, r$,

(b) $f_i(x^*) = 0$ *for* $i = r + 1, \ldots, m$, *and*

(c) $0 \in \partial f_0(x^*) + \lambda_1^* \partial f_1(x^*) + \cdots + \lambda_m^* \partial f_m(x^*)$.

Proof of Theorem C.9. See Rockafellar (1970, Theorem 28.3, p.281).

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References

Aiyagari, S.R. and D. Peled (1991) "Dominant root characterization of Pareto optimality and the existence of optimal equilibria in stochastic overlapping generations models," *Journal of Economic Theory* **54**, 69–83.

Balasko, Y. and K. Shell (1980) "The overlapping-generations Model, I: The case of pure exchange without money," *Journal of Economic Theory* **23**, 281–306.

Barbie, M., M. Hagedorn, and A. Kaul (2007) "On the interaction between risk sharing and capital accumulation in a stochastic OLG model with production," *Journal of Economic Theory* **137**, 568–579.

Bewley, T. (2002) "Knightian decision theory: Part I," *Decisions in Economics and Finance* **25**, 79–110.

Beissner, P. (2017) "Equilibrium prices and trade under ambiguous volatility," *Economic Theory* **64**, 213–238.

Billot, A., A. Chateauneuf, I. Gilboa, and J.-M. Tallon (2000) "Sharing beliefs: Between Agreeing and disagreeing," *Econometrica* **68**, 685–694.

Bloise, G. and F.L. Calciano (2008) "A characterization of inefficiency in stochastic overlapping generations economies," *Journal of Economic Theory* **143**, 442–468.

Carlier, G. and R.-A. Dana (2013) "Pareto optima and equilibria when preferences are incompletely known," *Journal of Economic Theory* **148**, 1606–1623.

Casadesus-Masanell, R., Klibanoff, P. and E. Ozdenoren (2000) "Maxmin expected utility over Savage acts with a set of priors," *Journal of Economic Theory* **92**, 35–65.

Cass, D. (1972) "On capital overaccumulation in the aggregative, neoclassical model of economic growth: A complete characterization," *Journal of Economic Theory* **4**, 200–223.

de Castro, L.I. and A. Chateauneuf (2011) "Ambiguity aversion and trade," *Economic Theory* **48**, 243–273.

de Castro, L.I., Z. Liu, and N.C. Yannelis (2017) "Ambiguous implementation: the partial model," *Economic Theory* **63**, 233–261.

Chambers, C.P. and F. Echenique (2018) “On multiple discount rates,” *Econometrica* **86**, 1325–1346.

Chateauneuf, A., R.-A. Dana, and J.-M. Tallon (2000) “Risk sharing rules and equilibria with non-additive expected utilities,” *Journal of Mathematical Economics* **61**, 953–957.

Chattopadhyay, S. (2001) “The unit root probyty and optimality: a simple proof,” *Journal of Mathematical Economics* **36**, 151–159.

Chattopadhyay, S. (2006) “Optimality in stochastic OLG models: Theory for tests,” *Journal of Economic Theory* **131**, 282–294.

Chattopadhyay, S. and P. Gottardi (1999) “Stochastic OLG models, market structure, and optimality,” *Journal of Economic Theory* **89**, 21–67.

Condie, S. and J.V. Ganguli (2011) “Ambiguity and rational expectations equilibriua,” *Review of Economic Studies* **78**, 821–845.

Dana, R.-A. (2004) “Ambiguity, uncertainty aversion and equilibrium welfare,” *Economic Theory* **23**, 569–587.

Dana, R.-A. and C. Le Van (2010) “Overlapping risk adjusted sets of priors and the existence of efficient allocations and equilibria with short-selling,” *Journal of Economic Theory* **145**, 2186–2202.

Dana, R.-A. and F. Riedel (2013) “Intertemporal equilibria with Knightian uncertainty,” *Journal of Economic Theory* **148**, 1582–1605.

Debreu, G. and I.N. Herstein (1953) “Nonnegative square matrix,” *Econometrica* **21**, 597–607.

Demange, G. and G. Laroque (1999) “Social security and demographic shocks,” *Econometrica* **67**, 527–542.

Demange, G. and G. Laroque (2000) “Social security, optimality, and equilibria in a stochastic overlapping generations economy,” *Journal of Public Economic Theory* **2**, 1–23.

Dow, J. and G. Gorton (1993) “Security market returns and the social discount rate: A note on welfare in the overlapping generations model,” *Economics Letters* **43**, 23–26.

Dow, J. and S.R.C. Werlang (1992) “Uncertainty aversion, risk aversion, and the optimal choice of portfolio,” *Econometrica* **60**, 197–204.

Drugeon, J.-P., T. Ha-Huy, and T.D.H. Nguyen(2018) “On maximin dynamic programming and the rate of discount,” *Economic Theory*, *forthcoming.*

Ellsberg, D. (1961) “Risk, ambiguity, and the Savage axioms,” *Quarterly Journal of Economics* **75**, 643–669.

Epstein, L.G. and T. Wang (1994) “Intertemporal asset pricing under Knightian uncertainty,” *Econometrica* **62**, 283–322.

Epstein, L.G. and T. Wang (1995) “Uncertainty, risk-nuetral measures and security price booms and crashes,” *Journal of Economic Theory* **67**, 40–82.

Epstein, L.G. and M. Schneider (2007) “Learning under ambiguity,” *Review of Economic Studies* **74**, 1275–1303.

Faro, J.H. (2013) “Cobb-Douglas preferences under uncertainty,” *Economic Theory* **54**, 273–285.

Fukuda, S.-i. (2008) “Knightian uncertainty and poverty trap in a model of economic growth,” *Review of Economic Dynamics* **11**, 652–663.

Geanakoplos, J. (1987) “The overlapping generations model of general equilibrium.” In J. Eatwell, M. Milgate, P. Newman (eds.), *The New Palgrave Dictionary of Economics*, Vol. 3., London: Macmillan Press, 1987, pp. 767-779.

Ghirardato, P. and M. Marinacci (2002) “Ambiguity made precise: A comparative foundation,” *Journal of Economic Theory* **102**, pp.251–289.

Ghirardato, P., F. Maccheroni, and M. Marinacci (2004) “Differentiating ambiguity and ambiguity attitude,” *Journal of Economic Theory* **118**, pp.133–173.

Gilboa, I. and D. Schmeidler (1989) “Maxmin expected utility with non-unique prior,” *Journal of Mathematical Economics* **18**, 141–153.

Gollier, C. (2011) “Portfolio choices and asset prices: The comparative statics of ambiguity aversion,” *Review of Economic Studies* **78**, 1329–1344.

Gottardi, P. (1996) “Stationary monetary equilibria in overlapping generations models with incomplete markets,” *Journal of Economic Theory* **71**, 75–89.

Gottardi, P. and F. Kübler (2011) “Social security and risk sharing,” *Journal of Economic Theory* **146**, 1078–1106.

Hansen, L.P. and T.J. Sargent (2001) “Robust control and model uncertainty,” *American Economic Review* **91**, 60–66.

Hiriart-Urruty, J.-B. and C. Lemarechal (2004) *Fundamentals of Convex Analysis*, New York: Springer-Verlag.

Kajii, A. and T. Ui (2005) "Incomplete information games with multiple priors," *Japanese Economic Review* **56**, 332–351.

Kajii, A. and T. Ui (2009) "Interim efficient allocations under uncertainty," *Journal of Economic Theory* **144**, 337–353.

Karni, E. (2009) "A reformulation of the maxmin expected utility model with application to agency theory," *Journal of Mathematical Economics* **2009**, 97–112.

Knight, Frank (1921) *Risk, Uncertainty and Profit*, Boston: Houghton Mifflin.

Labadie, P. (2004) "Aggregate risk sharing and equivalent financial mechanisms in an endowment economy of incomplete participation," *Economic Theory* **27**, 789–809.

Lopomo, G., L. Rigotti, and C. Shannon (2011) "Knightian uncertainty and moral hazard," *Journal of Economic Theory* **146**, 1148–1172.

Maccheroni, F., M. Marinacci, and A. Rustihici (2006): "Ambiguity aversion, robustness, and the variational representation of preferences," *Econometrica* **74**, 1447–1498.

Magill, M. and M. Quinzii (2003) "Indeterminacy of equilibrium in stochastic OLG models," *Economic Theory* **21**, 435–454.

Mandler, M. (2013) "Endogenous indeterminacy and volatility of asset prices under ambiguity," *Theoretical Economics* **8**, 729–750.

Manuelli, R. (1990) "Existence and optimality of currency equilibrium in stochastic overlapping generations models: The pure endowment case," *Journal of Economic Theory* **51**, 268–294.

Martins-da-Rocha, V.F. (2010) "Interim efficiency with MEU-preferences," *Journal of Economic Theory* **145**, 1987–2017.

McCandless, G.T. and N. Wallace (1992) *Introduction to Dynamic Macroeconomic Theory: An Overlapping Generations Approach*, Cambridge, Mass.: Harvard University Press.

Milgrom, P. and N. Stokey (1982) "Information, trade and common knowledge," *Journal of Economic Theory* **26**, pp.17–27.

Muench, T.J. (1977) "Optimality, the interaction of spot and futures markets, and the nonneutrality of money in the Lucas model," *Journal of Economic Theory* **15**, 325–344.

Mukerji, S. and J.-M. Tallon (1998) "Ambiguity aversion and incompleteness of contractual form," *American Economic Review* **88**, 1207–1231.

Mukerji, S. and J.-M. Tallon (2001) "Ambiguity aversion and incompleteness of financial markets," *Review of Economic Studies* **68**, 883–904.

Mukerji, S. and J.-M. Tallon (2004a) "Ambiguity aversion and the absence of indexed debt," *Economic Theory* **24**, 665–685.

Mukerji, S. and J.-M. Tallon (2004b) "Ambiguity aversion and the absence of wage indexation," *Journal of Monetary Economics* **51**, 653–670.

Nishimura, K.G. and H. Ozaki (2004) "Search and Knightian uncertainty," *Journal of Economic Theory* **119**, 299–333.

Nishimura, K.G. and H. Ozaki (2006) "An Axiomatic Approach to ε-contamination," *Economic Theory* **27**, 333–340.

Nishimura, K.G. and H. Ozaki (2007) "Irreversible investment and Knightian uncertainty," *Journal of Economic Theory* **136**, 668–694.

Ohtaki, E. (2013) "Golden rule optimality in stochastic OLG economies," *Mathematical Social Sciences* **65**, 60–66.

Ohtaki, E. (2014) "Tractable graphical device for analyzing stationary stochastic OLG economies," *Journal of Macroeconomics* **40**, 16–26.

Ohtaki, E. (2015) "A note on existence and uniqueness of stationary monetary equilibrium in a stochastic OLG Model," *Macroeconomic Dynamics* **19**, 701–707.

Ohtaki, E. and H. Ozaki (2015) "Monetary equilibria and Knightian uncertainty," *Economic Theory* **59**, 435–459.

Peled, D. (1984) "Stationary Pareto optimality of stochastic asset equilibria with overlapping generations," *Journal of Economic Theory* **34**, 396–403.

Rigotti, L. and C. Shannon (2005) "Uncertainty and risk in financial markets," *Econometrica* **73**, 203–243.

Rigotti, L. and C. Shannon (2012) "Sharing risk and ambiguity," *Journal of Economic Theory* **147**, 2028–2039.

Rigotti, L., C. Shannon, and T. Strzalecki (2008) "Subjective beliefs and ex-ante trade," *Econometrica* **76**, 1167–1190.

Rinaldi, F. (2009) "Endogenous incompleteness of financail markets: The role of ambiguity and ambiguity aversion," *Journal of Mathematical Economics* **45**, 872–893.

Rockafellar, R.T. (1970) *Convex Analysis*, Princeton University Press: Princeton, NJ.

Sakai, Y. (1988) "Conditional Pareto optimality of stationary equilibrium in a stochastic overlapping generations model," *Journal of Economic Theory* **44**, 209–213.

Schmeidler, D. (1989) “Subjective probability and expected utility without additivity,” *Econometrica* **57**, 571–587. (Its working paper is first published in 1982)

Strzalecki, T. (2011) “Axiomatic foundation of multiplier preferences,” *Econometrica* **79**, 47–73.

Takayama, A. (1974) *Mathematical Economics*. Hinsdale, IL.: The Dryden Press.

Strzalecki, T. and J. Werner (2011) “Efficient allocations under ambiguity,” *Journal of Economic Theory* **146**, 1173–1194.

Wakai, K. (2008) “A model of Utility smoothing,” *Econometrica* **76**, 137–153.

Xue, J. (2018) “Preferences with changing ambiguity aversion,” *Economic Theory*, *forthcoming*.