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Abstract

This study examines the optimal tax structure in an endogenous fertility model with non-cooperative couples. In the model, both child quality and quantity are suboptimal due to the non-cooperative behavior of couples. Moreover, we consider the external effects of children on society and center-based childcare services. In such a unified model, we characterize the formulas for optimal income tax rates, child tax/subsidy rates, and tax/subsidy rates on center-based childcare services. We find that income taxation, not child subsidy, corrects the suboptimal low fertility level caused by the non-cooperative behavior of couples, and that a child tax can be optimal to alleviate the deadweight loss from income taxation. We also identify the condition under which a child subsidy is needed. The subsidy for external childcare services corrects the external effects of children on society, not the non-cooperative behavior of couples.

Keywords: Non-Cooperative Couple, Child Quality, Child Quantity, Optimal Income Tax, Optimal Child Tax/Subsidy

Classification: H21, J13, J16

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1 Introduction

Most member countries of the Organisation for Economic Co-operation and Development (OECD) are facing a sharp decrease in fertility rates: on average, the total fertility rate (TFR) was on a declining trend until around the year 2000 and has remained at that particular low level ever since, as shown in Figure 1.[1] Since this demographic trend may have a substantial negative impact on economic growth, OECD governments have designed various pro-natalist policies, such as direct child subsidy, subsidy for center-based childcare services, income tax deduction, childbearing leave program, and enhancement of childcare facilities, to encourage families to raise children (e.g., Eydal and Rostgaard, 2018).

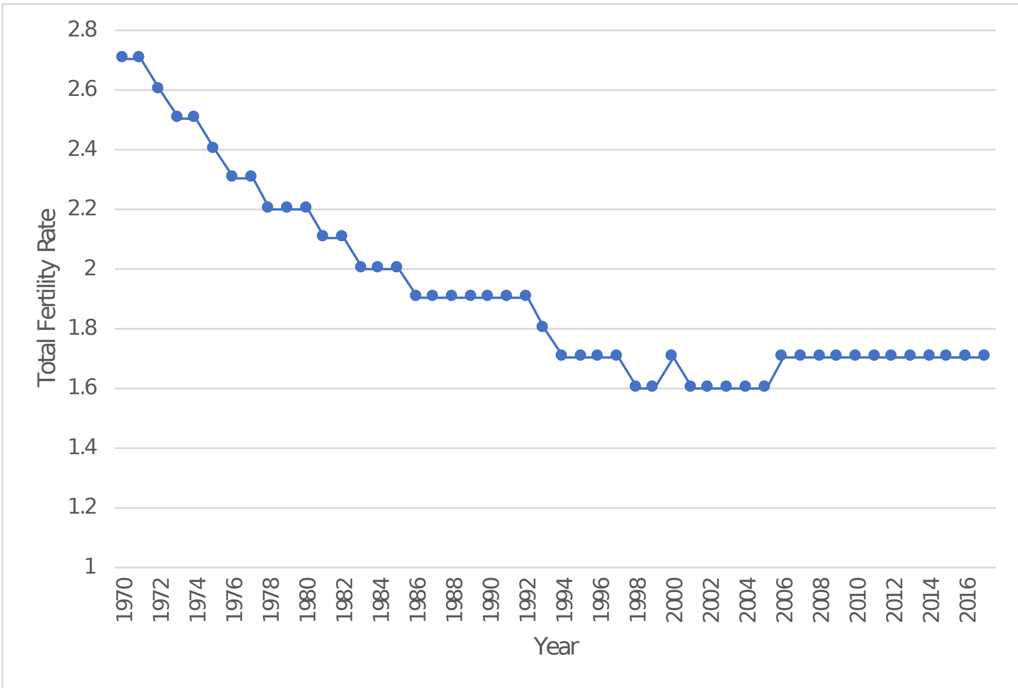

Figure 1: OECD average total fertility rates from 1970 to 2017

If the demographic trend stems from the sub-optimality of a family's fertility choice, it is crucial to clarify the mechanism of inefficient fertility choices in order to implement effective family policies. This study elucidates two driving forces underlying the downward pressure on a household's fertility choice: the external effects of children on society, and the non-cooperative behavior of couples for the provision of childcare. The study shows which family policies are the most appropriate for correcting fertility decisions that yield downward pressure on the fertility rate.

The externality of children on society has been treated as a major driving force underlying the inefficiently low fertility in the modern economy. The existing literature mainly considered the external

[1]TFR refers to the total number of children who would be born to a woman if she were to live till the end of her childbearing years and give birth to children according to the current age-specific fertility rates. The TFR data used in Figure 1 are taken from OECD Data (https://data.oecd.org/pop/fertility-rates.htm).

effect of children on society to be a positive fiscal externality generated under pay-as-you-go (PAYG) pension systems (e.g., Cigno, 1992; Sinn, 2001).[2] Existing literature on this subject concludes that child subsidies have a significant capacity to achieve the first-best investment in fertility. However, the fertility rate has remained at a much lower level and has not yet recovered, even in countries that provide high child subsidies, such as Germany, Spain, and Japan.[3] It seems there is a very limited effect of child subsidies on fertility choice since the demographic transition has not reversed over time.

This study focuses on the non-cooperative behavior of spouses as another driving force behind suboptimal low fertility.[4] In our model, parents care about child quality and quantity, which are treated as household public goods.[5] We consider a sequential decision setting where a couple collectively decides child quantity and, in the next stage, each spouse non-cooperatively provides his/her childcare time to enhance child quality.[6] Collective decision making regarding child quantity, which maximizes the weighted sum of spouses' utilities, is a cooperative agreement between spouses; hence, it supposes Pareto efficiency. In contrast, the strategic interaction between a husband and wife in providing childcare leads to underinvestment in childcare due to the free-rider problem. Under such a sequential decision setting, we find that even though fertility choice is a collective decision, suboptimal low fertility arises due to non-cooperative behavior regarding the amount of childcare provided by the spouses. This is consistent with the empirical result of Doepke and Kindermann (2019), who conclude that non-cooperation between spouses leads to a low fertility rate. To the best of our knowledge, this study is the first to theoretically derive that non-cooperative behavior of spouses toward childcare leads to inefficiently low fertility in the model, distinguishing between child quality and quantity. Based on this finding, we propose a novel channel through which the government can improve the low fertility rate by employing an appropriate choice of family policies.

The strategic interaction between spouses is supported by recent econometric evidence from Del Boca and Flinn (2012), who show that one-fourth of couples under-provide household public goods because of non-cooperative behavior. Regarding childcare decisions, Rasul (2008) empirically proves that spouses cannot commit to household chores because a couple cannot reach legally enforceable agreements about their investments in children because of non-observability by third parties; moreover, Pailhé and Solaz (2008) show that childcare provision is not always observable, which is attributable to a lack of effective monitoring between partners. Thus, the commitment of previously determined time investment in childcare is not credible, and each partner's child-caring decisions are unobservable to the

[2]Cigno (1992) states that one of the motives for having children is to secure the risk of old-age consumption. Since PAYG pension systems secure this purpose, public insurance induces people to have fewer children; thus, fertility rates decline. Sinn (2001) estimates that an additional child in Germany brings a net benefit of approximately 90,000 euros to the pension system.

[3]The fertility rates in Germany, Spain, and Japan remain below the replacement level.

[4]Our model is applicable to both married couples and couples under common-law marriage if there is a household public good (here, children).

[5]Consistent with de la Croix and Doepke (2003) and Gobbi (2018), we consider both child quality and quantity as household public goods.

[6]The justification of this setting is that even though fertility and time devoted to childcare are collectively and simultaneously determined, it is possible to change the amount of time devoted to childcare from the current plan, indicating a lack of commitment (Rasul, 2008). Moreover, there is no way of monitoring a certain amount of childcare duties performed by another partner (Pailhé and Solaz, 2008). Therefore, we assume such a sequential decision setting.

other. Allowing for this fact, the present study theoretically considers that households do not commit to decisions regarding the time supplied toward childcare and non-cooperatively determine the time.[7] In our model, childcare time includes the time parents devote to improving the quality of their children's non-cognitive and cognitive skills, as well as the time spent raising their children. Therefore, childcare time includes the qualitative aspect of educational investment in children (Del Boca et al., 2014).[8]

The previous literature considers the externality of the number of children as a driving force behind the inefficiently low fertility. In our model, child quality as well as child quantity have external effects on society. Heckman (2006) and Heckman and Masterov (2007) empirically demonstrate that an increase in child quality improves health conditions in the local area, promotes social skills, and reduces both crime rate and high school dropout rate.

In addition, we introduce external childcare services offered by centers, which can be substituted for spousal childcare time. Examples of such services are external early childhood education facilities, preschools, and cram schools. In this extended model, we compare the effectiveness of the subsidy for center-based childcare services to that of direct child subsidies.

We allow the government to employ commodity tax, linear income tax, (direct) tax/subsidy (tax/subsidy on/for child quantity), and tax/subsidy on/for center-based childcare services in order to correct the suboptimal low fertility levels. We note that many countries face the issue of securing tax revenue due to cumulative budget deficits and increasing social security expenditure. Further, a revenue source of subsidies for childcare should be collected in the absence of a lump-sum tax in a real-world tax system. Therefore, this study adopts the revenue-constrained optimal tax framework, originally contributed by Ramsey (1927) and extended by Diamond and Mirrlees (1971a, b) and Mirrlees (1971). This study allows policymakers to employ the differential income tax rates for a couple (husband and wife), that is, the so-called gender-based taxation system.[9] The allowance of gender-based taxation makes the model richer and gives us interesting numerical results and suggestions.[10] We also theoretically analyze the case of a common income tax rate on the spouses with different wages, which is a more realistic tax system, and show that our main results remain unchanged whether the government employs gender-based income taxation or not.[11] This study focuses on the case where the government implements linear tax instruments since we are concerned with the structure of tax burden on children, labor supply, and childcare facility to correct the suboptimal low fertility level caused by the non-cooperative behavior of couples (efficiency consideration) rather than the redistributive tax

[7]Browning et al. (2014) state that spousal behavior must be observable to each other to achieve a Pareto-efficient allocation.

[8]Del Boca et al. (2014) indicate that the time inputs of both parents are extremely important in the cognitive development process, particularly for young children.

[9]See the last paragraph of Section 2 for several prior studies examining gender-based taxation.

[10]Gender-based taxation has received some negative reactions (e.g., political infeasibility). However, in a Vox column (https://voxeu.org/article/gender-based-taxation-response-critics#fn1), Alesina, Ichino, and Karabarbounis mention that gender-based taxation has been intensely discussed in many European countries such as Spain, Italy, Germany, Austria, France, and Denmark. In particular, the opposition party in Spain has proposed gender-based taxation in its campaign platform.

[11]Several countries tend toward positive assortative mating; that is, the difference in wages between spouses declines (e.g., Eika et al., 2019). If the wage rates of the two spouses are equal, the gender-based taxation requires the common income tax rate on the two spouses.

structure (equity consideration). Thus, a theoretical analysis of non-linear policies is beyond the scope of this study.

We demonstrate that an increase in labor income tax enhances fertility, as a result of comparative statics of couples' behavior due to a change in tax. This theoretical result is consistent with the empirical result of Baughman and Dickert-Conlin (2009), showing that a reduction in income taxes decreases the fertility rate. Under the optimal tax framework, income taxes, not the child tax/subsidy, play a vital role in improving the low level of fertility caused by the non-cooperative behavior of spouses. In other words, income taxes have a double dividend in that they increase tax revenue and correct the low fertility level caused by non-cooperative behavior. Child tax/subsidy mitigates the deadweight loss induced by income tax and corrects the external effects of children on society, in addition to allowing for own price-induced deadweight loss. Specifically, under the availability of lump-sum taxes, absence of externality of children on society, and identical bargaining power across spouses, the optimal intervention for children is to ambiguously impose a tax to alleviate the distortion on labor supply induced by income taxation for correcting the non-cooperative behavior. Even if lump-sum taxes are unavailable, the child tax is likely to be optimal as the required tax revenue becomes larger or the degree of external effects of children on society reduces. Based on this result, it appears that child subsidy tends to become optimal as the required tax revenue is reduced or the degree of external effects increases. However, in our model, it is analytically unclear whether this relationship is valid. This question is clarified through our numerical analysis. As other important results, the subsidy for center-based childcare services becomes optimal if there is an externality of children on society and the difference in the bargaining power of spouses is not significant. The role of the subsidy is to correct the externality of children on society, not to improve the low fertility associated with spousal non-cooperative behavior.

Our numerical analysis offers useful policy suggestions by investigating the impact of changes in several parameters on optimal tax rates. We observe that the optimal intervention on children tends to provide a subsidy as the revenue requirement reduces or the degree of the external effects increases. Notably, we also investigate the ranking of the direct child subsidy and subsidy for center-based childcare services. The result shows that the subsidy rate for center-based childcare services is more likely to be higher (lower) than the direct child subsidy rate as the required tax revenue increases (decreases). We discuss the intuition underlying this result. We also numerically prove that the introduction of childcare facilities always improves welfare, increases child quantity, and raises child quality under the optimal tax framework. In addition, we examine how a difference in spousal wage rates and that in the bargaining power between spouses affect the income tax rate of both husband and wife, which corresponds to the analysis of gender-based taxation.

Based on the theoretical and numerical results, we suggest the following policy implications for family policies to improve the suboptimal low fertility rate under a revenue constraint. First, we show that income taxation is effective in improving the fertility rate rather than direct child subsidy when the non-cooperative behavior of couples is the key factor underlying the low fertility rate. This policy suggestion is supported by empirical evidence; for example, Jones and Tertilt (2008) and Jones et al. (2010) show that fertility is negatively related to the wage rate in most countries at most times. Consequently, if the low fertility rates in the OECD countries that adopt direct child subsidies arise due to the non-cooperative behavior of households, we recommend an upward shift in the income tax rate and a downward shift in direct child subsidies as a policy reform, which may lead to a direct child tax. This conclusion is novel relative to the findings of prior studies emphasizing that Pigouvian (or corrective) child subsidies are desirable. Second, although child subsidy is generally not a useful method for enhancing the low fertility rate caused by households' non-cooperative behavior, it is required if the degree of the externality of children on society is crucial. When policymakers aim to improve the TFR by correcting the under-provision of child quality that arises from the non-cooperative behavior and external effect of children on society, they may employ a combination of income taxation and child subsidy. Third, the theoretical and numerical results show that a childcare facility enhances the fertility rate, and child subsidy is not an effective device for improving the low fertility rate caused by the non-cooperative behavior of couples. These findings explain why pro-natalist policies implemented by France, Belgium, and Norway are successful, while those executed by Germany seem to be ineffective: fertility rate has improved in countries that provide more public childcare, such as France, Belgium, and Norway, while it continues to remain low in countries with high subsidies for childbearing, such as Germany (Doepke and Kindermann, 2019). The government has an option to introduce childcare facilities rather than direct child subsidies. Fourth, childcare policies under the fulfillment of the provision of childcare facilities depend on the government's required tax revenue. Given that the subsidy for center-based childcare services is more likely to be higher (lower) than direct child subsidy as the required tax revenue becomes larger (smaller), we suggest that a subsidy for center-based childcare services is desirable for countries that can collect large tax revenues (e.g., developed countries), while the direct child subsidy is suitable for countries that cannot collect large tax revenues (e.g., developing counties).

The remainder of the paper is structured as follows. The next section discusses related literature. Section 3 describes our model, and Section 4 provides solutions for our model. The approach of optimal taxation is analyzed in Section 5, and a childcare facility is introduced as an extension to the model in Section 6. A numerical analysis is undertaken in Section 7. Section 8 concludes the study.

2 Related Literature

This study constructs a model based on the non-cooperative behavior of couples who underinvest in child quality, leading to a suboptimal low child quantity, and examines the optimal tax structure that plays a corrective role as an efficiency-enhancing device under a revenue constraint. In this respect, this study is mainly related to three strands of research. First, several previous works investigate the structure of a household's decision-making style. In the traditional framework, households are considered as a single decision making agent, which is known as the "unitary" approach initiated by Samuelson (1956) and Becker (1974). Due to a lack of empirical support for the unitary model of households, Apps and Rees (1988) and Chiappori (1988, 1992) propose a "collective" approach, allowing for bargaining power between spouses and assuming that households achieve a Pareto-efficient allocation.[12] A common assumption of the unitary and collective approach is that intra-household behavior is efficient. However, recent studies have increasingly employed the non-cooperative model in which allocation is not fully efficient (Konrad and Lommerud, 1995; Cigno, 2012; Gobbi, 2018).[13] The non-cooperative model is supported by empirical evidence. For example, Del Boca and Flinn (2012) estimate household time allocation between the production of a public good and labor market work, and find that about one-fourth households act non-cooperatively. Our analysis builds on the literature studying the non-cooperative model of households.

The second relevant strand of literature concerns analysis of the optimal tax for households consisting of two or more agents.[14] In particular, using the self-selection approach (Stiglitz, 1982), Balestrino et al. (2002) develop a two-type model with non-linear labor income taxation, non-linear child taxes or subsidies, and linear commodity taxation when households differ in their ability in household production as well as in the labor market. Corresponding to our model, Balestrino et al. (2002) consider that both fertility and child quality are endogenously determined.[15] However, their model falls within the "unitary" approach that supports Pareto efficiency and thus, the fertility rate is initially efficient. In their model, government intervention is justified by equity considerations (redistribution from rich to poor) and allocative efficiency considerations (specialization in domestic or market activities according to comparative advantage). By contrast, we consider another justification for the government's intervention, which is to correct the under-provision of household public goods that emerges from non-cooperative household behavior. Using the Ramsey tax framework, this study analyzes optimal tax policies for improving the suboptimal low fertility rate induced by non-cooperative couples. In a recent contribution, Meier and Rainer (2015) study gender-based income taxation in a model with a single household public good and find that marginal income tax rates should be differentiated by gender based on both the Pigou and Ramsey considerations. In the model, the household public good is under-provided due to the non-cooperative behavior. Even though their setting is similar to our model, this study differs from their framework in four ways. First, this study considers both child quality and quantity as household public goods. Importantly, the two household public goods are determined in different ways and stages; child quantity is collectively chosen first, and child quality is then decided non-cooperatively. Second, our model allows the government to employ a child tax/subsidy on child

[12]The unitary model ensures an income-pooling result in which a change in the source of household income does not affect demand if the total income is constant. However, it is empirically rejected by Browning and Chiappori (1998).

[13]Non-cooperative family decision making has been adopted in theoretical, empirical, and experimental literature. As with our model, Konrad and Lommerud (1995), Cigno (2012), and Gobbi (2018) use a non-cooperative model for childcare decisions. See other related literature on the non-cooperative model, for example, Lundberg and Pollak (1993), Anderberg (2007), Lechene and Preston (2011), Cochard et al. (2016), Doepke and Tertilt (2019), and Heath and Tan (2020).

[14]There is a growing body of literature analyzing the optimal family tax/subsidy scheme; see, for example, Cremer et al. (2003, 2011b, 2016, 2021), Schroyen (2003), Brett (2007), Kleven et al. (2009), Meier and Wrede (2013), Frankel (2014), Apps and Rees (2018), Bastani et al. (2020), and Ho and Pavoni (2020).

[15]Other related studies explore the optimal system of policy instruments under endogenous fertility and child quality (e.g., Cigno, 2001; Cigno and Pettini, 2002).

quantity as a direct intervention on the public good. Third, we introduce center-based childcare services, which can be substituted for spousal childcare time. Fourth, we allow for the externality of children on society.

The third strand of literature discusses the driving force underlying the low fertility rate in an economy and then establishes Pareto-improving family policies that may correct the inefficiency. A major explanation for the reduction in the number of children is that they involve a positive fiscal externality when the government redistributes from the young to the old (e.g., PAYG transfers). As argued by Cigno (1992), PAYG transfers lead to a suboptimal number of children since children, considered as assets by parents, are no longer required to secure consumption in retirement. Groezen et al. (2003) analyze the role of a child allowance scheme when fertility is socially inefficient owing to PAYG transfers.[16] They show that a child allowance system ensures the first-best outcome under lump-sum transfers. Another explanation for the suboptimal low fertility rate is that an increase in children's human capital enhances the local security level, promotes social skills, and reduces adverse health conditions as external effects (Heckman, 2006; Heckman and Masterov, 2007). Compared to these articles, our study proposes a theoretical framework that describes inefficiently low fertility due to the non-cooperative behavior of couples in addition to the external effects of children on society, and then provides the optimal structure of family policy measures.

In addition to the above three strands, our study also relates to the literature on gender-based taxation, which allows tax rates to differ between the husband and wife. Rosen (1977) is the first to argue about the efficiency gains from employing differential taxation based on gender, while Akerlof (1978) shows that the use of categorical information, such as age, gender, and disability status, known as "tagging," is welfare improving from the perspective of utilitarianism.[17] Hence, if the government reflects observable characteristics in the tax system, it can reinforce the redistributive tax system. Several studies explore the gender-based taxation system; see, for example, Boskin and Shesinski (1983), Piggott and Whalley (1996), Apps and Rees (1999a, b, 2011), Kleven and Kreiner (2007), Cremer et. al. (2010), Alesina et al. (2011), Bastani (2013), Meier and Rainer (2015), and Komura et al. (2019).

3 Model

Consider an economy comprising H identical households,[18] where the notation H denotes the number of households living in a range that the externality of children reaches across the households (see more details of the interpretation below equation (1)). Thus, the range of H may indicate the country, region, or local area, depending on the situation. Members of the household consist of a wife (f), husband (m), and children. The wife and husband collectively decide on the number of children they want, while each spouse non-cooperatively decides his/her two kinds of private consumption, labor supply

[16] Cigno et al. (2003), Fenge and Meier (2005), and Cremer et al. (2008, 2011a) are among the related literature on family policy in the presence of fiscal externalities.

[17] It is well known that tagging violates the principle of horizontal equity; thus, it is limited in practice.

[18] We consider that H to be an integer greater than or equal to 2.

in the external market, and time spent on childcare. Parental time investment in childcare enhances child quality, including non-cognitive and cognitive skills. A free-rider problem between the spouses generally occurs in the process of enhancing child quality. Both child quality and quantity positively affect the utility of spouses as household public goods. Furthermore, we allow child quality and quantity to positively affect society as externalities.

The government corrects the free-rider problem and externalities on society while facing a revenue constraint. It imposes linear taxes on income for each spouse and implements a commodity tax and a (direct) child tax/subsidy.[19] The child tax/subsidy is a tax/subsidy on/for a child; the amount of the child tax/subsidy proportionally increases with the number of children per couple. This study considers the case where the income tax rates on the husband and wife can differ: the so-called "gender-based taxation."[20] The case with a common income tax rate on the spouses is analyzed in Appendix E.

We consider the following sequential decisions of the government, the couple, and each spouse in the couple. First, the government determines the tax rates to collect a given level of tax revenue and to correct the suboptimal low fertility level. Second, the wife and husband collectively decide on child quantity. Third, each spouse non-cooperatively decides two kinds of private consumption, labor supply in the external market, and time spent on childcare.

3.1 Third Stage: Each Spouse in the Couple

For all identical households, each spouse non-cooperatively decides the amount of working time in the outside labor market l_i, time spent by each spouse on childcare activities h_i, and private consumption of the numeraire z_i and another commodity y_i. We suppose that children provide direct utility benefits; that is, children are a consumption good. Spouse i's utility u_i is given by[21]

$$u_m = z_m + \frac{y_m^{\varphi}}{\varphi} - \frac{(l_m + h_m)^{1+\phi}}{1+\phi} + nq + \mu N q, \tag{1}$$

$$u_f = z_f + \frac{y_f^{\varphi}}{\varphi} - \frac{\left(l_f + h_f\right)^{1+\phi}}{1+\phi} + nq + \mu N q - c(n),$$

[19] By allowing the government to employ a commodity tax, we can check if it is virtually facing a revenue constraint (see below Proposition 4 for more details). As a result, we can exclude meaningless results such that the child subsidy is optimal to return a tax revenue beyond the required tax level to the consumer by checking the sign of the optimal commodity tax rate.

[20] Gender-based taxation is equivalent to the combination of a common tax rate on both genders and a tax rate deduction for women. Only women bear the burden of reproductive responsibility during the fertility period, such as the time devoted toward pregnancy, childbirth, and lactation. It is plausible that a subsidy or tax deduction in allowance must be provided for these responsibilities.

[21] Although sub-utility functions in the model are specified, the generalization of the sub-utility function, such as

$$u_m = z_m + \varkappa_m(y_m) + \varpi_m(l_m + h_m) + \vartheta_m(nq) + \varrho_m(Nq),$$

and

$$u_f = z_f + \varkappa_f(y_f) + \varpi_f(l_f + h_f) + \vartheta_f(nq) + \varrho_f(Nq) - c(n),$$

does not affect the optimal tax/subsidy expressions provided in Propositions 4 to 8, which are the main theoretical results in this study, because our optimal tax/subsidy rates are expressed in terms of price elasticities.

where $\varphi(< 1)$ is the curvature of the utility of commodity y, $\phi(> 0)$ is that of the disutility of total time use, n is child quantity (i.e., the number of children per couple), q is child quality (i.e., quality per child), and $N(= Hn)$ represents the total number of children in the economy. The fourth term nq in (1) positively and equally affects the spouses as household public goods, while the fifth term μNq captures the positive externalities regarding child quality and quantity across H households.

We suppose that the source of the externality is $Nq(= Hnq)$, which is child quality q multiplied by child quantity n of H households, and $\mu(\geq 0)$ denotes the (constant) marginal external effects on society. In this study, the externalities, described by the fifth term, are called "the externality of children on society." Following Sandmo (1975), we posit that each couple considers μNq as a fixed parameter. The intuition is that they behave in an atomistic manner; that is, they consider the impact of their own child quality and quantity on H households to be extremely small. In contrast, the government allows for the external effects of children on society. $c(n)$ is a cost that only wives bear, depending on child quantity. This cost arises from the biology of child rearing during the fertility period, such as the time devoted to pregnancy, childbirth, and lactation (Rasul, 2008).[22] The cost function is assumed to satisfy $c' > 0$ and $c'' > 0$.

Here, we provide some examples of the externalities of children on society. Improving q enhances the local security level, promotes social skills, and reduces adverse health conditions as external effects (Heckman, 2006; Heckman and Masterov, 2007). In addition, as q improves, the peer effects that children produce positive learning spillovers in school life increase. As the external effects of the total number of children in the economy N, children can learn sociality from the community of children, and parents can also learn about childcare and receive information about education and medical care from other couples with children. The increase in N generates synergy effects if q has peer effects.[23]

As with Alesina et al. (2011) and Meier and Rainer (2015), the quality function is given by

$$q = \frac{\left(s_m \frac{h_m}{n}\right)^{\sigma}}{\sigma} + \frac{\left(s_f \frac{h_f}{n}\right)^{\sigma}}{\sigma} = n^{-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{(s_f h_f)^{\sigma}}{\sigma}\right], \tag{2}$$

where s_i denotes the productivity of spouse i for child quality, and σ, which satisfies $0 < \sigma < 1$, is the curvature of the quality function.[24] h_i/n represents childcare time allotted to each child. We may interpret that childcare time h_i includes the time spent raising a child, as well as the time for improving children's non-cognitive and cognitive skills, such as the time spent reading books to children, time spent on early childhood education at home, and the cost of the effort to discipline children. Childcare time h_i can be divided into two components: $h_i = \widetilde{h}_i + \Upsilon n$, where $\widetilde{h}_i$ is the time spent enhancing child

[22]Rasul (2008) also assumes that only the wife bears the cost function.

[23]The additional interpretations are as follows. As the external effects of child quality q, children's human capital formation potentially increases the future tax base, which reduces the tax burden on future generations. Moreover, under PAYG social security systems, the size of a person's pension benefits depends on the number of children in all households N, as considered in many previous studies.

[24]Alesina et al. (2011) and Meier and Rainer (2015) use a similar function for the household public good. However, in contrast to their setting, we consider both child quality and quantity as household public goods, which are determined in different stages.

quality, and Υ is the (constant) minimal amount of time spent raising a child; hence, Υn is the total minimal amount of time spent raising children. If Υ is exogenous, the theoretical results obtained in this study remain unaffected as long as $\widetilde{h}_i > 0$.[25] Thus, this setting presented by (1) and (2) is not restrictive.

In our model, the childcare time provided by both spouses affects child quality. Del Boca et al. (2014) and Lundborg et al. (2014) empirically prove that the time invested by both husband and wife is important for the human capital accumulation in children. In particular, Del Boca et al. (2014) find that in the cognitive development process, the father's time is almost as important as the mother's time, especially for young children, and thus, we assume that the time a father spends with his children positively affects child quality. In addition, the quality function (2) does not include some commodities as inputs. This assumption also follows Del Boca et al. (2014), who empirically find that the impact of money on child quality is much more limited than the effect of parental time with children. As with our theoretical model, Gobbi (2018) does not include some commodities as inputs in the production of child quality, in line with Del Boca et al. (2014).[26]

Each spouse has a different budget constraint, which is based on the non-cooperative couple model (Lundberg and Pollak, 1993; Konrad and Lommerud, 1995; Anderberg, 2007; Lechene and Preston, 2011; Cigno, 2012; Meier and Rainer, 2015; Doepke and Tertilt, 2019; Heath and Tan, 2020).[27] The budget constraint for each spouse is

$$z_i + (1 + t_y)p_y y_i + \gamma_{xi} p_n x + \gamma_{ni} \kappa_n n = (1 - t_i) w_i l_i, \quad i = m, f, \tag{3}$$

where t_y is the commodity tax rate on y_i, p_y is the price of y_i, γ_{xi} is the share of spouse i on the purchase of the fertility good, x is the amount of a fertility good that a couple purchases, p_n is the price of x, γ_{ni} is the share of spouse i in the child tax payment or child subsidy receipt, κ_n is the child tax/subsidy, t_i (for $i = m, f$) is the income tax rate on the labor income of spouse i, and w_i is the wage rate of spouse i. Before-tax prices of the numeraire good are normalized by one without loss of generality. The share of the purchased fertility good and the share of child tax payment (child subsidy receipt) are given for each spouse at a certain level, satisfying $\gamma_{xm} + \gamma_{xf} = 1$ and $\gamma_{nm} + \gamma_{nf} = 1$.[28]

The required amount of the fertility good is given by the following function:

$$x = \upsilon n, \tag{4}$$

[25]Note that the first-order conditions of h_i and n in this setting are identical to (12) and (27), respectively, as long as $\widetilde{h}_i$ is positive, that is, $h_i > \Upsilon n$.

[26]Kleven et al. (2000) and Kleven (2004) assume that household production requires not only time spent in the home but also services bought in the market. The services bought in the market may be necessary in providing household public goods other than the child quality. However, Del Boca et al. (2014) empirically shows that parental time spent with children is an important input in improving child quality, while services bought in the market are not as effective.

[27]Substantial evidence on the fact that each spouse has his/her own budget constraint has been documented by Pahl (1983, 1995, 2008), Kenney (2006), and Lauer and Yodanis (2014).

[28]The couple may in part modify their defaults given by γ_{xi} and γ_{ni} ($i = m, f$). However, to simplify the analysis, we assume that the cost shares of both spouses are constant.

for a scalar υ. The cost includes food, clothing, medical expenses, and overhead costs needed for compulsory education.[29] The ratio of expenditure on nursery schools, tutors, and cram education to the cost of bringing up a child seems to be large, particularly in developed countries. This expenditure is related to improving child quality. The extensive case in which such expenditure affects child quality is discussed in Section 6. To simplify the analysis, we assume that one unit of the fertility good is required to raise a child, that is, $\upsilon = 1$ (Groezen et al., 2003),[30] and that the shares of the purchase of a fertility good and child tax payment (child subsidy receipt) are equal, that is, $\gamma_{xi} = \gamma_{ni} (\equiv \gamma_i)$ for $i = m, f$. Under these assumptions, (3) can be rewritten as

$$z_i + (1 + t_y) p_y y_i + \gamma_i (1 + t_n) p_n n = (1 - t_i) w_i l_i, \quad i = m, f, \tag{5}$$

where $t_n (\equiv \kappa_n / p_n)$ is the child tax/subsidy rate on child quantity.[31]

Denoting γ_m as γ and hence, γ_f as $1 - \gamma$ as well as making use of (2) and (5), (1) can be rewritten as

$$\begin{aligned} u_m \quad = \quad & (1 - t_m) w_m l_m - (1 + t_y) p_y y_m - \gamma (1 + t_n) p_n n + \frac{y_m^{\varphi}}{\varphi} \\ & - \frac{(l_m + h_m)^{1+\phi}}{1 + \phi} + (1 + \mu H) n^{1-\sigma} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{(s_f h_f)^{\sigma}}{\sigma} \right], \end{aligned} \tag{6}$$

$$\begin{aligned} u_f \quad = \quad & (1 - t_f) w_f l_f - (1 + t_y) p_y y_f - (1 - \gamma)(1 + t_n) p_n n + \frac{y_f^{\varphi}}{\varphi} \\ & - \frac{(l_f + h_f)^{1+\phi}}{1 + \phi} + (1 + \mu H) n^{1-\sigma} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{(s_f h_f)^{\sigma}}{\sigma} \right] - c(n). \end{aligned}$$

Each spouse decides his/her own labor supply and time invested for childcare, taking their partner's childcare time, child quantity, and external effects of children on society as given. Spouse i does not consider that his/her own childcare time positively affects their partner's utility. As shown by Rasul (2008), partners do not commit to supplying a certain amount of childcare duties (i.e., there is no clause in the marriage contract regarding how much time each parent should spend with their children). Moreover, actions are unobservable, and there is no way of monitoring the childcare time supplied by the other partner (Pailhé and Solaz, 2008). Hence, we resort to a Cournot–Nash non-cooperative game to model the third stage, which leads to a suboptimal low child quality owing to the free-rider

[29] Compulsory education includes fees for lunch, material, stationery, field trips, study tours, and school excursions.

[30] This simple assumption is also adopted by Groezen et al. (2003).

[31] If these two assumptions, $\upsilon = 1$ and $\gamma_{xi} = \gamma_{ni}$, are relaxed, the third and fourth terms on the left-hand side of (3) can be rewritten as $\gamma_{xi} \left(1 + \frac{\gamma_{ni} \kappa_n}{\gamma_{xi} p_n \upsilon} \right) p_n \upsilon n$. Defining $\widetilde{t_n} \equiv \kappa_n / p_n \upsilon$ and $\widetilde{p_n} \equiv p_n \upsilon$, the expression becomes $\gamma_{xi} \left(1 + \frac{\gamma_{ni}}{\gamma_{xi}} \widetilde{t_n} \right) \widetilde{p_n} n$, in which the additional term $\frac{\gamma_{ni}}{\gamma_{xi}}$ appears relative to the left-hand side of (5). That is, the assumption $\gamma_{xi} \neq \gamma_{ni}$ allows the effect of a child tax/subsidy rate to be different across spouses. Consequently, our theoretical results may be somewhat modified under $\gamma_{xi} \neq \gamma_{ni}$, although relaxing the assumption of $\upsilon = 1$ does not affect the results.

problem.[32]

3.2 Second Stage: The Couple

In the second stage, child quantity n is collectively determined as a decision made by the couple. In this decision, the couple takes the income tax rate, child tax/subsidy rate, and external effects of children on society as given. The couple's utility function is a weighted average of the utility of spouses:

$$u = \rho u_m + (1 - \rho) u_f, \tag{7}$$

where ρ is the bargaining power of the husband and satisfies $0 < \rho < 1$. The value of the bargaining power ρ is assumed to be constant in our model.[33] If the couple considers cost $c(n)$ as an important factor, ρ would be less than 0.5.[34] The couple maximizes u, allowing for l_i and h_i to be functions of n, which is formulated in the third decision stage.[35] Even though child quantity and childcare time are collectively determined, it is possible to deviate from current plans, and to non-cooperatively determine the amount of childcare due to lack of commitment and effective monitoring, as explained above. Thus, we postulate that the couple collectively determines child quantity prior to childcare time made non-cooperatively by both the husband and wife. In this setting, the determination process of n is efficient since the couple collectively decides child quantity. However, child quantity is at the suboptimal low level because the couple knows that child quality q is under-provided in the next stage even if there is no externality of children on society; that is, $\mu = 0$. This result is analytically provided in Subsection 4.3.

This setting is applicable to housing and healthcare. For example, a couple collectively determines the design, floor plan, and floor space for a house, and each spouse then non-cooperatively provides housing maintenance. As an alternative example, the couple collectively decides their medical insurance, and each spouse then non-cooperatively maintains their own health.

The number of children per couple can be divided into two components, $n = \overline{n} + \widetilde{n}$, where $\overline{n}$ is the initially determined number of children, which can be the number desired by the spouse who wants to have fewer children,[36] and $\widetilde{n}$ is the endogenously determined number. For example, $\overline{n}$ is the minimum number of children that a couple determines or promises before marriage, and $\widetilde{n}$ is the number after marriage. If each spouse intends to have at least one child before marriage (i.e., $\overline{n} = 1$), $\widetilde{n}$ can be interpreted as the number of subsequent children determined by the spouses. Throughout the study, the change in n may be interpreted as that in $\widetilde{n}$.

[32]This assumption is supported by recent econometric evidence from Del Boca and Flinn (2012), showing that one-fourth of couples under-provide household public goods because of non-cooperative behavior.

[33]In line with Basu (2006), Komura et al. (2019) consider endogenous bargaining power depending on the relative income difference between spouses. For simplicity, as per Cremer et al. (2016), we assume that weights are exogenous.

[34]As long as $\rho = 0.5$, even if the husband bears this type of cost as well or it is shared by the spouses, the theoretical results obtained in this study are unaffected because child quantity is collectively determined.

[35]Note that this optimization allows for the budget constraint of each spouse because u_i $(i = m, f)$ in (7) corresponds to each spouse's utility given by (6).

[36]Let n^i (for $i = m, f$) denote the number of children that spouse i wants. $\overline{n}$ can be regarded as $\min[n^m, n^f]$.

3.3 First Stage: The Government

The government maximizes its social welfare under a revenue constraint by manipulating the commodity tax, income tax, and child tax/subsidy. We presume that the government's objective function is the utilitarian optimum based on equal weights between the husband and wife. Owing to the assumption that couples are identical, the social welfare function is given by $W = H(u_m + u_f)$. Since H is constant, we consider the following objective function of the government, which is given by

$$\frac{W}{H} = u_m + u_f. \tag{8}$$

The revenue constraint of the government is

$$g = t_m w_m l_m + t_f w_f l_f + t_y p_y (y_m + y_f) + t_n p_n n, \tag{9}$$

where g is the required tax revenue per household and its level is assumed to be constant. The government maximizes (8) with respect to t_m, t_f, t_y, and t_n, subject to (9).[37] Since couples are identical, the result of this maximization problem coincides with the outcome of the optimization problem that the government maximizes the sum of all individuals' utilities subject to the revenue constraint derived by multiplying both sides of (9) by H. Since we assume that the social welfare function is utilitarian with equal weights across spouses, the optimal marginal tax rates only depend on efficiency considerations. Thus, unlike Alesina et al. (2011) and Meier and Rainer (2015), lump-sum transfers between spouses to resolve distributional concerns are not required. To make the analysis more meaningful, throughout this study, we assume that the required tax revenue exceeds the revenue collected from the tax systems that correct the under-provision of child quality and quantity.

4 Model Solutions

4.1 Spouse

In this section, we analyze the solutions to the utility maximization problem of each spouse in the couple, and the properties of the labor supply function and childcare function, given μNq.[38] Before doing so, we introduce the parameter τ that describes the degree of cooperation within a couple, where $0 \leq \tau \leq 1$. The increase in τ implies more cooperative behavior of the couple and reduces the free-rider problem for childcare. If $\tau = 0$, each spouse behaves non-cooperatively in providing childcare time, while each

[37]We implicitly assume that the government uses its tax revenue to purchase a public good G, satisfying $G = Hg$, and provides it to consumers. Moreover, we assume that the public good is additively separable in each spouse's utility; that is, $u_i + G$. Thus, the precise expressions for the couple's utility function and government's objective function are $u + G$ and $\frac{W}{H} + 2G$, respectively. From these functional forms and constant G, due to the fixed revenue requirement, the optimal conditions presented hereafter are not affected by G. Therefore, our results remain valid even if the constant public good is explicitly introduced into the utility functions.

[38]In (6), μNq corresponds to $\mu H n^{1-\sigma}\left[\frac{(s_m h_m(t_m,n))^{\sigma}}{\sigma} + \frac{(s_f h_f(t_f,n))^{\sigma}}{\sigma}\right]$.

spouse completely takes cooperative behavior if $\tau = 1$. The detailed explanation concerning τ is given below (12). From (6), we obtain the first-order conditions for the utility maximization problem of each spouse with respect to y_i, l_i, and h_i:

$$0 = \frac{\partial u_i}{\partial y_i} = -(1 + t_y)p_y + y_i^{\varphi-1}, \quad i = m, f, \tag{10}$$

$$0 = \frac{\partial u_i}{\partial l_i} = (1 - t_i)w_i - (l_i + h_i)^{\phi}, \quad i = m, f, \tag{11}$$

$$0 = \frac{\partial u_i}{\partial h_i} = -(l_i + h_i)^{\phi} + (1 + \tau)n^{1-\sigma} s_i^{\sigma} h_i^{\sigma-1}, \quad i = m, f. \tag{12}$$

If $\tau = 0$, spouse i decides h_i, allowing only for nq in his/her own utility. In this case, the marginal utility of h_i is $n^{1-\sigma} s_i^{\sigma} h_i^{\sigma-1}$, as shown by (12). In the fully cooperative case (i.e., if $\tau = 1$), each spouse allows for not only nq in his/her own utility but also nq in their partner's utility; that is, $2nq (= nq + nq)$. This case induces an efficient outcome. If $\tau = 1$, the marginal utility of h_i in (12) becomes $2n^{1-\sigma} s_i^{\sigma} h_i^{\sigma-1}$, which leads to the conclusion that the first-order condition of h_i is equivalent to that of the maximization of $u_m + u_f$ with respect to h_i. Indeed, we can confirm that equation (12) is equivalent to equation (D5) in Appendix D under $\rho = 0.5$. Given that equation (D5) coincides with the equations generating the Pareto-efficient allocation derived by equations (34), (35), (40), and (41) under $\mu = 0$ and no government's intervention, the case of $\tau = 1$ efficiently determines h_i. Hereafter, unless otherwise noted, we focus only on fully non-cooperative or partially cooperative cases (i.e., $0 \leq \tau < 1$).

Defining the after-tax wage rate as $\omega_i (\equiv (1 - t_i)w_i)$, (10), (11), and (12) immediately yield

$$y_i(t_y) = \left[\left(1 + t_y\right) p_y\right]^{\frac{1}{\varphi-1}}, \quad i = m, f, \tag{13}$$

$$h_i(t_i, n; w_i, s_i) = (1 + \tau)^{\frac{1}{1-\sigma}} \omega_i^{-\frac{1}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} n, \quad i = m, f, \tag{14}$$

$$l_i(t_i, n; w_i, s_i) = \omega_i^{\frac{1}{\phi}} - (1 + \tau)^{\frac{1}{1-\sigma}} \omega_i^{-\frac{1}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} n, \quad i = m, f, \tag{15}$$

$$h_i(t_i, n; w_i, s_i) + l_i(t_i, n; w_i, s_i) = \omega_i^{\frac{1}{\phi}}, \quad i = m, f. \tag{16}$$

The aggregate time for the external labor market and domestic childcare, as given by (16), depends only on the after-tax wage rate ω_i and the parameter of the sub-utility function ϕ due to a quasi-linear utility functional form. From (14) and (15), the time spent on domestic childcare and the external labor market is affected by the productivity of the household production s_i, child quantity n, as well as the after-tax wage rate ω_i. From (13), the commodity y_i depends only on the tax-inclusive price and the parameter of the sub-utility function φ.

From (14) and (15), we obtain

$$h_{in} \left(\equiv \frac{\partial h_i}{\partial n}\right) = -l_{in} \left(\equiv \frac{\partial l_i}{\partial n}\right) = (1 + \tau)^{\frac{1}{1-\sigma}} \omega_i^{-\frac{1}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} > 0, \quad i = m, f, \tag{17}$$

$$h_{is_i}\left(\equiv \frac{\partial h_i}{\partial s_i}\right) = -l_{is_i}\left(\equiv \frac{\partial l_i}{\partial s_i}\right) = \left(\frac{\sigma}{1-\sigma}\right)(1+\tau)^{\frac{1}{1-\sigma}}\omega_i^{-\frac{1}{1-\sigma}}s_i^{\frac{-1+2\sigma}{1-\sigma}}n > 0, \quad i = m, f, \tag{18}$$

$$h_{i\omega_i}\left(\equiv \frac{\partial h_i}{\partial \omega_i}\right) = -\left(\frac{1}{1-\sigma}\right)(1+\tau)^{\frac{1}{1-\sigma}}\omega_i^{\frac{-2+\sigma}{1-\sigma}}s_i^{\frac{\sigma}{1-\sigma}}n < 0, \quad i = m, f, \tag{19}$$

$$l_{i\omega_i}\left(\equiv \frac{\partial l_i}{\partial \omega_i}\right) = \frac{1}{\phi}\omega_i^{\frac{1-\phi}{\phi}} + \left(\frac{1}{1-\sigma}\right)(1+\tau)^{\frac{1}{1-\sigma}}\omega_i^{\frac{-2+\sigma}{1-\sigma}}s_i^{\frac{\sigma}{1-\sigma}}n > 0, \quad i = m, f, \tag{20}$$

$$h_{i\tau}\left(\equiv \frac{\partial h_i}{\partial \tau}\right) = -l_{i\tau}\left(\equiv \frac{\partial l_i}{\partial \tau}\right) = \left(\frac{1}{1-\sigma}\right)(1+\tau)^{\frac{\sigma}{1-\sigma}}\omega_i^{-\frac{1}{1-\sigma}}s_i^{\frac{\sigma}{1-\sigma}}n > 0, \quad i = m, f. \tag{21}$$

Equations (17) and (18) show that time spent on childcare increases while time spent on the external labor market decreases with child quantity n and childcare productivity s_i. These results are intuitive. The increase in n obviously requires more time to be spent on childcare. The increase in s_i enhances the marginal utility of h_i through a change in q; hence, the time spent on childcare increases with s_i. The amount of increase in h_i is the same as that of a decrease in l_i because n and s_i do not affect aggregate time $h_i + l_i$; that is, $h_{in} + l_{in} = 0$ and $h_{is_i} + l_{is_i} = 0$, as shown in (17) and (18). This is also confirmed by (16). From (19) and (20), ω_i has the opposite effects on h_i and l_i: time spent on childcare decreases while time spent on the external labor market increases with the after-tax wage ω_i. However, the increase in l_i exceeds the decrease in h_i. From (19) and (20), we have

$$h_{i\omega_i} + l_{i\omega_i} = \frac{1}{\phi}\omega_i^{\frac{1-\phi}{\phi}} > 0, \quad i = m, f, \tag{22}$$

which implies that income taxation yields price distortions.

Equation (21) shows that time spent on childcare increases while time spent on the external labor market decreases with the degree of cooperation within a couple τ. Since the improvement in cooperation increases the marginal benefit of childcare activities for each spouse, the free-rider problem is mitigated. Hence, toward the efficient level, time spent on childcare activities by each spouse increases, while time spent on the external labor market decreases.

A comparison between the time allocation of the wife and that of the husband is also obtained from (17)–(21). The following proposition summarizes the results.

Proposition 1. *Suppose that at least one of $\omega_i \geq \omega_j$ and $s_i \leq s_j$ is strict. Then, (i) $l_i > l_j$, (ii) $h_i < h_j$, (iii) $h_{in} < h_{jn}$, (iv) $-l_{in} < -l_{jn}$, (v) $h_{i\tau} < h_{j\tau}$, and (vi) $-l_{i\tau} < -l_{j\tau}$.*

Proposition 1(i) is obtained from (18) and (20), and 1(ii) from (18) and (19). We also confirm these results from (14) and (15). Propositions 1(i) and 1(ii) show that the couple's time allocation is similar to Ricardo's comparative advantage in the theory of international trade. Propositions 1(iii) and 1(iv) are obtained from (17), while Propositions 1(v) and 1(vi) are obtained from (21). They indicate that an increase in child quantity or more cooperative behavior within a couple creates more childcare activities and less labor supply in the external market for the spouse with higher s and lower ω. In other words, the presence of children or the situation of cooperation strengthens the movement toward a complete division of labor between domestic childcare and the external labor market if there are gender differences in productivity, w_i and s_i. In our model, a corner solution in child quantity (i.e., $n = 0$) is possible; however, to obtain meaningful suggestions, we assume that $n > 0$ under optimal taxation. The numerical examples in Section 7 ensure that $n > 0$.

Finally, we show that income taxation yields price distortions on time allocation between h_i and l_i. By noting that $h_{it_i} = -w_i h_{i\omega_i}$ and $l_{it_i} = -w_i l_{i\omega_i}$ for $i = m, f$, (19), (20), and (22) lead to

$$h_{it_i} \left(\equiv \frac{\partial h_i}{\partial t_i}\right) = \left(\frac{1}{1-\sigma}\right) w_i (1+\tau)^{\frac{1}{1-\sigma}} \omega_i^{\frac{-2+\sigma}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} n > 0, \quad i = m, f, \tag{23}$$

$$l_{it_i} \left(\equiv \frac{\partial l_i}{\partial t_i}\right) = -\frac{1}{\phi} w_i \omega_i^{\frac{1-\phi}{\phi}} - \left(\frac{1}{1-\sigma}\right) w_i (1+\tau)^{\frac{1}{1-\sigma}} \omega_i^{\frac{-2+\sigma}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} n < 0, \quad i = m, f, \tag{24}$$

$$h_{it_i} + l_{it_i} = -\frac{1}{\phi} w_i \omega_i^{\frac{1-\phi}{\phi}} < 0, \quad i = m, f. \tag{25}$$

Income taxes can change the time allocation between domestic childcare provision and external labor market: the income tax rate on spouse i increases their supply of childcare time and decreases their labor supply. Thus, income taxes can play the role of correcting the non-cooperative behavior of spouses, and then improving child quality. In other words, optimal income taxation would involve the Pigouvian tax consideration. However, since income tax reduces the total amount of time spent on domestic childcare and labor market, as shown in (25), it inevitably yields price distortions.

4.2 Couple

In this subsection, we consider the couple's decision on child quantity. Allowing for (13)–(15), the couple maximizes (7) with respect to n, given $\mu N q$ and all tax rates; this is the collective optimization problem of the couple.[39] Equation (7) is represented by

$$\begin{aligned} u \;=\; & \rho \left[(1-t_m) w_m l_m(t_m, n) - (1+t_y) p_y y_m(t_y) + \frac{(y_m(t_y))^{\varphi}}{\varphi} - \frac{(l_m(t_m,n) + h_m(t_m,n))^{1+\phi}}{1+\phi} \right] \\ & + (1-\rho) \left[(1-t_f) w_f l_f(t_m, n) - (1+t_y) p_y y_f(t_y) + \frac{(y_f(t_y))^{\varphi}}{\varphi} - \frac{\left(l_f(t_f,n) + h_f(t_f,n)\right)^{1+\phi}}{1+\phi} - c(n) \right] \\ & - \left[(1-\rho)(1-\gamma) + \rho\gamma\right] (1+t_n) p_n n + (1+\mu H) n^{1-\sigma} \left[\frac{(s_m h_m(t_m,n))^{\sigma}}{\sigma} + \frac{\left(s_f h_f(t_f,n)\right)^{\sigma}}{\sigma} \right]. \end{aligned} \tag{26}$$

[39] In (26), $\mu N q$ corresponds to $\mu H n^{1-\sigma} \left[\frac{(s_m h_m(t_m,n))^{\sigma}}{\sigma} + \frac{(s_f h_f(t_f,n))^{\sigma}}{\sigma} \right]$.

Allowing for (11), (12), (14), and (17), the first-order condition with respect to n is given by[40]

$$\begin{aligned} 0 \;=\; \frac{\partial u}{\partial n} &= -\left[(1-\rho)(1-\gamma)+\rho\gamma\right](1+t_n)p_n - (1-\rho)c'(n) \qquad (27) \\ &+ \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + 1 - \rho\right)(1-t_m)w_m h_{mn}(t_m) + \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \rho\right)(1-t_f)w_f h_{fn}(t_f). \end{aligned}$$

See Appendix A for the derivation of this condition. (27) implies that

$$n = n(t_n, t_m, t_f). \qquad (28)$$

Although child quantity is collectively determined, it downwardly deviates from an efficient level. This result has been analytically proven in Subsection 4.3. Here, we provide an intuitive explanation for this result. From (2), we observe that $nq = (n^{1-\sigma}/\sigma)\left[(s_m h_m)^{\sigma} + (s_f h_f)^{\sigma}\right]$, which shows that a smaller h_i lowers the marginal utility of n. With the spouses non-cooperatively taking care of their children in the third stage, the amount of h_i is under-provided. Thus, the child quantity is also under-provided.

Totally differentiating (27) with respect to n, t_m, t_f, and t_n, and using (17) yields the following results:

$$n_{t_m}\left(\equiv \frac{\partial n}{\partial t_m}\right) = \frac{\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + 1 - \rho\right)\left(\frac{\sigma}{1-\sigma}\right) w_m (1+\tau)^{\frac{1}{1-\sigma}} \omega_m^{-\frac{1}{1-\sigma}} s_m^{\frac{\sigma}{1-\sigma}}}{(1-\rho)c''} > 0, \qquad (29)$$

$$n_{t_f}\left(\equiv \frac{\partial n}{\partial t_f}\right) = \frac{\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \rho\right)\left(\frac{\sigma}{1-\sigma}\right) w_f (1+\tau)^{\frac{1}{1-\sigma}} \omega_f^{-\frac{1}{1-\sigma}} s_f^{\frac{\sigma}{1-\sigma}}}{(1-\rho)c''} > 0, \qquad (30)$$

$$n_{t_n}\left(\equiv \frac{\partial n}{\partial t_n}\right) = -\frac{\left[(1-\rho)(1-\gamma)+\rho\gamma\right]p_n}{(1-\rho)c''} < 0. \qquad (31)$$

From (29)–(31), we obtain the following proposition.

Proposition 2. *(i)* $n_{t_i} > 0$ *for* $i = m, f$, *and* $n_{t_n} < 0$. *(ii) Suppose that* $\rho = 0.5$. *Then, if* $w_i \geq w_j$ *and* $s_i \leq s_j$ *with at least one strict inequality,* $n_{t_i} < n_{t_j}$. *(iii) Suppose that* $w_m = w_f$ *and* $s_m = s_f$. *Then, if* $\rho \gtreqless 0.5$, $n_{t_m} \lesseqgtr n_{t_f}$.

Proposition 2(i) shows that child quantity increases with a rise in income tax rates. As mentioned above, children are under-provided because both spouses are aware of the non-cooperative behavior toward childcare in the next stage. Since the time spent on childcare increases with income tax, as shown by (23), child quality is improved with income tax rates and then child quantity is also improved. This result has an interesting policy implication; that is, income taxation raises tax revenue and improves the low fertility level. There is overwhelming empirical evidence on the fact that fertility is negatively related to the wage rate in most countries at most times (Jones and Tertilt, 2008; Jones et al., 2010), which supports the theoretical results in this study. Although income effects due to a reduction in income have a negative impact the fertility rate, the decrease in after-tax wage lowers the opportunity

[40]From (17), note that $h_{in}(t_i)$ (for $i = m, f$) is independent of n.

cost of having children. The empirical evidence implies that income effects are not significantly large; therefore, the increase in income tax can raise the fertility rate.

As discussed in the last part of Subsection 3.2, if $\overline{n} = 1$, the change in n can be interpreted as a change in the subsequent number of children after the first child. Baughman and Dickert-Conlin (2009) empirically show that income tax deductions decrease the number of subsequent children after the first child, which supports the first result in Proposition 2(i).

The second result in Proposition 2(i) shows that direct child subsidy unambiguously raises the fertility rate. The intuition behind the result is straightforward. Proposition 2(i) shows that both the high income tax rate and low child tax (or child subsidy) rates increase child quantity. In this context, an important question arises: which of these two instruments plays a vital role in correcting the low fertility rate caused by non-cooperative behavior in a revenue-constrained optimal tax framework? This is examined in Section 5.

From Proposition 2(ii), we observe that the income tax imposed on the spouse with lower productivity in the external labor market and higher childcare productivity yields a higher birthrate-improvement effect. This is because, as shown in Proposition 1(iii), an increase in income tax on this spouse yields larger marginal effects on childcare time.

Proposition 2(iii) shows that an increase in the income tax rate on a spouse with lower bargaining power induces a couple to have more children. In other words, although income taxes improve child quantity, the impact of income taxes on a spouse with higher bargaining power is limited. Without loss of generality, we consider a case in which the husband's bargaining power is larger (i.e., $\rho > 0.5$). Note that an increase in t_m directly decreases the husband's disposable income, although the increase in t_f does not directly affect his disposable income. Given this fact and $l_{mn} < 0$, the husband desires fewer children to mitigate the reduction in his private consumption when t_m increases than when t_f increases. Thus, since the couple's decision about n considers the husband's utility as being more important, the increase in n is further mitigated when t_m increases than when t_f rises.

Before analyzing the government's optimization problem, we provide the functions of h_i and l_i, which allow for (14) and (15), and (28) as

$$h_i(t_i, n(t_n, t_m, t_f)), \quad l_i(t_i, n(t_n, t_m, t_f)), \quad i = m, f. \tag{32}$$

These functions involve information about the decision made in the second and third stages. Allowing for (32), the government maximizes social welfare subject to the tax revenue constraint.

4.3 Pareto-Efficient Allocation of Time and Number of Children

The objective of this subsection is to justify the government's intervention for correcting the inefficiently low fertility because of the non-cooperative behavior of couples. To this end, we compare two allocations without the government's intervention: a Pareto-efficient allocation and a household allocation in our non-cooperative decision-making model. If child quantity under the non-cooperative setting deviates from the socially efficient level, then an efficiency-enhancing policy intervention is desirable. First, we derive a Pareto-efficient allocation without the government's intervention, which corresponds to a maximization problem of one partner's utility subject to a given level of the other partner's utility and the resource constraint without taxes or subsidies. To focus on the inefficiently low fertility rate due to the couple's non-cooperative behavior, we assume that there is no externality of children on society; that is, $\mu = 0$.[41] The Lagrangian is expressed by

$$\begin{aligned}\max_{\substack{z_m, z_f, y_m, y_f, l_m,\\ l_f, h_m, h_f, n}} \quad \mathcal{L} \quad = \quad & z_m + \frac{y_m^{\varphi}}{\varphi} - \frac{(l_m + h_m)^{1+\phi}}{1+\phi} + n^{1-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma}\right] \\ & + \iota\left\{z_f + \frac{y_m^{\varphi}}{\varphi} - \frac{\left(l_f + h_f\right)^{1+\phi}}{1+\phi} + n^{1-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma}\right] - c(n) - \overline{u}_f\right\} \\ & + \zeta\left(w_m l_m + w_f l_f - z_m - z_f - p_y(y_m + y_f) - p_n n\right),\end{aligned} \tag{33}$$

where $\overline{u}_f$ is the reservation utility of a wife, and ι and ζ are Lagrange multipliers.[42] The first-order conditions are

$$0 = \frac{\partial \mathcal{L}}{\partial z_m} = 1 - \zeta, \tag{34}$$

$$0 = \frac{\partial \mathcal{L}}{\partial z_f} = \iota - \zeta, \tag{35}$$

$$0 = \frac{\partial \mathcal{L}}{\partial y_m} = y_m^{\varphi-1} - \zeta p_y, \tag{36}$$

$$0 = \frac{\partial \mathcal{L}}{\partial y_f} = \iota y_f^{\varphi-1} - \zeta p_y, \tag{37}$$

$$0 = \frac{\partial \mathcal{L}}{\partial l_m} = -(l_m + h_m)^{\phi} + \zeta w_m, \tag{38}$$

$$0 = \frac{\partial \mathcal{L}}{\partial l_f} = -\iota\left(l_f + h_f\right)^{\phi} + \zeta w_f, \tag{39}$$

$$0 = \frac{\partial \mathcal{L}}{\partial h_m} = -(l_m + h_m)^{\phi} + (1+\iota) n^{1-\sigma} s_m^{\sigma} h_m^{\sigma-1}, \tag{40}$$

$$0 = \frac{\partial \mathcal{L}}{\partial h_f} = -\iota\left(l_f + h_f\right)^{\phi} + (1+\iota) n^{1-\sigma} s_f^{\sigma} h_f^{\sigma-1}, \tag{41}$$

$$0 = \frac{\partial \mathcal{L}}{\partial n} = (1+\iota)(1-\sigma) n^{-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma}\right] - \iota c'(n) - \zeta p_n. \tag{42}$$

[41]Even if the couple is fully cooperative, a gap between a Pareto-efficient allocation and a household allocation regarding child quantity occurs in the presence of the externality of children on society. This is because a couple does not allow for the external effects of children on society, which leads to an inefficiently low fertility.

[42]Note that q is replaced by the right-hand side of (2).

Before comparing child quantity between the two cases, to avoid any confusion, we denote child quantity under the Pareto-efficient allocation by n^{PE} and that under the non-cooperative case by n^{NC}.[43] Using (34)–(41), (42) can be rewritten as

$$n^{PE} : 0 = 2^{\frac{\sigma}{1-\sigma}} \left(\frac{1-\sigma}{\sigma}\right) \left(w_m^{\frac{-\sigma}{1-\sigma}} s_m^{\frac{\sigma}{1-\sigma}} + w_f^{\frac{-\sigma}{1-\sigma}} s_f^{\frac{\sigma}{1-\sigma}}\right) - \frac{1}{2} c'(n^{PE}) - \frac{1}{2} p_n. \tag{43}$$

See Appendix B for an in-depth derivation. Equation (43) determines n^{PE}. We next derive the condition that determines child quantity under the non-cooperative case. Given $t_i = 0$ for $i = m, f$, and $t_n = 0$, substituting (17) for h_{in} in (27) yields

$$\begin{aligned} n^{NC} : 0 = &-(1-\rho)c'(n^{NC}) - [(1-\rho)(1-\gamma) + \rho\gamma]\, p_n \\ &+ \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + 1 - \rho\right)(1+\tau)^{\frac{1}{1-\sigma}} w_m^{\frac{-\sigma}{1-\sigma}} s_m^{\frac{\sigma}{1-\sigma}} \\ &+ \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \rho\right)(1+\tau)^{\frac{1}{1-\sigma}} w_f^{\frac{-\sigma}{1-\sigma}} s_f^{\frac{\sigma}{1-\sigma}}, \end{aligned} \tag{44}$$

which determines n^{NC}.

To clarify the effect of non-cooperative household behavior on child quantity, we consider $\rho = 0.5$; that is, we eliminate the difference between the bargaining power of the spouses. In this case, (44) can be rewritten as

$$n^{NC} : 0 = (1+\tau)^{\frac{1}{1-\sigma}} \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \frac{1}{2}\right) \left(w_m^{\frac{-\sigma}{1-\sigma}} s_m^{\frac{\sigma}{1-\sigma}} + w_f^{\frac{-\sigma}{1-\sigma}} s_f^{\frac{\sigma}{1-\sigma}}\right) - \frac{1}{2} c'(n^{NC}) - \frac{1}{2} p_n. \tag{45}$$

Note that $c(n)$ is a strictly convex function, from (43) and (45), we observe that $n^{PE} > n^{NC}$ if $\pi(\sigma,\tau) \equiv 2^{\frac{\sigma}{1-\sigma}} \left(\frac{1-\sigma}{\sigma}\right) - (1+\tau)^{\frac{1}{1-\sigma}} \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \frac{1}{2}\right) > 0$. We can prove that $\pi(\sigma,\tau) > 0$ holds for any $0 < \sigma < 1$ under $0 \leq \tau < 1$ (see Appendix C). In other words, $n^{PE} > n^{NC}$ holds under fully non-cooperative or partially cooperative cases. However, if a couple cooperatively acts (i.e., $\tau = 1$), $\pi(\sigma, 1) = 0$ holds for any $0 < \sigma < 1$, which means that $n^{PE} = n^{NC}$. These are summarized in the following proposition.

Proposition 3. *Consider $\rho = 0.5$ and $\mu = 0$. If a couple provides the childcare time under fully non-cooperative or partially cooperative cases, the number of children per couple is under-provided; that is, $n^{PE} > n^{NC}$. On the other hand, if a couple cooperatively provides the childcare time, the number of children per couple is efficient; that is, $n^{PE} = n^{NC}$.*[44]

Although n^{PE} in Proposition 3 is realized under a given situation in which childcare time and child quantity are collectively and simultaneously determined, the collective decisions with different

[43]Note that n^{NC} is child quantity under the non-cooperative case when there are no taxes or subsidies.

[44]Since couples are identical in our model, $n^{PE} > n^{NC}$ results in $Hn^{PE} > Hn^{NC}$. Thus, the total number of children in the economy deviates from the socially efficient level. Since we can apply the same procedure to the case under $n^{PE} = n^{NC}$, the total number of children in the economy attains the socially efficient level.

stages also attain n^{PE}. Indeed, even if child quantity is collectively determined prior to the collective decision concerning q, it achieves the same level as that under Pareto-efficient allocation, that is, $n^{PE} = n^{C}$, where n^{C} denotes child quantity under the collective case in the sequential decision making (see Appendix D). Clearly, the number of children per couple under $\tau = 1$ corresponds to that under the collective case in the sequential decision making. Thus, when $\rho = 0.5$ and $\mu = 0$ hold, the low fertility rate is attributable to only the non-cooperative household behavior.

Furthermore, this argument holds even under the introduction of a childcare facility in Section 6.[45] This implies that, the time children spend in a childcare facility does not solve parental underinvestment in childcare owing to the non-cooperative household behavior, although it improves child quality.

5 Optimal Taxation

In this section, we examine the optimal structures of both the income tax and child tax/subsidy. The income tax rates can be differentiated across genders, which is the so-called "gender-based taxation." The case with a common income tax rate for a couple, which is a more restrictive and realistic tax system, essentially yields similar results as the case with gender-based taxation (see Appendix E). By allowing for (13), (28), and (32), the government's objective function (i.e., the government's welfare function per household) and tax revenue constraint are represented by

$$\begin{aligned}\frac{W}{H} \;=\; & (1-t_m)w_m l_m(t_m, n(t_n, t_m, t_f)) + \frac{\left(y_m(t_y)\right)^{\varphi}}{\varphi} \\ & - \frac{\left(l_m(t_m, n(t_n, t_m, t_f)) + h_m(t_m, n(t_n, t_m, t_f))\right)^{1+\phi}}{1+\phi} \\ & + (1-t_f)w_f l_f(t_f, n(t_n, t_m, t_f)) + \frac{\left(y_f(t_y)\right)^{\varphi}}{\varphi} \\ & - \frac{\left(l_f(t_f, n(t_n, t_m, t_f)) + h_f(t_f, n(t_n, t_m, t_f))\right)^{1+\phi}}{1+\phi} \\ & - c(n(t_n, t_m, t_f)) - (1+t_n)p_n n(t_n, t_m, t_f) - (1+t_y)p_y(y_m(t_y) + y_f(t_y)) \\ & + 2(1+\mu H)(n(t_n, t_m, t_f))^{1-\sigma}\left[\frac{\left(s_m h_m(t_m, n(t_n, t_m, t_f))\right)^{\sigma}}{\sigma} + \frac{\left(s_f h_f(t_f, n(t_n, t_m, t_f))\right)^{\sigma}}{\sigma}\right],\end{aligned} \tag{46}$$

$$g = t_m w_m l_m(t_m, n(t_n, t_m, t_f)) + t_f w_f l_f(t_f, n(t_n, t_m, t_f)) + t_y p_y \left(y_m(t_y) + y_f(t_y)\right) + t_n p_n n(t_n, t_m, t_f). \tag{47}$$

The government maximizes social welfare (46) under the tax revenue constraint (47) by manipulating t_y, t_m, t_f, and t_n. We define the Lagrange function as L and the Lagrange multiplier on the revenue constraint as λ. Allowing for (17), the first-order conditions with respect to t_y, t_m, t_f, and t_n are given

[45] We provide an outline of the proof. First, we conclude that $n^{PE} > n^{NC}$ holds even in the presence of a childcare facility using $\pi(\sigma, \tau) > 0$ for any $0 < \sigma < 1$ under $0 \le \tau < 1$, which is shown in Appendix C. Furthermore, using a similar method in Appendix D, we can show that $n^{PE} = n^{C}$ holds even under a childcare facility.

by

$$0 = \frac{\partial L}{\partial t_y} = -p_y y_m - (1+t_y)p_y y_m' - p_y y_f - (1+t_y)p_y y_f' + y_m^{\varphi-1} y_m' + y_f^{\varphi-1} y_f' - \lambda[y_m + y_f + t_y(y_m' + y_f')]p_y, \tag{48}$$

$$\begin{aligned} 0 = \frac{\partial L}{\partial t_m} &= -w_m l_m + (1-t_m)w_m l_{mt_m} + (1-t_m)w_m l_{mn} n_{t_m} \\ &- (l_m + h_m)^{\phi}\left(l_{mt_m} + h_{mt_m}\right) + (1-t_f)w_f l_{fn} n_{t_m} - c' n_{t_m} \\ &- (1+t_n)p_n n_{t_m} + 2(1+\mu H)(1-\sigma)\, n^{-\sigma} n_{t_m} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{(s_f h_f)^{\sigma}}{\sigma}\right] \\ &+ 2(1+\mu H)n^{1-\sigma}\left[s_m^{\sigma} h_m^{\sigma-1}\left(h_{mt_m} + h_{mn} n_{t_m}\right) + s_f^{\sigma} h_f^{\sigma-1} h_{fn} n_{t_m}\right] \\ &- \lambda\left(w_m l_m + t_m w_m l_{mt_m} + t_m w_m l_{mn} n_{t_m} + t_f w_f l_{fn} n_{t_m} + t_n p_n n_{t_m}\right), \end{aligned} \tag{49}$$

$$\begin{aligned} 0 = \frac{\partial L}{\partial t_f} &= (1-t_m)w_m l_{mn} n_{t_f} - w_f l_f + (1-t_f)w_f l_{ft_f} + (1-t_f)w_f l_{fn} n_{t_f} \\ &- \left(l_f + h_f\right)^{\phi}\left(l_{ft_f} + h_{ft_f}\right) - c' n_{t_f} - (1+t_n)p_n n_{t_f} \\ &+ 2(1+\mu H)(1-\sigma)n^{-\sigma} n_{t_f}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{(s_f h_f)^{\sigma}}{\sigma}\right] \\ &+ 2(1+\mu H)n^{1-\sigma}\left[s_m^{\sigma} h_m^{\sigma-1} h_{mn} n_{t_f} + s_f^{\sigma} h_f^{\sigma-1}(h_{ft_f} + h_{fn} n_{t_f})\right] \\ &- \lambda\left(t_m w_m l_{mn} n_{t_f} + w_f l_f + t_f w_f l_{ft_f} + t_f w_f l_{fn} n_{t_f} + t_n p_n n_{t_f}\right), \end{aligned} \tag{50}$$

$$\begin{aligned} 0 = \frac{\partial L}{\partial t_n} &= (1-t_m)w_m l_{mn} n_{t_n} + (1-t_f)w_f l_{fn} n_{t_n} - c' n_{t_n} - p_n n - (1+t_n)p_n n_{t_n} \\ &+ 2(1+\mu H)(1-\sigma)n^{-\sigma} n_{t_n}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{(s_f h_f)^{\sigma}}{\sigma}\right] \\ &+ 2(1+\mu H)n^{1-\sigma}\left(s_m^{\sigma} h_m^{\sigma-1} h_{mn} n_{t_n} + s_f^{\sigma} h_f^{\sigma-1} h_{fn} n_{t_n}\right) \\ &- \lambda\left(t_m w_m l_{mn} n_{t_n} + t_f w_f l_{fn} n_{t_n} + p_n n + t_n p_n n_{t_n}\right), \end{aligned} \tag{51}$$

where $y_i' \equiv dy_i/dt_y$. From these conditions, we first provide the optimal tax expressions and then discuss the optimal tax structure.

First, we examine the optimal tax rate on commodity y. Using (10) and (48), we immediately observe that

$$r_y\left(\equiv \frac{t_y}{1+t_y}\right) = \frac{\beta}{\Xi}, \tag{52}$$

where $\beta \equiv \frac{1+\lambda}{\lambda}$ and $\Xi \equiv -\frac{(1+t_y)(y'_m+y'_f)}{(y_m+y_f)}$.[46] It is the standard Ramsey tax expression, and the optimal tax rate on commodity y follows the well-known inverse elasticity rule. Consider the case in which a lump-sum tax is available for the government; that is, the government does not virtually face a revenue constraint. We consider a lump-sum tax equal across spouses and denote it by t_{lump}. Since a couple comprises two spouses, $2t_{lump}$ is subtracted from the government's welfare and is added to the revenue constraint. Thus, the first-order condition with respect to t_{lump} is that $t_{lump} : 0 = -2 - 2\lambda$, which leads to $\lambda = -1$, and hence, $\beta(\equiv (1+\lambda)/\lambda) = 0$. Therefore, the optimal tax rate on commodity y is zero. This is a natural consequence of the optimal tax theory under a revenue constraint. However, this consequence does not hold for the optimal income tax and child tax/subsidy in our model, as shown below.

Next, we explore the optimal income tax rates for spouses. Using (11), (12), and (51), (49) and (50) can be rewritten as the following conditions (see Appendix F):

$$t_m : 0 = -(1+\lambda)w_m l_m - \lambda t_m w_m l_{mt_m} + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-t_m)w_m h_{mt_m} + (1+\lambda)p_n n n_{t_n}^{-1} n_{t_m}, \quad (53)$$

$$t_f : 0 = -(1+\lambda)w_f l_f - \lambda t_f w_f l_{ft_f} + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-t_f)w_f h_{ft_f} + (1+\lambda)p_n n n_{t_n}^{-1} n_{t_f}. \quad (54)$$

These conditions explain the impact of income taxes clearly and intuitively. The first two terms reflect the price-distortion effects on resource allocation between the working time and the consumption of the numeraire. These terms are related to a standard Ramsey tax implication. In our model, income taxes also alter the time allocation from the external labor market to childcare time. This effect is described by the third term, which involves corrective taxes for non-cooperative behavior, taking into account the externality of children on society. The fourth term reflects the impact on child quantity. Although child quantity is inefficiently under-provided in our model, this term does not reflect the correction of the suboptimal number of children but is related to the tax-induced price distortions under a revenue constraint. To confirm this, we consider the case in which a lump-sum tax is available. As shown above, the availability of a lump-sum tax leads to $\lambda = -1$, and hence, the fourth term vanishes.

Before presenting the optimal income tax expression, we define some elasticities as

$$\eta_i \equiv \frac{\omega_i l_{i\omega_i}}{l_i} = -\frac{\omega_i l_{it_i}}{l_i w_i} = -\frac{(1-t_i)l_{it_i}}{l_i} > 0, \quad i = m, f, \quad (55)$$

$$\varepsilon_i \equiv -\frac{\omega_i h_{i\omega_i}}{h_i} = \frac{\omega_i h_{it_i}}{h_i w_i} = \frac{(1-t_i)h_{it_i}}{h_i} > 0, \quad i = m, f,$$

$$\theta_i \equiv -\frac{\omega_i n_{\omega_i}}{n} = \frac{\omega_i n_{t_i}}{n w_i} = \frac{(1-t_i)n_{t_i}}{n} > 0, \quad i = m, f,$$

$$\delta \equiv -\frac{(1+t_n)n_{t_n}}{n} > 0,$$

[46]Note that $y'_i = p_y(dy_i/d[(1+t_y)p_y])$. Thus, Ξ is the own-price elasticity of commodity y.

where we use the definition of $\omega_i (\equiv (1-t_i)w_i)$. η_i is the (after-tax) wage elasticity of labor supply and ε_i is the (after-tax) wage elasticity of childcare time. θ_i is the wage elasticity of child quantity and involves the effect of ω_i on n, determined in the second stage. δ can be interpreted as the price elasticity of child quantity.[47] Note that all elasticities are defined as positive values in this study. In addition, we adopt the following definitions:

$$\alpha^i_{hl} \equiv \frac{(1-t_i)w_i h_i}{(1-t_i)w_i l_i}, \quad \alpha^i_{nl} \equiv \frac{(1+t_n)p_n n}{(1-t_i)w_i l_i}, \quad i = m, f. \tag{56}$$

α^i_{hl} is the ratio between the after-tax labor income and the value of childcare evaluated by the opportunity cost, and α^i_{nl} is the expenditure share of childcare expenses on after-tax labor income. The tax rates are defined by

$$r_i \equiv \frac{t_i}{1-t_i}, \;\; i = m, f, \quad r_n \equiv \frac{t_n}{1+t_n}. \tag{57}$$

Note the following three points concerning the definitions of r_i. First, from the definition of r_i, we observe that $dr_i/dt_i = 1/(1-t_i)^2 > 0$ and $dr_n/dt_n = 1/(1+t_n)^2 > 0$. Second, allowing for the first property, we observe that $t_m \gtreqless t_f \Longleftrightarrow r_m \gtreqless r_f$. Third, the sign of r_i is the same as that of t_i because $t_i < 1$ for $i = m, f$, while the sign of r_n is the same as that of t_n because $t_n > -1$. Given that the optimal tax expressions of r_i and r_n are very simple and intuitive, we treat them to examine the properties and structure of t_i and t_n at the optimum.

Using (55)–(57), (53) and (54) are transformed as the following optimal tax formula, respectively (see Appendix G).

Proposition 4. *In the endogenous fertility model, the optimal income tax rates are given by*

$$r_m = \frac{\beta\left(1 + \alpha^m_{nl}\frac{\theta_m}{\delta}\right) + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-\beta)\,\alpha^m_{hl}\varepsilon_m}{\eta_m}, \tag{58}$$

$$r_f = \frac{\beta\left(1 + \alpha^f_{nl}\frac{\theta_f}{\delta}\right) + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-\beta)\,\alpha^f_{hl}\varepsilon_f}{\eta_f}, \tag{59}$$

where $\beta \equiv \frac{1+\lambda}{\lambda}$*, and hence,* $1-\beta \equiv -\frac{1}{\lambda}$.

We first discuss the sign for optimal income tax rates. Although $1-\beta > 0$ holds from $\lambda < 0$, the sign of β is unclear. However, equation (52) shows that $\beta < 0$ holds if and only if the optimal commodity tax rate is negative. This condition implies that revenue from the tax systems, which correct the under-provision of child quality and quantity, exceeds the required level g, and thus, tax revenue beyond the required level is returned to the consumer through the negative commodity tax.[48] This

[47] Note that $n_{t_n} (\equiv \partial n/\partial t_n) = p_n(\partial n/\partial[(1+t_n)p_n])$. Thus, δ is the price elasticity of child quantity.

[48] We should also rigorously consider elements other than the component correcting the under-provision of child quality and quantity, which appear in the optimal child tax/subsidy formula. As shown in (62), the other elements mitigate income-tax induced distortions and correct the difference in bargaining power between the couple's utility and government's social welfare.

case is meaningless since the government does not virtually face a revenue constraint. Therefore, we assume that $\beta > 0$ holds; that is, the revenue from the tax systems correcting the under-provision of child quality and quantity does not satisfy the required level.[49] Given that $\frac{2}{1+\tau} \geq 1$, the optimal income tax rates are positive under $\beta > 0$, from (58) and (59).[50]

We next provide an interpretation of elasticities η_i, ε_i, θ_i, and δ in the optimal income tax expression, in relation to gender-based taxation. First, elasticity η_i, which is in the denominator, is related to price distortions between the consumption of the numeraire and working time in the outside labor market. The optimal income tax rate r_i is inversely proportional to η_i, given that the other elasticities and expenditure shares are constant. To clarify this, let us consider the case in which $\theta_m = \theta_f$, $\varepsilon_m = \varepsilon_f$, $\alpha^m_{nl} = \alpha^f_{nl}$, and $\alpha^m_{hl} = \alpha^f_{hl}$. In this case, from (58) and (59), we observe that $r_m \gtreqless r_f \Longleftrightarrow \eta_m \lesseqgtr \eta_f$: a higher tax rate should be imposed on the income of the spouse with smaller wage elasticities of labor supply, which implies that the optimal gender-based taxation involves the Ramsey inverse elasticity rule (Boskin and Sheshinski, 1983).

Second, elasticity ε_i relates to the corrective effects regarding underinvestment in childcare and the suboptimal low fertility level. As shown in (58) and (59), the optimal income tax rate r_i increases as ε_i increases, *ceteris paribus*. Regarding relative tax rates, we observe that $r_m \gtreqless r_f \Longleftrightarrow \varepsilon_m \gtreqless \varepsilon_f$ if the other elasticities and all expenditure shares are equal between a wife and husband. Notice the coefficient of ε_i. As shown by (23), income taxation corrects the inefficiently low childcare time arising from not only the non-cooperative behavior of a couple (which is related to τ) but also the externality from children on society (which is related to μH), where μH denotes the degree of the external effects of children on society.[51] The corrective effect of income taxes should be considered as being more important as τ decreases or μH increases because the effect of underinvestment in childcare on each spouse in a couple or society exacerbates. Thus, as the value of τ is smaller or that of μH is larger, more time spent on childcare should be induced by higher income taxes to improve q and then n, which leads to an increase in $N(= Hn)$. As a result, when τ decreases or μH increases, higher income taxes must be recommended to improve N. Note that the corrective role of income taxes vanishes if $\tau = 1$ and $\mu = 0$ hold because the free-rider problem does not occur due to the couple's cooperative behavior and there is no externality from children on society. Consequently, income taxation is required to correct the effect of underinvestment in childcare because of not only non-cooperative behavior but also the externality from children on society. This argument implies that income taxation has a double dividend: it can increase tax revenue as well as correct the low fertility level caused by not only non-cooperative behavior but also the externality from children.

Even if we allow for other elements, if the negative commodity tax holds, the fact that the government does not virtually face a revenue constraint remains true.

[49]We numerically confirm that β is positive in the numerical examples provided in Section 7, regardless of the availability of a childcare facility.

[50]Even if $\beta < 0$ holds, the optimal income tax rates are positive when spouses are symmetric; that is, $r_m = r_f > 0$. In the symmetric case, if $r_m = r_f < 0$ holds, the optimal child tax rate r_n is also negative from (62). However, this cannot satisfy the revenue constraint (47) because the signs of all tax rates are negative. Thus, even when $\beta < 0$ holds, the optimal income tax rates are positive under the symmetric cases.

[51]In (58) and (59), μH is multiplied by 2 because the government's weights on each spouse are equal to one.

Finally, we discuss the relationship between the optimal income tax rates and the ratio of the elasticities θ_i/δ. Since θ_i/δ includes $-n_{t_i}/n_{t_n}$ $(i = m, f)$, we observe that it reflects the impact of income tax on the child tax/subsidy through a change in child quantity. Given that $-n_{t_i}/n_{t_n} > 0$ from (29)–(31), the increase in t_i $(i = m, f)$ raises t_n.[52] The increase in t_n reduces n and then the decrease in n increases labor supply, as shown in (17) and (31). Thus, an increase in t_n mitigates the reduction in labor supply induced by income taxes. Therefore, if the other elasticities and all expenditure shares are equal between a wife and husband, a higher income tax rate should be imposed on the spouse with a higher θ_i/δ; that is, $r_m \gtreqless r_f \Longleftrightarrow \theta_m \gtreqless \theta_f$. Note that this term is a Ramsey tax consideration under a revenue constraint: if a lump-sum tax is available (i.e., $\beta = 0$), the consideration is not needed under optimal taxation.

Here, we provide the optimal child tax/subsidy expressions. First, we define

$$\chi_i \equiv -\frac{nl_{in}}{l_i} > 0, \quad i = m, f. \tag{60}$$

χ_i denotes the elasticity of working time in the outside labor market with respect to child quantity. Using (11), (12), (14), (17), and (27), (51) can be rewritten as

$$\begin{aligned} t_n \quad : \quad 0 = & -(1+\lambda)p_n n n_{t_n}^{-1} - \lambda(t_m w_m l_{mn} + t_f w_f l_{fn} + t_n p_n) + (1-t_m)w_m l_{mn} + (1-t_f)w_f l_{fn} \\ & - (1+t_n)p_n - 2(1+\mu H)\rho(1-t_m)w_m l_{mn} - 2(1+\mu H)(1-\rho)(1-t_f)w_f l_{fn} \\ & + [2(1+\mu H)(1-\rho) - 1]c' + 2(1+\mu H)[(1-\rho)(1-\gamma) + \rho\gamma](1+t_n)p_n. \end{aligned} \tag{61}$$

See Appendix H for the derivation of this condition. Applying (55)–(57) and (60) to (61), we obtain the optimal child tax/subsidy expression in the following proposition (see Appendix I).

Proposition 5. *In the endogenous fertility model, the optimal child tax/subsidy is given by*

$$r_n = \frac{\beta}{\delta} + \frac{r_m \chi_m}{\alpha_{nl}^m} + \frac{r_f \chi_f}{\alpha_{nl}^f} + (1-\beta)\Lambda, \tag{62}$$

where

$$\begin{aligned} \Lambda \quad \equiv \quad & \left\{ [1 - 2(1+\mu H)\rho] \frac{\chi_m}{\alpha_{nl}^m} + [1 - 2(1+\mu H)(1-\rho)] \frac{\chi_f}{\alpha_{nl}^f} \right\} \\ & + \frac{[1 - 2(1+\mu H)(1-\rho)]\, c'}{(1+t_n)p_n} + \{1 - 2(1+\mu H)\,[(1-\rho)(1-\gamma) + \rho\gamma]\}\,. \end{aligned} \tag{63}$$

The first term β/δ in (62) shows its own price distortion on child quantity that is in line with the Ramsey tax implication: the direct child tax/subsidy rate should be inversely proportional to the own-price elasticity δ. The second and third terms are related to the deadweight loss created by income

[52] The mechanism through which t_i increases t_n is as follows: an increase in t_i raises n and then the increase in n induces an increase in t_n owing to a rise in the tax base of t_n (i.e., the rise in $p_n n$).

taxes. Noticing that $\chi_i(\equiv -nl_{in}/l_i)$ includes l_{in} and that χ_i is multiplied by the income tax rate r_i, we observe that $r_i\chi_i/\alpha^i_{nl}$ reflects the effects of t_n on income-tax induced distortions on the labor supply of spouse i through a change in child quantity. Since a larger χ_i reflects a larger response of l_i caused by the change in n, the larger χ_i implies that an increase in the child tax leads to a larger reduction in the income-tax induced deadweight loss. Thus, as the second and third terms increase, the child tax tends to become more desirable. The last term Λ allows for the bargaining power between the spouses ρ and degree of external effects μH.

To obtain an intuition of the optimal child tax/subsidy more clearly, let us consider the case in which the bargaining power is equal across spouses ($\rho = 0.5$), there is no externality of children on society ($\mu = 0$), and a lump-sum tax is available ($\beta = 0$). We will discuss ρ, μ, and β in the optimal child tax/subsidy later. In this case, the first term in (62) vanishes and $\Lambda|_{\rho=0.5,\,\mu=0} = 0$; that is, Λ is generated when the weights are different between the spouses in the couple's utility function ($\rho \neq 0.5$) or when there is an externality of children on society ($\mu \neq 0$). Assuming that $\mu = 0$, $\rho = 0.5$, and $\beta = 0$, (62) can be rewritten as

$$r_n = \frac{r_m\chi_m}{\alpha^m_{nl}} + \frac{r_f\chi_f}{\alpha^f_{nl}} > 0. \tag{64}$$

It shows that the child tax/subsidy is not zero even if a lump-sum tax is available: the optimal intervention for a child is to unambiguously impose a tax. The income tax acts as a device to correct underinvestment in childcare and, hence, improves the suboptimal low fertility by enhancing child quality. However, income taxation reduces the aggregate working time $l_i + h_i$, which implies the deadweight loss. To partially repress the distortions, the optimal intervention for child quantity is to impose a tax because the child tax lowers n, as shown by (31), and the decrease in n raises l_i, as shown by (17).

Here, by providing optimal income taxes when $\beta = 0$, $\mu = 0$, and $\rho = 0.5$, we further clarify the explicit role of the direct child tax/subsidy. Under these conditions, (58) and (59) can be rewritten as

$$r_i = \left[\frac{2}{1+\tau} - 1\right]\frac{\alpha^i_{hl}\varepsilon_i}{\eta_i} > 0, \quad i = m, f. \tag{65}$$

From (64), (65), and Proposition 2(i), we undoubtedly observe that income taxation, not direct child subsidy, plays the role of correcting the low fertility rate arising from underinvestment in childcare due to the non-cooperative behavior of the spouses. Income taxes can directly correct the inefficient decision on childcare time and, hence, enhance the low fertility rate by improving child quality, given that the decision on child quantity is made prior to the determination of childcare time. However, the direct child subsidy does not create such effects since it cannot directly improve child quality. Thus, income taxation is a more effective policy instrument for improving low fertility arising from the non-cooperative behavior of spouses regarding the provision of childcare. Note that if $\tau = 1$, indicating that a couple cooperatively behaves, income taxation is not required from (65) because suboptimal low fertility arising from underinvestment in childcare does not occur. Thus, since the deadweight loss stemming from income taxes is not generated, the child tax is also not required from (64). The results obtained from (64) and (65) are summarized as the following corollary.

Corollary 1. *Suppose that the lump-sum tax is available, there is no externality of children on society, and the bargaining power is equal across spouses. Then, income taxes are required to improve the inefficiently low fertility caused by non-cooperative behavior of the spouses, and the child tax is required to mitigate the income-tax induced price distortion. If the couple is completely cooperative in improving child quality, income taxes are not required, and hence, the child tax is also not required.*

In addition, we elucidate the optimal design of both income tax rates and the direct child tax/subsidy when $\tau = 1$, $\beta = 0$, and $\rho = 0.5$. Under these conditions, (58), (59), and (62) can be rewritten as

$$r_i = \frac{\mu H \alpha_{hl}^i \varepsilon_i}{\eta_i} > 0, \quad i = m, f, \quad r_n = \frac{r_m \chi_m}{\alpha_{nl}^m} + \frac{r_f \chi_f}{\alpha_{nl}^f} - \mu H \left[\frac{\chi_m}{\alpha_{nl}^m} + \frac{\chi_f}{\alpha_{nl}^f} + 1 + \frac{c'}{(1+t_n)\, p_n} \right]. \tag{66}$$

These results provide the characterization of income tax rates and the direct child/tax subsidy to correct the low fertility stemming from the external effects of children on society. Note that the third term on the right hand side of r_n in (66) is negative. From Proposition 2(i) and (66), we observe that both income taxation and direct child subsidy play the role of correcting child quantity that deviates from a socially desirable level due to the external effects of children on society. However, given that the first and second terms on the right hand side of r_n in (66) is positive, a decrease in the direct child subsidy is required to mitigate the deadweight loss caused by income taxes, which may lead to the desirability of the direct child tax. Consequently, in our model, the direct child subsidy is not necessarily always optimal to correct the externality of children on society. The following corollary summarizes these statements.

Corollary 2. *Suppose that the spouses are completely cooperative, the lump-sum tax is available, and the bargaining power is equal across spouses. Income taxes are required in the presence of the externality of children on society. Whether the optimal intervention on children becomes the child tax or child subsidy depends on the relative size between the income-tax induced distortion and the externality of children on society.*

Now, we turn to exploring the implication of the last term Λ, which relates to the bargaining power between both spouses ρ and the degree of the external effects μH. First, to focus on the role of the spousal bargaining power, consider that $\mu = 0$. Then, (63) can be rewritten as

$$\Lambda|_{\mu=0} = (2\rho - 1) \left[\left(\frac{\chi_f}{\alpha_{nl}^f} - \frac{\chi_m}{\alpha_{nl}^m} \right) + \frac{c'}{(1+t_n)p_n} + (1 - 2\gamma) \right]. \tag{67}$$

Totally differentiating (27) with respect to ρ and n and making use of (17), (56), and (60), we obtain[53]

$$\frac{\partial n}{\partial \rho} = \left[\frac{(1+t_n)p_n}{c''(1-\rho)} \right] \left[\left(\frac{\chi_f}{\alpha_{nl}^f} - \frac{\chi_m}{\alpha_{nl}^m} \right) + \frac{c'}{(1+t_n)p_n} + (1 - 2\gamma) \right]. \tag{68}$$

[53] $\partial n / \partial \rho$ is independent of μ because n is determined by the couple ignoring μ.

See Appendix J. Noting that $1-\rho>0$, $1+t_n>0$, and $c''>0$, from (67) and (68), we observe that

$$\Lambda|_{\mu=0} \gtreqless 0 \Longleftrightarrow (\rho-0.5)\left(\frac{\partial n}{\partial \rho}\right) \gtreqless 0. \tag{69}$$

First, we clarify the meaning of (69). Without loss of generality, we assume that $\rho>0.5$: the bargaining power of the husband is larger than that of the wife. If $\partial n/\partial \rho>(<)0$, the husband wants to increase (decrease) in child quantity. Under $\rho>0.5$, the children are over-born (under-born) for the government, because the government places equal weights on the spouses in the welfare function. Thus, the government increases (decreases) the child tax rate to decrease (increase) the child quantity. Hence, from (62), the optimal child tax (subsidy) increases with the absolute value of Λ if $\Lambda>(<)0$. This argument holds even in the case where $\rho<0.5$; that is, the wife has more bargaining power than the husband. When $\rho=0.5$, since the weights on the spouses are equal between the couple's utility and the government's welfare function, the government does not need to adjust the suboptimal low fertility level caused by a difference in the weights.[54] Thus, if $\rho=0.5$, $\Lambda|_{\mu=0}=0$.

Next, we explore the determinants of the sign of $\partial n/\partial \rho$. The sign of $\partial n/\partial \rho$ depends on the three terms $(\chi_f/\alpha_{nl}^f-\chi_m/\alpha_{nl}^m)$, $c'/(1+t_n)p_n$, and $(1-2\gamma)$ in (68) because $(1+t_n)p_n/c''(1-\rho)>0$. First, we consider the meaning of the term $\chi_f/\alpha_{nl}^f-\chi_m/\alpha_{nl}^m$. Roughly speaking, as the effect of child quantity on labor supply of spouse i increases, χ_i/α_{nl}^i becomes large because χ_i includes l_{in}. The reduction in labor supply of each spouse is harmful to them because it reduces their private consumption, while a part of the labor supply reduction is used for childcare and leads to improvements in child quality, which is beneficial to both spouses. Consider the condition that $\chi_f/\alpha_{nl}^f-\chi_m/\alpha_{nl}^m>0$ holds. Given (60), it implies a larger reduction in the wife's labor supply. Under this condition, the husband wants to increase child quantity in the second stage because the increase in n benefits him without a large reduction in his private consumption. Thus, an increase in the husband's bargaining power ρ under the condition that $\chi_f/\alpha_{nl}^f-\chi_m/\alpha_{nl}^m>0$ contributes toward an increase in child quantity. The second term $c'/(1+t_n)p_n$ describes the allowance for the cost incurred by the wife. As the bargaining power of the husband ρ is large, child quantity increases because the cost $c(n)$ is irrelevant to the husband.[55] The final term $(1-2\gamma)$ is related to the cost burden of raising children. If $0.5>\gamma$ (i.e., if a smaller cost burden toward the expenditure of bringing up children is imposed on the husband), then the husband aims to increase the child quantity and, thus, the final term contributes to becoming $\frac{\partial n}{\partial \rho}>0$.

Finally, we discuss the external effects of children on society, reflected by μH, and the required tax revenue, represented by β, in the optimal child tax/subsidy. To clarify the effects of μH and β in (62),

[54] In addition to the non-cooperative behavior of couples and the external effects of children on society, the government must allow for the difference in social and private welfare as the third factor that causes the suboptimal low fertility rate.

[55] As shown in the fourth term in (70), the marginal cost of the child quantity c' contributes to the lower tax rate (higher subsidy rate). To improve the fertility rate, the direct child tax should be reduced. This is true even under childcare facilities. Indeed, (92) under $\rho=0.5$ yields the same expression as (70).

we consider the case where $\rho = 0.5$. In this case, (62) is reduced to

$$r_n = \frac{\beta}{\delta} + \frac{r_m \chi_m}{\alpha_{nl}^m} + \frac{r_f \chi_f}{\alpha_{nl}^f} - (1-\beta)\mu H \left[\frac{\chi_m}{\alpha_{nl}^m} + \frac{\chi_f}{\alpha_{nl}^f} + 1 + \frac{c'}{(1+t_n)\, p_n} \right]. \tag{70}$$

We observe that the direct child tax/subsidy depends on Ramsey consideration that allows for price distortion in the revenue-constrained optimal taxation framework, and Pigou consideration that allows for correcting the externality of children on society. The former is shown by the first term, which includes β, and the latter is shown by the fourth term, which includes $1-\beta$ and μH. Moreover, the direct child tax/subsidy allows for income-tax induced deadweight loss, which correspond to the second and third terms that include the income tax rates depending on β, $1-\beta$, and μH from (58) and (59). Allowing for these facts, we analyze the effects of β and μH on optimal child tax/subsidy. The first, second, and third terms are positive, while the fourth term is negative. Given the sign of each term, as the required tax revenue increases (as indicated by a larger β) or as the external effects of children on society reduce (as indicated by a smaller μH), the first term tends to be larger than the fourth term (i.e., Ramsey consideration dominates Pigou consideration), *ceteris paribus*. Given that the second and third terms are positive irrespective of the changes in β and μH, the direct child tax is likely to be optimal as β increases or μH decreases.

However, as the required tax revenue decreases or the degree of the external effects of children on society increases, we cannot conclude that the direct child subsidy tends to be optimal. As the required tax revenue reduces, the fourth term grows larger than the first term (i.e., Ramsey consideration is dominated by Pigou consideration), while the change in the second and third terms in response to a decrease in β is ambiguous from (58) and (59). Moreover, as the degree of external effects of children on society increases, we observe that the second and third terms take larger positive values from (58) and (59), whereas the fourth term takes a larger negative value, *ceteris paribus*. Hence, under the two conditions, it is unclear if the direct child subsidy is desirable. We numerically examine how the changes in the required tax revenue and the degree of the external effects of children on society affect the optimal direct child tax/subsidy in Section 7.

6 Childcare Facility

In this section, we introduce center-based childcare services, such as facilities for early childhood education, preschools, and cram schools. These services can substitute for the childcare time each spouse contributes. Let us denote the number of hours that children per couple spend in a childcare facility by h_c. In this model, although the couple collectively decides the time children spend in a childcare facility as well as child quantity, such decisions are not made simultaneously.[56] We modify

[56]Due to the long-term nature of bringing up children, the decision on child quantity that a couple will have is made prior to using services at a childcare facility. Therefore, we consider that the couple decides the time they plan to use the childcare facility after they choose the child quantity. Moreover, a motivation for allowing cooperation regarding the use of the childcare facility stems from the fact that parents can observe the amount of time children spend at the center.

the sequential decisions of the government, the couple, and each partner in the couple, as follows: first, the government determines the tax rates; second, the couple collectively decides on child quantity; third, the couple collectively decides the amount of time children spend in the childcare facility; and finally, each spouse non-cooperatively decides his/her two kinds of private consumption, labor supply in the external market, and time spent on domestic childcare.[57]

The function of child quality q is modified by

$$q = \frac{\left(s_m \frac{h_m}{n}\right)^{\sigma}}{\sigma} + \frac{\left(s_f \frac{h_f}{n}\right)^{\sigma}}{\sigma} + \frac{\left(s_c \frac{h_c}{n}\right)^{\sigma}}{\sigma} = n^{-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right], \tag{71}$$

where s_c is the productivity of a childcare facility.[58] To simplify the analysis, we assume that the curvature of the quality function for the time children spend in the childcare facility, σ, is the same as that on childcare time spent by each spouse.[59] The budget constraint of each spouse is modified by

$$z_i + (1 + t_y)p_y y_i + \nu_i\{(1 + t_n)p_n n + (1 + t_c)p_c h_c\} = (1 - t_i)w_i l_i, \quad i = m, f. \tag{72}$$

The expenditure on the childcare facility is given by $(1 + t_c)p_c h_c$, where p_c is the hourly price and t_c is the tax/subsidy rate for using a childcare facility. ν_i is spouse i's share of the total expenditure on the fertility good and childcare facility. Generally, spouse i's share of childcare facility expenditure may differ from that of fertility good expenditure. In Appendix K, we examine the optimal tax structure under the case in which the two types of cost shares of spouse i differ.

Defining $\nu_m \equiv \nu$ (and hence, $\nu_f \equiv 1 - \nu$) and substituting (71) for q in (1) and (72) for z_i in (1) yield

$$\begin{aligned} u_m \quad = \quad & (1 - t_m)w_m l_m - (1 + t_y)p_y y_m - \nu\{(1 + t_n)p_n n + (1 + t_c)p_c h_c\} + \frac{y_m^{\varphi}}{\varphi} \\ & - \frac{(l_m + h_m)^{1+\phi}}{1 + \phi} + (1 + \mu H)n^{1-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right], \end{aligned} \tag{73}$$

$$\begin{aligned} u_f \quad = \quad & (1 - t_f)w_f l_f - (1 + t_y)p_y y_f - (1 - \nu)\{(1 + t_n)p_n n + (1 + t_c)p_c h_c\} + \frac{y_f^{\varphi}}{\varphi} \\ & - \frac{\left(l_f + h_f\right)^{1+\phi}}{1 + \phi} + (1 + \mu H)n^{1-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right] - c(n). \end{aligned}$$

[57]Even if the order of the third (h_c) and fourth decisions (l_i and h_i) is reversed, the main qualitative results are unaffected because l_i and h_i ($i = m, f$) are separable from h_c in the utility functions (see equation (73)).

[58]Bastani et al. (2020) consider the quality of a childcare facility as a choice variable of parents. For simplicity, the quality of the childcare facility is given at an exogenous certain level in our model.

[59]The difference between home care productivity for each spouse and the quality of the childcare facility is indicated by the difference between s_i ($i = m, f$) and s_c. Although we checked the numerical results for how an increase in s_c affects optimal tax/subsidy structures while keeping s_m and s_f constant, we omit the results because the implication is straightforward; the rise in s_c increases the optimal subsidy rate on the use of the childcare facility because the use of childcare facilities becomes more beneficial.

The differences in each spouse's utility function between the cases with and without the childcare facility are the expenditure on the childcare facility (second term in braces) and contribution of childcare facilities to child quality (third term in square brackets). Note that because y_i, l_i, and h_i are additively separable with respect to h_c in each spouse's utility function, the first-order conditions of each spouse with respect to y_i, l_i, and h_i are identical to (10), (11), and (12). Hence, equations (13)–(25) hold even in the model with the childcare facility.[60] This fact is used in the analysis in this section.

Substituting (73) for u_i in (7) and allowing for (13)–(15), we obtain the couple's utility function:

$$\begin{aligned} u \;=\; & \rho\left[(1-t_m)w_m l_m(t_m,n) - (1+t_y)p_y y_m(t_y) + \frac{\left(y_m(t_y)\right)^{\varphi}}{\varphi} - \frac{\left(l_m(t_m,n)+h_m(t_m,n)\right)^{1+\phi}}{1+\phi}\right] \\ & + (1-\rho)\left[(1-t_f)w_f l_f(t_f,n) - (1+t_y)p_y y_f(t_y) + \frac{\left(y_f(t_y)\right)^{\varphi}}{\varphi} - \frac{\left(l_f(t_f,n)+h_f(t_f,n)\right)^{1+\phi}}{1+\phi} - c(n)\right] \\ & - \left[(1-\rho)(1-\nu)+\rho\nu\right]\left\{(1+t_n)p_n n + (1+t_c)p_c h_c\right\} \\ & + (1+\mu H)n^{1-\sigma}\left[\frac{\left(s_m h_m(t_m,n)\right)^{\sigma}}{\sigma} + \frac{\left(s_f h_f(t_f,n)\right)^{\sigma}}{\sigma} + \frac{\left(s_c h_c\right)^{\sigma}}{\sigma}\right]. \end{aligned} \tag{74}$$

As mentioned above, the spouses collectively maximize u first with respect to n and next with respect to h_c. First, we show the couple's determination of h_c. Given μNq,[61] the first-order condition of (74) with respect to h_c is

$$0 = \frac{\partial u}{\partial h_c} = -\left[(1-\rho)(1-\nu)+\rho\nu\right](1+t_c)p_c + n^{1-\sigma}s_c^{\sigma}h_c^{\sigma-1}. \tag{75}$$

Solving this equation with respect to h_c, we immediately obtain the following function:

$$h_c(t_c,n) = \left\{\left[(1-\rho)(1-\nu)+\rho\nu\right](1+t_c)p_c\right\}^{-\frac{1}{1-\sigma}} s_c^{\frac{\sigma}{1-\sigma}} n. \tag{76}$$

From (76), we obtain

$$h_{ct_c}\left(\equiv \frac{\partial h_c}{\partial t_c}\right) = -\left(\frac{1}{1-\sigma}\right) h_c (1+t_c)^{-1} < 0, \tag{77}$$

$$h_{cn}\left(\equiv \frac{\partial h_c}{\partial n}\right) = \left\{\left[(1-\rho)(1-\nu)+\rho\nu\right](1+t_c)p_c\right\}^{-\frac{1}{1-\sigma}} s_c^{\frac{\sigma}{1-\sigma}} > 0. \tag{78}$$

The intuitions for the two results are highly straightforward.

We now turn to the couple's decision about child quantity. Allowing for $h_c = h_c(t_c,n)$, the couple maximizes the utility function (74) with respect to n. Considering μNq as given, the first-order condition

[60]Note that each spouse takes μNq as given in the optimization with with respect to y_i, l_i, and h_i. In (73), μNq corresponds to $\mu H n^{1-\sigma}\left[\frac{(s_m h_m(t_m,n))^{\sigma}}{\sigma} + \frac{(s_f h_f(t_f,n))^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right]$.

[61]In (74), μNq corresponds to $\mu H n^{1-\sigma}\left[\frac{(s_m h_m(t_m,n))^{\sigma}}{\sigma} + \frac{(s_f h_f(t_f,n))^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right]$.

with respect to n is that

$$
\begin{aligned}
0 \quad = \quad & \frac{\partial u}{\partial n} = -\left[(1-\rho)(1-\nu)+\rho\nu\right](1+t_n)p_n - (1-\rho)c'(n) \\
& + \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + 1 - \rho\right)(1-t_m)w_m h_{mn}(t_m) + \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \rho\right)(1-t_f)w_f h_{fn}(t_f) \\
& + \left(\frac{1-\sigma}{\sigma}\right)\left[(1-\rho)(1-\nu)+\rho\nu\right](1+t_c)p_c h_{cn}(t_c),
\end{aligned} \tag{79}
$$

where we use (11), (12), (14), (17), (75), (76), and (78) to derive this equation (see Appendix L). Equation (79) implies

$$n = n(t_c, t_n, t_m, t_f). \tag{80}$$

Here, we propose the impact of each tax rate on child quantity. Totally differentiating (79) with respect to n, t_m, t_f, t_n, and t_c yields

$$n_{t_m}\left(\equiv \frac{\partial n}{\partial t_m}\right) = \frac{\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + 1 - \rho\right)\left(\frac{\sigma}{1-\sigma}\right) w_m (1+\tau)^{\frac{1}{1-\sigma}} \omega_m^{-\frac{1}{1-\sigma}} s_m^{\frac{\sigma}{1-\sigma}}}{(1-\rho)c''} > 0, \tag{81}$$

$$n_{t_f}\left(\equiv \frac{\partial n}{\partial t_f}\right) = \frac{\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \rho\right)\left(\frac{\sigma}{1-\sigma}\right) w_f (1+\tau)^{\frac{1}{1-\sigma}} \omega_f^{-\frac{1}{1-\sigma}} s_f^{\frac{\sigma}{1-\sigma}}}{(1-\rho)c''} > 0, \tag{82}$$

$$n_{t_n}\left(\equiv \frac{\partial n}{\partial t_n}\right) = -\frac{\left[(1-\rho)(1-\nu)+\rho\nu\right]p_n}{(1-\rho)c''} < 0, \tag{83}$$

$$n_{t_c}\left(\equiv \frac{\partial n}{\partial t_c}\right) = -\frac{\left[(1-\rho)(1-\nu)+\rho\nu\right]^{-\frac{\sigma}{1-\sigma}} p_c \omega_c^{-\frac{1}{1-\sigma}} s_c^{\frac{\sigma}{1-\sigma}}}{(1-\rho)c''} < 0, \tag{84}$$

where $\omega_c \equiv (1+t_c)p_c$. To derive these four equations, we use (17) and (78). Equations (81), (82), and (83) coincide with (29), (30), and (31), respectively. The intuition behind the results is discussed below (29), (30), and (31), respectively. The intuition for (84) is extremely straightforward. The increase in t_c reduces the time children spend in the childcare facility, which means that it worsens child quality and then induces a lower fertility rate.

Substituting (80) for n in (14), (15), and (76) yields

$$h_i(t_i, n(t_c, t_n, t_m, t_f)), \quad l_i(t_i, n(t_c, t_n, t_m, t_f)), \quad \text{for } i = m, f, \quad \text{and} \quad h_c(t_c, n(t_c, t_n, t_m, t_f)). \tag{85}$$

These functions involve information regarding the decision process in the second, third, and fourth stages.

Substituting (73) for u_i in (8) and allowing for (13), (80), and (85), we obtain the government's welfare function:

$$
\begin{aligned}
\frac{W}{H} = & (1-t_m)w_m l_m(t_m, n(t_c, t_n, t_m, t_f)) + \frac{(y_m(t_y))^{\varphi}}{\varphi} \\
& - \frac{(l_m(t_m, n(t_c, t_n, t_m, t_f)) + h_m(t_m, n(t_c, t_n, t_m, t_f)))^{1+\phi}}{1+\phi} + (1-t_f)w_f l_f(t_f, n(t_c, t_n, t_m, t_f)) \\
& + \frac{(y_f(t_y))^{\varphi}}{\varphi} - \frac{(l_f(t_f, n(t_c, t_n, t_m, t_f)) + h_f(t_f, n(t_c, t_n, t_m, t_f)))^{1+\phi}}{1+\phi} \\
& - (1+t_y)p_y(y_m(t_y) + y_f(t_y)) - c(n(t_c, t_n, t_m, t_f)) - (1+t_n)p_n n(t_c, t_n, t_m, t_f) \\
& - (1+t_c)p_c h_c(t_c, n(t_c, t_n, t_m, t_f)) \\
& + 2(1+\mu H)(n(t_c, t_n, t_m, t_f))^{1-\sigma}\left[\frac{(s_m h_m(t_m, n(t_c, t_n, t_m, t_f)))^{\sigma}}{\sigma}\right. \\
& \left. + \frac{(s_f h_f(t_f, n(t_c, t_n, t_m, t_f)))^{\sigma}}{\sigma} + \frac{(s_c h_c(t_c, n(t_c, t_n, t_m, t_f)))^{\sigma}}{\sigma}\right].
\end{aligned} \tag{86}
$$

The government's revenue constraint is modified by

$$
\begin{aligned}
g = & t_m w_m l_m(t_m, n(t_c, t_n, t_m, t_f)) + t_f w_f l_f(t_f, n(t_c, t_n, t_m, t_f)) \\
& + t_y p_y \left(y_m(t_y) + y_f(t_y)\right) + t_n p_n n(t_c, t_n, t_m, t_f) + t_c p_c h_c(t_c, n(t_c, t_n, t_m, t_f)),
\end{aligned} \tag{87}
$$

where the fifth term represents tax revenue from the tax/subsidy for using the childcare facility. From the government's social welfare maximization subject to a revenue constraint, we obtain the optimal tax expressions for t_y, t_m, t_f, t_n, and t_c. Before characterizing them, we define the following tax rate:

$$
r_c \equiv \frac{t_c}{1+t_c}. \tag{88}
$$

In the economy with a childcare facility, we derive the following optimal tax formulas (see Appendix M).

Proposition 6. *In the endogenous fertility model with a childcare facility, optimal taxes are characterized by*

$$
r_y = \frac{\beta}{\Xi}, \tag{89}
$$

$$
r_m = \frac{\beta\left(1 + \alpha_{nl}^m \frac{\theta_m}{\delta}\right) + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-\beta)\,\alpha_{hl}^m \varepsilon_m}{\eta_m}, \tag{90}
$$

$$
r_f = \frac{\beta\left(1 + \alpha_{nl}^f \frac{\theta_f}{\delta}\right) + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-\beta)\,\alpha_{hl}^f \varepsilon_f}{\eta_f}, \tag{91}
$$

$$
r_n = \frac{\beta}{\delta} + \frac{r_m \chi_m}{\alpha_{nl}^m} + \frac{r_f \chi_f}{\alpha_{nl}^f} + (1-\beta)\Omega, \tag{92}
$$

$$r_c = (1-\beta)\{1 - 2(1+\mu H)[(1-\rho)(1-\nu) + \rho\nu]\}, \tag{93}$$

where

$$\Omega \equiv \left\{[1-2(1+\mu H)\rho]\frac{\chi_m}{\alpha^m_{nl}} + [1-2(1+\mu H)(1-\rho)]\frac{\chi_f}{\alpha^f_{nl}}\right\} + \frac{[1-2(1+\mu H)(1-\rho)]\,c'}{(1+t_n)p_n} + \{1-2(1+\mu H)\,[(1-\rho)(1-\nu)+\rho\nu]\}\,. \tag{94}$$

Comparing (89)–(91) with (52), (58), and (59), we observe that the optimal commodity and income tax expressions are identical to those in the case without a childcare facility. Noting that Ω is obtained by replacing γ in (63) with ν, we can observe that the expression for optimal child tax/subsidy (92) is identical to (62). Hence, the optimal child tax/subsidy takes the same formula irrespective of whether the childcare facility is available or not. The interpretation for the optimal child tax/subsidy schemes remains unchanged, regardless of the availability of a childcare facility. The expression for the optimal tax/subsidy for center-based childcare services (93) reflects the corrections for h_c deviating from a socially desirable level because of the parameters ρ, ν, and μH. The intuition is similar to that of the third term in Λ, which is discussed below Proposition 5.[62] If $\rho = 0.5$ or $\nu = 0.5$ holds, the formula of (93) reduces to

$$r_c = -(1-\beta)\mu H \leq 0. \tag{95}$$

The optimal intervention in the childcare facility is to unambiguously provide a subsidy to correct the external effects of children on society, provided $\mu > 0$. As the subsidy for center-based childcare services increases the time devoted to a childcare facility, it improves child quality and, hence, enhances child quantity. If there is no externality of children on society (i.e., $\mu = 0$), then r_c is zero; the tax/subsidy on the use of center-based childcare services is not needed. The intuition is given below equation (97).

Next, as one of the primary concern, we examine the ranking of r_n and r_c at the optimum. From (92) and (93), we obtain

$$r_n - r_c = \frac{\beta}{\delta} + \frac{r_m\chi_m}{\alpha^m_{nl}} + \frac{r_f\chi_f}{\alpha^f_{nl}} + (1-\beta)\left\{[1-2(1+\mu H)\rho]\frac{\chi_m}{\alpha^m_{nl}} + [1-2(1+\mu H)(1-\rho)]\frac{\chi_f}{\alpha^f_{nl}} + \frac{[1-2(1+\mu H)(1-\rho)]c'}{(1+t_n)p_n}\right\}. \tag{96}$$

We observe that the optimal ranking of r_n and r_c depends on Ramsey consideration, which corresponds

[62] First, suppose $\mu = 0$ to focus on the role of bargaining power. In this case, the term in (93) is $(1-\beta)(2\rho-1)(1-2\nu)(\equiv \Theta)$. Furthermore, differentiating (76) with respect to ρ yields $\frac{\partial h_c}{\partial \rho} = \frac{1-2\nu}{1-\sigma}\frac{h_c}{(1-\rho)(1-\nu)+\rho\nu}$, which leads to $\frac{\partial h_c}{\partial \rho} \gtreqless 0$ if $0.5 \gtreqless \nu$. It indicates that the effects of the difference in the bargaining power on h_c occurs only if the cost shares of spouses are different ($\nu \neq 0.5$). Using these results, we observe that $\Theta \gtreqless 0 \iff (\rho - 0.5)\frac{\partial h_c}{\partial \rho} \gtreqless 0$, which is similar to the meaning of (69). Consequently, ρ and ν are critical to determine the sign of Θ. Second, we consider either $\rho = 0.5$ or $\nu = 0.5$ to examine the effects of μH. This case is discussed below (95).

to the first term including β, and Pigou consideration, which corresponds to the fourth term including $1-\beta$ and μH. To focus on the impact of changes in β and μH, we assume $\rho = 0.5$. Under $\rho = 0.5$, the fourth term other than the weight $(1-\beta)$ can be rewritten as $-\mu H[\frac{\chi_m}{\alpha^m_{nl}} + \frac{\chi_f}{\alpha^f_{nl}} + \frac{c'}{(1+t_n)p_n}] < 0$. Furthermore, the optimal ranking of r_n and r_c depends on the second and third terms, which allow for the income-tax induced deadweight loss and include β, $1-\beta$, and μH from (90) and (91). Given that the first term is positive and the fourth term is negative, as the required tax revenue increases (as indicated by a larger β), the first term is likely to be larger than the absolute value of the fourth term, *ceteris paribus*. Additionally, it is likely to hold as the external effects of children on society reduce (as indicated by a smaller μH), *ceteris paribus*. Thus, since the second and third terms are positive, irrespective of the changes in β and μH, $r_n > r_c$ is likely to hold as β increases or μH decreases. These cases imply that Ramsey consideration dominates Pigou consideration.

However, we cannot conclude that the optimal tax/subsidy structure is such that $r_n < r_c$ as the required tax revenue decreases or the degree of the external effects of children on society increases. First, as the required tax revenue reduces, the absolute value of the fourth term under $\rho = 0.5$ is likely to be larger than the first term, *ceteris paribus*. However, since it is unclear how the second and third terms change in response to a decrease in β from (90) and (91), it is ambiguous if $r_n < r_c$ holds. Next, as the degree of the external effects of children on society becomes larger, we observe that the second and the third terms increase with μH from (90) and (91), while the fourth term decreases with μH under $\rho = 0.5$, *ceteris paribus*. Given that the second and third terms are positive and the fourth term is negative, the two effects work in opposite directions in terms of an increase in μH; thus, we cannot conclude that $r_n < r_c$ holds. These results are confirmed in the numerical analysis in Section 7.

Finally, we clarify the role of each policy instrument to correct the inefficiently low fertility arising from the non-cooperative behavior of a couple. To this end, we assume that $\mu = 0$, $\rho = 0.5$, and $\beta = 0$. Then, from Proposition 6, we obtain

$$r_i = \left[\frac{2}{1+\tau} - 1\right]\frac{\alpha^i_{hl}\varepsilon_i}{\eta_i} > 0, \quad i = m, f, \quad r_n = \frac{r_m\chi_m}{\alpha^m_{nl}} + \frac{r_f\chi_f}{\alpha^f_{nl}} > 0, \quad r_c = 0. \tag{97}$$

The optimal income tax and direct child tax/subsidy are identical to (64) and (65), and the intuitions are given below each equation. Note that no intervention in the childcare facility is optimal. First, we examine the reason why a subsidy is not required for center-based childcare services. Although such a subsidy can improve fertility from (84), it is not required from (95) if there is no externality of children on society. This is because the subsidy for center-based childcare services yields price distortion on the choice of h_c, which is efficiently decided by the couple, while income taxes directly adjust h_m and h_f, which were under-provided in our model. Thus, income taxes are more effective than the subsidy for center-based childcare services to correct the non-cooperative behavior of couples. Next, we clarify the reason why the tax for center-based childcare services is not required. The intuitive interpretation for this result is obtained by comparing the tax on external childcare services with the direct child tax. As shown by (97), a direct child tax is needed to mitigate income-tax induced deadweight loss. From

(17) and (84), we can observe that the tax on external childcare services also induces the downward distortion on child quantity and, thus, it also mitigates price distortions on labor supply induced by income tax. However, from (77), the tax on external childcare services affects resource allocation other than child quantity n; it yields the price distortion on the time use of center-based childcare services. For this reason, the child tax is more effective than the tax on external childcare services to mitigate the deadweight loss induced by income taxes. Thus, the tax on external childcare services is not required. The results obtained from (97) are summarized as the following corollary.

Corollary 3. *Corollaries 1 and 2 hold even under the childcare facility. Suppose that the bargaining power is equal across spouses. Then, the subsidy for center-based childcare services is needed if and only if there is the externality of children on society, regardless of the degree of cooperation within a couple.*

7 Numerical Analysis

This section numerically examines the optimal tax structure in the presence of a childcare facility when some important and suggestive parameters vary. The overall objective of this numerical analysis is to illustrate and reinforce our theoretical results, which provide important policy implications. We consider the variations of parameters μH, τ, g, w_m, and ρ. The change in μH implies the variation in the degree of the external effects of children on society. The increase in τ induces more cooperative behavior of the couple and thus, mitigates the free-rider problem for childcare. Furthermore, since the required tax revenue per household g is positively correlated with β, an increase in g enables us to examine the effect of an increase in β in the optimal tax structure. The variations in μH, τ, and g clarify the properties of the optimal tax structure, and those of w_m and ρ are undertaken to examine the gender-based income taxation under the asymmetric spouses.

To make the analysis tractable, we specify the function $c(n)$ and the parameters as follows: $c(n) = n^2/2$, $\varphi = 0.2$, $\sigma = \nu = 0.5$, $\phi = 1.0$, $w_f = p_c = 4.2$, $s_m = s_f = 1.2$, $s_c = 1.2$, and $p_y = p_n = 1.0$.[63] Then, (79) yields

$$n = -\frac{1+t_n}{2(1-\rho)} + \frac{6\left(2(1+\tau)-\rho(1+\tau)^2\right)}{5(1-\rho)(1-t_m)w_m} + \frac{2\left(2(1+\tau)-(1+\tau)^2+\rho(1+\tau)^2\right)}{7(1-\rho)(1-t_f)} + \frac{4}{7(1-\rho)(1+t_c)}, \quad (98)$$

and (81)–(84) can be rewritten as

$$n_{t_m} = \frac{6\left(2(1+\tau)-\rho(1+\tau)^2\right)}{5(1-\rho)(1-t_m)^2 w_m}, \quad n_{t_f} = \frac{2\left(2(1+\tau)-(1+\tau)^2+\rho(1+\tau)^2\right)}{7(1-\rho)(1-t_f)^2}, \quad (99)$$
$$n_{t_n} = -\frac{1}{2(1-\rho)}, \quad n_{t_c} = -\frac{4}{7(1-\rho)(1+t_c)^2}.$$

As a benchmark case, we consider the symmetric spouses under fully non-cooperative case, in which

[63] Since labor intensity in center-based childcare services is very high, we assume that p_c equals the wage rate w_f.

$\mu H = 0.15$, $\tau = 0$, $g = 5.0$, $w_m = 4.2$, and $\rho = 0.5$.[64] Unless otherwise noted, we consider these values under which the spouses are symmetric. We use these numerical values of the parameters, (98), and (99), in numerically deriving the optimal tax rates t_y, t_i ($i = m, f$), t_n, and t_c. Notice that the signs of optimal tax rates and the relative size between them are unchanged, even if we employ r_y, r_i ($i = m, f$), r_n, and r_c (see below (57)).

7.1 Child Subsidy

We first investigate how the parameter μH, indicating the degree of external effects of children on society, affects the optimal tax structure. Table 1 presents the optimal tax rates when μH takes values from 0 to 0.3 with an interval of 0.05. Note that the optimal income tax rates are always the same between the spouses, $t_m = t_f$, because of spousal symmetry. As μH increases (i.e., the deviation of child quality and quantity from a socially desirable level is greater), the optimal income tax rates increase in order to increase childcare time and then improve child quality and quantity, and the optimal subsidy rate for center-based childcare services increases to promote time use of childcare facilities and, hence, improves child quality and quantity. The child tax (subsidy) becomes optimal if the external effect of children is relatively small (large). More interestingly, the optimal child tax rate decreases to $\mu H = 0.25$ and then increases; it takes a U-shaped pattern with respect to the degree of the external effects of children on society. The intuition for this change is as follows. In the former part, the optimal child tax rate decreases to directly enhance child quantity. Before explaining the latter part, note that the tax-induced deadweight loss sharply increases with its tax rate. To mitigate the price distortions induced by income taxation, the optimal child tax rate increases in the latter part because the optimal income tax rates are sufficiently high.[65] The results differ from the optimal structure of child tax/subsidy obtained in the previous literature; the child subsidy ($t_n < 0$) is optimal under $\mu > 0$ and increases with μH. To secure tax revenues for subsidies for external childcare services and to compensate for the deficit in revenue due to a decrease in the child tax, both the commodity and income taxes increase with μH.

Table 1. Optimal Tax Rates: Change in μH

μH	0	0.05	0.1	0.15	0.2	0.25	0.3
t_y	0.08355	0.08357	0.087	0.094	0.106	0.128	0.161
t_m	0.151	0.167	0.189	0.219	0.256	0.301	0.352
t_f	0.151	0.167	0.189	0.219	0.256	0.301	0.352
t_n	0.299	0.161	0.031	−0.085	−0.174	−0.215	−0.185
t_c	0	−0.043	−0.083	−0.118	−0.150	−0.177	−0.199

[64]Note that since the marginal utility of parents' own children is 1, as shown in (1), we largely discount the marginal utility of children through the externality across couples by μH.

[65]Even if μH is greater than 0.3, the optimal child tax rate continues to increase. In particular, the sign of the tax rate changes between 0.35 and 0.4 (i.e., $t_n > 0$). However, the tax revenue from income taxes decreases with μH when μH is greater than 0.3. To focus on the left of the Laffer curve that an increase in taxes would raise the tax revenue, we omit the numerical results in this range.

Table 1 shows that the optimal intervention for children tends to be a subsidy as μH increases. A larger μH implies that the externality of children on society becomes a more important determinant of suboptimal low fertility level than the non-cooperative behavior of the couple. Table 1 suggests that if the non-cooperative behavior of the spouses is the main cause of under-provision for children (i.e., if μH is relatively small), the direct child subsidy worsens welfare. Furthermore, Table 1 clarifies how the sign of $t_n - t_c$ changes in response to μH. As discussed below (96), the theoretical analysis demonstrates that, under the optimal taxation, t_c tends to be lower than t_n as μH becomes smaller, whereas it is unclear how the ranking of these subsidies holds as μH becomes larger, *ceteris paribus*. Table 1 shows that t_c tends to be lower than t_n as μH becomes smaller, which is consistent with the theoretical result. Meanwhile, as μH increases, the ranking of these subsidies is switched at $\mu H = 0.2$, that is, t_n tends to be lower than t_c, and it is switched again at $\mu H = 0.3$, that is, t_n is higher than t_c. This double switching stems from a U-shaped pattern of the optimal child tax/subsidy.

Table 2 shows the impact of the proportional changes in parameter τ, reflecting the degree of non-cooperative behavior, on the optimal tax structure. As τ increases (i.e., a couple behaves more cooperatively), the optimal income tax rates decrease, except for $\tau = 0$ to $\tau = 0.2$. Income taxation is not likely to be required as Piguvian corrective taxes because child quality and quantity are improved owing to more cooperative behavior. However, in the case of $\tau = 0$ to $\tau = 0.2$, income tax rates increase. From (97), the optimal income tax rates given by $\left[\frac{2}{1+\tau} - 1\right] \frac{\alpha^i_{hl}\varepsilon_i}{\eta_i}$ consist of Pigou term $\left[\frac{2}{1+\tau} - 1\right] \varepsilon_i$ and Ramsey term $\frac{\alpha^i_{hl}}{\eta_i}$. As τ increases, the Pigou term decreases and the Ramsey term increases, *ceteris paribus*.[66] According to the numerical results in Table 2, we can interpret that since Ramsey consideration puts more weight than Pigou consideration, optimal income tax rates increase in the case of $\tau = 0$ to $\tau = 0.2$. As τ becomes larger, child quality is enhanced and the number of children increases. Thus, t_n and t_c increase with τ.

Table 2. Optimal Tax Rates: Change in τ

τ	0	0.2	0.4	0.6	0.8	1.0
t_y	0.094	0.108	0.137	0.176	0.223	0.278
t_m	0.219	0.223	0.215	0.202	0.188	0.175
t_f	0.219	0.223	0.215	0.202	0.188	0.175
t_n	−0.085	0.004	0.131	0.266	0.400	0.526
t_c	−0.118	−0.116	−0.113	−0.109	−0.104	−0.098

Next, we examine the sensitivity of the optimal tax structure to changes in revenue requirement. Table 3 demonstrates the optimal tax rates when g takes values from 4.1 to 5.9 with an interval of 0.3. Note that both the optimal income taxes and the direct child tax increase, and the subsidy rate for external childcare services decreases with the revenue requirement. The tax changes are simply because of the relatively high revenue requirement.

[66] We can confirm that the Ramsey term increases with τ from (14), (15), (23), (24), (55), and (56).

The numerical results in Tables 1 and 3 yield important policy implications for the child tax/subsidy. As shown in the theoretical part, the optimal child tax/subsidy is given by (92), which coincides with (62) and then (70) under $\rho = 0.5$. As discussed below (70), the direct child tax is likely to be optimal as the required tax revenue increases or the degree of the external effects of children on society decreases. However, it is theoretically unclear if the direct child subsidy tends to be desirable as the required tax revenue decreases or the degree of the external effects of children on society increases. The numerical results in Tables 1 and 3 show that the direct child tax becomes optimal as g becomes smaller, which is consistent with the theoretical part. Meanwhile, they demonstrate that the direct child subsidy becomes optimal as μH increases, since the corrective effect on the suboptimal low fertility level arising from the externality of children on society (fourth term in (92)) is more likely to dominate the effect of price distortions under a revenue constraint (first term) and the income-tax induced distortions on labor supply (second and third terms).

Table 3. Optimal Tax Rates: Change in g

g	4.1	4.4	4.7	5.0	5.3	5.6	5.9
t_y	0.059	0.070	0.082	0.094	0.107	0.121	0.136
t_m	0.194	0.202	0.210	0.219	0.227	0.236	0.245
t_f	0.194	0.202	0.210	0.219	0.227	0.236	0.245
t_n	−0.221	−0.178	−0.133	−0.085	−0.036	0.014	0.067
t_c	−0.122	−0.121	−0.120	−0.118	−0.117	−0.115	−0.113

Another important finding is as follows. The ranking of the direct child subsidy rate and the subsidy rate for center-based childcare services are switched with the required tax revenue. As mentioned below (96), the theoretical analysis demonstrates that, under the optimal tax framework, t_c tends to be lower than t_n as g increases, whereas it is theoretically unclear how the ranking of these subsidies holds as g decreases, *ceteris paribus*. Table 3 shows that, under the optimal taxation model, t_c tends to be lower than t_n as g increases, whereas t_n tends to be lower than t_c as g decreases. Based on (96) under $\rho = 0.5$, the result $t_n < t_c$ suggests that the optimal taxes/subsidies must emphasize correcting the external effect of children on society (fourth term in (96)) more than mitigating both price distortions under a revenue constraint (first term) and income-tax induced distortions on labor supply (second and third terms). Thus, as the required tax revenue increases, the ranking order of these subsidy rates is switched. As a policy recommendation, a welfare state in a developed country, where a huge amount of tax revenue is required because the government size is generally large, should design its tax/subsidy system such that the subsidy rate for center-based childcare services would be higher than the direct child subsidy rate. However, in developing countries, where the government's size is generally small, the direct child subsidy rate would be higher than the subsidy rate for center-based childcare services.

Finally, Table 4 shows the rate of welfare gain and the change in fertility rate owing to the availability of childcare facilities, compared to a situation where childcare facilities are not available to households. Unambiguously, they are improved by the introduction of childcare facilities, as confirmed by almost all the parameter values. The improvement and expansion of childcare facilities are effective for enhancing the fertility rate and welfare. Allowing for the result in Table 1, a direct child subsidy worsens welfare if the non-cooperative behavior of spouses is the main cause of under-provision for children and, hence, the government has the option to introduce or improve childcare facilities rather than direct child subsidies.

Table 4. Impact of Childcare Facilities on Welfare and Fertility Rates

μH	0	0.05	0.1	0.15	0.2	0.25	0.3
$\frac{\widehat{W}}{H}$	0.083	0.101	0.122	0.146	0.173	0.204	0.238
$\dot{n}$	1.074	1.195	1.322	1.450	1.573	1.678	1.737

$\frac{\widehat{W}}{H}$: the rate of welfare gain, $\dot{n}$: the difference in child quantity

7.2 Gender-Based Income Taxation

Here, we consider the asymmetric cases between spouses to examine gender-based taxation. The wage rates and bargaining powers of spouses vary. First, we consider the variation in the wage rate of the husband, while keeping the wife's wage rate constant: w_m takes the values from 3.6 to 4.8 with an interval of 0.2. The case of $w_m = 4.2$ is the benchmark case, as shown in the fourth column from the right side of Table 1. The optimal tax rates in this case are given in Table 5.

Table 5. Optimal Tax Structure under Different Wage Rates

w_m	3.6	3.8	4.0	4.2	4.4	4.6	4.8
t_y	0.107	0.101	0.097	0.094	0.090	0.088	0.085
t_m	0.332	0.285	0.248	0.219	0.196	0.177	0.162
t_f	0.243	0.233	0.225	0.219	0.213	0.208	0.203
t_n	0.080	−0.003	−0.052	−0.085	−0.110	−0.130	−0.146
t_c	−0.1165	−0.1173	−0.1178	−0.1181	−0.1185	−0.1188	−0.1192

All the tax rates decrease with the husband's wage rate. The increase in wage rate w_m implies the expansion of the tax base; hence, the required tax revenue can be attained at a lower tax rate. Moreover, the optimal income tax rate on the husband is lower (higher) than that on the wife if $w_m > (<) w_f$. It is contrary to the Ramsey inverse elasticity rule, which implies that a higher tax rate should be imposed on the spouse with a smaller wage elasticity, that is, with higher productivity (Boskin and Sheshinski, 1983). In the model with time spent on childcare, the income taxation motivates workers to reduce more labor supply in the external market. If the husband has higher productivity than the wife, the government has an incentive for the husband to work more in the external labor market to enhance economic efficiency, while the wife engages more in childcare activities (Meier and Rainer, 2015).[67]

Next, we consider the variation in the husband's bargaining power ρ in the decision about child

[67] We also examine the optimal tax structure when the husband's childcare productivity s_m varies from 0.9 to 1.5 with an interval of 0.1, keeping s_f constant. The increase (decrease) in s_m has the reverse impact of the increase (decrease) in w_m on time allocation of h_i and l_i. Therefore, the relative size between the optimal tax rates t_m and t_f is opposite to that in Table 5.

quantity. Table 6 shows the optimal tax rates in the case in which ρ takes from 0.65 to 0.35 with an interval of 0.05. As the value of ρ decreases, the optimal income tax rate on both spouses increases, while the optimal child tax rate decreases.

Table 6. Optimal Tax Structure under Higher Bargaining Powers of the Wife

ρ	0.65	0.6	0.55	0.5	0.45	0.4	0.35
t_y	0.018	0.043	0.069	0.094	0.116	0.136	0.153
t_m	0.160	0.183	0.203	0.219	0.231	0.240	0.246
t_f	0.161	0.184	0.204	0.219	0.229	0.236	0.241
t_n	0.400	0.208	0.047	−0.085	−0.196	−0.291	−0.375
t_c	−0.128	−0.125	−0.121	−0.1181	−0.115	−0.113	−0.111

As ρ decreases (i.e., $1 - \rho$ becomes larger), the cost $c(n)$ becomes a more important factor in the couple's decision about child quantity. As children impose a burden on the wife, her higher bargaining power results in fewer children.[68] To increase child quantity, the optimal child tax rate decreases with $1 - \rho$ and the subsidy tends to be optimal as $1 - \rho$ increases. The optimal income tax rates on both spouses increase with $1 - \rho$ to improve child quantity based on the above reason and to secure funds for the child subsidy.

Another feature of the income tax rates is as follows: a spouse with higher bargaining power should be taxed at a lower rate. Without loss of generality, consider a situation in which the husband's bargaining power ρ is higher than 0.5. In this case, the government intends to reduce child quantity since the couple opts to have more children because they disregard the cost $c(n)$ for the wife. To this end, it is more effective to reduce the income tax rates on the wife more than that on the husband, as shown by Proposition 2(iii). However, there is a disadvantage of imposing a lower income tax on the wife: Propositions 2(iii) and (17) imply that the couple decides to have fewer children and then the parents' childcare time is significantly reduced. Thus, a lower income tax rate on the wife brings stronger downward pressure on child quality than a lower income tax rate on the husband. Hence, the government must allow for the two forces working in opposite directions when differentiating the income tax rates between spouses. Table 6 suggests that the government should set a lower income tax rate on the husband under $\rho > 0.5$, which implies that the government should emphasize maintaining child quality over correcting the intra-family distribution through the decline in fertility rates. Lise and Yamada (2019) empirically show that the bargaining power of men is higher than that of women, $\rho > 0.5$. In this case, our numerical result shows that a higher income tax rate should be imposed on the wife rather than the husband.

[68]This is consistent with the empirical results of Ashraf et al. (2014).

8 Conclusion

This study analyzes the optimal taxation system in an economy with non-cooperative couples, including gender-based income taxation, commodity tax, child tax/subsidy, and tax/subsidy on/for external childcare services. The number of children is at a suboptimal low level in the economy for two reasons: non-cooperative household behavior and the externality of children on society. To model our scenario, we consider both child quality and child quantity as household public goods. As the time spent on childcare cannot be credibly committed between spouses and, hence, a couple's behavior becomes non-cooperative, child quality is sub-optimally low. Meanwhile, child quantity and the time children spend in a childcare facility are collectively decided by a couple. This study proves that non-cooperative household behavior regarding the amount of time spent on childcare leads to a suboptimal low fertility rate despite the collective determination of child quantity. This observation is consistent with the existing empirical evidence (Doepke and Kindermann, 2019).

From our findings, we recommend the following suggestions to improve the low fertility rate under a revenue constraint. First, the suboptimal low fertility rate stemming from non-cooperative behavior should be corrected by income taxation and not through the implementation of a child subsidy. If the external effects of children on society are relatively small, a child tax becomes desirable to mitigate the distortionary impact of income taxes on labor supply. In this situation, the child subsidy should be reduced or removed since it impairs welfare. Second, from the numerical analysis, as the external effects of children on society increase, the optimal income tax rate and subsidy rate on external childcare services increase, while the optimal child tax rate decreases and becomes negative (i.e., child subsidy is optimal) at first and then increases beyond a certain point (i.e., child subsidy decreases). According to the first and second arguments, if a low fertility rate is caused by both the couple's non-cooperative behavior and externality of children on society, the government faces the problem of designing appropriate family policies corresponding to the two driving forces underlying the inefficiently low fertility. Third, the numerical analysis shows that the full utilization of childcare facilities is an effective policy to improve the fertility rate. This finding supports policies that provide public childcare facilities, which are notably implemented by countries with higher fertility rates (e.g., France, Norway, and Belgium). Thus, the government has an option to introduce childcare facilities rather than direct child subsidies. Finally, we recommend that countries collecting large tax revenues (e.g., developed countries) should employ higher subsidy rates for center-based childcare services than direct child subsidy rates, while countries that do not collect large tax revenues (e.g., developing countries) should implement the opposite policy.

Some extensions are left for future research. First, since the model considers identical households and linear tax/subsidy instruments, it does not clarify whether all tax/subsidy instruments, including gender-based income taxation, direct child tax/subsidy, and tax/subsidy on/for center-based childcare services, should be regressive, proportional, or progressive regarding family size and earnings. To explore the optimal design of such policies under a couple's non-cooperative behavior, we aim to extend our model to the Mirrleesian framework with heterogenous households and non-linear schedules of these tax/subsidy instruments, which is beyond the scope of the present paper. Additionally, by incorporating cooperative households into the setting, it would be interesting to examine the optimal tax/subsidy policies when the government cannot observe whether households are cooperative or non-cooperative. Second, we abstract from the impact of parents' human capital accumulation on the amount of time they spend with their children. As mentioned in Gobbi (2018), the American Time Use Survey for the period of 2003 to 2013 shows that the amount of time parents invest increases with their education. This may imply that subsidies for higher education bring significant returns on children's human capital being inefficiently low due to the non-cooperative behavior of the couple. Thus, it may be valuable to consider the impact of education subsidies through such a channel on children's human capital in order to suggest implications for applied tax/subsidy policies. Third, our model does not consider the government's equity consideration since all the couples are identical and the government places equal welfare weights on spouses. Allowing for heterogeneous couples and unequal welfare weights on spouses, it will be important to investigate both intra- and inter-couple redistribution. Finally, we adopt a quasilinear utility function to avoid analytical complexity. It would be interesting to derive policy implications under a utility function with an income effect, which may reduce the fertility rate by increasing the income tax rate.[69]

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[69]The negative relationship between wages and fertility is widespread across time and regions (Jones and Tertilt, 2008; Baughman and Dickert-Conlin, 2009; Jones et al., 2010). However, it is not universal. Several studies have reported exceptional findings. It is sometimes argued that in the early stage of the development process, there is a positive income-fertility relationship (e.g., Vogl, 2016). The cross-sectional relationship between fertility and women's education in the United States has recently become U-shaped (Hazan and Zoabi, 2015).

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Appendix A: The Derivation of (27)

Allowing for (17), the first-order condition of (26) with respect to n is

$$0 = -\rho(1-t_m)w_m h_{mn} - (1-\rho)(1-t_f)w_f h_{fn} - [(1-\rho)(1-\gamma)+\rho\gamma](1+t_n)p_n \\ -(1-\rho)c' + \left(\frac{1-\sigma}{\sigma}\right) n^{-\sigma}\left[(s_m h_m)^{\sigma} + \left(s_f h_f\right)^{\sigma}\right] + n^{1-\sigma}\left(s_m^{\sigma} h_m^{\sigma-1} h_{mn} + s_f^{\sigma} h_f^{\sigma-1} h_{fn}\right). \quad \text{(A1)}$$

From (11) and (12), we have

$$\frac{(1-t_i)w_i}{1+\tau} = n^{1-\sigma} s_i^{\sigma} h_i^{\sigma-1}, \quad i=m,f, \quad \text{that is,} \quad \frac{(1-t_i)w_i}{1+\tau} h_i = n^{1-\sigma} s_i^{\sigma} h_i^{\sigma}, \quad i=m,f. \quad \text{(A2)}$$

From (14) and (17), we observe that $h_{in} = h_i n^{-1}$. Using this relationship and (A2), (A1) can be rewritten as (27).

Appendix B: The Derivation of (43)

Using (34)–(41), we obtain

$$w_i = 2n^{1-\sigma} s_i^{\sigma} h_i^{\sigma-1}, \quad i=m,f, \quad \text{that is,} \quad w_i h_i = 2n^{1-\sigma} s_i^{\sigma} h_i^{\sigma}, \quad i=m,f. \quad \text{(B1)}$$

Then, it yields

$$h_i = 2^{\frac{1}{1-\sigma}} w_i^{-\frac{1}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} n, \quad i=m,f, \quad \text{(B2)}$$

$$l_i = w_i^{\frac{1}{\phi}} - 2^{\frac{1}{1-\sigma}} w_i^{-\frac{1}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} n, \quad i=m,f. \quad \text{(B3)}$$

Substituting (B1) into (42) and using $\zeta = \iota = 1$ from (34) and (35) yields

$$\left(\frac{1-\sigma}{2\sigma}\right) n^{-1}\left(w_m h_m + w_f h_f\right) - \frac{1}{2}c' - \frac{1}{2}p_n = 0. \quad \text{(B4)}$$

Moreover, substituting (B2) for h_i in (B4) yields (43).

Appendix C: The Proof of $\pi(\sigma, \tau) > 0$

To show that $\pi(\sigma,\tau) > 0$ for any $0 < \sigma < 1$ under $0 \le \tau < 1$, we take the following three steps: First, we show that $\lim_{\sigma\to 0}\left[\frac{(1-\sigma)}{\sigma}2^{\frac{\sigma}{1-\sigma}} - \frac{(1-\sigma)}{\sigma}\right] = \ln 2 \approx 0.69$. Define $f(\sigma) \equiv (1-\sigma)(2^{\frac{\sigma}{1-\sigma}} - 1)$, and $g(\sigma) \equiv \sigma$. Obviously, $f(0) = g(0) = 0$ and $g'(\sigma) = 1$. Furthermore, $\lim_{\sigma\to 0}\left[\frac{f'(\sigma)}{g'(\sigma)}\right] = \ln 2$. Hence, using L'Hôpital's rule, $\lim_{\sigma\to 0}\left[\frac{f(\sigma)}{g(\sigma)}\right] = \lim_{\sigma\to 0}\left[\frac{f'(\sigma)}{g'(\sigma)}\right]$. As a result, $\lim_{\sigma\to 0}\pi(\sigma,0) > 0$.

Second, we show that $\pi_{\sigma}(\sigma,0) > 0$ for any $0 < \sigma < 1$, where $\pi_{\sigma}(\sigma,0) = \frac{1+2^{\frac{\sigma}{1-\sigma}}\left(\frac{\sigma}{1-\sigma}\ln 2 - 1\right)}{\sigma^2}$. Noticing that $2^{\frac{\sigma}{1-\sigma}}\left(\frac{\sigma}{1-\sigma}\ln 2 - 1\right)$ ($\equiv \psi(\sigma)$) is -1 at $\sigma = 0$, it is sufficient to show that $\psi'(\sigma) > 0$ for any $0 < \sigma < 1$. After the computation, we obtain $\psi'(\sigma) = 2^{\frac{\sigma}{1-\sigma}}(\ln 2)^2 \frac{\sigma}{(1-\sigma)^3}$, which is positive for any $0 < \sigma < 1$. Therefore, $\pi_\sigma(\sigma, 0)$ is positive for any $0 < \sigma < 1$.

Third, we show that, for any $0 < \sigma < 1$, $\pi(\sigma, \tau)$ is a strictly decreasing function in the interval $\tau \in [0, 1)$. Pick any $\sigma \in (0, 1)$, $\pi_\tau(\sigma, \tau) = -\frac{(1+\tau)^{\frac{\sigma}{1-\sigma}}}{1-\sigma}\left[\frac{1}{1+\tau} - \frac{1}{2}\right] < 0$ under $0 \leq \tau < 1$. From the first and second steps, we obtain $\pi(\sigma, 0) > 0$, for any $0 < \sigma < 1$. Moreover, by the definition of $\pi(\sigma, \tau)$, $\pi(\sigma, 1) = 0$ for any $0 < \sigma < 1$. Based on these results, we conclude that $\pi(\sigma, \tau) > 0$ holds for any $0 < \sigma < 1$ under $0 \leq \tau < 1$.

Appendix D: The Proof of $n^{PE} = n^C$

In a cooperative setting for time allocation and consumption choices, the total after-tax income for each spouse is shared between the husband and wife such that i's consumption is

$$z_i + (1 + t_y)p_y y_i + \gamma_i(1 + t_n)p_n n = \varsigma_i \left[(1 - t_m)w_m l_m + (1 - t_f)w_f l_f\right], \tag{D1}$$

where $\varsigma_m + \varsigma_f = 1$.

Given this sharing rule and the assumption that $\mu = 0$, a cooperative couple maximizes joint utility (7). Using (D1), the maximization problem at the third stage can be rewritten as

$$\begin{aligned}\max_{y_m,y_f,l_m,l_f,h_m,h_f} \quad u = {} & (\rho\varsigma_m + (1-\rho)\varsigma_f)\left[(1 - t_m)w_m l_m + (1 - t_f)w_f l_f\right] \\ & + \rho\frac{y_m^\varphi}{\varphi} + (1-\rho)\frac{y_f^\varphi}{\varphi} - (1 + t_y)p_y(\rho y_m + (1-\rho)y_f) \\ & - [(1-\rho)(1-\gamma) + \rho\gamma](1 + t_n)p_n n - \rho\frac{(l_m + h_m)^{1+\phi}}{1+\phi} \\ & - (1-\rho)\frac{\left(l_f + h_f\right)^{1+\phi}}{1+\phi} + n^{1-\sigma}\left[\frac{(s_m h_m)^\sigma}{\sigma} + \frac{\left(s_f h_f\right)^\sigma}{\sigma}\right] - (1-\rho)c(n).\end{aligned} \tag{D2}$$

Denoting $\rho_m \equiv \rho$ and $\rho_f \equiv 1 - \rho$, the first-order conditions with respect to y_i, l_i, and h_i are

$$0 = \frac{\partial u}{\partial y_i} = y_i^{\varphi-1} - (1 + t_y)p_y, \quad i = m, f, \tag{D3}$$

$$0 = \frac{\partial u}{\partial l_i} = (\rho_m\varsigma_m + \rho_f\varsigma_f)(1 - t_i)w_i - \rho_i\left(l_i + h_i\right)^\phi, \quad i = m, f, \tag{D4}$$

$$0 = \frac{\partial u}{\partial h_i} = -\rho_i\left(l_i + h_i\right)^\phi + n^{1-\sigma}s_i^\sigma h_i^{\sigma-1}, \quad i = m, f. \tag{D5}$$

(D4) and (D5) yield

$$(1 - t_i)w_i = \Gamma n^{1-\sigma}s_i^\sigma h_i^{\sigma-1}, \quad i = m, f, \tag{D6}$$

$$\text{that is, } (1 - t_i)w_i h_i = \Gamma n^{1-\sigma}s_i^\sigma h_i^\sigma, \quad i = m, f,$$

where $\Gamma \equiv \frac{1}{\rho\varsigma_m+(1-\rho)\varsigma_f}$. Then, (D3), (D4), and (D6) yield

$$y_i^*(t_y) = \left[\left(1+t_y\right)p_y\right]^{\frac{1}{\varphi-1}}, \quad i = m, f, \tag{D7}$$

$$h_i^*(t_i, n) = \omega_i^{-\frac{1}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} \Gamma^{\frac{1}{1-\sigma}} n, \quad i = m, f, \tag{D8}$$

$$l_i^*(t_i, n) = (\Theta_i \omega_i)^{\frac{1}{\phi}} - \omega_i^{-\frac{1}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} \Gamma^{\frac{1}{1-\sigma}} n, \quad i = m, f, \tag{D9}$$

where $\Theta_i \equiv \frac{\rho\varsigma_m+(1-\rho)\varsigma_f}{\rho_i}$. Note that a similar condition to (17) holds:

$$h_{in}^* = -l_{in}^* = \omega_i^{-\frac{1}{1-\sigma}} s_i^{\frac{\sigma}{1-\sigma}} \Gamma^{\frac{1}{1-\sigma}} > 0, \quad i = m, f. \tag{D10}$$

In the second stage, the cooperative couple maximizes (D2), which is evaluated by y_i^*, h_i^*, and l_i^*, with respect to n. Using (D10), we obtain the following first-order condition with respect to n:

$$\begin{aligned} n \quad : \quad 0 = & \left[\rho\varsigma_m + (1-\rho)\varsigma_f\right]\left[(1-t_m)w_m l_{mn}^* + (1-t_f)w_f l_{fn}^*\right] \\ & - [(1-\rho)(1-\gamma)+\rho\gamma](1+t_n)p_n - (1-\rho)c'(n) \\ & + (1-\sigma)n^{-\sigma}\left[\frac{\left(s_m h_m^*\right)^\sigma}{\sigma} + \frac{(s_f h_f^*)^\sigma}{\sigma}\right] + n^{1-\sigma}\left[s_m^\sigma (h_m^*)^{\sigma-1} h_{mn}^* + s_f^\sigma (h_f^*)^{\sigma-1} h_{fn}^*\right]. \end{aligned} \tag{D11}$$

By allowing for (D6) and (D10), this condition can be rewritten as

$$n : 0 = -(1-\rho)c'(n) - [(1-\rho)(1-\gamma)+\rho\gamma](1+t_n)p_n + \Gamma^{-1}\frac{(1-\sigma)n^{-1}}{\sigma}\left(\omega_m h_m^* + \omega_f h_f^*\right). \tag{D12}$$

Using (D8), (D12) can be rewritten as

$$n : 0 = -(1-\rho)c'(n) - [(1-\rho)(1-\gamma)+\rho\gamma](1+t_n)p_n + \frac{(1-\sigma)}{\sigma}\Gamma^{\frac{\sigma}{1-\sigma}}\left(\omega_m^{\frac{-\sigma}{1-\sigma}} s_m^{\frac{\sigma}{1-\sigma}} + \omega_f^{\frac{-\sigma}{1-\sigma}} s_f^{\frac{\sigma}{1-\sigma}}\right), \tag{D13}$$

which yields child quantity under the cooperative case in the sequential decision, which is denoted by n^C. If $\rho = 1/2$ and $t_i = t_n = 0$, (D13) coincides with (43), which means that $n^{PE} = n^C$.

Appendix E: Common Income Tax Rate

We consider a common income tax rate on the husband and wife instead of the gender-based taxation. Let us denote the common income tax rate by t. Hence, $t_m = t_f (\equiv t)$ and $dt_m = dt_f (\equiv dt)$. Except for (29) and (30), note that all conditions and equations obtained in Sections 3 and 4 are valid, provided the index i of t_i is deleted. Equations (29) and (30) are replaced by the following equation:

$$n_t = \frac{\left[\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + 1 - \rho\right]\left(\frac{\sigma}{1-\sigma}\right) w_m \omega_m^{-\frac{1}{1-\sigma}} s_m^{\frac{\sigma}{1-\sigma}} + \left[\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \rho\right]\left(\frac{\sigma}{1-\sigma}\right) w_f \omega_f^{-\frac{1}{1-\sigma}} s_f^{\frac{\sigma}{1-\sigma}}}{(1+\tau)^{\frac{-1}{1-\sigma}}(1-\rho)c''} > 0. \tag{E1}$$

Given these facts, the government maximizes (46) subject to (47) by choosing t and t_n. Using (17), the first-order conditions with respect to t and t_n are[70]

$$
\begin{aligned}
t \quad : \quad 0 = & -w_m l_m + (1-t)w_m l_{mt} + (1-t)w_m l_{mn} n_t - (l_m + h_m)^{\phi}\,(l_{mt} + h_{mt}) \\
& - w_f l_f + (1-t)w_f l_{ft} + (1-t)w_f l_{fn} n_t - \left(l_f + h_f\right)^{\phi} \left(l_{ft} + h_{ft}\right) - (1+t_n)p_n n_t \\
& - c' n_t + 2(1+\mu H)\,(1-\sigma)\,n^{-\sigma} n_t \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} \right] \\
& + 2(1+\mu H) n^{1-\sigma} \left[s_m^{\sigma} h_m^{\sigma-1}\,(h_{mt} + h_{mn} n_t) + s_f^{\sigma} h_f^{\sigma-1} \left(h_{ft} + h_{fn} n_t\right) \right] \\
& - \lambda \left(w_m l_m + w_f l_f + t w_m l_{mt} + t w_f l_{ft} + t w_m l_{mn} n_t + t w_f l_{fn} n_t + t_n p_n n_t \right),
\end{aligned} \tag{E2}
$$

$$
\begin{aligned}
t_n \quad : \quad 0 = & (1-t)w_m l_{mn} n_{t_n} + (1-t)w_f l_{fn} n_{t_n} - p_n n - (1+t_n)p_n n_{t_n} \\
& - c' n_{t_n} + 2(1+\mu H)(1-\sigma) n^{-\sigma} n_{t_n} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} \right] \\
& + 2(1+\mu H) n^{1-\sigma} \left(s_m^{\sigma} h_m^{\sigma-1} h_{mn} n_{t_n} + s_f^{\sigma} h_f^{\sigma-1} h_{fn} n_{t_n} \right) \\
& - \lambda \left(t w_m l_{mn} n_{t_n} + t w_f l_{fn} n_{t_n} + p_n n + t_n p_n n_{t_n} \right).
\end{aligned} \tag{E3}
$$

By noting that $t_m = t_f = t$, (27) is reduced to

$$
\begin{aligned}
n \quad : \quad 0 = & -\left[(1-\rho)(1-\gamma) + \rho\gamma\right](1+t_n)p_n - (1-\rho)c'(n) \\
& + \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + 1 - \rho \right)(1-t)w_m h_{mn}(t) + \left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \rho \right)(1-t)w_f h_{fn}(t).
\end{aligned} \tag{E4}
$$

Before providing the proof, we define the following expressions:

$$
\begin{aligned}
& \alpha_{nl} \equiv \frac{(1+t_n)p_n n}{a_m + a_f}, \quad \theta_{\omega} \equiv \frac{(1+t)n_t}{n}, \quad \eta \equiv \frac{a_m \eta_m + a_f \eta_f}{a_m + a_f}, \\
& \varepsilon \equiv \frac{a_m \alpha_{hl}^m \varepsilon_m + a_f \alpha_{hl}^f \varepsilon_f}{a_m + a_f}, \quad r \equiv \frac{t}{1-t}, \quad a_i \equiv (1-t)w_i l_i, \quad i = m, f.
\end{aligned} \tag{E5}
$$

Multiplying each term in (E3) by $-n_{t_n}^{-1} n_t$ and applying the resulting equation to (E2) yields

$$
\begin{aligned}
t \quad : \quad 0 = & -w_m l_m + (1-t)w_m l_{mt} - (l_m + h_m)^{\phi}\,(l_{mt} + h_{mt}) - w_f l_f + (1-t)w_f l_{ft} \\
& - \left(l_f + h_f\right)^{\phi} \left(l_{ft} + h_{ft}\right) + p_n n n_{t_n}^{-1} n_t + 2(1+\mu H) n^{1-\sigma} \left(s_m^{\sigma} h_m^{\sigma-1} h_{mt} + s_f^{\sigma} h_f^{\sigma-1} h_{ft} \right) \\
& - \lambda \left(w_m l_m + w_f l_f + t w_m l_{mt} + t w_f l_{ft} - p_n n n_{t_n}^{-1} n_t \right).
\end{aligned} \tag{E6}
$$

[70]We omit the derivation of the optimal commodity tax rate on y_i since the same expression as (52) obviously holds even under the common income tax rate.

Using (11), (A2), and the definition of β, (E6) can be rewritten as

$$t \quad : \quad 0 = -\beta\left(w_m l_m + w_f l_f\right) - (1-\beta)\left(\frac{2(1+\mu H)}{1+\tau} - 1\right)(1-t)\left(w_m h_{mt} + w_f h_{ft}\right) \qquad \text{(E7)}$$
$$- t\left(w_m l_{mt} + w_f l_{ft}\right) + \beta p_n n n_{t_n}^{-1} n_t,$$

which yields

$$\frac{t}{1-t} = -\frac{w_m l_m + w_f l_f}{(1-t)\left(w_m l_{mt} + w_f l_{ft}\right)}\left\{\beta\left(1 - \frac{p_n n n_{t_n}^{-1} n_t}{w_m l_m + w_f l_f}\right)\right. \qquad \text{(E8)}$$
$$\left. +(1-\beta)\left[\frac{\left(\frac{2(1+\mu H)}{1+\tau} - 1\right)(1-t)\left(w_m h_{mt} + w_f h_{ft}\right)}{\left(w_m l_m + w_f l_f\right)}\right]\right\}.$$

Using (55), (56), and (E5), we obtain

$$\frac{w_m l_m + w_f l_f}{(1-t)\left(w_m l_{mt} + w_f l_{ft}\right)} = -\frac{1}{\frac{a_m \eta_m + a_f \eta_f}{a_m + a_f}}, \qquad \text{(E9)}$$

$$\frac{(1-t)\left(w_m h_{mt} + w_f h_{ft}\right)}{w_m l_m + w_f l_f} = \frac{a_m \alpha_{hl}^m \varepsilon_m + a_f \alpha_{hl}^f \varepsilon_f}{a_m + a_f}, \qquad \text{(E10)}$$

$$\frac{p_n n n_{t_n}^{-1} n_t}{w_m l_m + w_f l_f} = -\frac{\alpha_{nl}\theta_\omega}{\delta}. \qquad \text{(E11)}$$

Using (E5) and (E9)–(E11), (E8) can be rewritten as

$$r = \frac{\beta\left(1 + \frac{\alpha_{nl}\theta_\omega}{\delta}\right) + (1-\beta)\left[\frac{2(1+\mu H)}{1+\tau} - 1\right]\varepsilon}{\eta}. \qquad \text{(E12)}$$

The optimal income tax expression under the common tax rate (E12) is similar to the optimal income tax expressions under the gender-based taxation system, which is provided in Proposition 4. η is the weighted average of the wage elasticities of the spouses, and ε is the weighted average of $\alpha_{hl}^i \varepsilon_i$, with the weight being the disposable income share of the spouses. The intuition is similar to Proposition 4.

Multiplying each term in (E3) by $-n_{t_n}^{-1}$, we obtain

$$t_n \quad : \quad 0 = -(1-t) w_m l_{mn} - (1-t) w_f l_{fn} + p_n n n_{t_n}^{-1} + (1+t_n) p_n \qquad \text{(E13)}$$
$$+ c' - 2(1+\mu H)(1-\sigma) n^{-\sigma}\left[\frac{(s_m h_m)^\sigma}{\sigma} + \frac{\left(s_f h_f\right)^\sigma}{\sigma}\right]$$
$$- 2(1+\mu H) n^{1-\sigma}\left(s_m^\sigma h_m^{\sigma-1} h_{mn} + s_f^\sigma h_f^{\sigma-1} h_{fn}\right) + \lambda\left(t w_m l_{mn} + t w_f l_{fn} + p_n n n_{t_n}^{-1} + t_n p_n\right).$$

Using (17) and (A2), this expression can be rewritten as:

$$
\begin{aligned}
t_n \quad : \quad 0 = & -(1-t)w_m l_{mn} - (1-t)w_f l_{fn} + p_n n n_{t_n}^{-1} + (1+t_n)p_n + c' \\
& - 2(1+\mu H)\left(\frac{1-\sigma}{\sigma(1+\tau)}\right) n^{-1}(1-t)(w_m h_m + w_f h_f) \\
& + 2(1+\mu H)\frac{1}{1+\tau}(1-t)\left(w_m l_{mn} + w_f l_{fn}\right) + \lambda\left(t w_m l_{mn} + t w_f l_{fn} + p_n n n_{t_n}^{-1} + t_n p_n\right).
\end{aligned} \tag{E14}
$$

Multiplying each term in (E4) by $2(1+\mu H)$ and making use of (14) and (17) yields

$$
\begin{aligned}
n \quad : \quad 0 = & -2(1+\mu H)\left[(1-\rho)(1-\gamma)+\rho\gamma\right](1+t_n)p_n - 2(1+\mu H)(1-\rho)c' \\
& + 2(1+\mu H)\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)}\right) n^{-1}(1-t)(w_m h_m + w_f h_f) \\
& - 2(1+\mu H)(1-\rho)(1-t)w_m l_{mn} - 2(1+\mu H)\rho(1-t)w_f l_{fn}.
\end{aligned}
$$

Applying this expression to (E14), we obtain

$$
\begin{aligned}
t_n \quad : \quad 0 = & (1+\lambda)p_n n n_{t_n}^{-1} + \left\{1 - 2(1+\mu H)\left[(1-\rho)(1-\gamma)+\rho\gamma\right]\right\}(1+t_n)p_n \\
& + \left[1 - 2(1+\mu H)(1-\rho)\right]c' - \left[1 - 2(1+\mu H)\rho\right](1-t)w_m l_{mn} \\
& - \left[1 - 2(1+\mu H)(1-\rho)\right](1-t)w_f l_{fn} + \lambda\left(t w_m l_{mn} + t w_f l_{fn} + t_n p_n\right).
\end{aligned} \tag{E15}
$$

Using the definitions of β, (55), (56), and (57), (E15) can be rewritten as

$$
r_n = \frac{\beta}{\delta} + \frac{r\chi_m}{\alpha_{nl}^m} + \frac{r\chi_f}{\alpha_{nl}^f} + (1-\beta)\Lambda. \tag{E16}
$$

The expression of the optimal child tax/subsidy takes the same form as in (62), regardless of whether income tax rates are differentiable or not.

Appendix F: The Derivations of (53) and (54)

Multiplying each term in (51) by $n_{t_n}^{-1} n_{t_m}$ and subtracting the resulting equation from (49) yield

$$
\begin{aligned}
t_m \quad : \quad 0 = & -w_m l_m + (1-t_m)w_m l_{mt_m} - (l_m + h_m)^{\phi}\left(l_{mt_m} + h_{mt_m}\right) \\
& + 2(1+\mu H)n^{1-\sigma} s_m^{\sigma} h_m^{\sigma-1} h_{mt_m} - \lambda\left(w_m l_m + t_m w_m l_{mt_m}\right) + (1+\lambda)p_n n n_{t_n}^{-1} n_{t_m}.
\end{aligned} \tag{F1}
$$

Using (11) and (12), (F1) can be rewritten as (53). Similarly, we obtain (54).

Appendix G: The Proof of Proposition 4

Equation (53) can be rewritten as

$$\begin{aligned}
\frac{t_m}{1-t_m} &= -\left(\frac{1+\lambda}{\lambda}\right)\frac{l_m}{(1-t_m)l_{mt_m}} + \frac{1}{\lambda}\frac{\left[\frac{2(1+\mu H)}{1+\tau}-1\right] w_m h_{mt_m}}{w_m l_{mt_m}} + \left(\frac{1+\lambda}{\lambda}\right)\frac{p_n n n_{t_n}^{-1} n_{t_m}}{(1-t_m)w_m l_{mt_m}} \quad \text{(G1)}\\
&= -\left(\frac{1+\lambda}{\lambda}\right)\frac{1}{\frac{(1-t_m)l_{mt_m}}{l_m}} + \frac{1}{\lambda}\left[\frac{2(1+\mu H)}{1+\tau}-1\right]\frac{(1-t_m)w_m h_m \frac{(1-t_m)h_{mt_m}}{h_m}}{(1-t_m)w_m l_m \frac{(1-t_m)l_{mt_m}}{l_m}}\\
&\quad + \left(\frac{1+\lambda}{\lambda}\right)\frac{(1+t_n)p_n n\frac{(1-t_m)n_{t_m}}{n}}{(1-t_m)w_m l_m \frac{(1-t_m)l_{mt_m}}{l_m}\frac{(1+t_n)n_{t_n}}{n}}.
\end{aligned}$$

Using (55)–(57), (G1) can be rewritten as (58). Similarly, we obtain (59) by rewriting (54).

Appendix H: The Derivation of (61)

By multiplying each term in (51) by $n_{t_n}^{-1}$ and making use of (11), (12), and the fact that $h_{in}n = h_i$ ($i = m, f$), which is derived by (14) and (17), after some manipulations, (51) can be rewritten as

$$\begin{aligned}
0 &= (1-t_m)w_m l_{mn} + (1-t_f)w_f l_{fn} - c' - p_n n n_{t_n}^{-1} - (1+t_n)p_n \quad \text{(H1)}\\
&\quad + 2(1+\mu H)\left(\frac{1}{\sigma(1+\tau)}\right)\left\{(1-t_m)w_m h_{mn} + (1-t_f)w_f h_{fn}\right\}\\
&\quad - \lambda\left(t_m w_m l_{mn} + t_f w_f l_{fn} + p_n n n_{t_n}^{-1} + t_n p_n\right).
\end{aligned}$$

Multiplying each term in (27) by $2(1+\mu H)$, subtracting the resulting equation from (H1), and making use of (17), we obtain (61).

Appendix I: The Proof of Proposition 5

From (61), we obtain

$$\begin{aligned}
\frac{t_n}{1+t_n} &= -\left(\frac{1+\lambda}{\lambda}\right)\frac{1}{\frac{(1+t_n)n_{t_n}}{n}} - \frac{t_m}{1+t_m}\frac{(1+t_m)w_m l_m}{(1+t_n)\,p_n n}\frac{nl_{mn}}{l_m} - \frac{t_f}{1+t_f}\frac{(1+t_f)w_f l_f}{(1+t_n)\,p_n n}\frac{nl_{fn}}{l_f} \quad \text{(I1)}\\
&\quad - \frac{1}{\lambda}\left[1-2(1+\mu H)(1-\rho)\right]\frac{c'}{(1+t_n)\,p_n} + \frac{1}{\lambda}\left[1-2(1+\mu H)\rho\right]\frac{(1-t_m)w_m l_m}{(1+t_n)\,p_n n}\frac{nl_{mn}}{l_m}\\
&\quad + \frac{1}{\lambda}\left[1-2(1+\mu H)(1-\rho)\right]\frac{(1-t_f)w_f l_f}{(1+t_n)\,p_n n}\frac{nl_{fn}}{l_f} - \frac{1}{\lambda}\left\{1-2(1+\mu H)\left[(1-\rho)(1-\gamma)+\rho\gamma\right]\right\}.
\end{aligned}$$

Using (55)–(57) and (60), (I1) can be rewritten as (62).

Appendix J: The Derivations of (68)

Substituting (17) for h_{in} in (27) yields

$$\begin{aligned} 0 = & -\left[(1-\rho)(1-\gamma)+\rho\gamma\right](1+t_n)p_n-(1-\rho)c'(n) \\ & +\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)}+1-\rho\right)(1+\tau)^{\frac{1}{1-\sigma}}\omega_m^{\frac{-\sigma}{1-\sigma}}s_m^{\frac{\sigma}{1-\sigma}} \\ & +\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)}+\rho\right)(1+\tau)^{\frac{1}{1-\sigma}}\omega_f^{\frac{-\sigma}{1-\sigma}}s_f^{\frac{\sigma}{1-\sigma}}. \end{aligned} \tag{J1}$$

By totally differentiating (J1) with respect to ρ and n, we obtain

$$0=(1-2\gamma)(1+t_n)p_n+c'(n)-(1-\rho)c''\frac{\partial n}{\partial \rho}+(1+\tau)^{\frac{1}{1-\sigma}}\left(\omega_f^{\frac{-\sigma}{1-\sigma}}s_f^{\frac{\sigma}{1-\sigma}}-\omega_m^{\frac{-\sigma}{1-\sigma}}s_m^{\frac{\sigma}{1-\sigma}}\right). \tag{J2}$$

Using (17), (56), and (60), we obtain the following result:

$$\frac{\chi_i}{\alpha_{nl}^i}=\frac{(1+\tau)^{\frac{1}{1-\sigma}}\omega_i^{\frac{-\sigma}{1-\sigma}}s_i^{\frac{\sigma}{1-\sigma}}}{(1+t_n)\,p_n}, \quad i=m,f. \tag{J3}$$

Substituting (J3) into the fourth term in (J2) and dividing the resulting equation by $(1-\rho)c''$ yields (68).

Appendix K: The Optimal Taxation under the Case in Which the Two Types of Cost Shares of Spouse i Differ

Here, we examine the optimal taxation in the case of two types of cost shares of spouse i that differ. In the setting, the budget constraint of each spouse is modified by

$$z_i+(1+t_y)p_y y_i+\gamma_i(1+t_n)p_n n+\hat{\gamma}_i(1+t_c)p_c h_c=(1-t_i)w_i l_i, \quad i=m,f. \tag{K1}$$

The expenditure on the childcare facility is given by the fourth term on the left-hand side, where $\hat{\gamma}_i$ is the share of spouse i toward the childcare facility expenditure, p_c is the hourly price, and t_c is the tax/subsidy rate for using the childcare facility.[71] Defining $\hat{\gamma}_m \equiv \hat{\gamma}$ (and hence, $\hat{\gamma}_f \equiv 1-\hat{\gamma}$) and substituting (71) for q in (1) and (K1) for z_i in (1) yield

$$\begin{aligned} u_m = & (1-t_m)w_m l_m-(1+t_y)p_y y_m-\gamma(1+t_n)p_n n+\frac{y_m^{\varphi}}{\varphi}-\hat{\gamma}(1+t_c)p_c h_c \\ & -\frac{(l_m+h_m)^{1+\phi}}{1+\phi}+(1+\mu H)n^{1-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma}+\frac{(s_f h_f)^{\sigma}}{\sigma}+\frac{(s_c h_c)^{\sigma}}{\sigma}\right], \end{aligned} \tag{K2}$$

[71] As mentioned in footnote 28, for simplicity, we assume that the cost shares of a wife and husband are exogenous. Thus, $\hat{\gamma}_m$ and $\hat{\gamma}_f$ are fixed parameters.

$$
\begin{aligned}
u_f \quad = \quad & (1-t_f)w_f l_f - (1+t_y)p_y y_f - (1-\gamma)(1+t_n)p_n n + \frac{y_f^{\varphi}}{\varphi} - (1-\hat{\gamma})(1+t_c)p_c h_c \\
& - \frac{\left(l_f + h_f\right)^{1+\phi}}{1+\phi} + (1+\mu H)n^{1-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right] - c(n).
\end{aligned}
$$

Substituting (K2) for u_i in (7) and allowing for (13)–(15), we obtain the couple's utility function:

$$
\begin{aligned}
u \quad = \quad & \rho\left[(1-t_m)w_m l_m(t_m,n) - (1+t_y)p_y y_m(t_y) + \frac{\left(y_m(t_y)\right)^{\varphi}}{\varphi}\right. \\
& \left. - \frac{(l_m(t_m,n) + h_m(t_m,n))^{1+\phi}}{1+\phi}\right] \\
& + (1-\rho)\left[(1-t_f)w_f l_f(t_f,n) - (1+t_y)p_y y_f(t_y) + \frac{\left(y_f(t_y)\right)^{\varphi}}{\varphi}\right. \\
& \left. - \frac{\left(l_f(t_f,n) + h_f(t_f,n)\right)^{1+\phi}}{1+\phi} - c(n)\right] \\
& - \left[(1-\rho)(1-\gamma) + \rho\gamma\right](1+t_n)p_n n - \left[(1-\rho)(1-\hat{\gamma}) + \rho\hat{\gamma}\right](1+t_c)p_c h_c \\
& + (1+\mu H)n^{1-\sigma}\left[\frac{(s_m h_m(t_m,n))^{\sigma}}{\sigma} + \frac{\left(s_f h_f(t_f,n)\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right].
\end{aligned}
\tag{K3}
$$

As mentioned above, the spouses collectively maximize u first with respect to n and next with respect to h_c. We first show the couple's determination of h_c. Given $\mu N q$,[72] the first-order condition of (K3) with respect to h_c is

$$
0 = \frac{\partial u}{\partial h_c} = -\left[(1-\rho)(1-\hat{\gamma}) + \rho\hat{\gamma}\right](1+t_c)p_c + n^{1-\sigma}s_c^{\sigma}h_c^{\sigma-1}. \tag{K4}
$$

Solving this equation with respect to h_c, we immediately obtain the following function:

$$
h_c(t_c,n) = \left\{\left[(1-\rho)(1-\hat{\gamma}) + \rho\hat{\gamma}\right](1+t_c)p_c\right\}^{-\frac{1}{1-\sigma}} s_c^{\frac{\sigma}{1-\sigma}} n. \tag{K5}
$$

From (K5), we obtain

$$
h_{ct_c}\left(\equiv \frac{\partial h_c}{\partial t_c}\right) = -\left(\frac{1}{1-\sigma}\right)h_c(1+t_c)^{-1} < 0, \tag{K6}
$$

$$
h_{cn}\left(\equiv \frac{\partial h_c}{\partial n}\right) = \left\{\left[(1-\rho)(1-\hat{\gamma}) + \rho\hat{\gamma}\right](1+t_c)p_c\right\}^{-\frac{1}{1-\sigma}} s_c^{\frac{\sigma}{1-\sigma}} > 0. \tag{K7}
$$

The intuitions for the two results are highly straightforward.

We now turn to the couple's decision about child quantity. Allowing for $h_c = h_c(t_c,n)$, the couple

[72] In (K3), $\mu N q$ corresponds to $\mu H n^{1-\sigma}\left[\frac{(s_m h_m(t_m,n))^{\sigma}}{\sigma} + \frac{\left(s_f h_f(t_f,n)\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right]$.

maximizes the utility function (K3) with respect to n. The first-order condition with respect to n is

$$\begin{aligned} 0 \;=\; \frac{\partial u}{\partial n} =& -[(1-\rho)(1-\gamma)+\rho\gamma](1+t_n)p_n - (1-\rho)c'(n) \\ &+\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + 1 - \rho\right)(1-t_m)w_m h_{mn}(t_m) \\ &+\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \rho\right)(1-t_f)w_f h_{fn}(t_f) \\ &+\left(\frac{1-\sigma}{\sigma}\right)[(1-\rho)(1-\hat{\gamma})+\rho\hat{\gamma}](1+t_c)p_c h_{cn}(t_c), \end{aligned} \tag{K8}$$

where we use (11), (12), (14), (17), (K4), (K5), and (K7) to derive the equation. The derivation is the same procedure as that found in Appendix L. Equation (K8) implies

$$n = n(t_c, t_n, t_m, t_f). \tag{K9}$$

Here, we propose the effects of each tax rate on child quantity. Totally differentiating (K8) with respect to n, t_m, t_f, t_n, and t_c yields

$$n_{t_m}\left(\equiv \frac{\partial n}{\partial t_m}\right) = \frac{\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + 1 - \rho\right)\left(\frac{\sigma}{1-\sigma}\right) w_m (1+\tau)^{\frac{1}{1-\sigma}} \omega_m^{-\frac{1}{1-\sigma}} s_m^{\frac{\sigma}{1-\sigma}}}{(1-\rho)c''} > 0, \tag{K10}$$

$$n_{t_f}\left(\equiv \frac{\partial n}{\partial t_f}\right) = \frac{\left(\frac{1-\sigma(1+\tau)}{\sigma(1+\tau)} + \rho\right)\left(\frac{\sigma}{1-\sigma}\right) w_f (1+\tau)^{\frac{1}{1-\sigma}} \omega_f^{-\frac{1}{1-\sigma}} s_f^{\frac{\sigma}{1-\sigma}}}{(1-\rho)c''} > 0, \tag{K11}$$

$$n_{t_n}\left(\equiv \frac{\partial n}{\partial t_n}\right) = -\frac{[(1-\rho)(1-\gamma)+\rho\gamma]\,p_n}{(1-\rho)c''} < 0, \tag{K12}$$

$$n_{t_c}\left(\equiv \frac{\partial n}{\partial t_c}\right) = -\frac{[(1-\rho)(1-\hat{\gamma})+\rho\hat{\gamma}]^{-\frac{\sigma}{1-\sigma}} p_c \omega_c^{-\frac{1}{1-\sigma}} s_c^{\frac{\sigma}{1-\sigma}}}{(1-\rho)c''} < 0, \tag{K13}$$

where $\omega_c \equiv (1+t_c)p_c$. To derive these four equations, we use (17) and (K7).

Substituting (K9) for n in (14), (15), and (K5) yields

$$h_i(t_i, n(t_c,t_n,t_m,t_f)),\quad l_i(t_i, n(t_c,t_n,t_m,t_f)),\quad \text{for } i = m, f,\quad \text{and}\quad h_c(t_c, n(t_c,t_n,t_m,t_f)). \tag{K14}$$

These functions involve information regarding the decision process in the second, third, and fourth stages.

Substituting (K2) for u_i in (8) and allowing for (13), (K9), and (K14), we obtain the government's welfare function:

$$
\begin{aligned}
\frac{W}{H} &= (1-t_m)w_m l_m(t_m, n(t_c,t_n,t_m,t_f)) + \frac{\left(y_m(t_y)\right)^{\varphi}}{\varphi} \\
&\quad - \frac{\left(l_m(t_m, n(t_c,t_n,t_m,t_f)) + h_m(t_m, n(t_c,t_n,t_m,t_f))\right)^{1+\phi}}{1+\phi} \\
&\quad + (1-t_f)w_f l_f(t_f, n(t_c,t_n,t_m,t_f)) + \frac{\left(y_f(t_y)\right)^{\varphi}}{\varphi} \\
&\quad - \frac{\left(l_f(t_f, n(t_c,t_n,t_m,t_f)) + h_f(t_f, n(t_c,t_n,t_m,t_f))\right)^{1+\phi}}{1+\phi} \\
&\quad - (1+t_y)p_y(y_m(t_y)+y_f(t_y)) - c(n(t_c,t_n,t_m,t_f)) \\
&\quad - (1+t_n)p_n n(t_c,t_n,t_m,t_f) - (1+t_c)p_c h_c(t_c, n(t_c,t_n,t_m,t_f)) \\
&\quad + 2(1+\mu H)(n(t_c,t_n,t_m,t_f))^{1-\sigma}\left[\frac{\left(s_m h_m(t_m, n(t_c,t_n,t_m,t_f))\right)^{\sigma}}{\sigma}\right. \\
&\quad \left. + \frac{\left(s_f h_f(t_f, n(t_c,t_n,t_m,t_f))\right)^{\sigma}}{\sigma} + \frac{\left(s_c h_c(t_c, n(t_c,t_n,t_m,t_f))\right)^{\sigma}}{\sigma}\right].
\end{aligned} \tag{K15}
$$

The revenue constraint of the government is modified by

$$
\begin{aligned}
g &= t_m w_m l_m(t_m, n(t_c,t_n,t_m,t_f)) + t_f w_f l_f(t_f, n(t_c,t_n,t_m,t_f)) \\
&\quad + t_y p_y\left(y_m(t_y)+y_f(t_y)\right) + t_n p_n n(t_c,t_n,t_m,t_f) + t_c p_c h_c(t_c, n(t_c,t_n,t_m,t_f)),
\end{aligned} \tag{K16}
$$

where the fifth term represents the tax revenue from the tax/subsidy for using the childcare facility. From the government's social welfare maximization subject to the revenue constraint, we obtain the optimal tax expressions for t_y, t_m, t_f, t_n, and t_c. Before characterizing them, we provide the following definitions:

$$
\begin{aligned}
\delta_c &\equiv -\frac{(1+t_c)h_{ct_c}}{h_c} > 0, \quad \xi \equiv -\frac{(1+t_c)n_{t_c}}{n} > 0, \\
\alpha_{nh_c} &\equiv \frac{(1+t_n)p_n n}{(1+t_c)p_c h_c} > 0, \quad \chi_c \equiv \frac{h_{cn}n}{h_c} = 1, \quad r_c \equiv \frac{t_c}{1+t_c},
\end{aligned} \tag{K17}
$$

where δ_c is the price elasticity of the time use of a childcare facility, ξ is the elasticity of child quantity with respect to the price of external childcare services, α_{nh_c} is the ratio between the expenditure on the fertility good and childcare expenditure, and χ_c is the elasticity of time use of a childcare facility with respect to child quantity.[73]

From the conditions and equations shown in this Appendix, we provide the following optimal tax formulas for the case with a childcare facility.

[73]Note that $h_{ct_c}(\equiv \partial h_c/\partial t_c) = p_c(\partial h_c/\partial[(1+t_c)p_c])$. Thus, δ_c is the price elasticity of external childcare services. Similarly, $n_{t_c}(\equiv \partial n/\partial t_c) = p_c(\partial n/\partial[(1+t_c)p_c])$. Thus, ξ can be interpreted as the elasticity of child quantity with respect to the price of external childcare services.

Proposition 7. *In the endogenous fertility model with a childcare facility, the optimal taxes are characterized by*

$$r_y = \frac{\beta}{\Xi}, \tag{K18}$$

$$r_m = \frac{\beta\left(1 + \alpha_{nl}^m \frac{\theta_m}{\delta}\right) + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-\beta)\,\alpha_{hl}^m \varepsilon_m}{\eta_m}, \tag{K19}$$

$$r_f = \frac{\beta\left(1 + \alpha_{nl}^f \frac{\theta_f}{\delta}\right) + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-\beta)\,\alpha_{hl}^f \varepsilon_f}{\eta_f}, \tag{K20}$$

$$r_n = \frac{\beta}{\delta} + \frac{r_m \chi_m}{\alpha_{nl}^m} + \frac{r_f \chi_f}{\alpha_{nl}^f} - \frac{r_c \chi_c}{\alpha_{nh_c}} + (1-\beta)\tilde{\Omega}, \tag{K21}$$

$$r_c = \frac{\beta\left(1 - \alpha_{nh_c}\frac{\xi}{\delta}\right)}{\delta_c} + (1-\beta)\left\{1 - 2(1+\mu H)\left[(1-\rho)(1-\hat{\gamma}) + \rho\hat{\gamma}\right]\right\}, \tag{K22}$$

where

$$\tilde{\Omega} \equiv \Lambda + \left\{1 - 2(1+\mu H)\left[(1-\rho)(1-\hat{\gamma}) + \rho\hat{\gamma}\right]\right\}\frac{\chi_c}{\alpha_{nh_c}}. \tag{K23}$$

Before interpreting these equations, we provide the derivations of these formulas. By defining the Lagrange function as L and the Lagrange multiplier on the revenue constraint as λ, and by making use of (17), the first-order conditions of the government's social welfare maximization (K15) subject to the revenue constraint (K16) with respect to t_y, t_m, t_f, t_n, and t_c are given by

$$\begin{aligned} 0 \;=\; \frac{\partial L}{\partial t_y} &= -p_y y_m - (1+t_y)p_y y_m' - p_y y_f - (1+t_y)p_y y_f' \\ &\quad + y_m^{\varphi-1} y_m' + y_f^{\varphi-1} y_f' - \lambda[y_m + y_f + t_y(y_m' + y_f')]p_y, \end{aligned} \tag{K24}$$

$$\begin{aligned} 0 \;=\; \frac{\partial L}{\partial t_m} &= -w_m l_m + (1-t_m)w_m l_{mt_m} + (1-t_m)w_m l_{mn} n_{t_m} \\ &\quad - (l_m + h_m)^{\phi}\left(l_{mt_m} + h_{mt_m}\right) + (1-t_f)w_f l_{fn} n_{t_m} - c' n_{t_m} \\ &\quad - (1+t_n)p_n n_{t_m} - (1+t_c)p_c h_{cn} n_{t_m} \\ &\quad + 2(1+\mu H)(1-\sigma)\,n^{-\sigma} n_{t_m}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right] \\ &\quad + 2(1+\mu H)n^{1-\sigma}[s_m^{\sigma} h_m^{\sigma-1}\left(h_{mt_m} + h_{mn} n_{t_m}\right) \\ &\quad + s_f^{\sigma} h_f^{\sigma-1} h_{fn} n_{t_m} + s_c^{\sigma} h_c^{\sigma-1} h_{cn} n_{t_m}] \\ &\quad - \lambda(w_m l_m + t_m w_m l_{mt_m} + t_m w_m l_{mn} n_{t_m} \\ &\quad + t_f w_f l_{fn} n_{t_m} + t_n p_n n_{t_m} + t_c p_c h_{cn} n_{t_m}), \end{aligned} \tag{K25}$$

$$0 = \frac{\partial L}{\partial t_f} = (1-t_m)w_m l_{mn} n_{t_f} - w_f l_f + (1-t_f)w_f l_{ft_f} + (1-t_f)w_f l_{fn} n_{t_f} \tag{K26}$$
$$- \left(l_f + h_f\right)^{\phi} \left(l_{ft_f} + h_{ft_f}\right) - c' n_{t_f} - (1+t_n)p_n n_{t_f} - (1+t_c)p_c h_{cn} n_{t_f}$$
$$+ 2(1+\mu H)(1-\sigma)n^{-\sigma} n_{t_f} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right]$$
$$+ 2(1+\mu H)n^{1-\sigma}[s_m^{\sigma} h_m^{\sigma-1} h_{mn} n_{t_f} + s_f^{\sigma} h_f^{\sigma-1}(h_{ft_f} + h_{fn} n_{t_f}) + s_c^{\sigma} h_c^{\sigma-1} h_{cn} n_{t_f}]$$
$$- \lambda(t_m w_m l_{mn} n_{t_f} + w_f l_f + t_f w_f l_{ft_f} + t_f w_f l_{fn} n_{t_f} + t_n p_n n_{t_f} + t_c p_c h_{cn} n_{t_f}),$$

$$0 = \frac{\partial L}{\partial t_n} = (1-t_m)w_m l_{mn} n_{t_n} + (1-t_f)w_f l_{fn} n_{t_n} - c' n_{t_n} \tag{K27}$$
$$- p_n n - (1+t_n)p_n n_{t_n} - (1+t_c)p_c h_{cn} n_{t_n}$$
$$+ 2(1+\mu H)(1-\sigma)n^{-\sigma} n_{t_n} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right]$$
$$+ 2(1+\mu H)n^{1-\sigma} \left(s_m^{\sigma} h_m^{\sigma-1} h_{mn} n_{t_n} + s_f^{\sigma} h_f^{\sigma-1} h_{fn} n_{t_n} + s_c^{\sigma} h_c^{\sigma-1} h_{cn} n_{t_n}\right)$$
$$- \lambda\left(t_m w_m l_{mn} n_{t_n} + t_f w_f l_{fn} n_{t_n} + p_n n + t_n p_n n_{t_n} + t_c p_c h_{cn} n_{t_n}\right),$$

$$0 = \frac{\partial L}{\partial t_c} = (1-t_m)w_m l_{mn} n_{t_c} + (1-t_f)w_f l_{fn} n_{t_c} - c' n_{t_c} \tag{K28}$$
$$- (1+t_n)p_n n_{t_c} - p_c h_c - (1+t_c)p_c h_{ct_c} - (1+t_c)p_c h_{cn} n_{t_c}$$
$$+ 2(1+\mu H)(1-\sigma)n^{-\sigma} n_{t_c} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right]$$
$$+ 2(1+\mu H)n^{1-\sigma}[s_m^{\sigma} h_m^{\sigma-1} h_{mn} n_{t_c}$$
$$+ s_f^{\sigma} h_f^{\sigma-1} h_{fn} n_{t_c} + s_c^{\sigma} h_c^{\sigma-1}(h_{ct_c} + h_{cn} n_{t_c})]$$
$$- \lambda(t_m w_m l_{mn} n_{t_c} + t_f w_f l_{fn} n_{t_c} + t_n p_n n_{t_c}$$
$$+ p_c h_c + t_c p_c h_{ct_c} + t_c p_c h_{cn} n_{t_c}).$$

First, given that (K24) coincides with (48), we obtain (K18), which is the same as (52). Moreover, using (11), (12), and (K27), after some manipulation, (K25) and (K26) can be rewritten as

$$t_m : 0 = -(1+\lambda)w_m l_m - \lambda t_m w_m l_{mt_m} + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-t_m)w_m h_{mt_m} + (1+\lambda)p_n n n_{t_n}^{-1} n_{t_m},$$

$$t_f : 0 = -(1+\lambda)w_f l_f - \lambda t_f w_f l_{ft_f} + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-t_f)w_f h_{ft_f} + (1+\lambda)p_n n n_{t_n}^{-1} n_{t_f}.$$

These two equations are the same as (53) and (54), respectively. Using the process in Appendix G, we observe that the two equations lead to (K19) and (K20), respectively.

Next, we derive (K22). Multiplying each term in (K27) by $n_{t_n}^{-1} n_{t_c}$ and subtracting the resulting equation from (K28), we obtain

$$
\begin{aligned}
0 &= \beta p_n n n_{t_n}^{-1} n_{t_c} + (1-\beta) p_c h_c + (1-\beta)(1+t_c) p_c h_{ct_c} \\
&\quad - 2(1-\beta)(1+\mu H) n^{1-\sigma} s_c^{\sigma} h_c^{\sigma-1} h_{ct_c} - p_c h_c - t_c p_c h_{ct_c}.
\end{aligned}
$$

Using (K4), it can be rewritten as

$$
\begin{aligned}
\frac{t_c}{1+t_c} &= -\beta \frac{(1+t_n) p_n n}{(1+t_c) p_c h_c} \frac{-\frac{(1+t_c) n_{t_c}}{n}}{-\frac{(1+t_c) h_{ct_c}}{h_c}} \frac{1}{-\frac{(1+t_n) n_{t_n}}{n}} - (1-\beta) \frac{1}{-\frac{(1+t_c) h_{ct_c}}{h_c}} \\
&\quad + (1-\beta) - 2(1-\beta)(1+\mu H) \left[(1-\rho)(1-\hat{\gamma}) + \rho \hat{\gamma} \right] + \frac{1}{-\frac{(1+t_c) h_{ct_c}}{h_c}}.
\end{aligned}
\tag{K29}
$$

Using (55) and (K17), after some manipulations, it can be rewritten as (K22).

Finally, we derive the optimal child tax/subsidy expression. By multiplying each term in (K27) by $n_{t_n}^{-1}$ and using (A2), (K4), and the fact that $h_{in} n = h_i$ ($i = m, f, c$), after some manipulations, (K27) can be rewritten as

$$
\begin{aligned}
0 &= (1-t_m) w_m l_{mn} + (1-t_f) w_f l_{fn} - c' - p_n n n_{t_n}^{-1} - (1+t_n) p_n \\
&\quad - (1+t_c) p_c h_{cn} + 2(1+\mu H) \left(\frac{1}{\sigma} \right) \left\{ \frac{(1-t_m) w_m}{1+\tau} h_{mn} + \frac{(1-t_f) w_f}{1+\tau} h_{fn} \right. \\
&\quad \left. + \left[(1-\rho)(1-\hat{\gamma}) + \rho \hat{\gamma} \right] (1+t_c) p_c h_{cn} \right\} \\
&\quad - \lambda \left(t_m w_m l_{mn} + t_f w_f l_{fn} + p_n n n_{t_n}^{-1} + t_n p_n + t_c p_c h_{cn} \right).
\end{aligned}
\tag{K30}
$$

Multiplying each term in (K8) by $2(1+\mu H)$, subtracting the resulting equation from (K30), and using (17), we obtain

$$
\begin{aligned}
0 &= \left[1 - 2(1+\mu H)(1-\rho) \right] c' + (1+\lambda) p_n n n_{t_n}^{-1} \\
&\quad + \left\{ 1 - 2(1+\mu H) \left[(1-\rho)(1-\hat{\gamma}) + \rho \hat{\gamma} \right] \right\} (1+t_c) p_c h_{cn} \\
&\quad + \lambda \left(t_m w_m l_{mn} + t_f w_f l_{fn} + t_n p_n + t_c p_c h_{cn} \right) \\
&\quad + \left\{ 1 - 2(1+\mu H) \left[(1-\rho)(1-\gamma) + \rho \gamma \right] \right\} (1+t_n) p_n \\
&\quad - \left[1 - 2(1+\mu H) \rho \right] (1-t_m) w_m l_{mn} - \left[1 - 2(1+\mu H)(1-\rho) \right] (1-t_f) w_f l_{fn},
\end{aligned}
\tag{K31}
$$

which yields

$$\frac{t_n}{1+t_n} = \left(-\frac{1}{\lambda}\right)[1-2(1+\mu H)(1-\rho)]\frac{c'}{(1+t_n)p_n} + \left(\frac{1+\lambda}{\lambda}\right)\frac{1}{-\frac{(1+t_n)n_{t_n}}{n}}$$
$$+\left(-\frac{1}{\lambda}\right)\{1-2(1+\mu H)[(1-\rho)(1-\hat{\gamma})+\rho\hat{\gamma}]\}\frac{\frac{nh_{cn}}{h_c}}{\frac{(1+t_n)p_n n}{(1+t_c)p_c h_c}}$$
$$+\left(\frac{t_m}{1-t_m}\right)\frac{-\frac{nl_{mn}}{l_m}}{\frac{(1+t_n)p_n n}{(1-t_m)w_m l_m}} + \left(\frac{t_f}{1-t_f}\right)\frac{-\frac{nl_{fn}}{l_f}}{\frac{(1+t_n)p_n n}{(1-t_f)w_f}}$$
$$-\left(\frac{t_c}{1+t_c}\right)\frac{\frac{nh_{cn}}{h_c}}{\frac{(1+t_n)p_n n}{(1+t_c)p_c h_c}} + \left(-\frac{1}{\lambda}\right)\{1-2(1+\mu H)[(1-\rho)(1-\gamma)+\rho\gamma]\}$$
$$+\left(-\frac{1}{\lambda}\right)[1-2(1+\mu H)\rho]\frac{-\frac{nl_{mn}}{l_m}}{\frac{(1+t_n)p_n n}{(1-t_m)w_m l_m}} + \left(-\frac{1}{\lambda}\right)[1-2(1+\mu H)(1-\rho)]\frac{-\frac{nl_{fn}}{l_f}}{\frac{(1+t_n)p_n n}{(1-t_f)w_f l_f}}. \quad \text{(K32)}$$

Using the definitions of β, (55), (56), (57), (60), and (K17), (K32) can be rewritten as (K21).

Now, we turn to the interpretations of equations (K18)–(K22). Comparing (K18)–(K20) with (52), (58), and (59), we observe that the optimal commodity and income tax expressions are identical to those in the case without a childcare facility. In the optimal child tax/subsidy expression (K21), the fourth term newly appears compared with (62). By noting the definition of χ_c in (K17), the term reflects the impact of t_n on the t_c-induced distortions on h_c through the change in n. More precisely, because $r_c < (>)0$ leads to overconsumption (underconsumption) of h_c, the decrease (increase) in n is desirable to mitigate overconsumption (to improve underconsumption) of h_c; thus, the child tax (subsidy) becomes desirable from the perspective of this distortion. If $r_c < (>)0$, as the absolute value of the fourth term is larger, the child taxes (subsidies) tend to be desirable.[74]

The second term in $\tilde{\Omega}$ also newly appears in the optimal child tax/subsidy expression in the presence of center-based childcare services, which corresponds to the corrections for child quantity that deviates from a socially desirable level due to the externalities of children on society and the difference in the weights on the spouses between the social welfare function and the couple's utility function. Regarding the bargaining power, the second term in $\tilde{\Omega}$ allows for the cost burden of center-based childcare services $\hat{\gamma}$. If $0.5 > \hat{\gamma}$ (i.e., if a smaller cost burden for the expense of center-based childcare services is imposed on the husband), then the husband wants to increase child quantity since the increase in n is beneficial to him due to the smaller cost burden. Thus, the condition $0.5 > \hat{\gamma}$ contributes to becoming $\frac{\partial n}{\partial \rho} > 0$.[75] With regard to the external effects of children, the second term in $\tilde{\Omega}$ reflects the deviation of child quantity from a socially desirable level, which stems from the fact that a couple disregards the effect

[74]Given that $\chi_c = 1$, the absolute value of the fourth term depends on r_c and α_{nh_c}.

[75]Suppose $\mu = 0$ to focus on the role of the bargaining power. In this case, using (67) and (K23),

$$\tilde{\Omega} = (2\rho-1)\left[\left(\frac{\chi_f}{\alpha^f_{nl}} - \frac{\chi_m}{\alpha^m_{nl}}\right) + \frac{c'}{(1+t_n)p_n} + (1-2\gamma) + (1-2\hat{\gamma})\frac{\chi_c}{\alpha_{nh_c}}\right].$$

of the improvement of child quality due to center-based childcare services on society. More precisely, given that a couple determines the level of n without taking account of the external effects of children on society μNq and that (K7) holds, the couple does not consider the benefit of the enhancement of μNq due to an increase in h_c via an increase in n. Thus, the existence of h_c induces an inefficient lower level of n. To correct the externality of children on society, the optimal child tax/subsidy expression must consider the effect of h_c.[76]

Next, we discuss the expression for the optimal tax/subsidy for the center-based childcare services (K22). If $\xi = 0$, then the first term is reduced to the standard Ramsey expression β/δ_c. If $\xi \neq 0$, $-\alpha_{nh_c}\frac{\xi}{\delta}$ appears in the first term. Since $-\alpha_{nh_c}\frac{\xi}{\delta}$ includes $-n_{t_c}/n_{t_n}$, it reflects the effect of t_c on t_n through the change in n. Noting that $-n_{t_c}/n_{t_n} < 0$ from (K12) and (K13), an increase in t_c reduces t_n.[77] The decrease in t_n raises n and then the increase in n reduces labor supply from (17). Therefore, the increase in t_c exacerbates the deadweight loss in labor supply induced by income taxes. By allowing for this distortionary effect, the optimal tax (subsidy) for center-based childcare services becomes lower (higher) as ξ/δ becomes larger. The second term in (K22), which depends on ρ, $\hat{\gamma}$, and μH, reflects the corrections for h_c deviating from a socially desirable level because of these parameters. The intuition of this term is similar to that of the third term of Λ in equation (62), which is discussed below Proposition 5.[78]

Consider the case in which $\hat{\gamma} = \gamma(\equiv \nu)$, that is, the two kinds of child expenditure are equal for each spouse, which corresponds to (92) and (93) in Proposition 6. In this case, we obtain the following optimal tax expressions.

Moreover, totally differentiating (K8) with respect to n and ρ yields

$$\frac{\partial n}{\partial \rho} = \left[\frac{(1+t_n)p_n}{c''(1-\rho)}\right]\left[\left(\frac{\chi_f}{\alpha^f_{nl}} - \frac{\chi_m}{\alpha^m_{nl}}\right) + \frac{c'}{(1+t_n)p_n} + (1-2\gamma) + (1-2\hat{\gamma})\frac{\chi_c}{\alpha_{nh_c}}\right].$$

According to the two equations, we observe that $\tilde{\Omega} \gtreqless 0 \iff (\rho - 0.5)\frac{\partial n}{\partial \rho} \gtreqless 0$. The fourth term in the second brackets in the second equation above appears newly compared to (68) and is related to the cost burden of center-based childcare services. If $0.5 > \hat{\gamma}$, this term contributes to the positive sign of $\frac{\partial n}{\partial \rho}$. The interpretations for the other three terms are given below (69).

[76]Suppose $\rho = 0.5$ to clarify the effect of μH. In this case, we have

$$\tilde{\Omega} = -\mu H\left[\frac{\chi_m}{\alpha^m_{nl}} + \frac{\chi_f}{\alpha^f_{nl}} + \frac{\chi_c}{\alpha_{nh_c}} + 1 + \frac{c'}{(1+t_n)\,p_n}\right] < 0.$$

The third term in brackets exhibits the deviation of n from a socially desirable level, which is attributable to the fact that a couple underestimates the improvement in child quality through the use of center-based childcare services. As a result, the third term reduces r_n to correct the externality.

[77]The mechanism through which t_c decreases t_n is as follows: the increase in t_c reduces n, and the decrease in n then leads to the decrease in t_n due to the reduction in the tax base for t_n (i.e., the reduction in $p_n n$).

[78]First, suppose $\mu = 0$ to focus on the role of bargaining power. In this case, the second term in (K22) is $(1-\beta)(2\rho-1)(1-2\hat{\gamma})(\equiv \Theta)$. Furthermore, differentiating (K5) with respect to ρ yields $\frac{\partial h_c}{\partial \rho} = \frac{1-2\hat{\gamma}}{1-\sigma}\frac{h_c}{(1-\rho)(1-\hat{\gamma})+\rho\hat{\gamma}}$, which leads to $\frac{\partial h_c}{\partial \rho} \gtreqless 0$ if $0.5 \gtreqless \hat{\gamma}$; this indicates that the effects of the difference in the bargaining power on h_c occurs only if the cost shares of spouses for center-based childcare services are different ($\hat{\gamma} \neq 0.5$). Using these results, we observe that $\Theta \gtreqless 0 \iff (\rho - 0.5)\frac{\partial h_c}{\partial \rho} \gtreqless 0$, which is similar to the meaning of (69). Consequently, ρ and $\hat{\gamma}$ are crucial to determine the sign of Θ. Next, to examine the effect of μH, suppose that either $\rho = 0.5$ or $\hat{\gamma} = 0.5$ holds. In this case, the second term in (K22) is $-(1-\beta)\mu H$. This shows that r_c decreases with μH to enhance h_c, which deviates from a socially desirable level.

Proposition 8. *If $\hat{\gamma} = \gamma(\equiv \nu)$, the optimal taxes satisfy*

$$r_n = \frac{\beta}{\delta} + \frac{r_m \chi_m}{\alpha_{nl}^m} + \frac{r_f \chi_f}{\alpha_{nl}^f} + (1-\beta)\Omega, \tag{K33}$$

$$r_c = (1-\beta)\{1 - 2(1+\mu H)[(1-\rho)(1-\nu) + \rho\nu]\}. \tag{K34}$$

First, we provide the derivations of these formulas. Note that the following relationship holds:

$$1 - \alpha_{nh_c}\frac{\xi}{\delta} = 1 - \frac{(1+t_n)p_n n}{(1+t_c)p_c h_c}\frac{\frac{(1+t_c)n_{t_c}}{n}}{\frac{(1+t_n)n_{t_n}}{n}} = 1 - \frac{p_n n n_{t_n}^{-1}}{p_c h_c n_{t_c}^{-1}}. \tag{K35}$$

Equation (K12) yields

$$p_n n n_{t_n}^{-1} = -\frac{(1-\rho)nc''}{(1-\rho)(1-\gamma) + \rho\gamma}, \tag{K36}$$

and (K5) and (K13) yield

$$p_c h_c n_{t_c}^{-1} = -\frac{(1-\rho)nc''}{(1-\rho)(1-\hat{\gamma}) + \rho\hat{\gamma}}. \tag{K37}$$

Applying (K36) and (K37) to (K35), we obtain

$$1 - \alpha_{nh_c}\frac{\xi}{\delta} = \frac{(1-2\rho)(\hat{\gamma} - \gamma)}{(1-\rho)(1-\gamma) + \rho\gamma}. \tag{K38}$$

This shows that if $\hat{\gamma} = \gamma$, then $1 - \alpha_{nh_c}\frac{\xi}{\delta} = 0$. Hence, the first term in (K22) vanishes. Thus, we obtain (K34). Moreover, by substituting (K34) for r_c in (K21), the fourth term in (K21) offsets the second term in (K23), and thus, we obtain (K33) by using the relationship $\Omega = \Lambda$ under $\hat{\gamma} = \gamma(\equiv \nu)$.

Next, we interpret equations (K33) and (K34). Comparing Proposition 8 with Proposition 5, we observe that, if $\hat{\gamma} = \gamma$, the optimal child tax/subsidy expression (K33) is identical to (62) irrespective of whether there is a childcare facility. This is because the fourth term in (K21) and the second term in (K23) cancel out, given that the first term on the right-hand side of (K22) vanishes when the condition is satisfied; that is, the net revenue-raising effect of t_c (β) offsets the distortionary effect of t_c on labor supply through the change in n ($-\beta\alpha_{nh_c}\frac{\xi}{\delta}$).[79] Allowing for the result, we observe that the direct child subsidy (tax) under $\hat{\gamma} \neq \gamma$, which is given by (K21), alleviates the inefficiently downward (upward) distortion on h_c induced by t_c when the net revenue-raising effect is greater (lower) than the distortionary effect. However, the direct child tax/subsidy does not play a role in correcting h_c that deviates from a socially desirable level due to changes in ρ and μH.

Meanwhile, if $\hat{\gamma} = \gamma$ holds, the optimal tax formula for r_c is expressed by (K34) because the first term in (K22) disappears under the condition $\hat{\gamma} = \gamma$, as mentioned above.

[79]The former effect reflects the net benefit of tax-revenue-raising, taking into account the welfare losses due to distortionary taxes (Saez, 2001). As explained below Proposition 7, the latter effect of t_c indicates that an increase in t_c induces each spouse to supply less labor. As shown in this Appendix, if $\hat{\gamma} = \gamma$, then $\alpha_{nh_c}\frac{\xi}{\delta} = 1$. Thus, the two effects cancel out.

Appendix L: The Derivation of (79)

We substitute (76) for h_c in (74) and consider the maximization of u. Allowing for (17), the first-order condition with respect to n is

$$\begin{aligned}0 \;=\; \frac{\partial u}{\partial n} &= \rho(1-t_m)w_m l_{mn} + (1-\rho)(1-t_f)w_f l_{fn} - (1-\rho)c' \\ &- [(1-\rho)(1-\gamma)+\rho\gamma](1+t_n)p_n - [(1-\rho)(1-\nu)+\rho\nu](1+t_c)p_c h_{cn} \\ &+ n^{1-\sigma}\left(s_m^{\sigma}h_m^{\sigma-1}h_{mn} + s_f^{\sigma}h_f^{\sigma-1}h_{fn} + s_c^{\sigma}h_c^{\sigma-1}h_{cn}\right) \\ &+ (1-\sigma)n^{-\sigma}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{(s_f h_f)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right].\end{aligned} \tag{L1}$$

Using (11), (12), (14), (17), (75), (76), and (78), (L1) can be rewritten as

$$\begin{aligned}0 \;=\; &\rho(1-t_m)w_m l_{mn} + (1-\rho)(1-t_f)w_f l_{fn} - (1-\rho)c' \\ &- [(1-\rho)(1-\gamma)+\rho\gamma](1+t_n)p_n - [(1-\rho)(1-\nu)+\rho\nu](1+t_c)p_c h_{cn} \\ &+ \frac{(1-t_m)w_m}{1+\tau}h_{mn} + \frac{(1-t_f)w_f}{1+\tau}h_{fn} + [(1-\rho)(1-\nu)+\rho\nu](1+t_c)p_c h_{cn} \\ &+ \left(\frac{1-\sigma}{\sigma}\right)\{\frac{(1-t_m)w_m}{1+\tau}h_{mn} + \frac{(1-t_f)w_f}{1+\tau}h_{fn} + [(1-\rho)(1-\nu)+\rho\nu](1+t_c)p_c h_{cn}\}.\end{aligned} \tag{L2}$$

Using (17), (L2) can be rewritten as (79).

Appendix M: The Proof of Proposition 6

By defining the Lagrange function as L and the Lagrange multiplier on the revenue constraint as λ, and by making use of (17), the first-order conditions of the government's social welfare maximization (86) subject to the revenue constraint (87) with respect to t_y, t_m, t_f, t_n, and t_c are given by

$$\begin{aligned}0 \;=\; \frac{\partial L}{\partial t_y} &= -p_y y_m - (1+t_y)p_y y_m' - p_y y_f - (1+t_y)p_y y_f' \\ &+ y_m^{\varphi-1}y_m' + y_f^{\varphi-1}y_f' - \lambda[y_m + y_f + t_y(y_m' + y_f')]p_y,\end{aligned} \tag{M1}$$

$$\begin{aligned}0 \;=\; \frac{\partial L}{\partial t_m} &= -w_m l_m + (1-t_m)w_m l_{mt_m} + (1-t_m)w_m l_{mn}n_{t_m} - (l_m+h_m)^{\phi}\left(l_{mt_m} + h_{mt_m}\right) \\ &+ (1-t_f)w_f l_{fn}n_{t_m} - c'n_{t_m} - (1+t_n)p_n n_{t_m} - (1+t_c)p_c h_{cn}n_{t_m} \\ &+ 2(1+\mu H)(1-\sigma)n^{-\sigma}n_{t_m}\left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{(s_f h_f)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right] \\ &+ 2(1+\mu H)n^{1-\sigma}[s_m^{\sigma}h_m^{\sigma-1}\left(h_{mt_m} + h_{mn}n_{t_m}\right) + s_f^{\sigma}h_f^{\sigma-1}h_{fn}n_{t_m} + s_c^{\sigma}h_c^{\sigma-1}h_{cn}n_{t_m}] \\ &- \lambda(w_m l_m + t_m w_m l_{mt_m} + t_m w_m l_{mn}n_{t_m} + t_f w_f l_{fn}n_{t_m} + t_n p_n n_{t_m} + t_c p_c h_{cn}n_{t_m}),\end{aligned} \tag{M2}$$

$$0 = \frac{\partial L}{\partial t_f} = (1-t_m)w_m l_{mn} n_{t_f} - w_f l_f + (1-t_f)w_f l_{ft_f} + (1-t_f)w_f l_{fn} n_{t_f} \tag{M3}$$
$$- \left(l_f + h_f\right)^{\phi} \left(l_{ft_f} + h_{ft_f}\right) - c' n_{t_f} - (1+t_n)p_n n_{t_f} - (1+t_c)p_c h_{cn} n_{t_f}$$
$$+ 2(1+\mu H)(1-\sigma)n^{-\sigma} n_{t_f} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right]$$
$$+ 2(1+\mu H)n^{1-\sigma}[s_m^{\sigma} h_m^{\sigma-1} h_{mn} n_{t_f} + s_f^{\sigma} h_f^{\sigma-1}(h_{ft_f} + h_{fn} n_{t_f}) + s_c^{\sigma} h_c^{\sigma-1} h_{cn} n_{t_f}]$$
$$- \lambda(t_m w_m l_{mn} n_{t_f} + w_f l_f + t_f w_f l_{ft_f} + t_f w_f l_{fn} n_{t_f} + t_n p_n n_{t_f} + t_c p_c h_{cn} n_{t_f}),$$

$$0 = \frac{\partial L}{\partial t_n} = (1-t_m)w_m l_{mn} n_{t_n} + (1-t_f)w_f l_{fn} n_{t_n} - c' n_{t_n} \tag{M4}$$
$$- p_n n - (1+t_n)p_n n_{t_n} - (1+t_c)p_c h_{cn} n_{t_n}$$
$$+ 2(1+\mu H)(1-\sigma)n^{-\sigma} n_{t_n} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right]$$
$$+ 2(1+\mu H)n^{1-\sigma} \left(s_m^{\sigma} h_m^{\sigma-1} h_{mn} n_{t_n} + s_f^{\sigma} h_f^{\sigma-1} h_{fn} n_{t_n} + s_c^{\sigma} h_c^{\sigma-1} h_{cn} n_{t_n}\right)$$
$$- \lambda\left(t_m w_m l_{mn} n_{t_n} + t_f w_f l_{fn} n_{t_n} + p_n n + t_n p_n n_{t_n} + t_c p_c h_{cn} n_{t_n}\right),$$

$$0 = \frac{\partial L}{\partial t_c} = (1-t_m)w_m l_{mn} n_{t_c} + (1-t_f)w_f l_{fn} n_{t_c} - c' n_{t_c} \tag{M5}$$
$$- (1+t_n)p_n n_{t_c} - p_c h_c - (1+t_c)p_c h_{ct_c} - (1+t_c)p_c h_{cn} n_{t_c}$$
$$+ 2(1+\mu H)(1-\sigma)n^{-\sigma} n_{t_c} \left[\frac{(s_m h_m)^{\sigma}}{\sigma} + \frac{\left(s_f h_f\right)^{\sigma}}{\sigma} + \frac{(s_c h_c)^{\sigma}}{\sigma}\right]$$
$$+ 2(1+\mu H)n^{1-\sigma}[s_m^{\sigma} h_m^{\sigma-1} h_{mn} n_{t_c} + s_f^{\sigma} h_f^{\sigma-1} h_{fn} n_{t_c} + s_c^{\sigma} h_c^{\sigma-1}(h_{ct_c} + h_{cn} n_{t_c})]$$
$$- \lambda(t_m w_m l_{mn} n_{t_c} + t_f w_f l_{fn} n_{t_c} + t_n p_n n_{t_c} + p_c h_c + t_c p_c h_{ct_c} + t_c p_c h_{cn} n_{t_c}).$$

First, given that (M1) coincides with (48), we obtain (89), which is the same expression as (52). Moreover, using (11), (12), and (M4), after some manipulation, (M2) and (M3) can be rewritten as

$$t_m : 0 = -(1+\lambda)w_m l_m - \lambda t_m w_m l_{mt_m} + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-t_m)w_m h_{mt_m} + (1+\lambda)p_n n n_{t_n}^{-1} n_{t_m},$$

$$t_f : 0 = -(1+\lambda)w_f l_f - \lambda t_f w_f l_{ft_f} + \left[\frac{2(1+\mu H)}{1+\tau} - 1\right](1-t_f)w_f h_{ft_f} + (1+\lambda)p_n n n_{t_n}^{-1} n_{t_f}.$$

These two equations are the same as (53) and (54), respectively. Using the process in Appendix G, we observe that the two equations lead to (90) and (91), respectively.

Next, we derive (93). Multiplying each term in (M4) by $n_{t_n}^{-1} n_{t_c}$ and subtracting the resulting equation from (M5), we obtain

$$\begin{aligned} 0 &= \beta p_n n n_{t_n}^{-1} n_{t_c} + (1-\beta) p_c h_c + (1-\beta)(1+t_c) p_c h_{ct_c} \\ &\quad - 2(1-\beta)(1+\mu H) n^{1-\sigma} s_c^{\sigma} h_c^{\sigma-1} h_{ct_c} - p_c h_c - t_c p_c h_{ct_c}. \end{aligned}$$

Using (75), this expression can be rewritten as

$$\frac{t_c}{1+t_c} = \frac{\beta}{-\frac{(1+t_c)h_{ct_c}}{h_c}} \left[1 - \frac{p_n n n_{t_n}^{-1}}{p_c h_c n_{t_c}^{-1}} \right] + (1-\beta)\{1 - 2(1+\mu H)[(1-\rho)(1-\nu)+\rho\nu]\}. \quad \text{(M6)}$$

Equation (83) yields

$$p_n n n_{t_n}^{-1} = -\frac{(1-\rho) n c''}{(1-\rho)(1-\nu)+\rho\nu}, \quad \text{(M7)}$$

and (76) and (84) yield

$$p_c h_c n_{t_c}^{-1} = -\frac{(1-\rho) n c''}{(1-\rho)(1-\nu)+\rho\nu}. \quad \text{(M8)}$$

Allowing for (M7) and (M8), the first term on the right-hand side in (M6) vanishes. Thus, using (88), we obtain (93) from (M6).

Finally, we derive the optimal child tax/subsidy expression. By multiplying each term in (M4) by $n_{t_n}^{-1}$ and making use of (A2), (75), and the fact that $h_{in} n = h_i$ ($i = m, f, c$), after some manipulations, (M4) can be rewritten as

$$\begin{aligned} 0 &= (1-t_m) w_m l_{mn} + (1-t_f) w_f l_{fn} - c' - p_n n n_{t_n}^{-1} - (1+t_n) p_n - (1+t_c) p_c h_{cn} \quad \text{(M9)} \\ &\quad + 2(1+\mu H)\left(\frac{1}{\sigma}\right)\{\frac{(1-t_m)w_m}{1+\tau} h_{mn} + \frac{(1-t_f)w_f}{1+\tau} h_{fn} + [(1-\rho)(1-\nu)+\rho\nu](1+t_c) p_c h_{cn}\} \\ &\quad - \lambda \left(t_m w_m l_{mn} + t_f w_f l_{fn} + p_n n n_{t_n}^{-1} + t_n p_n + t_c p_c h_{cn} \right). \end{aligned}$$

Multiplying each term in (79) by $2(1+\mu H)$, subtracting the resulting equation from (M9), and using (17), we obtain

$$\begin{aligned} 0 &= [1 - 2(1+\mu H)(1-\rho)] c' + (1+\lambda) p_n n n_{t_n}^{-1} \quad \text{(M10)} \\ &\quad + \{1 - 2(1+\mu H)[(1-\rho)(1-\nu)+\rho\nu]\}(1+t_c) p_c h_{cn} \\ &\quad + \lambda \left(t_m w_m l_{mn} + t_f w_f l_{fn} + t_n p_n + t_c p_c h_{cn} \right) + \{1 - 2(1+\mu H)[(1-\rho)(1-\nu)+\rho\nu]\}(1+t_n) p_n \\ &\quad - [1 - 2(1+\mu H)\rho](1-t_m) w_m l_{mn} - [1 - 2(1+\mu H)(1-\rho)](1-t_f) w_f l_{fn}, \end{aligned}$$

which yields

$$
\begin{aligned}
\frac{t_n}{1+t_n} \quad = \quad & \left(-\frac{1}{\lambda}\right)[1-2(1+\mu H)(1-\rho)]\,\frac{c'}{(1+t_n)p_n} + \left(\frac{1+\lambda}{\lambda}\right)\frac{1}{-\frac{(1+t_n)n_{t_n}}{n}} \\
& + \left(-\frac{1}{\lambda}\right)\{1-2(1+\mu H)\,[(1-\rho)(1-\nu)+\rho\nu]\}\,\frac{\frac{nh_{cn}}{h_c}}{\frac{(1+t_n)p_n n}{(1+t_c)p_c h_c}} \\
& + \left(\frac{t_m}{1-t_m}\right)\frac{-\frac{nl_{mn}}{l_m}}{\frac{(1+t_n)p_n n}{(1-t_m)w_m l_m}} + \left(\frac{t_f}{1-t_f}\right)\frac{-\frac{nl_{fn}}{l_f}}{\frac{(1+t_n)p_n n}{(1-t_f)w_f}} \\
& - \left(\frac{t_c}{1+t_c}\right)\frac{\frac{nh_{cn}}{h_c}}{\frac{(1+t_n)p_n n}{(1+t_c)p_c h_c}} + \left(-\frac{1}{\lambda}\right)\{1-2(1+\mu H)\,[(1-\rho)(1-\nu)+\rho\nu]\} \\
& + \left(-\frac{1}{\lambda}\right)[1-2(1+\mu H)\rho]\frac{-\frac{nl_{mn}}{l_m}}{\frac{(1+t_n)p_n n}{(1-t_m)w_m l_m}} + \left(-\frac{1}{\lambda}\right)[1-2(1+\mu H)(1-\rho)]\frac{-\frac{nl_{fn}}{l_f}}{\frac{(1+t_n)p_n n}{(1-t_f)w_f l_f}}.
\end{aligned}
\tag{M11}
$$

Given $1-\beta=-\frac{1}{\lambda}$ and (88), if we apply (93) to the sixth term in (M11), the third and sixth terms in (M11) cancel out. Using the definition of β, (55), (56), (57), and (60), (M11) can be rewritten as (92).