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Eisei Ohtaki

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TOKYO CENTER FOR ECONOMIC RESEARCH
1-7-10-703 Iidabashi, Chiyoda-ku, Tokyo 102-0072, Japan

Abstract

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Eisei Ohtaki
TCER
and
Kanagawa University
Department of Economics
3-27-1 Rokkakubashi, Kanagawa-ku, Yokohama-shi, Kanagawa 221-8686, Japan
ohtaki@kanagawa-u.ac.jp

Does climate change lead financial instability?: A benchmark result

Eisei Ohtaki[†]

[†] Department of Economics, Kanagawa university, 3-27-1 Rokkakubashi, Kanagawa-ku, Yokohama-shi, Kanagawa 221-8686, Japan
email address: ohtaki@kanagawa-u.ac.jp

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Keywords: Climate change; Financial stability; Monetary policy; Golden rule optimality; Overlapping generations model.

JEL Classification Number: E40; E50; Q50.

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1 Introduction

Since the establishing of the Network of Central Banks and Supervisors for Greening the Financial System (NGFS) at the Paris "One Planet Summit" in December 2017, central banks and supervisors in a lot of countries have directed significant attention towards developing an understanding of the implications of climate change for the financial sector and financial stability. As mentioned by Brunetti et.al. (2021), however, 'analysis and research is at an early stage.' To contribute from Economics to this stream, therefore, it is worthwhile to build a monetary or financial model with climate change, which becomes a benchmark to argue financial risks related to climate change.

Climate change, or the environment in a broader sense, affects not only a certain generation but also many generations in a long period. To incorporate the effects of the environmental quality on many generations, the overlapping generations (OLG) model of the environment is already developed by John and Pecchenino (1994). In the model, the environment quality was modeled by a long-lasting accumulable public good. Then, most of the previous studies paid much attention to the market creation or fiscal policies such as tax-subsidy systems to internalize environmental externalities (see Section 4). However, they paid little attention to the role of monetary policy.

This study aims to explore implications of climate change on financial stability. For this aim, this study builds an OLG model of the environment and money by embedding money in a simplified version of the OLG model of John and Pecchenino (1994). In this model, the environmental quality is modeled by the accumulable public good, similar to John and Pecchenino, and is interpreted as a level of climate change. The lump-sum money transfers are also introduced. We can observe, in this model, the conflicts between market activities and environmental quality. Then, financial stability is represented as stability of equilibria.

This study shows the equilibrium dynamics drastically different from the standard monetary OLG model. It is well-known in the standard argument that a stationary monetary equilibrium, which is a stationary equilibrium with valued money, is locally unstable when the elasticity of the second-period utility index function is less than one. Contrary to the existing result, we demonstrate that a stationary monetary equilibrium might be a saddle point even when the elasticity of the utility index function for the second-period consumptions is less than one. Therefore, in such a case, we can find, under a certain initial condition, a unique equilibrium path converging to the stationary monetary equilibrium.

This study also explores a role of monetary policy in the model. When the equilibrium path converging to a stationary monetary equilibrium, it is reasonable to argue the efficiency of the equilibrium and optimal monetary policy improving welfare at the equilibrium. We adopt the golden rule optimality as an efficiency criterion of stationary allocations. As the tradition from Friedman (1969), it is well-known that the constant or deflationary money supply (such that the nominal interest rate will be zero) will be optimal when there is no market friction. In fact, in the standard monetary OLG model, the optimal money growth is equal to one.[1] Contrary to the standard argument in the monetary OLG model, it is shown that the optimal gross rate of money growth, which maximizes the welfare at stationary monetary equilibrium, exists and is greater than one.[2] Further, we show that the optimal money growth actually achieves the golden rule optimality.

The remainder of this paper is organized as follows: Section 2 presents an OLG model of the environment and money. The golden rule optimality is also defined. Section 3 presents the main results. Subsection 3.1 shows the existence and uniqueness of stationary monetary equilibrium. Furthermore, the local stability of the stationary monetary equilibrium is shown. Subsection 3.2 argues on the global dynamics of equilibrium paths. Subsection 3.3 provides some results on the equilibrium efficiency and the optimal monetary policy. Section 4 presents some of the relevant results from the existing literature. Section 5 provides concluding remarks. Proofs of Propositions are presented in the Appendix.

2 The Model

An overlapping generations model of the environment and money is considered. Time is discrete and runs from $t = 1$ to infinity. The consumption good in each period is perishable.

At each date, one new agent is born and lives for two period. Each agent is endowed with ω units of the consumption good in the first period of her life and nothing in the second period. She ranks pairs of the environmental quality and consumption in the second period of her life, $(E, c) \in \mathbb{R}^2_+$, according to a lifetime utility function $U : \mathbb{R}^2_{++} \to \mathbb{R}$. It is assumed that there exist real-valued functions u and v of $\mathbb{R}_+$ to $\mathbb{R}$ such that $U(E, c) = u(E) + v(c)$, where u and

[1] See (Wallace, 1980) for example.

[2] Suboptimality of the Friedman rule in the OLG framework has been reported, for example, by (Haslag and Martin, 2007). However, in their model, optimal gross rate of money growth is one, that is, the optimal monetary policy requires a constant money supply.

v are assumed to be strictly increasing, strictly concave, and continuously differentiable on the interior of its domain and satisfy that $\lim_{c\downarrow 0} u'(E) = \infty$ and $\lim_{E\downarrow 0} v'(c) = \infty$.

Similar to John and Pecchenino (1994), we assume the following law of motion on the environment quality:

$$E_{t+1} = (1-\alpha)E_t - \beta c_t + \gamma z_t$$

for each $t \geq 1$, where $\alpha \in (0,1)$, $\beta > 0$, and $\gamma > 0$. In the last equation, z_t and c_t represents the amount of maintenance for the environment initiated in period t and the consumption externalities in period t, respectively. The initial environmental quality, E_1, is treated as given.

We adopt golden rule optimality as a criterion of optimality. A *stationary feasible allocation* is a pair $x = (E, c) \in \mathbb{R}^2_{++}$ satisfying that $(1 + \beta/\gamma)c + (\alpha/\gamma)E = \omega$, where E and c are the environmental quality and consumption, respectively. It is a *golden rule optimal allocation* if it satisfies that $U(x) \geq U(\tilde{x})$ for each feasible stationary allocation $\tilde{x}$.

In this economy, an infinitely lived outside asset yielding no dividend, money, exists. Its stock at date t is denoted by M_t. The constant money growth is assumed, that is, $M_{t+1} = \sigma M_t$ for each date $t \geq 1$, where $\sigma > 0$. The newly issued money of period $t+1$, $\Delta_{t+1} := M_{t+1} - M_t$, is given to the old agent in the period. The initial money stock M_1 is given and held by the *initial old* who is a one-period-lived agent in period 1.

We define equilibrium. A *monetary equilibrium given money growth rate* σ is a pair (E^e, q^e) of the positive sequence of real money balances, q^e, and the positive sequence of the environment qualities, E^e, such that there exists a sequence m^e such that: for any date $t \geq 1$, (i) given $c_t = q^e_t$,

$$(E^e_{t+1}, m^e_t) \in \arg\max_{E,m} \; U(E, c)$$

subject to

$$(q^e_t / M_t)m + z = \omega,$$

$$c = (q^e_{t+1}/M_{t+1})(m + \Delta_{t+1}),$$

$$E = (1-\alpha)E^e_t - \beta c_t + \gamma z$$

and (ii) $m^e_t = M_t$. Moreover, it is *stationary* if there exists some $(E, q) \gg 0$ such that $(E^e_t, q^e_t) = (E, q)$ for each $t \geq 1$.

Finally, suppose that there exists a unique stationary monetary equilibrium given σ, $(\bar{E}(\sigma), \bar{q}(\sigma))$. Then, the equilibrium welfare, denoted by $W(\sigma)$, can be written in the form as

$$W(\sigma) = u(\bar{E}(\sigma)) + v(\bar{q}(\sigma)).$$

A money growth rate σ is *optimal* if $W(\sigma) \geq W(\tilde{\sigma})$ for each $\tilde{\sigma} > 0$.

3 Main Results

This section provides main results, whereas their proofs are provided in the Appendix.

3.1 Basic Properties of Stationary Monetary Equilibrium

This subsection examines existence, uniqueness, and the local dynamics of stationary monetary equilibrium.

In a standard OLG model of monetary economy, a monetary equilibrium is characterized by a single difference equation. On the other hand, we characterize a monetary equilibrium by a *system* of difference equations:

Proposition 1 *A pair (E^e, q^e) of positive sequences is a monetary equilibrium given money growth rate σ if and only if it holds that, for all $t \geq 1$,*

$$\sigma\gamma q_t^e u'(E_{t+1}^e) = q_{t+1}^e v'(q_{t+1}^e) \tag{1}$$

$$E_{t+1}^e = (1-\alpha)E_t^e + \gamma\omega - (\beta+\gamma)q_t^e. \tag{2}$$

In the previous proposition, the pair of equations (1) and (2) determines the equilibrium dynamics. It says that, given the period t equilibrium outcome (E_t^e, q_t^e), period $t+1$ environmental quality E_{t+1}^e is first determined following the law of motion of the environment quality, (2), and then, period $t+1$ real money balance q_{t+1}^e is derived from (1) given (E_t^e, q_t^e) and E_{t+1}^e. Following this characterization, we can find that a stationary monetary equilibrium $(\bar{E}, \bar{q})$ is, if any, characterized by the pair of equations

$$\frac{u'(\bar{E})}{v'(\bar{q})} = \frac{1}{\sigma\gamma} \tag{3}$$

and

$$\bar{E} = (\gamma/\alpha)\omega - (\beta+\gamma)\bar{q}/\alpha. \tag{4}$$

The next proposition guarantees the existence of a unique solution of the above equations.

Proposition 2 *A stationary monetary equilibrium $(\bar{E}(\sigma), \bar{q}(\sigma))$ given σ exists and is unique.*

Here, we argue the local equilibrium dynamics around the stationary monetary equilibrium. If $\alpha = \gamma = 1$ and $\beta = 0$, then the system of Eqs.(1) and (2) degenerates into

(5) $\quad q_t^e u'(\omega - q_t^e) = \sigma^{-1} q_{t+1}^e v'(q_{t+1}^e).$

One should find that this is the equilibrium difference equation in the standard monetary OLG model without the environment. As the previous proposition, we can find the unique stationary solution $\hat{q} \in (0, \omega)$ for Eq.(5). In addition to $\alpha = \gamma = 1$ and $\beta = 0$, suppose that $-cv''(c)/v'(c) < 1$ on an open neighborhood $\hat{C}$ of $\hat{q}$. Applying the implicit function theorem, we can summarize the local equilibrium dynamics of Eq.(5) on some open neighborhood $\hat{Q} \subset \hat{C}$ of $\hat{q}$ in the form that $q_{t+1} = f(q_t)$. One can easily find that $f'(q) > 1$ on $\hat{Q}$, provided that $\sigma \geq 1$. Therefore, when there is no environmental externalities, the stationary monetary equilibrium might be locally unstable.

Contrary to the argument in the previous paragraph, the following proposition provides a sufficient condition for the saddle-point property of the stationary monetary equilibrium with the environment quality.

Proposition 3 *The stationary monetary equilibrium* $(\bar{E}(\sigma), \bar{q}(\sigma))$ *given* σ *is a saddle point if* $\alpha < -cv''(c)/v'(c) < 1$ *on some open neighborhood* $\bar{C}$ *of* $\bar{q}(\sigma)$*.*

Therefore, for a certain initial condition, there exists a unique saddle path, that is, an equilibrium path with a certain condition converges to the stationary monetary equilibrium, provided that the elasticity of v lies on a certain interval. This is a remarkable difference from the standard monetary OLG model. Note that the presented sufficient condition is independent of the rate of money growth, σ. Therefore, under such a condition, we can find a saddle path for an arbitrary rate of money growth.

3.2 Optimal Monetary Policy

We now turn to discuss the optimal monetary policy.[3] Because the unique stationary monetary equilibrium given $\sigma > 0$ exists, the welfare function $W(\sigma)$ is well-defined. Therefore, we should next find an optimal money growth rate. The existence of such a rate is guaranteed by the following proposition. Furthermore, its exact solution is provided.

[3] Note that the following results does not impose any restrictions on the elasticity of u.

Proposition 4 *An optimal money growth rate σ^* exists and is unique. Further, it is characterized by $\sigma^* = (\beta + \gamma)/(\alpha\gamma) > 1$.*

Note that this proposition implies the fact that the constant money supply ($\sigma = 1$) does not maximize the welfare at the stationary monetary equilibrium.

By its definition, the optimal money growth rate maximizes the welfare at stationary monetary equilibrium. However, it is not so trivial that the equilibrium allocation under the optimal rate is actually optimal. To examine this issue, we first characterize the golden rule optimality of stationary feasible allocations.

Proposition 5 *A stationary feasible allocation (E^g, c^g) is golden rule optimal if and only if it satisfies that*

$$(6) \quad \frac{u'(E^g)}{v'(c^g)} = \frac{\alpha}{\beta + \gamma}$$

and

$$(7) \quad (1 + \beta/\gamma)c^g + (\alpha/\gamma)E^g = \omega.$$

Note that, similar to John and Pecchenino (1994), the environment qualities in our model play a role as the public good. Therefore, Eq.(6) can be then interpreted as an intertemporal variation of the Samuelson condition for the optimal provision of the public good. By comparing the pair of Eqs.(3) and (4) with the pair of Eqs.(6) and (7), one can easily find that they coincide under the optimal money growth rate σ^*. In fact, the stationary monetary equilibrium under σ^* achieves the golden rule optimality:

Proposition 6 *The optimal money growth rate achieves the golden rule optimality.*

As a corollary of this proposition, we can say that the constant money supply is not optimal in the sense of golden rule optimality. This is a remarkable difference from the standard argument in the monetary OLG model, which claims that the lump-sum money transfer will cause inefficiency. As shown in the proof of Proposition 4, $\bar{q}'(\sigma) < 0$ for $\sigma > 0$, which implies that $E^g > \bar{E}(1)$. This means that, in the equilibrium, agents invest more in money instead of the maintenance of the environment (the free-rider problem). On the other hand, the lump-sum money transfer decreases the rate of return of money. Therefore, the growth of money decreases the investment in money and remedies the inefficiency caused by the free-rider problem.

4 Related Literature

Even in economics, there is a lot of studies that tried to resolve conflicts between economic activities and the environment. Among them, John and Pecchenino (1994) is one of important studies because it succeeded to capture generational problems in conflicts between economic growth and the environment the environmental quality by applying the OLG structure.

After their study, studies that considered environmental issues in OLG frameworks are ever growing. For example, John, Pecchenino, Schimmelpfenning, and Schreft (1995), Ono (1996), Ono and Maeda (2002) studied about optimal tax-subsidy systems in the framework of John and Pecchenino (1994) . Jouvet, Michael, and Vidal (2000) showed that, in the presence of altruism, proportional taxes should be used in order to neutralize the external effects. Jouvet, Michael, and Rotillon (2005) introduced a market for permits to a variant of John and Pecchenino (1994) and showed that all permits should be auctioned. Ono (2007) studied about the effect of the environmental tax reform on economic growth and welfare in the OLG model with endogenous growth, unemployment, and pollution. Prieur (2009) showed that the emergence of the environmental Kuznets curve is no longer the rule when the assimilation capacity of nature is limited and vanishes beyond a critical level of pollution. Jouvet, Pestieau, and Ponthiere (2010) considered an OLG model with endogenous longevity and showed that the decentralization of the social optimum requires a tax not only on capital income but also on health expenditures. Bosi and Desmarchelier (2013) considered an OLG model with endogenous fertility and pollution externalities and showed that a raise in the cost of rearing children increases (decreases) consumption and decreases (increases) pollution under dominant income (substitution) effects. Dao and Dávila (2014) assumed that both consumption and production generates pollution and studied about optimal tax and transfer policies. Ponthiere (2016) characterized the optimal level of pollution when pollution deteriorates survival conditions. Fodha, Seegmuller, and Yamagami (2018) studied about the environmental tax reform in an OLG model with pollution and the government's debt constraint. Constant and Davin (2019) considered an open OLG economy and examined how the underlying costs can spread from a vulnerable to a non-vulnerable country through international trade. Cisco and Gatto (2021) provided several calibration results for a variant of John and Pecchenino (1994).

As reviewed in the previous paragraph, there seems no work that studied on the relation between the environment and financial stability.[4] To our best knowledge, therefore, this study

[4]Bloise, Currarini, and Kikidis (2002) explored optimal inflation rates in a pure-endowment OLG model with

is the first to explore implications of the environment on monetary economy and contributes to the literature by investigating stability of a stationary monetary equilibrium.

5 Concluding Remarks

Motivated by recent climate actions of central banks and supervisors in a lot of countries, this study has developed an overlapping generations model of the environment and money and examined equilibrium instability. It has been shown that a stationary monetary equilibrium exists, is unique, and is a locally saddle point. This study has also shown that the optimal gross rate of money growth, which maximizes the welfare at stationary monetary equilibrium, exists uniquely and is greater than one. Furthermore, we have demonstrated that the optimal money growth actually achieves the golden rule optimality. These results seem to imply that monetary policy should be utilized more positively in order to resolve environmental issues.

We close this study with two remarks. First, we should remark that the market mechanism is stable only on the saddle path converging to the monetary steady state.

Second, we should also remark that our results are obtained in a pure-endowment environment. Characterizing optimal policies in an environment with intertemporal production technologies should be one of future researches.[5]

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Appendix: Proofs of Propositions

Proof of Proposition 1. We can summarize each agent's optimization problem as the maximization problem of the real-valued function V defined by, for all $m \in \mathbb{R}$,

$$V(m) \quad := \quad u\left((1-\alpha)E_t^e - \beta c_t + \gamma\left(\omega - \frac{q_t^e}{M_t}m\right)\right) + v\left(\frac{q_{t+1}^e}{M_{t+1}}(m+\Delta_{t+1})\right).$$

Then, the first order condition at the equilibrium, $V'(M_t) = 0$, implies Eq.(1). Eq.(2) follows from the given law of motion of levels of the environment. This completes the *only if* part of Proposition 1. Concavity of u and v also ensures the *if* part of Proposition 1. Q.E.D.

accumulated public goods, whereas they paid no attention to environmental problems.

[5] For example, Ohtaki (2023) explores optimal combinations of money growth rates and tax instruments in an environmental OLG model with a linear production technology.

Proof of Proposition 2. By Proposition 1, a stationary monetary equilibrium $(\bar{E}, \bar{q})$ given σ is, if any, characterized by $u'(\gamma\omega/\alpha - (\beta+\gamma)\bar{q}/\alpha) = \sigma\gamma v'(\bar{q})$ and $\bar{E} = \gamma\omega/\alpha - (\beta+\gamma)\bar{q}/\alpha$. Then, define the function $f : \mathbb{R}_{++} \to \mathbb{R}$ by, for each $q \geq 0$,

$$f(q) := \sigma\gamma \frac{u'(\gamma\omega/\alpha - (\beta+\gamma)q/\alpha)}{v'(q)}.$$

It can be easily verified that f is continuous, $\lim_{q\downarrow 0} f(q) = 0$, and $\lim_{q\uparrow \frac{\gamma}{\beta+\gamma}\omega} f(q) = \infty$. Therefore, there exists some $\bar{q} \in (0, \gamma\omega/(\beta+\gamma))$ such that $f(\bar{q}) = 1$. Furthermore, such $\bar{q}$ is unique because $f'(q) > 0$ on the previous interval. Now define $\bar{E} = \gamma\omega/\alpha - (\beta+\gamma)\bar{q}/\alpha$, which is positive because $\bar{q} < \gamma\omega/(\beta+\gamma)$. Obviously, $(\bar{E}, \bar{q})$ is the unique stationary monetary equilibrium given σ. Q.E.D.

Proof of Proposition 3. For notational convenience, let $(\bar{E}, \bar{q}) = (\bar{E}(\sigma), \bar{q}(\sigma))$. Define the function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ by $\zeta(q) := qv'(q)$ for any $q \geq 0$. Suppose that there is an open neighborhood $\bar{C}$ of $\bar{q}$, on which it holds that $\alpha < -cv''(c)/v'(c) < 1$. Because $\zeta'(q) = v'(q) + qv''(q) > 0$ on $\bar{C}$, the inverse function ζ^{-1} on $\zeta(\bar{C})$ exists. Therefore, on a sufficiently small open neighborhood $\bar{X}$ of $(\bar{E}, \bar{q})$, we can rewrite Eqs.(1) and (2) as

$$\begin{aligned} q_{t+1} &= \phi(q_t, E_t) \\ &:= \zeta^{-1}(\sigma\gamma q_t u'(\psi(q_t, E_t))), \\ E_{t+1} &= \psi(q_t, E_t) \\ &:= (1-\alpha)E_t + \gamma\omega - (\beta+\gamma)q_t. \end{aligned}$$

By linearizing these two equations around $(\bar{E}, \bar{q})$, we can obtain that

$$\begin{bmatrix} q_{t+1} - \bar{q} \\ E_{t+1} - \bar{E} \end{bmatrix} = A \begin{bmatrix} q_t - \bar{q} \\ E_t - \bar{E} \end{bmatrix},$$

where

$$A = \begin{bmatrix} \bar{\phi}_1 & \bar{\phi}_2 \\ \bar{\psi}_1 & \bar{\psi}_2 \end{bmatrix} = \begin{bmatrix} \phi_1(\bar{E}, \bar{q}) & \phi_2(\bar{E}, \bar{q}) \\ \psi_1(\bar{E}, \bar{q}) & \psi_2(\bar{E}, \bar{q}) \end{bmatrix}.$$

Now let $T := \bar{\phi}_1 + \bar{\psi}_2$ and $D := \bar{\phi}_1\bar{\psi}_2 - \bar{\phi}_2\bar{\psi}_1$. It is well-known that the local equilibrium dynamics can be analyzed by the relationship between the trace, T, and determinant, D. By the definition of D and the assumption that $\alpha < -cv''(c)/v'(c) < 1$, it follows that

$$\begin{aligned} D &= (1-\alpha)(\zeta^{-1})'(\zeta(\bar{q}))\sigma\gamma u'(\bar{E}) \\ &= \frac{1-\alpha}{1 + \bar{q}v''(\bar{q})/v'(\bar{q})} \\ &> 1. \end{aligned}$$

Also by the definition of T, it also follows that

$$\begin{aligned} T &= D/(1-\alpha) + 1 - \alpha - \frac{\sigma\gamma\bar{q}(\beta+\gamma)u''(\bar{E})}{v'(\bar{q}) + \bar{q}v''(\bar{q})} \\ &> 2. \end{aligned}$$

Finally, we can obtain that

$$\begin{aligned} \frac{D}{T-1} &= \frac{D}{\frac{D}{1-\alpha} - \alpha - \frac{\sigma\gamma\bar{q}(\beta+\gamma)u''(\bar{E})}{v'(\bar{q})+\bar{q}v''(\bar{q})}} \\ &= \frac{1-\alpha}{1-\alpha\left(1+\frac{\bar{q}v''(\bar{q})}{v'(\bar{q})} - \frac{\sigma\gamma\bar{q}(\beta+\gamma)u''(\bar{E})}{v'(\bar{q})}\right)} \\ &< \frac{1-\alpha}{1-\alpha\left(1+\frac{\bar{q}v''(\bar{q})}{v'(\bar{q})}\right)} \\ &< 1, \end{aligned}$$

i.e., $D < T - 1$. Therefore, we can conclude that the stationary monetary equilibrium given σ is a saddle point. Q.E.D.

Proof of Proposition 4. We first claim that $\bar{q}\,'(\sigma) < 0$ for each $\sigma > 0$. To verify it, define the function $F : \mathbb{R}^2_{++} \to \mathbb{R}$ by, for all $(q, \sigma) \in \mathbb{R}^2_{++}$,

$$F(q,\sigma) := \sigma\gamma u'\left(\gamma\omega/\alpha - (\beta+\gamma)q/\alpha\right) - v'(q).$$

Because $D_q F(\bar{q}(\sigma), \sigma) = -v'' - (\beta+\gamma)u''/\alpha > 0$ and $D_\sigma F(\bar{q}(\sigma), \sigma) = \gamma u' > 0$, it follows from the implicit function theorem that $\bar{q}\,'(\sigma) < 0$.

By the proof of Proposition 2, we can now rewrite $W(\sigma)$ as

$$W(\sigma) = u\left(\frac{\gamma}{\alpha}\omega - \frac{\beta+\gamma}{\alpha}\bar{q}(\sigma)\right) + v(\bar{q}(\sigma)).$$

Because $v'(\bar{q}(\sigma)) = \sigma\gamma u'\left(\gamma\omega/\alpha - (\beta+\gamma)\bar{q}(\sigma)/\alpha\right)$ and $\bar{q}\,'(\sigma) < 0$ hold, it follows that, for each $\sigma > 0$,

$$\begin{aligned} W'(\sigma) = \bar{q}\,'(\sigma)u'(\bar{E}(\sigma))\left[\sigma\gamma - \frac{\beta+\gamma}{\alpha}\right] \\ \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} \quad 0 \quad \textit{iff} \quad \sigma \quad \left\{\begin{array}{c} < \\ = \\ > \end{array}\right\} \quad \frac{\beta+\gamma}{\alpha\gamma}. \end{aligned}$$

Therefore, $\sigma^* := (\beta+\gamma)/(\alpha\gamma) > 1$ is the unique optimal money growth rate. Q.E.D.

Proof of Proposition 5. This proposition immediately follows from the definition of the golden rule optimality. Q.E.D.

Proof of Proposition 6. Let $c^* := \bar{q}(\sigma^*)$ and $E^* := \bar{E}(\sigma^*)$. Then, one can easily verify that (c^*, E^*), which is the stationary monetary equilibrium allocation given σ^*, achieves the golden rule optimality. Q.E.D.

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