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Abstract

Which type of monetary policy rule best describes the policy conducted by the Bank of Japan during the period when the nominal interest rate is constrained at the zero lower bound (ZLB)? What are the economic fundamentals that explain Japan’s prolonged stagnation? How important is incorporating nonlinearities in the analysis? We answer these questions by estimating a small-scale nonlinear DSGE model. We find that: the Bank of Japan conducted a threshold-based forward guidance policy; adverse demand shocks explain Japan's experience; and nonlinear models are very useful in the analysis of the Japanese economy during the ZLB period.

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Estimating a Nonlinear New Keynesian Model with the Zero Lower Bound for Japan[*]

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Abstract

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Keywords: Bayesian inference; DSGE model; Particle filter; Sequential Monte Carlo Squared

JEL classification: C11, C13, C61, C63, E31, E43, E52

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1 Introduction

Japan has experienced a long stagnation, the so-called "lost decades," since the early 1990s. During the period of lost decades, Japan's output growth rate has been slow and the inflation rate has been low, with occasional mild deflation. To stimulate the economy, the Bank of Japan (BOJ) lowered the short-term nominal interest rate and set it at nearly zero for most of the time since 1995. In addition, the BOJ adopted a number of unconventional monetary policies, including a forward guidance policy, for the purpose of influencing the expectations of private agents about the future course of monetary policy, and in turn, their behaviors in the current period. However, despite the BOJ's efforts, the Japanese economy has not yet fully recovered from the period of stagnation.

In focusing on this experience, this paper intends to answer the following three questions. The first question is whether the forward guidance policy is most appropriate in describing the monetary policy conducted by the BOJ during the period when the nominal interest rate was constrained by the zero lower bound (ZLB). The second question is whether Japan's experience of a long duration of zero interest rates can be explained by economic fundamentals in the general equilibrium framework. The third question is how important is incorporating nonlinearities and ZLB in the analysis of the Japanese economy. We address these questions by estimating a nonlinear dynamic stochastic general equilibrium (DSGE) model, which incorporates both the ZLB of nominal interest rates and a forward guidance policy during the zero interest rate period. Unlike the typical analysis using linear DSGE models, however, we focus on a small-scale DSGE model because computationally intensive methods are required in solving and estimating the nonlinear DSGE model.

Let us provide in more detail the background behind each question. Regarding the first question on the type of monetary policy, the BOJ claims that the forward guidance policy was implemented during the ZLB period.[1] However, based on the observation that the

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[1] The BOJ announced in April 1999 that the "zero rate will be maintained until deflationary concerns are dispelled," and in March 2001 that the BOJ's easing policy continues to be in place "until the consumer price index (excluding perishables) registers stably a zero percent or an increase year on year."

inflation rate had increased to a certain level, the BOJ twice decided to end the zero-rate policy, first in 2000 and then in 2006. Since such policy changes have been followed by the recessions in 2001 and 2007, respectively, one may view that the BOJ's forward guidance policy was neither clearly stated nor consistently implemented. For this reason, several studies consider whether the term "forward guidance policy" appropriately describes the BOJ's actual policy (e.g., Ito and Mishkin 2004). In contrast, Hayashi and Koeda (2019) have estimated a regime-switching vector autoregressive (VAR) model and provide evidence that the BOJ implemented a type of forward guidance policy. In particular, their model incorporates the threshold-based forward guidance policy, in the terminology of Boneva, Harrison, and Waldron (2018), in which an inflation rate needs to be sufficiently high to exit from the zero-rate policy. We aim to provide further evidence on this issue by estimating a DSGE model that not only allows the ZLB but also several monetary policy rules, including a threshold-based forward guidance rule.[2]

Providing the evidence for the first question brings us to the second question. It is natural to ask why the zero interest rate period has continued for more than two decades, and why the inflation rate has remained low despite the BOJ's efforts to conduct unconventional monetary easing policies, including the forward guidance policy. According to the macroeconomic theory, the prolonged stagnation can be explained either by adverse shocks to economic fundamentals (e.g., Summers 2013, Eggertsson, Mehrotra, and Robbins 2019,

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[2] There are a number of studies on monetary policy during the ZLB period. Aoki and Ueno (2012) use forward rate curves to take into account the effects of the ZLB for Japan, while estimating a linear DSGE model. Kulish, Morley, and Robinson (2017) estimate the duration of the ZLB for the US by assuming that it is foreseen perfectly in each period and then assess the time-varying policy functions, given the estimated duration of the ZLB. In the finance literature, based on the partial-equilibrium approach, a shadow rate is estimated to evaluate the effects of unconventional monetary policy on the yield curve (see, e.g., Ichiue and Ueno 2006, Kim and Singleton 2012, Krippner 2013, Bauer and Rudebusch 2016, Wu and Xia 2016, and Ueno 2017). Kim and Pruitt (2017) use forecasters' surveys to estimate a monetary policy rule in the US. A partial-equilibrium approach and structural VAR model may enable us to identify a monetary policy rule by estimating the response of nominal interest rates to economic disturbances. In contrast, our estimation based on a DSGE model takes into account not only the response of nominal interest rates to economic disturbances but also the effects of monetary policy on the economy. Thus, if the forward guidance puzzle is significant (i.e., if the theoretical power of forward guidance is too strong; Del Negro, Giannoni, and Patterson 2015; McKay, Nakamura, and Steinsson 2016), then our estimation unlikely supports the threshold-based forward guidance rule.

and Katagiri, Konishi, and Ueda 2020) or by the presence of a sunspot equilibrium (e.g., Benhabib, Schmitt-Grohé, and Uribe 2001, Benigno and Fornaro 2018). For the latter explanation, several studies provide supporting evidence by estimating DSGE models of a sunspot equilibrium.[3] In contrast, evidence for the former explanation based on the DSGE model does not seem to be sufficient, partly because of technical reasons. Indeed, estimating a nonlinear DSGE model incorporating the ZLB can be very difficult when the duration of episodes of zero interest rates becomes longer. In this regard, the fact that our solution and estimation algorithm is successful in describing the Japanese economy is a benefit. Without relying on a sunspot equilibrium, our approach can demonstrate the extent to which economic fundamentals can account for Japan's experience. For the purpose of understanding the role of fundamentals, we focus specifically on the estimated natural rate of interest (see, e.g., Wicksell 1936, Krugman 1998, and Woodford 2003). We compare the estimated natural rate of interest with the real interest rate, and then seek the reason for Japan's prolonged stagnation by historically decomposing the natural rate of interest with structural shocks.

For the third question, we evaluate the usefulness of incorporating nonlinearities and ZLB to describe the Japanese economy over the past few decades. It is well-known that the presence of the ZLB on the nominal interest rate often changes the policy implications of DSGE models (see Eggertsson 2011, Fernández-Villaverde et al. 2015, Boneva, Braun, and Waki, 2016, and Nakata 2017, among others). However, for computational convenience, linearized DSGE models have commonly been employed in the analysis of the Japanese economy (e.g., Sugo and Ueda 2008, Fujiwara, Hirose, and Shintani 2011, and Hirakata et al. 2016). Indeed, Hirose and Inoue (2016) report no significant bias in parameter estimates in the linear DSGE model when the data is generated from a nonlinear DSGE model for the US economy, in which the average duration of ZLB spells is relatively short. Also, in the

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[3] See, e.g., Hirose (2008), Aruoba, Cuba-Borda, and Schorfheide (2018), and Cuba-Borda and Singh (2019). Aruoba, Cuba-Borda, and Schorfheide (2018) construct a model, in which the economy fluctuates between a normal determinate equilibrium and a deflationary indeterminate equilibrium. They estimate the model without the ZLB and then generate data and conduct simulations that incorporate the ZLB. Cuba-Borda and Singh (2019) estimate a model with the ZLB by assuming a permanent liquidity trap and estimating parameters only associated with shocks.

case of the US, Atkinson, Richter, and Throckmorton (2019) find supporting evidence in the use of a piecewise linear model, which incorporates only the nonlinearity from the ZLB. In a similar vein, we investigate whether the estimation of approximated models, such as linear and piecewise linear models, changes the implication of the nonlinear model in describing the Japanese economy where the duration of ZLB spells has been much longer.

We answer three questions in a unified framework by estimating a fully nonlinear DSGE model incorporating the ZLB. In general, however, it is computationally challenging to solve a rational expectations equilibrium in the model with the ZLB. Estimating the model is even more difficult because it involves solving the model many times over a large number of possible sets of parameter values. It should also be noted that Japan is a unique and important case that makes the problem of the ZLB particularly serious, given its long duration of stagnation and zero interest rate policy.

In this paper, we employ two techniques useful for analyzing the model with an occasionally binding constraint: the *time iteration with linear interpolation* (TL) method to solve for a rational expectations equilibrium and the *Sequential Monte Carlo Squared* (SMC^2) method to estimate the model. The TL method, which is recommended by Richter, Throckmorton, and Walker (2014), is useful for solving the rational expectations equilibrium. This method is categorized within the class of policy function iterations and, according to Richter, Throckmorton, and Walker (2014), is flexible, accurate, and speedy. Second, the SMC^2 method developed by Chopin, Jacob, and Papaspiliopoulos (2013) and applied to DSGE models by Herbst and Schorfheide (2015) is useful in estimating the model parameters. As Chopin, Jacob, and Papaspiliopoulos (2013), Herbst and Schorfheide (2015), and Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) argue, in comparison with the particle filters combined with Metropolis–Hastings (PFMH) algorithm, the SMC^2 method leads to a more reliable posterior inference, and thus we do not need to introduce large measurement errors. Although the TL and SMC^2 methods have many advantages, they are still computationally intensive. To facilitate the computations, we therefore use one of the simplest New Keynesian models, which abstracts many of the important features of standard DSGE models, such as capital formation and wage stickiness, but embeds consumption habits.

Our main findings are as follows. First, according to estimation results, the model combined with the threshold-based forward guidance rule provides the best empirical fit, compared to the model combined with other monetary policy rules. The commitment to continue the zero-rate policy until inflation becomes sufficiently high, as determined by the BOJ, has most likely been effective in making private agents expect a considerably long duration of ZLB spells. At the same time, however, the impulse response shows that a monetary easing policy shock can hardly increase output and inflation rates in the ZLB period compared to normal times when the nominal interest rate stays in the positive range.

Second, our model shows that the combination of adverse demand shocks and the ZLB explains Japan's experience of long duration of zero interest and prolonged stagnation. The natural rate of interest fell in the early 1990s and often became negative in the 2000s and 2010s, whereas the real interest rate did not fall much below the natural rate of interest. This evidence suggests that, despite the continuation of the zero-rate policy, inflation expectations did not sufficiently increase to lower the real interest rate or increase output and inflation. Moreover, the historical decomposition shows that the low natural rate of interest is mainly explained by adverse demand shocks, an outcome which is consistent with the secular stagnation view of Summers (2013).

Third, we show that nonlinear estimation is crucial for deriving implications for monetary policy. The fit of the model significantly worsens, especially when we ignore the nonlinearity coming from the ZLB. The parameter estimates may be biased when a model fails to incorporate nonlinearity, and thus it can result in incorrect inferences. When a linear model or a piecewise linear model is used, long episodes of zero interest rates cannot be replicated, and a very different estimate of the natural rate of interest is obtained.

Our analysis, which solves and estimates a nonlinear DSGE model incorporating the ZLB, is closely related to previous studies by Gust et al. (2017) and Plante, Richter, and

Throckmorton (2018). There are four main differences between our work and theirs. First, and most importantly, they use the data for the US, where the ZLB is relevant for only a few years. Our analysis is based on the data for Japan where the nominal interest rate has been almost zero for two decades, and the effect of the ZLB is therefore likely to be larger on the economy. Second, we consider several types of monetary policy rules, including the threshold-based forward guidance policy (which we call the Exit Condition Model). On the other hand, Gust et al. (2017) and Plante, Richter, and Throckmorton (2018) focus on a type of monetary policy rule that embeds an inertia in notional interest rates (which we call the Notional Rate Model). Although such a policy rule can also generate a longer expected duration of the zero interest rate than the standard Taylor rule with interest smoothing (which we call the Nominal Rate Model), we find that the threshold-based forward guidance policy fits better in the case of Japan. Third, Gust et al. (2017) estimate a richer medium-sized DSGE model than that of Plante, Richter, and Throckmorton (2018) or our model. Fourth, our analysis also differs from Gust et al. (2017) and Plante, Richter, and Throckmorton (2018) in the choice of the estimation method. These studies use the PFMH algorithm to estimate the model, while we use the SMC^2 method. As we explain in detail in Section 3, we do not need to assume large measurement errors in the SMC^2 method, and the posterior inference is likely to be more reliable.

The remainder of this paper is structured as follows. Section 2 briefly explains our model and Section 3 outlines our estimation methods. Sections 4, 5, and 6 discuss implications of our estimation results for monetary policy, prolonged stagnation, and linear approximation, respectively. Section 7 concludes.

2 Model

Our model is one of the simplest New Keynesian models. The economy consists of a representative household, firms, and a central bank. Firms consist of monopolistically competitive intermediate-good producers and a perfectly competitive final-good producer. The central bank facing the ZLB constraint is assumed to follow one of the three alternative monetary policy rules. The economy is subject to three types of exogenous shocks, namely, a discount factor (preference) shock, a technology shock, and a monetary policy shock.

2.1 Household

The representative household i ($\in [0,1]$) maximizes intertemporal utility given by

$$\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j Z^b_{t+j}\left\{\frac{(C_{i,t+j}-hC_{t+j-1})^{1-\sigma}}{1-\sigma}-\chi\frac{(A_{t+j})^{1-\sigma}\,l_{i,t+j}^{1+\omega}}{1+\omega}\right\}\right], \tag{1}$$

subject to the budget constraint $C_{i,t}+B_{i,t}/P_t \leq W_t l_{i,t}+R_{t-1}B_{i,t-1}/P_t+T_t$, where $C_{i,t}$, $l_{i,t}$, P_t, W_t, R_t, and T_t represent consumption, labor services, the aggregate price level, the real wage, the nominal rate of return, and the lump-sum transfer, respectively, in period t. In addition, $B_{i,t}$ represents the holding of one-period riskless bonds at the end of period t. We assume external habits as in Plante, Richter, and Throckmorton (2018) so that utility in the current period depends on aggregate consumption $C_t=\left\{\int_0^1 C_{i,t}^{\frac{\varepsilon-1}{\varepsilon}}di\right\}^{\frac{\varepsilon}{\varepsilon-1}}$ in the previous period, where ε is defined below. Parameter $\beta(\in(0,1))$ is the subjective discount factor; $\sigma(>0)$ is the inverse of the intertemporal elasticity of substitution of consumption; $h(\in[0,1))$ measures the importance of consumption habits; $\omega(>0)$ is the inverse of the labor supply elasticity; and $\chi(>0)$ is the scale parameter for the disutility from working. As in Erceg, Guerrieri, and Gust (2006), we allow preferences for leisure to shift with the level of technology, A_t, to ensure the existence of a balanced growth path. Finally, Z^b_t represents a shock to the discount factor (preference), generated from the first-order autoregressive (AR(1)) process:

$$\log(Z^b_t)=\rho_b\log(Z^b_{t-1})+\epsilon^b_t, \tag{2}$$

where $|\rho_b|<1$ and $\epsilon^b_t\sim i.i.d.\ N(0,\sigma_b^2)$. Hereafter, for simplicity, we omit subscript i.

2.2 Firms

The perfectly competitive final-good firm maximizes its profit by choosing the best combination of intermediate inputs $Y_{f,t}$ and by selling the final good Y_t, given the aggregate price P_t, the intermediate good price $P_{f,t}$, and the Dixit–Stiglitz form of aggregations $Y_t = \left\{\int_0^1 Y_{f,t}^{\frac{\varepsilon-1}{\varepsilon}} df\right\}^{\frac{\varepsilon}{\varepsilon-1}}$, where $\varepsilon(> 1)$ represents the elasticity of substitution between intermediate goods.

The monopolistically competitive intermediate-good firm $f(\in [0,1])$ produces output $Y_{f,t} = A_t l_{f,t}$ using labor input $l_{f,t}$. The technology A_t in log follows an I(1) process:

$$\log A_t = \log A_{t-1} + \gamma_a + \mu_t^a, \tag{3}$$

where γ_a is the mean growth rate and μ_t^a is the technology shock. The latter follows a stationary AR(1) process:

$$\mu_t^a = \rho_a \mu_{t-1}^a + \epsilon_t^a, \tag{4}$$

where $|\rho_a| < 1$ and $\epsilon_t^a \sim i.i.d.\ N(0, \sigma_a^2)$. The intermediate-good firm f maximizes its firm value by setting the optimal price $P_{f,t}$ in period t in the presence of a Rotemberg-type price adjustment cost:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j \frac{\Lambda_{t+j} Z_{t+j}^b}{\Lambda_t Z_t^b} \left(\frac{P_{f,t+j}}{P_{t+j}} - \frac{W_{t+j}}{A_{t+j}} - \frac{\phi}{2} \left(\frac{P_{f,t+j}}{P_{f,t+j-1}} - \pi^* \right)^2 \right) Y_{f,t+j} \right] \tag{5}$$

subject to downward-sloping demand, where Λ_t and π^* represent the stochastic discount factor and the target inflation rate, respectively, and ϕ is the parameter for the Rotemberg-type price adjustment cost.

2.3 Central Bank

Without ZLB constraint, the central bank is typically assumed to set nominal interest rate R_t following the standard Taylor rule given by

$$R_t = (R_{t-1})^{\rho_r} \left(r^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_\pi} \left(\frac{y_t}{y_t^*}\right)^{\psi_y} \right)^{1-\rho_r} e^{\epsilon_t^r} \tag{6}$$

where r^* is the steady-state natural rate of interest, $\pi_t(= P_t/P_{t-1})$ is the inflation rate, $y_t(= Y_t/A_t)$ is the detrended output, and y_t^* is the (detrended) natural level of output. Parameters ρ_r, ψ_π, and ψ_y capture interest rate smoothing ($0 < \rho_r < 1$), the sensitivity to the inflation rate, and the sensitivity to the output gap, respectively, and the monetary policy shock is given by $\epsilon_t^r \sim i.i.d.N(0, \sigma_r^2)$. Since the search for the best description of the monetary policy conducted by the Bank of Japan during the ZLB period is one of the main objectives of this study, we consider the following three alternative models of the central bank facing the ZLB constraint.

Nominal Rate Model

With ZLB, the nominal interest rate R_t cannot be less than one, so that the monetary policy rule can be described as

$$R_t = \begin{cases} R_t^* & \text{if } R_t^* > 1; \\ 1 & \text{otherwise,} \end{cases} \tag{7}$$

where R_t^* represents the notional interest rate, which can take a value of less than one. By an analogy to (6), the notional rate can follow the Taylor rule given by

$$R_t^* = (R_{t-1})^{\rho_r} \left(r^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_\pi} \left(\frac{y_t}{y_t^*}\right)^{\psi_y} \right)^{1-\rho_r} e^{\epsilon_t^r}. \tag{8}$$

This specification has been employed, for example, by Aruoba, Cuba-Borda, and Schorfheide (2018). Since the interest smoothing is given as a weighted sum of the target rate and the lagged nominal rate, we refer to this monetary policy rule as the *Nominal Rate Model.*

Notional Rate Model

The second model of the monetary policy rule we consider is given by

$$R_t = \begin{cases} R_t^* & \text{if } R_t^* > 1; \\ 1 & \text{otherwise,} \end{cases} \tag{9}$$

$$R_t^* = \left(R_{t-1}^*\right)^{\rho_r} \left(r^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_\pi} \left(\frac{y_t}{y_t^*}\right)^{\psi_y} \right)^{1-\rho_r} e^{\epsilon_t^r}. \tag{10}$$

Since the interest smoothing is now given as a weighted sum of the target rate and the lagged notional rate, we refer to this monetary policy rule as the *Notional Rate Model.* This model has been employed by Gust et al. (2017) and Plante, Richter, and Throckmorton (2018). The Notional Rate Model differs from the nominal rate model only in the choice of a lagged interest rate in the interest smoothing. With this change, the Notional Rate Model implies a stronger promise to continue the zero-rate policy in the future than does the Nominal Rate Model. Because R_t^* can be below one and depends on R_{t-1}^*, the experiences of adverse shocks in the past lower the future interest rate R_t for long periods. In other words, the central bank compensates for its inability to lower the policy rate below zero by promising to continue the zero-rate policy, ceteris paribus. This type of carry-over policy is essentially the one that Reifschneider and Williams (2000) propose, which indeed has influenced monetary policy-making at the BOJ. In April 1999, the BOJ introduced the first forward guidance policy by announcing that the "zero rate will be maintained until deflationary concerns are dispelled." Behind this decision was the theoretical work of Reifschneider and Williams (2000), as stated by a former member of the Policy Board (Ueda 2000) and as surveyed by an official of the BOJ (Ugai 2006).

Exit Condition Model

The third model of the monetary policy rule we consider is given by

$$R_t = \begin{cases} R_t^* & \text{if (i) } R_t^* > 1 \text{ and } R_{t-1} > 1; \text{or (ii) } R_t^* > 1, R_{t-1} = 1 \text{ and } \pi_t > \overline{\pi} \\ 1 & \text{otherwise,} \end{cases} \tag{11}$$

$$R_t^* = (R_{t-1})^{\rho_r} \left(r^* \pi^* \left(\frac{\pi_t}{\pi^*} \right)^{\psi_\pi} \left(\frac{y_t}{y_t^*} \right)^{\psi_y} \right)^{1-\rho_r} e^{\epsilon_t^r}. \tag{12}$$

It should be noted that the notional rate R_t^* is the same as that in the Nominal Rate Model. However, this model differs from the Nominal Rate Model regarding the condition to exit from the zero interest rate. For this reason, we refer to this monetary policy rule as the *Exit Condition Model.* The Exit Condition Model embeds threshold-based forward guidance policy in the spirit of Boneva, Harrison, and Waldron (2018) and Hayashi and Koeda (2019). The condition (ii) in equation (11) suggests that having $R_t^* > 1$ is not sufficient to exit from the zero-rate policy. If the interest rate is zero in the previous period, the current inflation rate needs to exceed a certain threshold level $\overline{\pi}$ to raise the interest rate above zero. Indeed, in September 2016, the BOJ announced an inflation-overshooting commitment policy, such that it would "continue expanding the monetary base until the year-on-year rate of increase in the observed CPI (all items less fresh food) exceeds the price stability target of 2 percent and stays above the target in a stable manner." This statement implies that threshold $\overline{\pi}$ should be around 2% at an annual rate.

2.4 Closing the Model

The goods market clearing condition is given by

$$Y_t = C_t + \phi \left(\pi_t - \pi^* \right)^2 Y_t/2. \tag{13}$$

The flexible-price equilibrium is defined as the equilibrium in which there is no cost of price adjustment ($\phi = 0$). The natural rate of interest r_t^* in the model is the real rate of return, and the natural level of output Y_t^* is the level of output in such a flexible-price economy. It should be noted that the representative household would take its actual consumption in the previous period as the reference value, rather than that in the flexible-price equilibrium in the previous period.

3 Methodology

In this section, we outline how we solve and estimate the nonlinear DSGE model with the ZLB. We then explain our data and prior specifications. Finally, we discuss the advantage of the SMC^2 method.[4]

3.1 Model Solution

To solve the rational expectations equilibrium of our model, we employ the *time iteration with linear interpolation* (TL) method, a type of policy function iteration. The TL method has been recommended by Richter, Throckmorton, and Walker (2014) as it provides the best balance between the speed and accuracy among competing methods. We solve for the rational expectations equilibrium or policy function for a given parameter θ. In our model, the policy function is expressed as a function of five variables $z_t = (\mu_t^a, Z_t^b, \epsilon_t^r, y_{t-1}, R_{t-1}^*)'$. Note that z_t consists of the minimum number of variables because there are three shocks and two state variables, y_{t-1} and R_{t-1}^*. Intuitively, the TL method begins with a time iteration for a policy function until intertemporal equations are satisfied at every node. Unlike the case of the fixed-point iteration, calling a nonlinear solver at each node is computationally costly. However, at the same time, it is more stable because the policy function is optimized at each node. This local approximation is then used to evaluate the global policy function by using

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[4] See the Online Appendix for the details. In Appendix A.1, we explain the method used to solve the rational expectations equilibrium. In Appendix A.2, we explain the method used to estimate our model.

linear interpolation. Compared to global approximation methods, such as the projection method using the Chebyshev polynomial basis, linear interpolation is expected to perform better when the ZLB produces kinks in the policy functions.

It should be noted that our solution method does not explicitly incorporate an indeterminate equilibrium, or the deflationary equilibrium shown by Benhabib, Schmitt-Grohé, and Uribe (2001). Gavin et al. (2015) show that the equilibrium never converges to the deflationary equilibrium in the TL method by changing the initial conjectures for inflation and consumption. Indeed, we find that the TL method fails to solve the equilibrium, for example, when monetary policy is passive ($\psi_\pi < 1$) or when the steady-state real interest rate and inflation rate (r^* and π^*, respectively) are low. In such cases, no stable intertemporal relationship can be obtained with the policy function of z_t. For this reason, we exclude the possibility of deflationary equilibrium in our analysis. See Aruoba, Cuba-Borda, and Schorfheide (2018) for an attempt to incorporate the deflationary equilibrium.

3.2 Estimation

To estimate the nonlinear DSGE model with the ZLB, we employ the SMC^2 method. The name of this method comes from the the fact that sequential Monte Carlo (SMC) methods are employed twice to evaluate two objectives. First, we evaluate the likelihood of a nonlinear model for a fixed parameter by generating particles of endogenous variables (this part is often referred to as the particle filter). Second, at the same time, we evaluate the posterior distribution by sampling the particles of the parameter.[5]

The method comprises the following four steps. In Step 1 (initialization), we draw N_θ particles for parameters θ. We then repeat Steps 2 to 4 below for N_ϕ stages. In Step 2

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[5] Before the SMC^2 method, use of the SMC method for parameter estimation was limited to linear state-space models, in which the Kalman filter could be applied to evaluate the likelihood. See, for example, Chopin (2002) and Herbst and Schorfheide (2014). Alternatively, when estimating nonlinear state-space models, past studies combine a particle filter with the MCMC technique developed by Andrieu, Doucet, and Holenstein (2010), often called the particle MCMC technique. See also Kitagawa (1996) and Fernández-Villaverde and Rubio-Ramírez (2005) for details on the particle filter.

(correction), given θ, we compute the likelihood $\hat{p}(\mathbf{Y}_t|\theta)$ and normalized weight $\tilde{W}$. In Step 3 (selection), we resample θ together with unnormalized weight W based on θ in the previous stage and $\tilde{W}$ in the previous step. Then, in Step 4 (mutation), we propagate θ and W using the Metropolis–Hastings algorithm.

Specifically, in Step 2, we solve the model for given θ using the TL method. Then, after drawing N_S particles for shock processes $(\mu_t^a, Z_t^b, \epsilon_t^r)'$, we generate the paths of variables $\mathbf{Y}_t$, compare them with observed variables $\mathbf{Y}_t$, and compute the likelihood $\hat{p}(\mathbf{Y}_t|\theta)$ by assuming the presence of the measurement error of $\mathbf{Y}_t$. Because the model is nonlinear, we cannot apply the Kalman filter. Thus, we use the particle filter, whereby we replace $p(\mathbf{Y}_t|\theta)$ by $\hat{p}(\mathbf{Y}_t|\theta)$ using a sufficiently large number of particles N_S with respect to shocks.

In our estimation, we use the particles of $N_S = 40,000$ and $N_\theta = 1,200$ and the stages of $N_\phi = 10$. For the number of particles of the shock processes, N_S, we follow Plante, Richter, and Throckmorton (2018). To the best of our knowledge, no previous study considers the optimal selection on the number of stages N_ϕ nor the number of parameter particles N_θ.[6] A single estimation takes about a week to process with a 32-core (Intel Xeon E5-2698v3) computer.

3.3 Data

We use data for Japan from 1983:Q2 to 2016:Q2. The starting point of the period was chosen to coincide with that of the output gap data, which we use for the robustness analysis. In the benchmark estimation, we use $\mathbf{Y}_t = \{\Delta\log Y_t,\, \pi_t,\, R_t\}'$, where $\Delta\log Y_t = \log Y_t - \log Y_{t-1}$ is the real per capita GDP growth rate; π_t is the consumer price index (CPI) inflation rate; and R_t is the overnight call rate. For Y_t, we divide real GDP by the population aged 15 years or over. For π_t, we exclude the effects of consumption tax changes using X-12-ARIMA. These two variables are quarterly changes from the previous quarter, and thus R_t is expressed in

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[6] More precisely, prior studies such as Herbst and Schorfheide (2014, 2015) may provide a clue as to their choice. However, because we use the likelihood tempering method for the importance sampling, whereas they use the data tempering method, our estimation is considered to be less subject to the number of stages.

a quarterly rate by dividing the annual rate by four. As an alternative to $\Delta\log Y_t$, we later use the output gap ($\log(Y_t/Y_t^*)$) constructed by the BOJ. The solid lines in Figure 1 show the time-series plots for the four variables.

3.4 Prior Specifications

In what follows, we choose the prior for the parameters based on Smets and Wouters (2007) and Sugo and Ueda (2008). We fix $\beta = 0.99875$, $\chi = 1$, and $\varepsilon = 6$ and set the prior distribution of other parameters as shown in Table 1, where κ represents the slope of the Phillips curve and is defined by $(\varepsilon - 1)\left(\omega + \sigma/(1 - he^{-\gamma_a})\right)/(\phi\pi^*)$. For convenience, prior distribution for parameters γ_a, π^*, and $\bar{\pi}$ are expressed in terms of $100\gamma_a$, $100(\pi^* - 1)$, and $100(\bar{\pi} - 1)$, respectively. In the following discussion, we also express the natural rate of interest r_t^* by deducting the value one. The prior mean of the target inflation rate π^* is set at 1% annually. This value corresponds to the midpoint of *The Understanding of Medium- to Long-Term Price Stability* (0 to 2%) clarified by the BOJ from March 2006 to January 2013,[7] and around 0.5% higher than the average of the actual inflation rate during the sample periods. Regarding the threshold inflation rate in the Exit Condition Model, we impose the restriction that $\bar{\pi}$ is strictly greater than π^* by assuming a Gamma distribution for $\bar{\pi} - \pi^*$. This restriction helps identifying parameters $\bar{\pi}$ and π^* and estimating actual forward guidance policy (e.g., the inflation-overshooting commitment policy).

For the measurement errors of $\Delta\log Y_t$, π_t, and R_t, we assume that their sizes are 3%, 3%, and 0.5% of their actual variances, respectively. Their sizes are much smaller than the 25% of Gust et al. (2017) and the 10% of Plante, Richter, and Throckmorton (2018). We assume that the measurement error of R_t is lower than those of $\Delta\log Y_t$ and π_t, but this difference has only a minor effect on the main results.

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[7] In March 2006, it was first announced in the policy statement that the BOJ's official view of inflation rate, which is consistent with the notion of price stability, is between 0 and 2% before it officially started the inflation targeting policy with the target of 2% in January 2013 (BOJ 2006).

3.5 Advantages of the SMC^2 Method over the PFMH Algorithm

We use the SMC^2 method by generating N_S and N_θ particles for the shock processes $(\mu_t^a, Z_t^b, \epsilon_t^r)'$ and for the parameter set θ, respectively. By contrast, Gust et al. (2017) and Plante, Richter, and Throckmorton (2018) use the PFMH algorithm, by which they derive the posterior distribution of the parameters using the MH algorithm while, as in the SMC^2 method, they generate particles for shock processes to approximate a likelihood. As Chopin, Jacob, and Papaspiliopoulos (2013), Herbst and Schorfheide (2015), and Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) argue, the SMC^2 method can lead to a more reliable posterior inference than the PFMH algorithm. In the SMC^2 method, since particles for parameters are uncorrelated between each stage, the sampling efficiency is high and a relatively small number of stages are required.[8] In contrast, in the PFMH algorithm, draws of parameters are highly correlated between each iteration (which is equivalent to the stage in the SMC^2 method). For this reason, the sampling efficiency is low and convergence tests are typically employed to make a judgement regarding the sufficient number of iterations. For example, Gust et al. (2017) use the PFMH algorithm with 40,000 iterations. In our estimation, we choose a relatively large number of parameter particles, $N_\theta = 1,200$, while keeping the number of stages small, $N_\phi = 10$. Within each stage, parallel computing can be employed in drawing parameter particles. In addition, unlike the PFMH algorithm, large measurement errors for observed variables are not required.

Unlike the estimation of linear models, the likelihood function of nonlinear models can not be exactly evaluated and is subject to an approximation error.[9] This approximation error can possibly result in an unreasonably high likelihood at a certain parameter value. For the evaluation of posterior distribution in the PFMH algorithm, a new candidate parameter

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[8] In particular, Herbst and Schorfheide (2015) argues that one of two types of tempering techniques is used in the SMC^2 method: likelihood tempering and data tempering. The computation for each stage is much faster in the data tempering than in the likelihood tempering. At the same time, required number of stages is much smaller in the likelihood tempering. In this paper, we employ the likelihood tempering.

[9] Note that approximation error in the evaluation of likelihood arises for two reasons: the finiteness of the number of grids in solving for the rational expectations equilibrium and finiteness of the number of particles in calculating the likelihood.

value is accepted by comparing its likelihood with that of the previously selected parameter value. However, once an outlier caused by the approximation error is selected, it will remain at this point. To maintain an acceptance probability of 25%, we need to assume relatively large measurement errors for the observed variables. This situation makes the likelihood less sensitive to changes in the parameter values and thus the difference in likelihood at an outlier and the other points decreases. However, large measurement errors imply that we discard information on the observable variables to estimate the model. As argued by Atkinson, Richter, and Throckmorton (2019) and Cuba-Borda et al. (2019), large measurement errors may reduce the accuracy of the parameter estimates. In contrast, the SMC^2 method is not subject to such an outlier problem, even when measurement errors are assumed to be small. Unlike the PFMH algorithm, the SMC^2 method not only compares two particles, but use many particles together to evaluate the likelihood. For this reason, it is unlikely to be stuck at an outlier.[10]

4 Which Specification Best Describes Japan's Monetary Policy?

4.1 Parameter Estimates and Model Selection

We first report the parameter estimates and compare the performance of the three models of monetary policy introduced in subsection 2.3, namely, (i) the Nominal Rate Model, (ii) the Notional Rate Model, and (iii) the Exit Condition Model. Columns 2 to 4 of Table 2 show the parameter estimates and marginal likelihood of each model. The marginal likelihood is the highest for the Exit Condition Model, followed by the Notional Rate Model and the

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[10] However, this example does not mean that the SMC^2 method is always superior to the PFMH algorithm. Herbst and Schorfheide (2015) find that the PFMH algorithm is stable and converges fast when using a linear model, providing an exact solution for the rational expectations equilibrium. In this setting, the problem that arises from a spurious evaluation of the likelihood is small. Hence, we may assume small measurement errors even when using the PFMH algorithm. Therefore, different posterior simulators, SMC^2 and PFMH, perform differently given the same level of accuracy of the particle filter approximation.

Nominal Rate Model. Thus, the data suggest that the Exit Condition Model is the preferred specification, and that the Bank of Japan has been conducting the threshold-based forward guidance policy. For this reason, in what follows, we provide discussion based mainly on the estimation results of the Exit Condition Model.

The estimates of the structural parameters such as σ and ω are within the range reported by earlier studies. The deterministic trend component of the technology shock A_t, γ_a, is $-0.18(= -0.046 \times 4)\%$ at an annual rate, which renders a steady-state natural rate of interest $r^* - 1 = e^{\sigma\gamma_a}/\beta - 1$ of $0.22(= 0.054 \times 4)\%$ at an annual rate. Regarding the two key parameters in the Exit Condition Model, namely the inflation target $\pi^* - 1$ and the threshold value to exit from the zero-rate policy $\overline{\pi} - 1$, we obtain $1.30(= 0.325 \times 4)\%$ and $1.36(= 0.339 \times 4)\%$ at an annual rate, respectively. These estimates are slightly below the price stability target of BOJ but within the official range of 0 to 2% annually (BOJ 2006). At the same time, our estimate of $\overline{\pi} - 1$ is higher than the mean of the threshold inflation rate (or target rate) obtained by Hayashi and Koeda (2019, Table 3), which is 0.53% per year. To evaluate these values further, we suppose that output equals the natural level of output ($Y_t = Y_t^*$); there is no policy inertia ($\rho_r = 0$); and the monetary policy shock is zero ($\epsilon_t^r = 0$) for simplicity. Without an exit condition, the nominal interest rate $R_t - 1$ equals $r^*\pi^*\left(\pi_t/\pi^*\right)^{\psi_\pi} - 1$. Substituting the parameter estimates, we find that the nominal interest rate exceeds zero if $\pi_t - 1$ exceeds 0.52% at an annual rate. This outcome by no means indicates a tight condition because the average inflation rate is 0.47% at an annual rate. Even during the ZLB period, the actual inflation rate often exceeds 0.47%. Therefore, it is difficult to explain a nearly two decades of a long duration of a ZLB spell unless the threshold value to exit is introduced in the model. Our estimate suggests that the nominal interest rate will not go above zero unless the inflation rate exceeds 1.36% per year.

4.2 The Notional Interest Rate

In addition to the observed series, Figure 1 includes the series of output growth, inflation, and the nominal interest rate predicted by the Exit Condition Model with 90% credible intervals. The model explains actual fluctuations well, especially for output growth and inflation. However, when the actual nominal interest rate is approximately zero but positive around 0.5% annually, the predicted series is often zero. This result implies that in the Exit Condition Model the difference is explained by measurement error.

Figure 2 shows the notional interest rate series, $R_t^* - 1$, predicted by the Exit Condition Model with 90% credible intervals. The model suggests that the notional interest rate most of the time takes negative values after 1995. Gust et al. (2017) report the notional interest rate for the US, using a similar DSGE model combined with the Notional Rate Model as the monetary policy rule. In the wake of the global financial crisis, the notional interest rate for the US fell to nearly -5.5% annually. However, within 12 months, it recovered quickly and increased to around zero percent. In contrast, the notional interest rate for Japan has been negative for more than two decades, although the size of the deviation from zero has been modest (around 2% annually), except for the period of the global financial crisis in 2008.[11]

In the finance literature, a shadow rate is often estimated using a term structure model. As with the notional interest rate in our model, the shadow rate can take a negative value, while the actual interest rate is nonnegative. When the Gaussian affine term structure model, or the Black (1995) model, is extended to incorporate the ZLB, the shadow rate captures the effect of unconventional policies to the extent that they influence the term structure. In contrast, the notional interest rate in our model can be interpreted as a desired rate that would be set by the central bank if there were no ZLB. In this respect, the wedge between the notional interest rate and shadow rate may be interpreted as the strength of the constraint faced by the BOJ during the ZLB periods.

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[11] See Online Appendix B for the notional interest rate based on the Notional Rate Model and the Nominal Rate Model.

We compare our estimated series of the notional interest rate with the shadow rate reported by Ueno (2017), which is shown by the dashed line in Figure 2. The notional interest rate in the Exit Condition Model has mostly been below the shadow rate since the mid-1990s. The wedge is particularly large from 1995 to 2003 and 2007 to 2011, which suggests that the ZLB has been a strong constraint for the BOJ's policy actions during these periods. This wedge between the notional interest rate and shadow rate disappeared around 2003 and in 2013–2016. It should be noted that the latter period corresponds to the timing of the QE policy with forward guidance conducted by the BOJ.

4.3 Expected Duration of the Zero-Rate Policy

As in the standard New Keynesian model, expectations play a crucially important role in our model in producing the real effect of monetary policy. If the agent in the face of ZLB believes that the zero-rate policy will continue for a long time, then higher inflation expectations result in an increase in real economic activity. In this regard, it is worth studying how long the zero-rate policy is expected to continue in our estimated model.

Figure 3 plots the average duration of ZLB spells in Japan. Following the analysis of Gust et al. (2017, Figure 6) in the US case, panel (a) shows the cumulative distribution function for the duration of staying at the zero interest rate, while panel (b) shows the right tail of the histogram for the same duration. The average durations of ZLB spells predicted by all three models of monetary policy functions are computed using 120 parameter particles from the posterior distribution. In particular, for each draw of the parameters, 100 simulated series of 1,000 observations are used to evaluate the duration.

Let us compare the expected durations of being at the ZLB among all three models. The Exit Condition Model generates by far the longest expected duration with the median at about 25 quarters. In contrast, the expected duration in the Notional Rate Model and the Nominal Rate Model, in terms of median, is only about five quarters. For the Exit Condition Model, panel (b) of Figure 3 shows that the duration of ZLB spells follows much fatter-tailed distribution compared to other two models. This outcome implies that the mean duration can be much longer than the median duration. Indeed, for the Exit Condition Model, the mean duration is about 35 quarters, which is around 10 years. Our model thus suggests that the threshold-based forward guidance policy was effective in managing the expectations of private agents, and the expected duration of the zero interest rate could be as long as 10 years.

Our estimate of the expected duration of ZLB spells in Japan is much longer than the one estimated by Gust et al. (2017) in the US case. They show that the median duration is just two quarters, and that long ZLB spells are unlikely for the US. This difference does not come solely from a different specification of the monetary policy function. Even when the same type of model, the Notional Rate Model in our classification, is employed in Japan, the median duration of ZLB spells is about five quarters and still longer than their estimates. Therefore, the difference may likely be explained by the combination of (1) the choice of solution and estimation algorithms (2) the specification of the model (Gust et al. (2017) employ a richer medium-sized DSGE model), and (3) the fact that the data contain considerably longer ZLB spells in Japan than in the US. We conjecture that the third point is particularly important in producing the clear difference between the two countries.

Our estimate of duration can also be compared with the expected duration of the negative shadow rate based on a term structure model. Using the estimation results of Ueno (2017), we obtain the median durations of 40 quarters and the mean duration of 50 quarters. Since our Exit Condition Model suggests the median duration of 25 quarters and the mean duration of 35 quarters, both our model and the term structure model explain a considerably long duration of ZLB spells in Japan.[12]

Professionals seem to make similar predictions regarding the stance of monetary policy. In a survey called the *ESP Forecast*, the Japan Center for Economic Research collects the forecasts of around 40 professionals every month. According to the survey conducted between

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[12] See Online Appendix C for detail.

December 25, 2015 and January 5, 2016, just before the introduction of Japan's negative interest rate policy, only seven of 35 professionals predicted a tight monetary policy and all seven answered that tightening would not start within a year. In December 2016, about one year later after the introduction of the negative interest rate, all 40 professionals predicted that, for the one-year-ahead horizon, the short-term interest rate would remain unchanged, or even decrease. This evidence is consistent with the prediction of our model that the probability of being at the ZLB is markedly high.

In summary, the Exit Condition Model best accounts for the observed duration of ZLB spells in the ZLB period of nearly two decades. Using our model, there is no need to appeal to indeterminacy or sunspots to account for the experience in Japan. However, it is not clear whether monetary policy shocks can be effective in the Exit Condition Model when the zero-rate policy is expected to continue for a long time. In the next subsection, we investigate this point by means of impulse response functions (IRFs) to monetary policy shocks.

4.4 Impulse Response Functions to Monetary Policy Shocks

Because our model is nonlinear, mainly due to the ZLB, the IRFs differ, depending on the state of the economy $z_t = (\mu_t^a, Z_t^b, \epsilon_t^r, y_{t-1}, R_{t-1}^*)'$ as well as the sign and size of the shock. To evaluate the typical shape of IRFs to monetary policy shocks, we show IRFs conditional on two historical episodes, 1985:Q1 and 2013:Q2, in the left and right panels of Figure 4, respectively. It should be noted that the nominal interest rate is well above zero in 1985:Q1, but the economy is constrained at ZLB in 2013:Q2. To be more specific, 2013:Q2 corresponds to the timing when the BOJ started the Quantitative and Qualitative Monetary Easing. At that time, the annual inflation rate was $0.88(= 0.22 \times 4)\%$, which is still lower than our estimate of $\overline{\pi} - 1$, but is higher than the average inflation rate during the ZLB periods.[13] We

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[13] We calculate the generalized IRFs following Koop, Pesaran, and Potter (1996). Using a set of the estimated parameter particles, we generate the paths of endogenous variables with and without a monetary policy shock. We then subtract the paths with a monetary policy shock from those without. We repeat this process by drawing 120 parameter particles of the 1,200 posterior distributions and take the mean.

fix the size of the shock at 0.025% and compute IRFs to both positive and negative shocks of the same magnitude. The top two panels show the IRFs of the inflation rate $\pi_t - 1$ and output y_t, whereas the bottom two panels show those of the nominal interest rate $R_t - 1$ and notional rate $R_t^* - 1$.

For 1985:Q1, the bottom-left panel of Figure 4 shows that the IRFs of the nominal interest rate are almost the same as those of the notional interest rate. Moreover, they represent symmetric responses to positive ("Pos" in the figure) and negative ("Neg" in the figure) monetary policy shocks. In contrast, for 2013:Q2, the IRFs differ significantly depending on the sign of the monetary policy shock. Negative responses of inflation and output to a positive monetary policy shock are clearly observed, while there are almost no response to a negative monetary policy shock. Because of the ZLB, the negative monetary policy shock can lower the notional interest rate $R_t^* - 1$, but has a small influence on the nominal interest rate. These results suggest that although the threshold-based forward guidance was effective in making private agents expect a considerably long duration of the zero interest rate, further policy easing (i.e., a negative interest rate shock) did not have strong effects on output and inflation rates.[14]

5 Can Fundamentals Explain Japan's Experience?

In the previous section, we showed that our model can generate a long duration of the zero interest rate without appealing to sunspots. However, it is not obvious in understanding why Japan's stagnation has been prolonged, even though the BOJ has continued an accommodative monetary policy over the past 20 years. In this section, we ask whether our model

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[14] However, the responses of π_t, y_t, and R_t to the monetary policy shock are not strictly zero. Depending on the draws of the parameter particles, $\pi_t - 1$ exceeds $\overline{\pi} - 1$ in some cases, causing $R_t - 1$ to be above zero and generating the effect of monetary policy. In addition, the responses of π_t, y_t, and R_t to monetary policy shocks become almost zero if $\pi_t - 1$ is well below $\overline{\pi} - 1$. In this case, even the positive monetary policy shock does not allow $R_t - 1$ to be above zero because the exit condition is not satisfied. In Online Appendix D, we show the IRFs in more detail.

Furthermore, in Online Appendix E, we examine the validity of the model by comparing the moments of the key economic variables with the data.

is useful in identifying the economic fundamentals that explain the long duration of the zero interest rate and prolonged stagnation in Japan.

5.1 Natural Rate of Interest in the Log-Linearized Model

To understand the role of economic fundamentals in our model, we utilize the notion of the natural rate of interest[15] and compare it with the real rate of interest. While we solve and estimate a system of nonlinear equations in the main analysis, it is convenient to utilize the linear approximation of the model for the purpose of understanding the dynamic relationship between the natural rate of interest, r_t^* and other variables in the model.[16] Our model can be approximated by using three key log-linearized equations:

$$\pi_t - \pi^* = \beta e^{(1-\sigma)\gamma_a} \mathbb{E}_t \left[\pi_{t+1} - \pi^*\right] + \frac{\varepsilon - 1}{\phi \pi^*} \left(\omega + \frac{\sigma}{1 - he^{-\gamma_a}}\right) (\hat{y}_t - \hat{y}_t^*), \tag{14}$$

$$\hat{y}_t - \hat{y}_t^* = \mathbb{E}_t \left[\hat{y}_{t+1} - \hat{y}_{t+1}^* - \frac{1 - he^{-\gamma_a}}{\sigma} \left(\frac{R_t - r^*\pi^*}{r^*\pi^*} - \frac{\pi_{t+1} - \pi^*}{\pi^*} - \frac{r_t^* - r^*}{r^*}\right)\right], \tag{15}$$

$$\begin{aligned} \frac{r_t^* - r^*}{r^*} &= \omega \hat{y}_t^* - \left(\omega + \frac{\sigma}{1 - he^{-\gamma_a}}\right)^{-1} \sigma \frac{he^{-\gamma_a}}{1 - he^{-\gamma_a}} \omega \hat{y}_t + (1 - \rho_b) \log Z_t^b \\ &\quad + \left\{\left(\omega + \frac{\sigma}{1 - he^{-\gamma_a}}\right)^{-1} \omega\sigma \frac{he^{-\gamma_a}}{1 - he^{-\gamma_a}} + \sigma\right\} \rho_a \mu_t^a, \end{aligned} \tag{16}$$

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[15] Studies on developments in the natural rate of interest include the works of Krugman (1998), Laubach and Williams (2003), Neiss and Nelson (2003), Andrés, López-Salido, and Nelson (2009), Kamada (2009), Hall (2011), Barsky, Justiniano, and Melosi (2014), Ikeda and Saito (2014), Cúrdia (2015), Cúrdia et al. (2015), Fujiwara et al. (2016), Del Negro et al. (2017b), Hirose and Sunakawa (2017), and Holston, Laubach, and Williams (2017). With the exception of the recent work by Hirose and Sunakawa (2017), none of these studies uses the DSGE model by explicitly considering the ZLB. Gust et al. (2017) and Plante, Richter, and Throckmorton (2018) do not model the flexible-price variables characterized by Y_t^* and r_t^*. Hirose and Sunakawa (2017) evaluate the natural rate of interest using the DSGE model with the ZLB for the US, but do not estimate the model with the ZLB. Instead, they estimate the model without the ZLB for the periods before the ZLB constrains the economy, and then evaluate the natural rate of interest using the estimated parameters for the extended periods.

[16] Moreover, the linearized system can be used to obtain an initial value of the equilibrium when we solve the model nonlinearly.

and

$$\hat{y}_t^* = \left(\omega + \frac{\sigma}{1 - he^{-\gamma_a}}\right)^{-1} \sigma \frac{he^{-\gamma_a}}{1 - he^{-\gamma_a}} (\hat{y}_{t-1} - \mu_t^a), \tag{17}$$

where variables with hat denote the log-deviations from steady states and $r^* = e^{\sigma\gamma_a}/\beta$ is the steady-state natural rate.

Equations (14) and (15) describe the role of the natural rate of interest r_t^* in aggregate fluctuations. When r_t^* decreases, both the output gap $\hat{y}_t - \hat{y}_t^*$ and the inflation rate π_t decrease, unless the monetary policy is sufficiently strong to offset this effect. The decrease in r_t^* also causes the nominal interest rate to fall. Therefore, the possibility of reaching the ZLB increases. Equation (16) implies that the natural rate of interest r_t^* depends on μ_t^a and Z_t^b as well as the state variables $\hat{y}_t$ and $\hat{y}_t^*$. Furthermore, in the absence of consumption habits ($h = 0$), r_t^* depends only on μ_t^a and Z_t^b and therefore we do not need to separately identify μ_t^a and Z_t^b to understand the dynamics of the output gap and the inflation rate. Equation (16) also shows that the monetary policy shock ϵ_t^r is irrelevant to r_t^*.

5.2 Developments in the Natural Rate of Interest

Figure 5 shows the time-series plot of the estimated natural rate of interest r_t^*. Although the steady-state value is $0.22(= 0.054 \times 4)\%$ at an annual rate, the natural rate seems to decline in the early 1990s and has often become negative since the mid-1990s. During this period, it has often fallen to around $-4(= -1 \times 4)\%$ at an annual rate.

The figure also shows the contribution of structural shocks, μ_t^a, Z_t^b, and ϵ_t^r in explaining the fluctuations of the natural rate. Similarly to Gust et al. (2017), we conduct the decomposition of the variable by using the prediction of the model, assuming that only one of the three shocks is present. Because of the nonlinearity, they do not sum up to the level of the natural rate of interest. The figure shows that the large fraction of the changes in the natural rate of interest can be explained by the discount factor shock Z_t^b. This result is consistent with the secular stagnation view of Summers (2013). In contrast, the contribution of the technology shock μ_t^a is relatively small, and the monetary policy shock ϵ_t^r does not contribute at all, which is consistent with equation (16).

Our result regarding the identification of fundamental shocks is in line with those of previous studies. Gust et al. (2017) find that the risk premium shock and marginal efficiency of the investment shock are more important in explaining the Great Recession in the US than the technology and monetary policy shocks. Sugo and Ueda (2008) estimate a medium-scale linear DSGE model for Japan, using the sample period before the ZLB constraint, and find that the contribution of the investment shock is the highest in accounting for aggregate fluctuations. It should be noted that shocks emphasized in these studies based on medium-scale models and the discount factor shock in our simpler model fall in the same class of the demand shock.

5.3 Real Interest Rate Gap

As equations (14) and (15) show, both the output gap and the inflation rate depend negatively on the difference between the real interest rate and the natural rate of interest, often called the real interest rate gap. Thus, although the natural rate of interest often turned negative after the mid-1990s, the Japanese economy would have escaped from the prolonged stagnation if the real interest rate had been sufficiently lower than the natural rate of interest.

Figure 6 shows the time-series paths of the estimated real interest rate $\mathbb{E}_t[R_t/\pi_{t+1}]$ and the estimated natural rate of interest r_t^*.[17] During the asset market bubble in the late 1980s, the real interest rate gap was negative, which explains the economic boom and inflation. Meanwhile, in the aftermath of the bubble in the early 1990s, the real interest rate gap is positive, which explains the recession and disinflation. However, it has been ambiguous as to whether the real interest rate gap has been positive or negative since the mid-1990s.

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[17] The volatility of the estimated natural rate of interest is much larger than that of the estimate by Iiboshi, Shintani, and Ueda (2018) because their model does not embed consumption habits. Since consumption in the flexible-price equilibrium is assumed to depend on actual y_{t-1}, which is highly volatile, both y_t^* and r_t^* become volatile, as shown in equations (16) and (17). Thus, following Kamada (2009), we display the centered three-quarter moving average of the natural rate of interest hereafter.

The bottom line is that the real interest rate did not fall sufficiently for the real side of the economy and the inflation rate to recover. Despite the effort of the BOJ by implementing the threshold-based forward guidance policy, as described in the Exit Condition Model, the ZLB prevented it from raising inflation expectations.

Evidence that supports our interpretation can be found in the aforementioned *ESP Forecast*. Each forecaster reports the distribution of the CPI inflation forecasts. For example, a respondent selects the subjective probability that the CPI inflation rate is between 0% and 0.25%. Taking the mean of each forecaster's forecast distribution for the fiscal year 2017 from the survey conducted in December 2016, we find that the probability that the inflation rate exceeds 2% is only 0.3%, while the probability that it is negative is 2%. Forecasters attach the highest probability of 30% to the range of inflation rate between 0.5% and 0.75%.

5.4 Other Measures of the Natural Rate of Interest

We compare our estimates of the natural rate of interest with those based on the Laubach–Williams (LW) model (Laubach and Williams 2003, and Holston, Laubach, and Williams 2017) as well as those based on the Hodrick–Prescott (HP) filter. The LW model is given by a system of linear equations, which include the backward-looking IS curve and Phillips curve. Because the natural rate of interest in the model is not directly observable, both unknown structural parameters and latent variables, including the natural rate of interest, are estimated by the Kalman filter. The *ex ante* real interest rate is obtained by using the one-year-ahead inflation expectation based on a univariate AR(3) model. In the LW model, the natural rate of interest r_t^* can be decomposed as

$$r_t^* = g_t + z_t, \tag{18}$$

where g_t and z_t are the trend growth rate in natural output and other determinants such as demand disturbances, respectively. Therefore, g_t and z_t in the LW model correspond to $\Delta \log A_t = \mu_t^a + \gamma_a$ and $\log Z_t^b$ in our model, respectively. However, in Laubach and Williams

(2003) and Holston, Laubach, and Williams (2017), g_t is assumed to be I(1), while z_t is assumed to be either I(0) or I(1). This assumption implies the natural rate of interest follows an I(1) process. In contrast, both $\Delta \log A_t$ and $\log Z_t^b$ in our model are I(0) so that the natural rate of interest follows an I(0) process. We report the one-sided (filtered) estimate of the natural rate of interest by fitting the LW model to the Japanese data. For the HP filter, we set the smoothing parameter λ to 1,600 and smooth the same *ex ante* real interest rate we used in the LW model.

Figure 7 shows that, on the whole, the estimated natural rate of interest from the LW model and that from our nonlinear DSGE model move closely together. While these are not shown in the figure, we confirm that most of the fluctuations in the natural rate of interest in the LW model can be explained by z_t, which is consistent with the fact that the discount factor shock Z_t^b is the dominant source of fluctuations in the natural rate of interest in our nonlinear DSGE model. The natural rate of interest from the HP filter is much smoother than the other two series, but all the series tend to move in the same direction. In particular, the natural rate of interest is positive until the mid-1990s and falls to zero or turns negative in the 2000s.

Finally, we check the robustness of our estimates of the natural rate of interest, using the output gap data. To be specific, we estimate the same DSGE model, either by using the output gap instead of the growth in real GDP or by using both the output gap and the growth in real GDP. The size of the measurement error for the output gap is set at 10% of its actual variance. Figure 8 shows that the three series of the natural rate of interest move in the same direction, while estimates based on the output gap become slightly higher in the early 1990s and in the 2000s.[18]

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[18] See Online Appendix F for the estimated parameter values.

6 Are Nonlinearities Important?

6.1 Linearized Models

Since it is simple and straightforward to solve and estimate linear DSGE models, they have been widely used in macroeconomic analysis. In contrast, it is in general costly to solve and estimate nonlinear DSGE models, and computation can take weeks, even if the model is as simple as ours. In this section, we aim to evaluate the performance of linearized models, which significantly reduce computational costs. In particular, we consider two variants of linear approximation: (i) the Linear Model, which ignores all nonlinearity; and (ii) the Piecewise Linear Model, which ignores nonlinearity except for the ZLB. We estimate the two models and compare their performance with our fully nonlinear model.

The Linear Model is expressed by equations (14) to (17) with the log-linearized monetary policy rule:

$$\frac{R_t - r^*\pi^*}{r^*\pi^*} = \rho_r \frac{R_{t-1} - r^*\pi^*}{r^*\pi^*} + (1-\rho_r)\left\{\psi_\pi \left(\frac{\pi_t - \pi^*}{\pi^*}\right) + \psi_y \left(\hat{y}_t - \hat{y}_t^*\right)\right\} + \epsilon_t^r. \tag{19}$$

The likelihood of the model can be evaluated using the Kalman filter without appealing to the particle filter. We estimate the Linear Model using Dynare, assuming away measurement errors. For this reason, the likelihood of the Linear Model is not directly comparable with that of the Exit Condition Model.

The Piecewise Linear Model uses the same log-linearized equations as in the case of the Linear Model, but incorporates the nonlinearity caused by the presence of the ZLB. To solve the model, we employ the piecewise linear solution algorithm proposed by Guerrieri and Iacoviello (2015). In the estimation, the likelihood function is evaluated using an inversion filter, as in Guerrieri and Iacoviello (2017), Atkinson, Richter, and Throckmorton (2019), and Cuba-Borda et al. (2019). The benefit of using the Piecewise Linear Model is that the solution can easily be obtained using the OccBin toolbox, and the estimation is nearly as fast as a linear model, even if the model incorporates the ZLB. One caveat is that we need to assume zero monetary policy shock for the likelihood evaluation in the ZLB regime. This assumption may not be desirable when applying the Piecewise Linear Model to Japan where ZLB spells have been much longer than the US.

6.2 Estimation Results

The last two columns of Table 2 show the parameter estimates and marginal likelihoods of the two linearized models. Comparisons with fully nonlinear estimates reveal the following three points. First, the marginal likelihood drops substantially by ignoring the nonlinearity. The three nonlinear models that take account of the ZLB, including the Nominal Rate Model, fit the data much better than do linearized models. Second, the parameter estimates differ, particularly for the standard deviations of the shocks. The size of the monetary policy shock tends to decrease when we ignore the ZLB because small monetary policy shocks help in explaining the continuation of the zero interest rate. In contrast, the size of the discount factor and the technology shocks increases. Third, compared with the findings of Guerrieri and Iacoviello (2015) and Atkinson, Richter, and Throckmorton (2019), the Piecewise Linear Model yields different estimates, not only for the standard deviations of the shocks but also for the structural parameters and monetary policy parameters. This result implies that the Piecewise Linear Model may not work well for an economy such as Japan, in which the zero interest rate has continued for a long time. It is interesting to note that the Linear Model yields similar estimates to the Exit Condition Model for the structural parameters and monetary policy parameters.

Different parameter estimates in linearized models alter the implications of monetary policy, the reasons behind Japan's stagnation, and how the natural rate of interest has evolved in Japan. We study the last point in the next subsection. It should also be noted that the Linear Model cannot account for a long duration of ZLB spells as the Exit Condition Model does.

6.3 Natural Rate of Interest under Different Monetary Policy Specifications

Having shown that analysis based on linearized models can lead to different implications, we now focus on the effect of using linearized models on the estimate of the natural rate of interest. Figure 9 shows the estimated natural rate of interest based on the linearized models, namely, the Linear Model and the Piecewise Linear Model, along with the one based on the Exit Condition Model. Although the parameter estimates differ between the Piecewise Linear Model and the Exit Condition Model, the movement of the natural rate of interest is somewhat similar. In contrast, the natural rate of interest based on the Linear Model is very different. Indeed, estimate is much more volatile, although the direction of the changes is similar.

Figure 10 shows the estimated natural rate of interest based on the Exit Condition Model, the Notional Rate Model and the nonlinear model without the ZLB.[19] The natural rate of interest based on the Notional Rate Model is similar to that based on the Exit Condition Model. The estimate based on the nonlinear model without the ZLB is also similar, particularly during the period of the ZLB since 2000.

The similarity of the natural rate of interest among these models is in sharp contrast to the result found by Hirose and Sunakawa (2017), who investigated the US data. They find that the natural rate of interest is substantially higher during the ZLB period when the ZLB is incorporated into the model. While the difference may partly be explained by the fact that our analysis is based on Japanese data, the estimation strategy also differs between two studies. Unlike our approach, they estimate a DSGE model using the subsample before the ZLB, and then use the estimated parameters in evaluating the natural rate of interest for the whole sample including the ZLB period.

To understand why there is so little variation in the estimated natural rate of interest

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[19] The nonlinear model without the ZLB is a nonlinear DSGE model without linear approximation but ignores the ZLB. See Online Appendix F for the estimated parameter values.

in our study, we separately consider three possible sources of differences between the Exit Condition Model and the nonlinear model without the ZLB: (1) the presence of the ZLB (2) estimated parameters, and (3) estimated shocks. We simulate the natural rate of interest based on the nonlinear model without the ZLB by changing one of three settings. First, Model 1 uses the same estimated parameters and shocks, but explicitly taking into account the ZLB. Second, Model 2 uses the estimated shocks in the Exit Condition Model. Third, Model 3 uses the estimated parameters in the Exit Condition Model. Here, Model 2 is analogous to the modeling strategy of Hirose and Sunakawa (2017).

Figure 11 shows the simulated paths of the counterfactual natural rate of interest. As for Model 1, we show that the presence of the ZLB does not influence the natural rate of interest per se. Because this rate rests on the flexible-price economy, by definition, the ZLB does not matter in its movements per se. For Models 2 and 3, the figure suggests that the parameter difference influences the natural rate of interest to a similar extent, but has the opposite effect to that of the shock difference. In Model 2, the natural rate of interest increases, as in Hirose and Sunakawa (2017). However, in Model 3, the natural rate of interest decreases by the same degree. This observation suggests that misleading results can be obtained if we do not simultaneously estimate the parameters and shocks.

7 Concluding Remarks

In this study, we estimated a nonlinear DSGE model with the ZLB using Japanese data, which include nearly 20 years of zero interest rate policy. Our estimated model provided fundamentals-based explanation for Japan's long period of the stagnation and the ZLB, illustrating that adverse demand shocks were the main culprit.

Our study contributes to the literature by incorporating important features, such as the ZLB and forward guidance policy, into a DSGE model. However, at the same time, there are several potential avenues for further work. The first is to estimate a richer DSGE model by, for example, embedding capital, wage stickiness, financial frictions, and so on. We are aware that the intrinsic persistence in our model is relatively low, which makes the economy return to the steady state rather within a short period. This could be one reason that we succeeded in estimating the nonlinear DSGE model with the ZLB, even though the duration of the ZLB is relatively long in Japan. However, incorporating more frictions further lengthens the duration of the ZLB, and makes it more likely that the equilibrium will be indeterminate, as argued by Aruoba, Cuba-Borda, and Schorfheide (2018).

This point leads to the second avenue for future research: estimating a regime-switching model. Specifically, we can consider three types of regime shifts. As in Aruoba, Cuba-Borda, and Schorfheide (2018), the equilibrium may fluctuate between a normal determinate equilibrium and a deflationary indeterminate equilibrium. Alternatively, the equilibrium may fluctuate between the regime of active monetary policy and passive fiscal policy and the regime of passive monetary policy and active fiscal policy. There could also be discontinuous changes in certain structural parameters, such as the steady-state growth rate of technology and the inflation target. Most importantly, the existence of a kink in the GDP growth rate has been pointed out when a large adverse shock hits the economy (e.g., around 1991 for Japan and 2008 for the US), which may call for a regime-switching model for the steady-state growth rate of technology.

Furthermore, pursuing the first and second avenues would enable us to embed QE. In our model, monetary policy influences nominal and real variables only through short-term nominal interest rates. A better model would incorporate money and financial intermediaries into a New Keynesian model, as in Gertler and Karadi (2011), Chen, Cúrdia, and Ferrero (2012), and Del Negro et al. (2017a). Hagedorn (2018) considers the implications of adding nominal bonds to standard DSGE models for inflation dynamics. Estimating such a DSGE model is a major challenge because it increases state variables (money, bonds, and the balance sheet of financial intermediaries) and requires regime switching between conventional monetary policy and the QE policy.

Finally, our method can be applied to other types of models, in which nonlinearity plays an important role. Examples include currency and financial crises, where crises occur as a tail-risk event and have significant impacts on the economy.

References

[1] Andrés, Javier, J. David López-Salido, and Edward Nelson. (2009) "Money and the Natural Rate of Interest: Structural Estimates for the United States and the Euro Area." *Journal of Economic Dynamics and Control*, 33(3), 758–776.

[2] Andrieu, Christophe, Arnaud Doucet, and Roman Holenstein. (2010) "Particle Markov Chain Monte Carlo Methods." *Journal of the Royal Statistical Society B*, 72(3), 269–342.

[3] Aoki, Kosuke and Yoichi Ueno. (2012) "Identification of Structural Shocks under the Zero Lower Bound on Nominal Interest Rates." Bank of Japan Working Paper 12-E-8.

[4] Aruoba, S. Borağan, Pablo Cuba-Borda, and Frank Schorfheide. (2018) "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries." *Review of Economic Studies*, 85, 87–118.

[5] Atkinson, Tyler, Alexander W. Richter, and Nathaniel A. Throckmorton. (2019) "The Zero Lower Bound and Estimation Accuracy." *Journal of Monetary Economics*, forthcoming.

[6] Bank of Japan. (2006) "The Introduction of a New Framework for the Conduct of Monetary Policy." March 9, 2006.

[7] Barsky, Robert, Alejandro Justiniano, and Leonardo Melosi. (2014) "The Natural Rate of Interest and Its Usefulness for Monetary Policy." *American Economic Review Papers and Proceedings*, 104(5), 37–43.

[8] Bauer, Michael D., and Glenn D. Rudebusch. (2016) "Monetary Policy Expectations at the Zero Lower Bound." *Journal of Money, Credit and Banking*, 48(7), 1439–1465.

[9] Benhabib, Jess, Stephanie Schmitt-Grohé, and Martín Uribe. (2001) "Monetary Policy and Multiple Equilibria." *American Economic Review*, 91(1), 167–186.

[10] Benigno, Gianluca, and Luca Fornaro. (2018) "Stagnation Traps." *Review of Economic Studies*, 85(3), 1425–1470.

[11] Black, Fischer. (1995) "Interest Rates as Options." *Journal of Finance*, 50(5), 1371–1376.

[12] Boneva, Lena, R. Anton Braun, and Yuichiro Waki. (2016) "Some Unpleasant Properties of Loglinearized Solutions When the Nominal Rate is Zero." *Journal of Monetary Economics*, 84, 216–232.

[13] Boneva, Lena, Richard Harrison, and Matt Waldron. (2018) "Threshold-based Forward Guidance." *Journal of Economic Dynamics and Control*, 90, 138–155.

[14] Chen, Han, Vasco Cúrdia, and Andrea Ferrero. (2012) "The Macroeconomic Effects of Large-Scale Asset Purchase Programmes." *Economic Journal*, 122(564), F289–315.

[15] Chopin, Nicolas. (2002) "A Sequential Particle Filter for Static Models." *Biometrika*, 89, 539–552.

[16] Chopin, Nicolas, P. E. Jacob, and O. Papaspiliopoulos. (2013) "SMC2: An Efficient Algorithm for Sequential Analysis of State Space Models." *Journal of the Royal Statistical Society B*, 75(3), 397–426.

[17] Cuba-Borda, Pablo, Luca Guerrieri, Matteo Iacoviello, and Molin Zhongy. (2019) "Likelihood Evaluation of Models with Occasionally Binding Constraints." draft, downloaded from http://lguerrieri.com/cuba_borda_guerrieri_iacovi.pdf.

[18] Cuba-Borda, Pablo, and Sanjay R. Singh. (2019) "Understanding Persistent Stagnation." Board of Governors of the Federal Reserve System International Finance Discussion Papers, 1243.

[19] Cúrdia, Vasco. (2015) "Why So Slow? A Gradual Return for Interest Rates." *Federal Reserve Bank of San Francisco Economic Letter*, 2015-32, 13 October.

[20] Cúrdia, Vasco, Andrea Ferrero, Ging Cee Ng, and Andrea Tambalotti. (2015) "Has U.S. Monetary Policy Tracked the Efficient Interest Rate?" *Journal of Monetary Economics*, 70, 72–83.

[21] Del Negro, Marco, Gauti Eggertsson, Andrea Ferrero, and Nobuhiro Kiyotaki. (2017a) "The Great Escape? A Quantitative Evaluation of the Fed's Liquidity Facilities." *American Economic Review*, 107(3), 824–857.

[22] Del Negro, Marco, Domenico Giannone, Marc P. Giannoni, and Andrea Tambalotti. (2017b) "Safety, Liquidity, and the Natural Rate of Interest." Brooking Papers on Economic Activity, Conference Drafts.

[23] Del Negro, Marco, Marc P. Giannoni, and Christina Patterson. (2015) "The Forward Guidance Puzzle." Federal Reserve Bank of New York Staff Reports, 574.

[24] Eggertsson, Gauti B. (2011) "What Fiscal Policy Is Effective at Zero Interest Rates?" in *NBER Macroeconomics Annual 2010*, 25, edited by D. Acemoglu and M. Woodford, University of Chicago Press: Chicago, 59–112.

[25] Eggertsson, Gauti B., Neil R. Mehrotra, and Jacob A. Robbins. (2019) "A Model of Secular Stagnation: Theory and Quantitative Evaluation." *American Economic Journal: Macroeconomics*, 11(1), 1–48.

[26] Erceg, Christopher J., Luca Guerrieri, and Christopher Gust. (2006) "SIGMA: A New Open Economy Model for Policy Analysis." *International Journal of Central Banking*, 2, 1–50.

[27] Fernández-Villaverde, Jesús, Grey Gordon, Pablo A. Guerron-Quintana, and Juan Rubio-Ramírez. (2015) "Nonlinear Adventures at the Zero Lower Bound." *Journal of Economic Dynamics and Control*, 57, 182–204.

[28] Fernández-Villaverde, Jesús, Juan F. Rubio-Ramírez, and Frank Schorfheide. (2016) "Solution and Estimation Methods for DSGE Models." in *Handbook of Macroeconomics*, Volume 2, edited by John B. Taylor and Harald Uhlig, Chapter 9, 527–724.

[29] Fernández-Villaverde, Jesús, and Juan Rubio-Ramírez. (2005) "Estimating Dynamic Equilibrium Economies: Linear Versus Nonlinear Likelihood." *Journal of Applied Econometrics*, 20(7), 891–910.

[30] Fujiwara, Ippei, Yasuo Hirose, and Mototsugu Shintani. (2011) "Can News be a Major Source of Aggregate Fluctuations? A Bayesian DSGE Approach." *Journal of Money, Credit and Banking*, 43(1), 1–29.

[31] Fujiwara, Shigeaki, Yuto Iwasaki, Ichiro Muto, Kenji Nishizaki, and Nao Sudo. (2016) "Supplementary Paper Series for the "Comprehensive Assessment" (2): Developments in the Natural Rate of Interest in Japan." Bank of Japan Review, 2016-E-12.

[32] Gavin, William T., Benjamin D. Keen, Alexander W. Richter, and Nathaniel A. Throckmorton. (2015) "The Zero Lower Bound, the Dual Mandate, and Unconventional Dynamics." *Journal of Economic Dynamics and Control*, 55, 14–38.

[33] Gertler, Mark, and Peter Karadi. (2011) "A Model of Unconventional Monetary Policy." *Journal of Monetary Economics*, 58(1), 17–34.

[34] Guerrieri, Luca, and Matteo Iacoviello. (2015) "OccBin: A Toolkit for Solving Dynamic Models with Occasionally Binding Constraints Easily." *Journal of Monetary Economics*, 70, 22–38.

[35] Guerrieri, Luca, and Matteo Iacoviello. (2017) "Collateral Constraints and Macroeconomic Asymmetries." *Journal of Monetary Economics*, 90, 28–49.

[36] Gust, Christopher, Edward Herbst, David López-Salido, and Matthew E. Smith. (2017) "The Empirical Implications of the Interest-Rate Lower Bound." *American Economic Review*, 107(7), 1971–2006.

[37] Hagedorn, Marcus. (2018) "Prices and Inflation When Government Bonds are Net Wealth." CEPR Discussion Paper DP12769.

[38] Hall, Robert E. (2011) "The Long Slump." *American Economic Review*, 101, 431–469.

[39] Hayashi, Fumio, and Junko Koeda. (2019) "Exiting from Quantitative Easing." *Quantitative Economics*, 10(3), 1069–1107.

[40] Herbst, Edward, and Frank Schorfheide. (2014) "Sequential Monte Carlo Sampling for DSGE Models." *Journal of Applied Econometrics*, 29(7), 1073–1098.

[41] Herbst, Edward, and Frank Schorfheide. (2015) *Bayesian Estimation of DSGE Models.* Princeton University Press: Princeton, NJ.

[42] Hills, Timothy, Taisuke Nakata, and Sebastian Schmidt. (2016) "The Risky Steady State and the Interest Rate Lower Bound." Federal Reserve Board, Finance and Economics Discussion Series, 2016-009.

[43] Hirakata, Naohisa, Nao Sudo, Ikuo Takei, and Kozo Ueda. (2016) "Japan's Financial Crises and Lost Decades." *Japan and the World Economy*, 40, 31–46.

[44] Hirose, Yasuo. (2008) “Equilibrium Indeterminacy and Asset Price Fluctuation in Japan: A Bayesian Investigation.” *Journal of Money, Credit, and Banking*, 40(5), 967–999.

[45] Hirose, Yasuo, and Atsushi Inoue. (2016) “The Zero Lower Bound and Parameter Bias in an Estimated DSGE Model.” *Journal of Applied Econometrics*, 31, 630–651.

[46] Hirose, Yasuo, and Takeki Sunakawa. (2017) “The Natural Rate of Interest in a Nonlinear DSGE Model.” CAMA Working Paper 38/2017.

[47] Holston, Kathryn, Thomas Laubach, and John C. Williams. (2017) “Measuring the Natural Rate of Interest: International Trends and Determinants.” *Journal of International Economics*, 108, S59–75.

[48] Ichiue, Hibiki, and Yoichi Ueno. (2006) “Monetary Policy and the Yield Curve at Zero Interest: The Macro-finance Model of Interest Rates as Options.” Bank of Japan Working Paper Series 06-E-16.

[49] Iiboshi, Hirokuni, Mototsugu Shintani, and Kozo Ueda. (2018) “Estimating a Nonlinear New Keynesian Model with the Zero Lower Bound for Japan.” CAMA Working Paper 37/2018.

[50] Ikeda, Daisuke, and Masashi Saito. (2014) “The Effects of Demographic Changes on the Real Interest Rate in Japan.” *Japan and the World Economy*, 32, 37–48.

[51] Ito, Takatoshi, and Frederic S. Mishkin. (2004) “Two Decades of Japanese Monetary Policy and the Deflation Problem.” NBER Working Paper 10878.

[52] Kamada, Koichiro. (2009) “Equilibrium Real Interest Rate for Japan.” in *Macroeconomy and Industry Structure,* edited by K. Fukao. Keio University Press: Tokyo, 387–427 (in Japanese).

[53] Katagiri, Mitsuru, Hideki Konishi, and Kozo Ueda. (2020) "Aging and Deflation from a Fiscal Perspective." *Journal of Monetary Economics*, 111, 1–15.

[54] Kim, Don H. and Kenneth J. Singleton. (2012) "Term Structure Models and the Zero Bound: an Empirical Investigation of Japanese Yields." *Journal of Econometrics*, 170, 32–49.

[55] Kim, Jinill, and Seth Pruitt. (2017) "Estimating Monetary Policy Rules When Nominal Interest Rates Are Stuck at Zero." *Journal of Money, Credit and Banking*, 49(4), 585–602.

[56] Kitagawa, Genshiro. (1996) "Monte Carlo Filter and Smoother for Non-Gaussian Nonlinear State Space Models." *Journal of Computational and Graphical Statistics*, 5(1), 1–25.

[57] Koop, Gary, M. Hashem Pesaran, and Simon M. Potter. (1996) "Impulse Response Analysis in Nonlinear Multivariate Models." *Journal of Econometrics*, 74(6), 119–147.

[58] Krippner, Leo. (2013) "Measuring the Stance of Monetary Policy in Zero Lower Bound Environments." *Economics Letters*, 118(1), 135–138.

[59] Krugman, Paul R. (1998) "It's Baaack: Japan's Slump and the Return of the Liquidity Trap." *Brookings Papers on Economic Activity*, 2, 137–187.

[60] Kulish, Mariano, James Morley, and Tim Robinson. (2017) "Estimating DSGE Models with Zero Interest Rate Policy." *Journal of Monetary Economics*, 88, 35–49.

[61] Laubach, Thomas, and John C. Williams. (2003) "Measuring the Natural Rate of Interest." *Review of Economics and Statistics*, 85(4), 1063–1070.

[62] McKay, Alisdair, Emi Nakamura, and Jon Steinsson. (2016) "The Power of Forward Guidance Revisited." *American Economic Review*, 106(10), 3133–3158.

[63] Nakata, Taisuke. (2017) "Uncertainty at the Zero Lower Bound." *American Economic Journal: Macroeconomics*, 9(3), 186–221.

[64] Neiss, Katharine S. and Edward Nelson. (2003) "The Real-Interest-Rate Gap as an Inflation Indicator." *Macroeconomic Dynamics*, 7(2), 239–262.

[65] Plante, Michael, Alexander W. Richter, and Nathaniel A. Throckmorton. (2018) "The Zero Lower Bound and Endogenous Uncertainty." *Economic Journal*, 128, 1730–1757.

[66] Reifschneider, David, and John C. Williams. (2000) "Three Lessons for Monetary Policy in a Low-Inflation Era." *Journal of Money, Credit, and Banking*, 32(4), 936–966.

[67] Richter, Alexander W., Nathaniel A. Throckmorton, and Todd B. Walker. (2014) "Accuracy, Speed and Robustness of Policy Function Iteration." *Computational Economics*, 44, 445–476.

[68] Sims, Christopher. (2002) "Solving Linear Rational Expectations Models." *Computational Economics*, 20, 1–20.

[69] Smets, Frank, and Raf Wouters. (2007) "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach." *American Economic Review*, 90, 586–606.

[70] Sugo, Tomohiro, and Kozo Ueda. (2008) "Estimating a Dynamic Stochastic General Equilibrium Model for Japan." *Journal of the Japanese and International Economies*, 22, 476–502.

[71] Summers, Lawrence. (2013),"Why Stagnation Might Prove to be the New Normal." The Financial Times (December 16).

[72] Ueda, Kazuo. (2000) "The Transmission Mechanism of Monetary Policy Near Zero Interest Rates: The Japanese Experience 1998–2000." Speech at Swedish Embassy in Tokyo on September 22, 2000.

[73] Ueno, Yoichi. (2017) “Term Structure Models with Negative Interest Rates.” Bank of Japan IMES Discussion Paper Series 2017-E-1.

[74] Ugai, Hiroshi. (2006) “Effects of the Quantitative Easing Policy: A Survey of Empirical Analyses.” Bank of Japan Working Paper Series, 06-E-10.

[75] Wicksell, Knut. (1936) *Interest and Prices*. Translation of 1898 edition by R. F. Kahn. Macmillan: London.

[76] Woodford, Michael. (2003) *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press: Princeton, NJ.

[77] Wu, Jing Cynthia, and Fan Dora Xia. (2016) “Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound.” *Journal of Money, Credit and Banking*, 48(2–3), 253–291.

Table 1: Prior Distribution

Parameter		Mean	S.D.	Shape
σ	inverse of intertemporal elasticity of substitution	1.5	0.3	Normal
h	consumption habit	0.5	0.2	Beta
γ_a	mean growth rate of technology	0	0.5	Normal
ω	inverse of labor supply elasticity	3	0.5	Normal
κ	slope of the Phillips curve	0.05	0.006	Normal
π^*	target inflation	0.25	0.1	Normal
$\bar{\pi} - \pi^*$	threshold inflation minus target inflation (Exit Condition Model only)	0.025	0.025	Gamma
ρ_r	interest rate smoothing (R)	0.5	0.2	Beta
ψ_π	sensitivity to inflation (R)	1.8	0.3	Normal
ψ_y	sensitivity to output gap (R)	0.125	0.025	Normal
ρ_a	technology shock	0.5	0.2	Beta
ρ_b	discount factor shock	0.5	0.2	Beta
σ_a	technology shock	$\sqrt{0.02}$	5 (d.f.)	Inv Gamma
σ_b	discount factor shock	$\sqrt{0.02}$	5 (d.f.)	Inv Gamma
σ_r	monetary policy shock	$\sqrt{0.02}$	5 (d.f.)	Inv Gamma

Table 2: Posterior Distribution and Marginal Likelihood

Method	Exit Condition Model SMC^2		Notional Rate Model SMC^2		Nominal Rate Model SMC^2		Linear Model Kalman filter		Piecewise Liner Model Inversion filter	
Parameter	Mean	[05, 95]	Mean	[05, 95]	Mean	[05, 95]	Mean	[05, 95]	Mean	[05, 95]
σ	1.548	[1.507, 1.621]	1.629	[1.504, 1.825]	1.450	[1.238, 1.767]	1.542	[1.042, 1.937]	1.091	[1.090, 1.093]
h	0.641	[0.602, 0.665]	0.618	[0.601, 0.635]	0.496	[0.402, 0.648]	0.502	[0.271, 0.776]	0.252	[0.251, 0.255]
γ_a	-0.046	[-0.055, -0.030]	0.027	[0.015, 0.039]	0.019	[-0.016, 0.044]	-0.002	[-0.084, 0.081]	0.165	[0.132, 0.202]
ω	3.922	[3.833, 4.086]	3.481	[3.296, 3.615]	3.969	[3.320, 4.257]	3.505	[2.765, 4.235]	4.938	[4.857, 4.995]
κ	0.051	[0.050, 0.053]	0.048	[0.044, 0.051]	0.052	[0.049, 0.054]	0.050	[0.040, 0.059]	0.070	[0.070, 0.070]
r^*	0.054	[0.039, 0.080]	0.168	[0.149, 0.184]	0.150	[0.100, 0.188]	0.123	[0.038, 0.281]	0.306	[0.269, 0.346]
π^*	0.325	[0.292, 0.348]	0.274	[0.260, 0.282]	0.236	[0.206, 0.311]	0.249	[0.166, 0.329]	0.460	[0.431, 0.488]
$\bar{\pi}$	0.339	[0.300, 0.366]	–		–		–		–	
ρ_r	0.394	[0.268, 0.483]	0.360	[0.335, 0.410]	0.430	[0.180, 0.566]	0.496	[0.156, 0.825]	0.847	[0.841, 0.856]
ψ_π	2.070	[1.872, 2.251]	1.998	[1.922, 2.170]	1.695	[1.591, 1.851]	1.995	[1.511, 2.475]	1.537	[1.528, 1.546]
ψ_y	0.137	[0.132, 0.142]	0.112	[0.101, 0.118]	0.139	[0.115, 0.151]	0.127	[0.086, 0.166]	0.058	[0.056, 0.061]
ρ_a	0.437	[0.403, 0.474]	0.256	[0.186, 0.291]	0.355	[0.223, 0.433]	0.499	[0.167, 0.823]	0.256	[0.252, 0.264]
ρ_b	0.163	[0.094, 0.278]	0.102	[0.045, 0.153]	0.348	[0.112, 0.512]	0.512	[0.183, 0.841]	0.89	[0.886, 0.894]
σ_a	1.567	[1.490, 1.674]	1.872	[1.673, 1.968]	1.682	[1.321, 2.447]	2.85	[2.58, 3.15]	1.497	[0.797, 2.523]
σ_b	4.407	[4.047, 4.685]	3.242	[2.929, 3.439]	4.427	[3.551, 6.367]	9.15	[8.22, 10.07]	5.238	[4.750, 5.717]
σ_r	1.849	[1.773, 1.966]	1.772	[1.647, 1.871]	1.981	[1.192, 3.218]	0.35	[0.31, 0.38]	0.810	[0.554, 1.094]
Marginal Likelihood	-204.72		-269.48		-295.14		-369.91		-555.19	

Note: See Table 1 for the descriptions of the parameters, and r^* represents a steady-state natural rate of interest defined as $e^{\sigma\gamma_a}/\beta - 1$.

Figure 1: Data

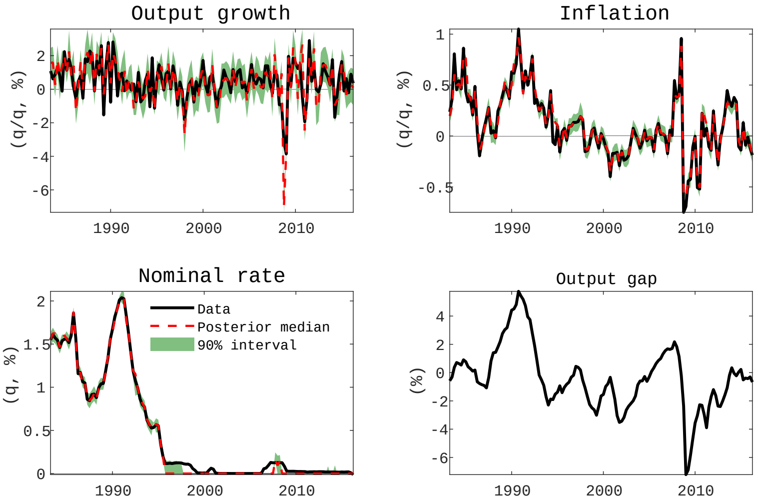


Figure 2: Comparison of the Notional Interest Rate and the Shadow Rate

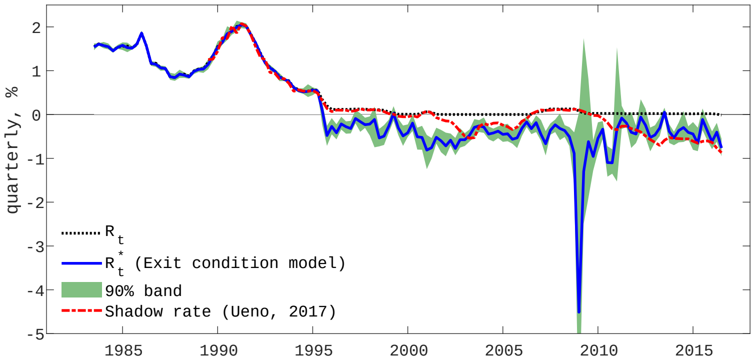
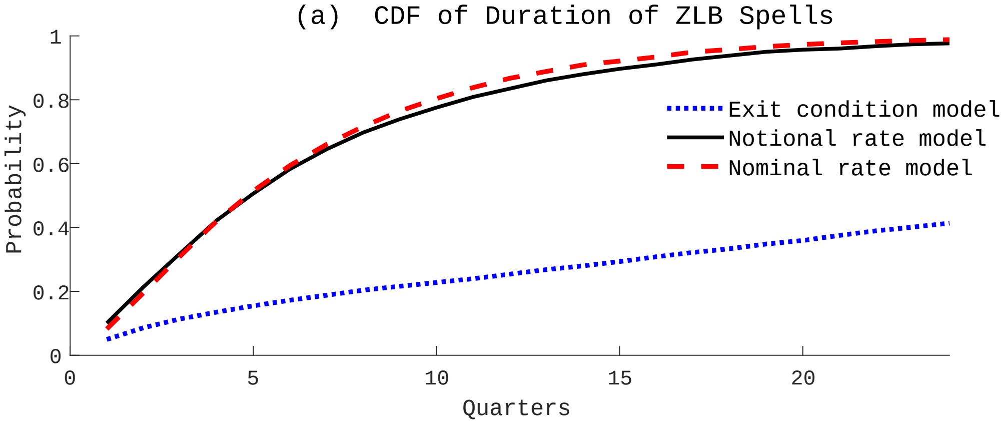
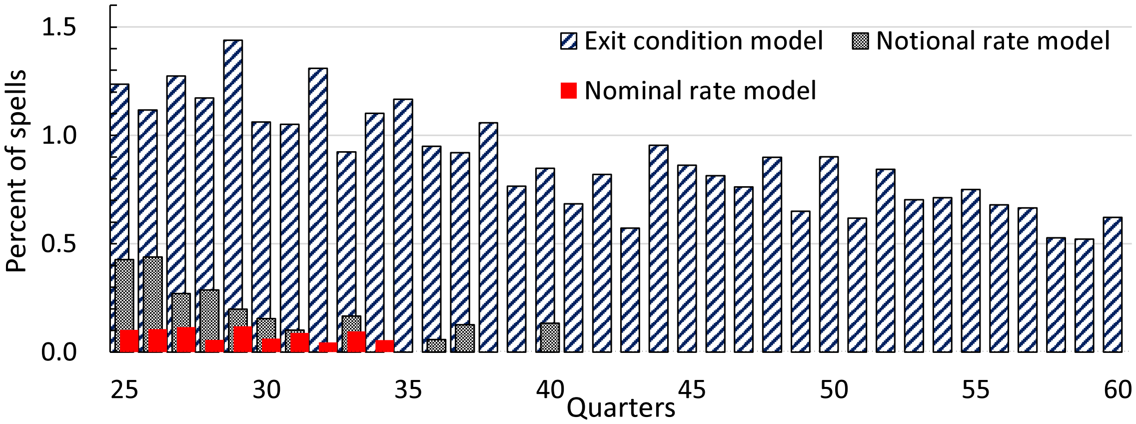


Figure 3: Duration of Being at the Zero Lower Bound

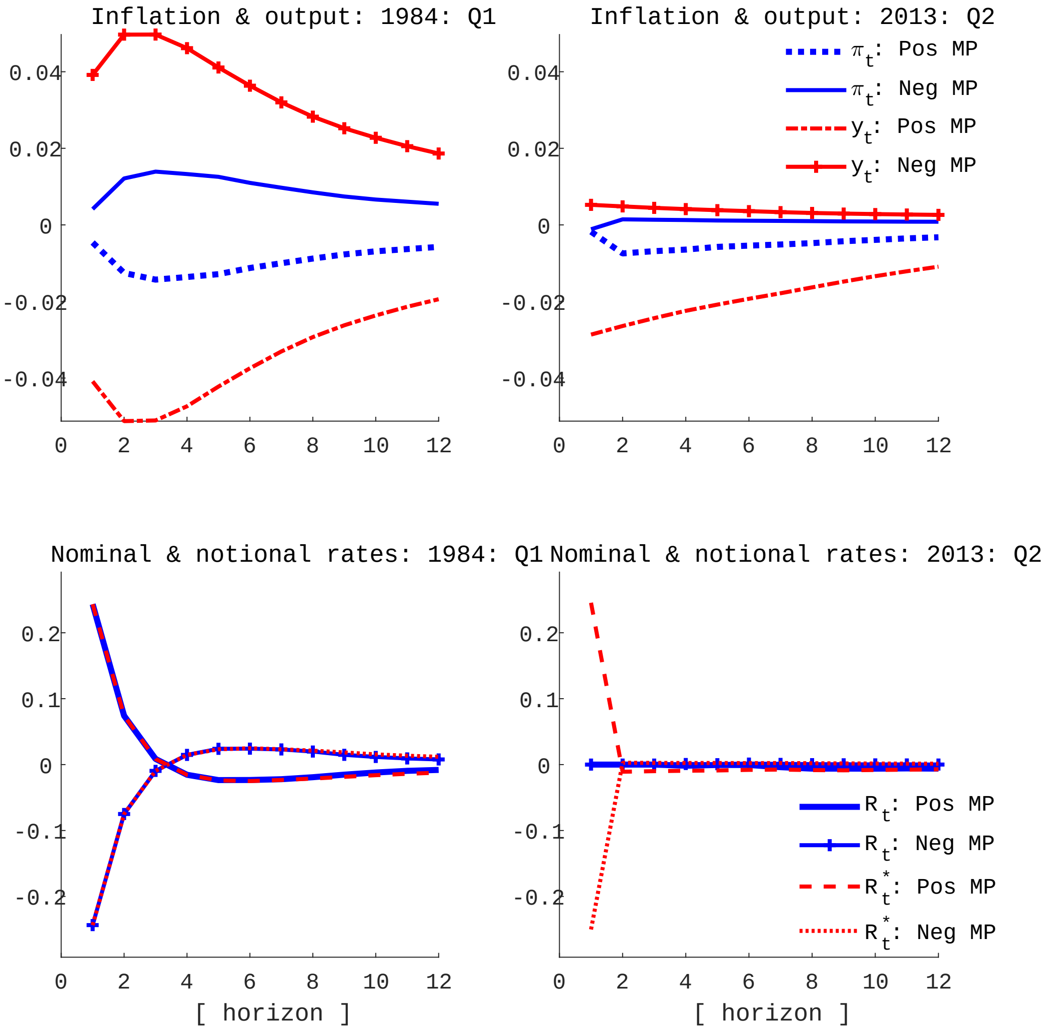


(a) CDF of Duration of ZLB Spells



(b) Right Tail of Histogram of Duration of ZLB Spells

Figure 4: Impulse Responses to a Monetary Policy Shock



Note: "Pos MP" and "Neg MP" represent positive (tightening) and negative (easing) monetary policy shocks, respectively.

Figure 5: Natural Rate of Interest and the Contribution of the Estimated Shocks

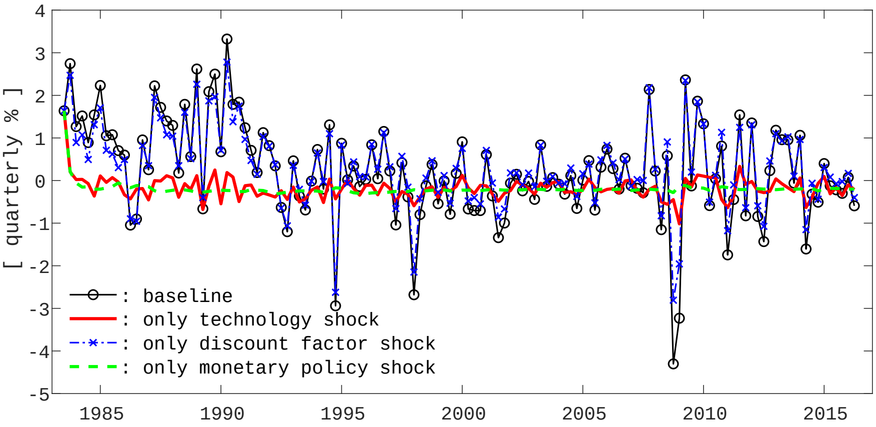


Figure 6: Comparison of the Real Interest Rate and the Natural Rate of Interest

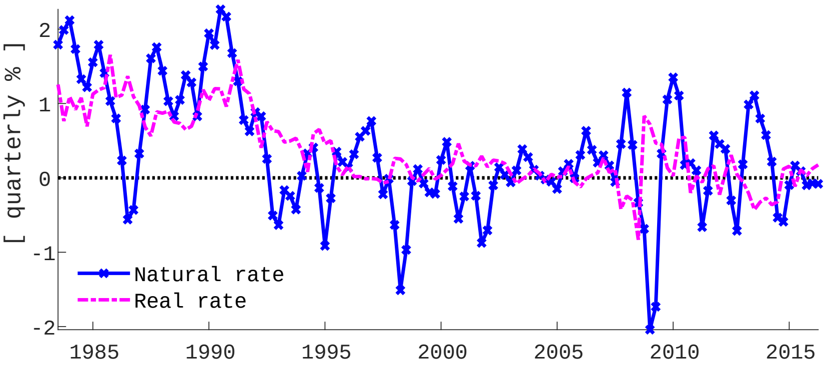


Figure 7: Natural Rate of Interest: Model Comparison

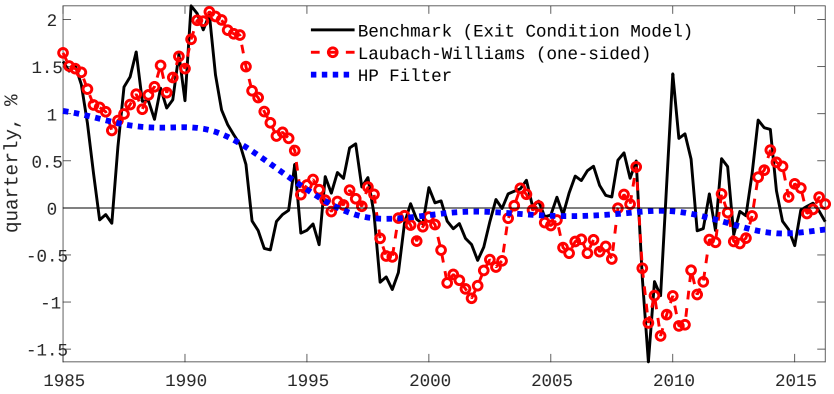


Figure 8: Natural Rate of Interest When Using the Output Gap Data

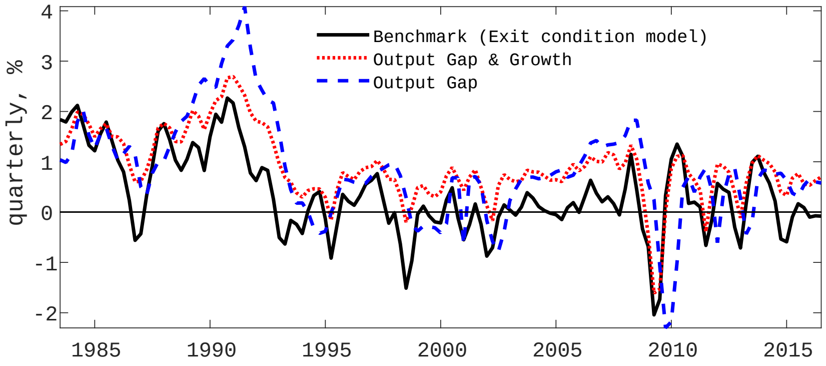


Figure 9: Natural Rate of Interest: Model Comparison (2)

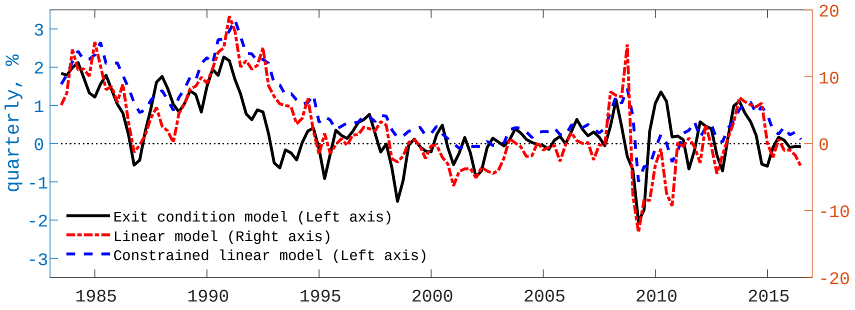


Figure 10: Natural Rate of Interest: Model Comparison (3)

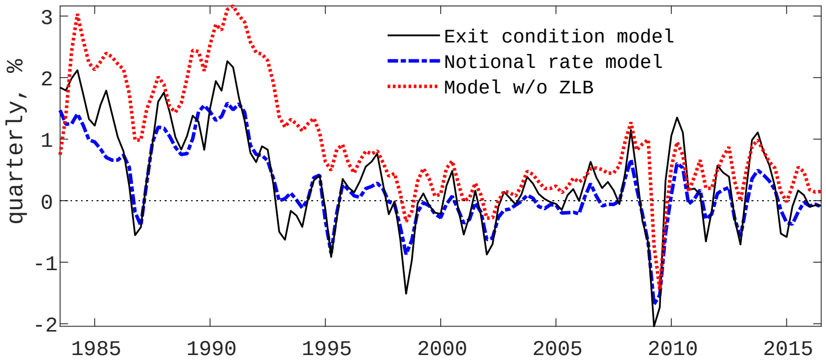
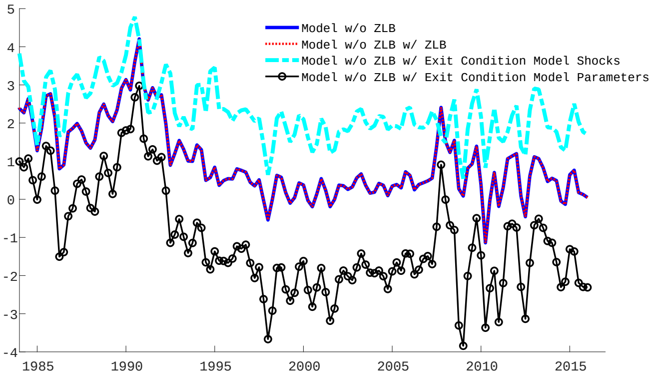


Figure 11: Natural Rate of Interest: Counterfactual Simulation



Note: “Model w/o ZLB” represents the natural rate of interest based on the nonlinear model without the ZLB. (1) “Model w/o ZLB w/ ZLB,” (2) “Model w/o ZLB w/ ... Shocks,” and (3) “Model w/o ZLB w/ ... Parameters” represent the simulated natural rate of interest using (1) estimated parameters and shocks in the nonlinear model without the ZLB, but now explicitly taking into account the ZLB, (2) estimated shocks in the Exit Condition Model, and (3) estimated parameters in the Exit Condition Model, respectively.