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Estimating Flexible Functional Forms using Macroeconomic Data

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Abstract

The paper estimates a flexible functional form for a joint cost function using US aggregate data for the years 1970-2022. There are four outputs (consumption, government, investment and exports) and six inputs (imports, labour, machinery and equipment services, structure services, other capital services and land services). Curvature conditions on the joint cost function are imposed without destroying the flexibility of the functional form. Various elasticities of supply and demand are estimated along with technical progress bias terms for each input. When using aggregate time series data based on the System of National Accounts, the paper shows that it is probably better to estimate a joint cost function rather than a gross output function or a GDP function. The paper also shows that assuming that an aggregate production function has constant elasticities of substitution is not appropriate for the US. Finally, the importance of including land as an aggregate input is stressed.

Keywords

Production theory, duality theory, production functions, joint cost functions, gross output functions, GDP functions, Shephard's Lemma, Hotelling's Lemma, Samuelson's Lemma, flexible functional forms, estimation of technical progress, user costs, modeling monopolistic behavior, land as a factor of production.

JEL Classification Numbers

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1 Introduction

It is important to be able to estimate production functions for a large number of policy purposes. If we attempt to estimate a twice continuously differentiable production function by regressing an output of a production unit on other outputs and inputs used by the production unit over a time period, we run into multicollinearity problems if the number of other outputs and inputs is large. If we want the estimated production function to be able to provide a second order Taylor series approximation to the true production function, we require a large number of parameters and we soon run into degrees of freedom problems. A partial solution to these problems is to assume that the production unit is either minimizing the cost of producing a vector of outputs (which leads to a joint cost function) or maximizing profits subject to one or more inputs being fixed (which leads to a gross output function if all inputs are "fixed") or leads to a GDP function (if all primary inputs are "fixed"). The assumption of optimizing behavior on the part of the production unit leads to many additional estimating equations and thus helps to overcome the degrees of freedom problem. Moreover, the additional estimating equations can be obtained by simply differentiating the joint cost function or profit function which is defined by the form of optimization. Thus the search for functional forms for *joint cost* and *gross output* functions that can provide second order approximations to arbitrary differentiable joint cost and gross output functions began about 50 years ago.[*1]

These dual representations of technology sets are important for policy purposes since they lead to estimates of technical change and measures of the biases in technical progress. They also lead to estimates for elasticities of output supply and for input demand. These elasticities are useful for a wide range of purposes. Index number methods are available for measuring Total Factor Productivity and technical progress, as are nonparametric methods, but these methods cannot estimate elasticities or biases in technical change.

The early literature on finding flexible functional forms for joint cost functions or gross output functions found that the estimated functional forms did not satisfy the curvature conditions that optimizing behavior imposes on the dual representations of technology. For example, a joint cost function, $C(\boldsymbol{y}, \boldsymbol{w})$ where $\boldsymbol{y}$ is a vector of outputs that the production unit produces and $\boldsymbol{w}$ is a vector of input prices that it faces, must be a concave function in input prices $\boldsymbol{w}$. Similarly, a gross output function, $G(\boldsymbol{p}, \boldsymbol{x})$ where $\boldsymbol{p}$ is vector of output prices that the production unit faces and $\boldsymbol{x}$ is a vector of input quantities, must be convex in output prices $\boldsymbol{p}$. In this paper, we will adapt the Normalized Quadratic functional form used by Diewert and Wales (1987)[23] (1992)[25] to construct functions $C(\boldsymbol{y}, \boldsymbol{w})$ and $G(\boldsymbol{p}, \boldsymbol{x})$ that are flexible and satisfy curvature conditions.[*2] In section 2 below, we will define these functions and derive alternative sets of estimating equations that can be used to estimate technologies.

In section 3, we restrict our attention to the estimation of aggregate production functions and technologies using macroeconomic data that are available from the national accounts of a country. We show that using aggregate data, it is better (in terms of fitting the data) to use a Normalized Quadratic Joint Cost Function rather than using a Normalized Quadratic Gross Output Function. Appendix B provides a proof of the flexibility of the NQ Joint Cost Function.

[*1] This approach to estimating technologies (and preferences) was developed by researchers at the University of California at Berkeley in the late 1960s and early 1970s; see McFadden (1966)[47] (1978)[48], Diewert (1971)[9] (1973)[10] (1974a)[11] (1974b)[12] and Christensen, Jorgenson and Lau (1973)[8].

[*2] In 1989, Kohli was the first person to adapt the Normalized Quadratic functional form to the GDP function. His paper was later published as Kohli (1993a)[41]. See also Kohli (1991)[40] for another early application of the NQ functional form.

Section 4 estimates a Normalized Quadratic Joint Cost Function for the US using aggregate data for the years 1970-2022. We have the usual $C + G + I + X$ macro aggregates as our 4 outputs and we constructed data for 6 inputs: M (imports), L (quality adjusted labour), and 4 types of capital services, $K_{\mathrm{M\&E}}, K_{\mathrm{S}}, K_{\mathrm{O}}$ and K_{L} (Machinery and Equipment, Structures, Other Capital (mainly R&D and Inventories) and Land). The data are described more fully in Appendix A. We show that land is an important input[*3] and that the aggregate US production function is very far from being a Cobb-Douglas or CES function. We also show that just using linear trends in each estimating equation to approximate the effects of technological change is a poor approximation.

Section 5 lists our econometric estimates for technical progress and for biases in technical change. Our technical progress estimates can be compared with index number methods for computing Total Factor Productivity growth. The Diewert and Morrison (1986)[22] and Kohli (1990)[39] (2003)[43] TFP growth accounting estimates for TFP are listed at the end of Appendix A. The Diewert and Fox (2018)[19] Nonparametric estimates of US TFP are explained and listed in Appendix C and compared with the econometric estimates for technical progress.[*4] Section 5 also lists the average price elasticities of input demand for our 6 aggregate inputs and the average elasticities of inverse output supply over 1970-2022.

For some purposes, it is useful to switch to estimating a GDP function or a Gross Output function or a Variable Profit function where only a few inputs are fixed and all outputs and variable inputs are free to vary. It turns out to be possible to get elasticity estimates for these alternative functions using our estimated joint cost function. We show how this can be done in Appendix D.

Section 6 lists some possible extensions (and possible problems) with our models and Section 7 concludes.

2 Alternative Representations for Production Possibilities Sets

Consider a production unit that can produce M outputs using N inputs over a period of time, which we take to be a year. Let $\boldsymbol{x} \equiv [x_1, \ldots, x_N] \geq \mathbf{0}_N$ be a nonnegative vector of annual inputs and let $\boldsymbol{y} \equiv [y_1, \ldots, y_M] \geq \mathbf{0}_M$ be a nonnegative vector of annual outputs. Define the production possibilities set S to be the set of output vectors $\boldsymbol{y}$ that could be produced by the technology given the availability of the vector of inputs $\boldsymbol{x}$. Thus $(\boldsymbol{y}, \boldsymbol{x}) \in S$ if the output vector $\boldsymbol{y}$ and be produced by the input vector $\boldsymbol{x}$. We assume that S is a nonempty, convex, closed cone[*5] that exhibits free disposal of inputs and outputs.

We consider alternative ways for estimating the technology set S using time series data. Suppose we have information on outputs produced and inputs used for year t, $[\boldsymbol{y}^t, \boldsymbol{x}^t] \equiv [y_1^t, \ldots, y_M^t, x_1^t, \ldots, x_N^t]$ for $t = 1, \ldots, T$. We could attempt to estimate a production function f which gives the maximum amount of output 1 y_1^t that could be produced conditional on having available the vector of year t inputs $\boldsymbol{x}^t$ and conditional on producing the amounts

[*3] For the importance of including land in production functions, see Kumhof, Tideman, Hudson and Goodhart (2021)[45] and Muellbauer (2024)[50].

[*4] The advantage of the nonparametric approach is that it distinguishes between technical progress and inefficiency. Of particular interest is the amount of inefficiency that occurred in the US economy during the oil shocks of the 1970s and the Covid shocks of 2020-2022.

[*5] Thus we are assuming that the technology exhibits constant returns to scale. The reason for this strong assumption is that it proves to be too difficult to distinguish technical progress from returns to scale using time series data for a production unit. Our goal in this paper is to estimate aggregate technologies using time series data.

$y_2^t, y_3^t, \ldots, y_M^t$ of other outputs:

$$y_1^t = f(y_2^t, y_3^t, \ldots, y_M^t, x_1^t, \ldots, x_N^t); \qquad t = 1, \ldots, T. \tag{1}$$

If f is twice continuously differentiable with respect to its arguments, then we would like f to be a *flexible functional form*; i.e., we would like f to be able to provide a second order Taylor series approximation to an arbitrary twice continuously differentiable production function that satisfies the appropriate regularity conditions.[*6] In the present context where we have assumed constant returns to scale in production, a candidate function to be a flexible functional form would require at least $(M + N)(M + N - 1)/2$ free parameters. If $M + N = 10$, a flexible functional form for a production function would have to have at least 45 parameters and so we would require over 45 years of data on the outputs and inputs produced by the production unit.

In order to gain more degrees of freedom to enable the econometric estimation of a technology, it is necessary to assume some form of optimizing behavior on the part of the producer. We consider some alternative assumptions below.

Suppose the production unit faces the strictly positive output price vector $\boldsymbol{p} \equiv [p_1, \ldots, p_M]$ and has the strictly positive input quantity vector $\boldsymbol{x} \equiv [x_1, \ldots, x_N]$ at its disposal. Let $\boldsymbol{y} \equiv [y_1, \ldots, y_M]$ be a nonnegative output quantity vector and let S be the production unit's production possibilities set that satisfies the above regularity conditions. The *gross output function*, $G(\boldsymbol{p}, \boldsymbol{x})$ for this production unit is defined as follows:[*7]

$$G(\boldsymbol{p}, \boldsymbol{x}) \equiv \max_y \{\boldsymbol{p} \cdot \boldsymbol{y} : (\boldsymbol{y}, \boldsymbol{x}) \in S\}.^{*8} \tag{2}$$

Thus $G(\boldsymbol{p}, \boldsymbol{x})$ is the maximum revenue the production unit can generate if it faces output prices $\boldsymbol{p}$ and uses the input vector $\boldsymbol{x}$ to produce the revenue maximizing output $\boldsymbol{y}(\boldsymbol{p}, \boldsymbol{x})$ which solves the constrained optimization problem (2). If intermediate inputs are included in the vector $\boldsymbol{y}$ (indexed by negative signs), then $G(\boldsymbol{p}, \boldsymbol{x})$ is a *value added function* or at the national level, it is a *GDP function*. The properties of this function were studied by Samuelson (1953)[54], McFadden (1966)[47] (1978)[48], Diewert (1973)[10] (1974a)[11] (1974b)[12] (2022)[17] and others. We assume constant returns to scale in production so that $G(\boldsymbol{p}, \boldsymbol{x})$ is linearly homogenous in $\boldsymbol{p}$ for fixed $\boldsymbol{x}$ and is linearly homogeneous in $\boldsymbol{x}$ for fixed $\boldsymbol{p}$. If $G(\boldsymbol{p}, \boldsymbol{x})$ is differentiable at a point $(\boldsymbol{p}, \boldsymbol{x})$, Hotelling's Lemma (1932; 594)[28] implies that the vector of *output supply functions* regarded as functions of $\boldsymbol{p}$ and $\boldsymbol{x}$, $\boldsymbol{y}(\boldsymbol{p}, \boldsymbol{x})$, can be obtained by differentiating $G(\boldsymbol{p}, \boldsymbol{x})$ with respect to the components of $\boldsymbol{p}$:

$$\boldsymbol{y}(\boldsymbol{p}, \boldsymbol{x}) = \nabla_p G(\boldsymbol{p}, \boldsymbol{x}). \tag{3}$$

Suppose further that the production unit faces the strictly positive input price vector $\boldsymbol{w} \equiv [w_1, \ldots, w_N]$. Samuelson's Lemma (1953; 10)[57][*9] implies that the producer's system of *inverse input demand functions* regarded as functions of $\boldsymbol{p}$ and $\boldsymbol{x}$, $\boldsymbol{w}(\boldsymbol{p}, \boldsymbol{x})$, can be obtained by differentiating $G(\boldsymbol{p}, \boldsymbol{x})$ with respect to the components of $\boldsymbol{x}$:

$$\boldsymbol{w}(\boldsymbol{p}, \boldsymbol{x}) = \nabla_x G(\boldsymbol{p}, \boldsymbol{x}). \tag{4}$$

[*6] See Diewert (1974a;113)[11] on the concept of a flexible functional form.

[*7] Alternative names for this function are the *national product function* Samuelson (1953; 10)[54], the *gross profit function* Gorman (1968)[27], the *conditional profit function* McFadden (1966)[47] (1978)[48], the *variable profit function* Diewert (1973)[10], the *revenue function* McFadden (1966)[47], Diewert (1974b)[12] and the *GDP function* Samuelson (1953)[54], Kohli (1978)[33] (1982)[36] (1991)[40]. These functions differ from the gross output function by taking some variable inputs (like imports or labour services) out of the list of inputs and treating them as negative outputs.

[*8] Notation: $\boldsymbol{p} \cdot \boldsymbol{y} \equiv \sum_{m=1}^{M} p_m y_m \equiv \boldsymbol{p}^T \boldsymbol{y}$ is the inner product between the vectors $\boldsymbol{p}$ and $\boldsymbol{y}$.

[*9] See also Diewert (1974a; 140)[11] (2022)[17].

Instead of conditioning on output prices $\boldsymbol{p}$ and input quantities $\boldsymbol{x}$, we can condition on output quantities $\boldsymbol{y}$ and input prices $\boldsymbol{w}$. Define the production unit's *joint cost function*, $C(\boldsymbol{y}, \boldsymbol{w})$, as the minimum cost of producing a given output vector $\boldsymbol{y}$:

$$C(\boldsymbol{y}, \boldsymbol{w}) \equiv \min_x\{\boldsymbol{w} \cdot \boldsymbol{x} : (\boldsymbol{y}, \boldsymbol{x}) \in S\}. \tag{5}$$

Under our strong regularity conditions on the set S, it can be shown that $C(\boldsymbol{y}, \boldsymbol{w})$ is linearly homogeneous in the components of $\boldsymbol{y}$ holding $\boldsymbol{w}$ constant and is linearly homogeneous in the components of $\boldsymbol{w}$ holding $\boldsymbol{y}$ constant. It is also a convex function in the components of $\boldsymbol{y}$ holding $\boldsymbol{w}$ fixed and a concave function in the components of $\boldsymbol{w}$ holding $\boldsymbol{y}$ fixed.[*10]

If $C(\boldsymbol{y}, \boldsymbol{w})$ is differentiable at a point $\boldsymbol{y}, \boldsymbol{w}$, then differentiating $C(\boldsymbol{y}, \boldsymbol{w})$ with respect to the components of $\boldsymbol{y}$ will generate the vector of marginal costs. If the producer takes output prices as being fixed and there are competitive markets, then this vector of marginal costs will be equal to the vector of selling prices $\boldsymbol{p}$. Thus the production unit's system of *inverse output supply functions*, $\boldsymbol{p}(\boldsymbol{y}, \boldsymbol{w})$, can be obtained by differentiating $C(\boldsymbol{y}, \boldsymbol{w})$ with respect to the components of $\boldsymbol{y}$:

$$\boldsymbol{p}(\boldsymbol{y}, \boldsymbol{w}) = \nabla_y C(\boldsymbol{y}, \boldsymbol{w}). \tag{6}$$

Shephard's Lemma (1953; 11)[57] (1970)[58] implies that the production unit's system of *input demand functions* regarded as functions of $\boldsymbol{y}$ and $\boldsymbol{w}$, $\boldsymbol{x}(\boldsymbol{y}, \boldsymbol{w})$, can be obtained by differentiating $C(\boldsymbol{y}, \boldsymbol{w})$ with respect to the components of $\boldsymbol{w}$:

$$\boldsymbol{x}(\boldsymbol{y}, \boldsymbol{w}) = \nabla_w C(\boldsymbol{y}, \boldsymbol{w}). \tag{7}$$

Thus we have two alternative methods for estimating a representation of the technology set S: the first representation assumes a functional form for the gross output function, $G(\boldsymbol{p}, \boldsymbol{x})$, and uses equations (3) and (4) as estimating equations and the second representation assumes a functional form for the joint cost function, $C(\boldsymbol{y}, \boldsymbol{w})$, and uses equations (6) and (7) as estimating equations. Note that both representations lead to $(M + N)T$ estimating equations if we have data on prices and quantities for T periods. This greatly increases degrees of freedom compared to the degrees of freedom that are available when estimating a production function.

In the following section, we ask whether we can choose between (3) plus (4) or (6) plus (7) when working with macroeconomic data.

3 Should We Estimate Gross Output Functions or Joint Cost Functions?

Consider the following specialization of the general joint cost function $C(\boldsymbol{y}, \boldsymbol{w})$ defined by (5):

$$C(\boldsymbol{y}, \boldsymbol{w}) \equiv \textstyle\sum_{m=1}^{M} c^m(\boldsymbol{w}) y_m. \tag{8}$$

The function $c^m(\boldsymbol{w})$ is the unit cost function that is dual to the single output constant returns to scale production function for sector m, $y_m = f^m(\boldsymbol{x}^m)$ for $m = 1, \ldots, M$ where $\boldsymbol{x}^m$ is the vector of inputs used by sector m. This is the small open country framework considered by

[*10] See McFadden (1966)[47] (1978)[48] and Diewert (1974a)[11] (2022)[17] for references to the literature. The first flexible functional form for a joint cost function that was estimated empirically was the translog joint cost function introduced by Burgess (1974)[6]. The problem with this functional form is the difficulty in imposing curvature conditions on it without destroying its flexibility.

Samuelson (1953)[54] in his seminal paper. It is sometimes called the Nonjoint Production Model: each sector of the economy produces only a single product (or a single group of products that does not overlap with the products produced in other sectors) and it does not use the products of other sectors as intermediate inputs. Note that the aggregate input vector $\boldsymbol{x}$ is equal to $\sum_{m=1}^{M} \boldsymbol{x}^m$ and the mth output price is equal to $p_m = c^m(\boldsymbol{w})$ for $m = 1, \ldots, M$. If each unit cost function is differentiable, then by applying Shephard's Lemma to each sectoral cost function, we can deduce that $\boldsymbol{x}^m = \nabla_w c^m(\boldsymbol{w}) y_m$ for $m = 1, \ldots, M$. Making use of these equalities, differentiate both sides of (8) with respect to the components of $\boldsymbol{w}$. We obtain:

$$\nabla_w C(\boldsymbol{y}, \boldsymbol{w}) = \textstyle\sum_{m=1}^{M} \nabla_w c^m(\boldsymbol{w}) y_m = \sum_{m=1}^{M} \boldsymbol{x}^m \equiv \boldsymbol{x}. \tag{9}$$

Now further specialize the unit cost functions $c^m(\boldsymbol{w})$ to be linear functions of $\boldsymbol{w}$:

$$c^m(\boldsymbol{w}) \equiv \textstyle\sum_{n=1}^{N} w_n d_{nm}; \qquad m = 1, \ldots, M \tag{10}$$

where the d_{nm} are NM constants. This means that the sectoral production functions are Leontief (no substitution) production functions. Define the N by M matrix of the d_{nm} as $\mathbf{D} \equiv [d_{nm}]$. Substitute definitions (10) into (8) and we obtain the following expression for the overall joint cost function:

$$C(\boldsymbol{y}, \boldsymbol{w}) \equiv \textstyle\sum_{m=1}^{M} \sum_{n=1}^{N} w_n d_{nm} y_m = \boldsymbol{w} \cdot \mathbf{D}\boldsymbol{y}. \tag{11}$$

Thus the joint cost function for this very special case of Leontief sectoral production functions turns out to be a *bilinear form* in the vectors of input prices $\boldsymbol{w}$ and of output quantities $\boldsymbol{y}$. Equations (6) and (7) for this special case turn out to be the following estimating equations for $t = 1, \ldots, T$:

$$\boldsymbol{p}^t = \mathbf{D}^T \boldsymbol{w}^t; \tag{12}$$

$$\boldsymbol{x}^t = \mathbf{D}\boldsymbol{y}^t \tag{13}$$

where $\boldsymbol{p}^t, \boldsymbol{w}^t, \boldsymbol{y}^t$ and $\boldsymbol{x}^t$ are the period t vectors of observed prices and quantities and $\mathbf{D}^T$ is the transpose of the matrix $\mathbf{D}$. The estimating equations defined by (12) and (13) are linear in the unknown MN parameters but of course, there are cross equation equality restrictions. However, this model can readily be estimated using standard nonlinear regression packages.

Consider the following specializations of the general gross output function $G(\boldsymbol{p}, \boldsymbol{x})$ defined by (2):

$$G(\boldsymbol{p}, \boldsymbol{x}) \equiv \textstyle\sum_{m=1}^{M} g^m(\boldsymbol{x}) p_m; \tag{14}$$

$$G(\boldsymbol{p}, \boldsymbol{x}) \equiv \textstyle\sum_{n=1}^{N} h^n(\boldsymbol{p}) x_n; \tag{15}$$

$$G(\boldsymbol{p}, \boldsymbol{x}) \equiv \textstyle\sum_{m=1}^{M} \sum_{n=1}^{N} x_n e_{nm} p_m = \boldsymbol{x} \cdot \mathbf{E}\boldsymbol{p} \tag{16}$$

where $\mathbf{E} \equiv [e_{nm}]$ is an N by M matrix of constants. The production model defined by (14) implies that output y_m is equal to the function $g^m(\boldsymbol{x})$ of aggregate input $\boldsymbol{x}$ for each output $m = 1, \ldots, M$. In the context where outputs are produced by sectoral production functions, this model is not very sensible if $M > 1$: the output of sector m is produced by the sector m vector of inputs, $\boldsymbol{x}^m$; not by the aggregate input vector $\boldsymbol{x}$.

The production model defined by (15) implies that each unit of aggregate input n, x_n, produces the vector of outputs $\nabla_p h^n(\boldsymbol{p})$ independently of all other inputs. This is also not a very sensible assumption if $N > 1$. Hence the bilinear model defined by (16), which is a special case of the models defined by (14) and (15), is also not a sensible model of production in the sectoral production function context.*11

Using (3) and (4), the estimating equations for the bilinear Gross Output function defined by (16) are as follows for $t = 1, \ldots, T$:

$$\boldsymbol{y}^t = \mathbf{E}^T \boldsymbol{x}^t; \tag{17}$$

$$\boldsymbol{w}^t = \mathbf{E}\boldsymbol{p}^t. \tag{18}$$

When we are estimating joint cost functions, it is convenient to start by estimating the bilinear function defined by (11), $C(\boldsymbol{y}, \boldsymbol{w}) = \boldsymbol{w} \cdot \mathbf{D}\boldsymbol{y}$. When we are estimating Gross Output Functions, it proves to be convenient to start by estimating the bilinear function defined by (16), $G(\boldsymbol{p}, \boldsymbol{x}) = \boldsymbol{x} \cdot \mathbf{E}\boldsymbol{p}$. The analysis above suggests that the Bilinear Joint Cost Function option will fit the data better than the Bilinear Gross Output Function option when we are using aggregate national accounts data. The bilinear starting functional form defined by (11) will probably provide a much better *global approximation* to the national technology if we are using national macroeconomic data than will be provided by the starting bilinear functional form for a gross output function defined by (16).

We will use US data for the years 1970-2022 to investigate which bilinear functional form does better at describing the data: the Gross Output Bilinear Function or the Joint Cost Bilinear Function.

Our data are based on the Augmented Productivity Database (APDB) for the US constructed by the Asian Productivity Organization and Keio University.*12 The data on outputs for the US economy are based on the recent historical revision of the US national accounts made by the Bureau of Economic Analysis (2023)[5]. The four outputs are the usual C, G, I and X macroeconomic aggregates and the six inputs are Imports, Labour, Machinery and Equipment, Structures, Other Capital (mainly R&D and Inventories) and Land ($M, L, K_{\mathrm{M\&E}}, K_{\mathrm{S}}, K_{\mathrm{O}}$ and K_{L}); see Appendix A for the details of the data construction. Denote the 4 dimensional output price and quantity vectors for year t by $\boldsymbol{p}^t$ and $\boldsymbol{y}^t$ and the 6 dimensional input price and quantity vectors for year t by $\boldsymbol{w}^t$ and $\boldsymbol{x}^t$ for $t = 1970, \ldots, 2022$.

We first estimated the parameters in the 10 equations that allow us to the Bilinear Gross Output Function. The first 6 estimating equations were equations (18) and the last 4 equations were equations (17). Although each set of equations is linear in the unknown parameters, there are cross equation restrictions on the parameters which need to be taken into account. We used the Nonlinear Estimation econometric package in Shazam*13 to do the estimation. The R^2 (between observed and predicted) were as follows for the 10 equations: 0.9159 0.9845 0.3454 0.9045 0.8703 0.5250 0.9981 0.4708 0.9675 0.9662. The equation with the lowest R^2 (0.3454) was the third equation in (18) which regressed the user cost of Machinery and Equipment on the 4 prices of the outputs (the prices of C, G, I and X). The equation with the highest R^2 (0.9981) was the first equation in (17) which regressed the quantity of consumption on the 6 aggregate quantities of inputs.

*11 There is an extensive literature on alternative specifications of joint and nonjoint production; see Samuelson (1966)[56] and Kohli (1981a)[34] (1983a)[37] (1991; 42-46)[40] (1993b)[42] (2005)[44]. Our purpose is not to test for the validity of alternative specifications of joint production: we simply wish to determine whether a joint cost function or a gross output function will fit the data better, given that a bilinear functional form is the initial functional form. Our paper does not answer this question definitively; i.e., which specification of technology is "best".

*12 See the Asian Productivity Organization (2022)[1] for a description of the data base. The detailed data base is not available to the public. However, the US data used in this study is listed in Appendix A.

*13 See White (2004)[59].

We also estimated the 10 equations that estimate the parameters in the Bilinear Joint Cost Function. The first 6 estimating equations were equations (13) and the last 4 equations were equations (12). The R^2 were as follows for the 10 estimating equations: 0.9908 0.9777 0.9740 0.9707 0.9506 0.9809 0.9895 0.9980 0.9930 0.9906. The equation with the lowest R^2 (0.9506) was the fifth equation in equations (13) which regressed the quantity of input 5 (the quantity of Other Capital Services) on the quantities of the 4 outputs. The equation with the highest R^2 (0.9980) regressed the price of Government output on the 6 input prices.

It is clear that the Bilinear Joint Cost function fits the US data much better than the Bilinear Gross Output function. These two bilinear functional forms are building blocks for the Normalized Quadratic Joint Cost and Gross Output functional forms which are flexible functional forms as we shall see. Thus it is likely that we will end up with a functional form that fits the data better if we estimate the Normalized Quadratic Joint Cost Function rather than the Normalized Quadratic Gross Output Function (because the bilinear special case for the joint cost function provides a global fit which is much better than the fit for the bilinear special case of the Normalized Quadratic Gross Output Function). In the following sections, we will estimate several variants of the Normalized Quadratic Joint Cost Function.

4 Estimating the Normalized Quadratic Joint Cost Function for the US

In the previous section, we defined $\boldsymbol{p}^t$ and $\boldsymbol{y}^t$ as the year t price and quantity vectors for our 4 gross outputs $(C+G+I+X)$ and $\boldsymbol{w}^t$ and $\boldsymbol{x}^t$ were defined as the year t price and quantity vectors for our 6 inputs (Imports, Labour, M&E services, Structure services, Other Capital services (mainly R&D and Inventories) and Land). We normalized the prices and quantities of the 4 outputs so that $y_m^{1970}=1$ for $m=1,2,3,4$. We also normalized the prices and quantities of the 6 inputs so that $w_n^{1970}=1$ for $n=1,\ldots,6$. The Normalized Quadratic Joint Cost Function requires a vector of fixed nonnegative weights for input prices, $\alpha \equiv [\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6]$. Define $\boldsymbol{x}^* \equiv \sum_{t=1970}^{2022}\boldsymbol{x}^t=[x_1^*,x_2^*,x_3^*,x_4^*,x_5^*,x_6^*]$. Thus $\boldsymbol{x}^*$ is simply the sample wide sum of the observed input quantity vectors. Define the sum of the x_n^* as x_{sum}^*. Finally define the components of the α vector as $\alpha_n \equiv x_n^*/x_{\text{sum}}^*$ for $n=1,\ldots,6$. Since all input prices w_n^t equal 1 when t equals 1, we have $\alpha \cdot \boldsymbol{w}^{1970}=1$. Thus $\alpha \cdot \boldsymbol{w}^t$ as t varies is a *fixed base index of input prices* that starts out at the level 1 in 1970.

The Normalized Quadratic Joint Cost Function also requires a vector of fixed nonnegative weights for output quantities, $\boldsymbol{\beta}\equiv[\beta_1,\beta_2,\beta_3,\beta_4]$. Define $\boldsymbol{p}^*\equiv\sum_{t=1970}^{2022}\boldsymbol{p}^t=[p_1^*,p_2^*,p_3^*,p_4^*]$. Thus $\boldsymbol{p}^*$ is simply the sample wide sum of the observed output price vectors. Define the sum of the p_m^* as p_{sum}^*. Finally define the components of the $\boldsymbol{\beta}$ vector as $\beta_m\equiv p_m^*/p_{\text{sum}}^*$ for $m=1,\ldots,4$. Since all output quantities y_m^t equal 1 when t equals 1, we have $\boldsymbol{\beta}\cdot\boldsymbol{y}^{1970}=1$. Thus $\boldsymbol{\beta}\cdot\boldsymbol{y}^t$ as t varies is a fixed base index of output quantities that starts out at the level 1 in 1970.

In order to condense the notation, we now let time $t=0,1,2,\ldots,52$ instead of $t=1970,1971,\ldots,2022$. The *Linear Time Trends Normalized Quadratic Joint Cost Function* at time period t, $C(\boldsymbol{y}^t,\boldsymbol{w}^t,t)$, is defined as follows:

$$C(\boldsymbol{y}^t,\boldsymbol{w}^t,t)\equiv(1/2)(\boldsymbol{w}^t\cdot\mathbf{A}\boldsymbol{w}^t)(\alpha\cdot\boldsymbol{w}^t)^{-1}(\boldsymbol{\beta}\cdot\boldsymbol{y}^t)+(1/2)(\boldsymbol{y}^t\cdot\mathbf{B}\boldsymbol{y}^t)(\alpha\cdot\boldsymbol{w}^t)(\boldsymbol{\beta}\cdot\boldsymbol{y}^t)^{-1} + \boldsymbol{w}^t\cdot\mathbf{D}\boldsymbol{y}^t+(\boldsymbol{b}\cdot\boldsymbol{w}^t)(\boldsymbol{\beta}\cdot\boldsymbol{y}^t)t \quad (19)$$

where $\mathbf{A}$ is a symmetric positive semidefinite matrix that satisfies the restrictions $\mathbf{A}\boldsymbol{w}^0=\mathbf{0}_6$, $\mathbf{B}$ is a symmetric negative semidefinite matrix that satisfies the restrictions $\mathbf{B}\boldsymbol{y}^0=\mathbf{0}_4$ and

$\boldsymbol{b} \equiv [b_1, \ldots, b_6]$ is a vector of parameters which allow for biased technical change. Using the algebra in Diewert and Wales (1987)[23], it can be shown that these conditions on the $\mathbf{A}$ and $\mathbf{B}$ matrices are sufficient to imply the global convexity of $C(\boldsymbol{y}, \boldsymbol{w}, t)$ in the components of $\boldsymbol{y}$ and the global concavity of $C(\boldsymbol{y}, \boldsymbol{w}, t)$ in the components of $\boldsymbol{w}$. The year t estimating equations (7) and (6) become for $t = 0, 1, 2, \ldots, 52$:

$$\begin{aligned}\boldsymbol{x}^t &= \mathbf{D}\boldsymbol{y}^t + \mathbf{A}\boldsymbol{w}^t(\alpha \cdot \boldsymbol{w}^t)^{-1}(\boldsymbol{\beta} \cdot \boldsymbol{y}^t) + (1/2)(\boldsymbol{y}^t \cdot \mathbf{B}\boldsymbol{y}^t)(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)^{-1}\alpha \\ &\quad - (1/2)(\boldsymbol{w}^t \cdot \mathbf{A}\boldsymbol{w}^t)(\alpha \cdot \boldsymbol{w}^t)^{-2}(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)\alpha + \boldsymbol{b}(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)t; \qquad (20)\\ \boldsymbol{p}^t &= \mathbf{D}^T\boldsymbol{w}^t + \mathbf{B}\boldsymbol{y}^t(\alpha \cdot \boldsymbol{w}^t)(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)^{-1} + (1/2)(\boldsymbol{w}^t \cdot \mathbf{A}\boldsymbol{w}^t)(\alpha \cdot \boldsymbol{w}^t)^{-1}\boldsymbol{\beta} \\ &\quad - (1/2)(\boldsymbol{y}^t \cdot \mathbf{B}\boldsymbol{y}^t)(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)^{-2}(\alpha \cdot \boldsymbol{w}^t)\boldsymbol{\beta} + \boldsymbol{\beta}(\boldsymbol{b} \cdot \boldsymbol{w}^t)t. \qquad (21)\end{aligned}$$

The unknown parameters in the above equations are the components of the 6 by 4 matrix $\mathbf{D}$, the components of the 6 by 6 and 4 by 4 symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ and the 6 components of the vector of technical progress parameters $\boldsymbol{b}$ which is 51 parameters in all, taking into account the symmetry restrictions on the $\mathbf{A}$ and $\mathbf{B}$ matrices. There are 53 times 10 observations so we have 530 degrees of freedom. Note that all parameters appear in a linear fashion but there are cross equations constraints on the parameters which means that we have to use nonlinear regression techniques to estimate the unknown parameters. Note that when $t = 0$, the technical progress terms involving the $\boldsymbol{b}$ vector on the right hand sides of (20) and (21) vanish.

It proved to be too difficult for the nonlinear estimation command in Shazam to estimate the model defined by equations (20) and (21). Nonlinear estimation requires good starting values for the parameters in the nonlinear regression. In the previous section, we essentially set $\mathbf{A}, \mathbf{B}$ and $\boldsymbol{b}$ equal to zero matrices and estimated the components of the $\mathbf{D}$ matrix. In this section, we used the estimates for the components of the $\mathbf{D}$ matrix as starting coefficients for the estimation of (20) and (21) with $\mathbf{A}$ and $\mathbf{B}$ set equal to zero matrices. The starting values for the components of the $\boldsymbol{b}$ vector were zeros. The log likelihood for the new model that allowed for technical progress increased by 238.85 log likelihood points for adding 6 technical progress parameters. Thus it is extremely important to allow for technical progress. The R^2 for the 10 equations were as follows:

0.9891 0.9576 0.9868 0.9952 0.9967 0.8291 0.9980 0.9964 0.9705 0.9466.

We turn our attention to the estimation of the components of the input substitution matrix $\mathbf{A}$ and the (inverse) output substitution matrix $\mathbf{B}$. We need to ensure that our estimated $\mathbf{A}$ matrix is a negative semidefinite symmetric matrix that satisfies $\mathbf{A}\boldsymbol{w}^0 = \mathbf{A}\mathbf{1}_6 = \mathbf{0}_6$ where $\mathbf{1}_6$ is a vector of ones of dimension 6.

The imposition of symmetry and negative semidefiniteness on $\mathbf{A}$ can be accomplished using a technique due to Wiley, Schmidt and Bramble (1973)[60]: simply replace the matrix $\mathbf{A}$ by

$$\mathbf{A} \equiv -\mathbf{U}\mathbf{U}^T \qquad (22)$$

where $\mathbf{U}$ is a 6 by 6 *lower triangular matrix*; i.e., $u_{ij} = 0$ if $i < j$.

The restrictions $\mathbf{A}\boldsymbol{w}^0 = \mathbf{A}\mathbf{1}_6 = \mathbf{0}_6$ on $\mathbf{A}$ can be imposed if we impose the following restrictions on $\mathbf{U}$:

$$\mathbf{U}^T\mathbf{1}_6 = \mathbf{0}_6. \qquad (23)$$

The imposition of symmetry and positive semidefiniteness on $\mathbf{B}$ can be accomplished in a similar fashion: set $\mathbf{B}$ equal to:

$$\mathbf{B} \equiv \mathbf{V}\mathbf{V}^T \qquad (24)$$

where $\mathbf{V}$ is a 4 by 4 *lower triangular matrix*; i.e., $v_{ij} = 0$ if $i < j$.

The restrictions $\mathbf{B}\boldsymbol{y}^0 = \mathbf{B}\mathbf{1}_4 = \mathbf{0}_4$ on $\mathbf{B}$ can be imposed if we impose the following restrictions on $\mathbf{V}$:

$$\mathbf{V}^T\mathbf{1}_4 = \mathbf{0}_4. \tag{25}$$

The restrictions (23) and (24) imply that the maximum rank for the $\mathbf{A}$ and $\mathbf{B}$ matrices is 5 and 3 respectively. Once the matrices $\mathbf{A}$ and $\mathbf{B}$ in the estimating equations (20) and (21) are replaced by $-\mathbf{U}\mathbf{U}^T$ and $\mathbf{V}\mathbf{V}^T$ respectively, the resulting estimating equations are no longer linear in the unknown parameters and a nonlinear regression package must be used. For more details on how to implement this method for imposing the restrictions on the substitution matrices, see Diewert and Wales (1987)[23] (1988)[24] (1992)[25].

Setting $\mathbf{A} \equiv -\mathbf{U}\mathbf{U}^T$ and $\mathbf{B} \equiv \mathbf{V}\mathbf{V}^T$ can lead to difficulties in estimation due to the addition of a large number of new parameters at the same time. A more fool proof method of proceeding is to introduce the columns of $\mathbf{U}$ and $\mathbf{V}$ into the regression one column at a time. Thus we set $\mathbf{A} \equiv -\boldsymbol{u}\boldsymbol{u}^T$ where $\boldsymbol{u}^T \equiv [u_1, u_2, \ldots, u_6]$ and u_1 is set equal to $-\sum_{n=2}^{6} u_n$. Similarly, we set $\mathbf{B} = \boldsymbol{v}\boldsymbol{v}^T$ where $\boldsymbol{v}^T \equiv [v_1, v_2, v_3, v_4]$ and v_1 is set equal to $-\sum_{m=2}^{4} v_m$. We used the final coefficients in the previous regression as starting values in this new regression along with the starting values for the u_n and v_m equal to 0. Thus this new nonlinear regression estimates rank 1 substitution matrices.[*14] The log likelihood for the new model increased by 76.088 log likelihood points for adding 5 input and 3 output substitution parameters. The R^2 for the 10 equations were as follows:

0.9954 0.9852 0.9866 0.9925 0.9984 0.9214 0.9717 0.9971 0.9852 0.9624.

Note that the R^2 for equation 6 (the demand for land services) was 0.8291 in the previous nonlinear regression model but has now increased to 0.9214 when we allow for some substitution between the 6 inputs.

Looking at the components of the output substitution matrix, it turned out that the components of the estimated $\boldsymbol{v}$ vector were as follows $(v_2^*, v_3^*, v_4^*) = (-0.0114, -0.1454, 0.0108)$ and $v_1^* = -(v_2^* + v_3^* + v_4^*) = 0.1460$. If the products produced by the 4 sectors in our model (the C, G, I and X sectors) were unique to each sector (i.e., if the Nonjoint Production Model defined by (8) above held), then the estimated $\boldsymbol{v}$ vector should be a zero vector. Since the estimated (v_2^*, v_3^*, v_4^*) were significantly different from zero, it is likely that there is some joint production in the US economy.[*15]

The process of adding an additional column to the input and output substitution matrices $\mathbf{U}$ and $\mathbf{V}$ (and then running a new nonlinear regression using the final coefficients in the previous regression as starting values in the new regression) continued until a rank 5 input substitution matrix and a rank 3 output substitution matrix were estimated. The resulting model is the model described by equations (20) - (25). The log likelihood for this 51 parameter model was 27.867 points higher than the final log likelihood for the model that had rank 1 input and output substitution matrices. We have added 13 new parameters to the previous rank 1 model. However, 4 of the new substitution parameters converged to 0: the data only supported the estimation of a rank 4 input substitution matrix (maximum rank is 5) and a rank 1 output substitution matrix (maximum rank is 3).

We need to develop formulae for various elasticities of substitution once we have estimates for the components of $\mathbf{A}, \mathbf{B}, \mathbf{D}$ and $\boldsymbol{b}$. Define the vector of input demand functions $\boldsymbol{x}(\boldsymbol{y}^t, \boldsymbol{w}^t, t)$ as the right hand side of equations (20). The 6 by 6 matrix of derivatives of the input demand

[*14] There are now 38 parameters in our model.

[*15] The standard errors for (v_2^*, v_3^*, v_4^*) were $(0.0229, 0.0187, 0.0052)$.

functions with respect to the components of the input price vector $\boldsymbol{w}$ evaluated at the year t values for $\boldsymbol{y}$ and $\boldsymbol{x}$, $\nabla_w \boldsymbol{x}(\boldsymbol{y}^t, \boldsymbol{w}^t, t)$, can be obtained by differentiating the right hand side of (20) with respect to the components of $\boldsymbol{w}$:

$$\begin{aligned}\nabla_w \boldsymbol{x}(\boldsymbol{y}^t, \boldsymbol{w}^t, t) = {} & \mathbf{A}(\alpha \cdot \boldsymbol{w}^t)^{-1}(\boldsymbol{\beta} \cdot \boldsymbol{y}^t) - (\mathbf{A}\boldsymbol{w}^t)\alpha^T(\alpha \cdot \boldsymbol{w}^t)^{-2}(\boldsymbol{\beta} \cdot \boldsymbol{y}^t) \\ & - \alpha(\mathbf{A}\boldsymbol{w}^t)^T(\alpha \cdot \boldsymbol{w}^t)^{-2}(\boldsymbol{\beta} \cdot \boldsymbol{y}^t) + \alpha\alpha^T(\boldsymbol{w}^t \cdot \mathbf{A}\boldsymbol{w}^t)(\alpha \cdot \boldsymbol{w}^t)^{-3}(\boldsymbol{\beta} \cdot \boldsymbol{y}^t). \end{aligned} \quad (26)$$

Define the *year t elasticity of the demand for input n* with respect to a change in the price of input j, Ex^t_{nj}, as follows for $n = 1, \ldots, 6$ and $j = 1, \ldots, 6$:

$$Ex^t_{nj} \equiv \partial \ln x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial \ln w_j \equiv [\partial x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial w_j][w^t_j/x^t_n] \quad (27)$$

where $x^t_n \equiv x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t)$ is the fitted demand for input n in year t defined by the nth component on the right hand side of equation (20) for year t and $\partial x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial w_j$ is defined in equations (26).

Recall that when $t = 0$, $\boldsymbol{w}^0$ and $\mathbf{A}$ satisfy the equation $\mathbf{A}\boldsymbol{w}^0 = \mathbf{0}_6$. Thus when $t = 0$, the right hand side of (26) when $t = 0$ collapses to $\nabla_w \boldsymbol{x}(\boldsymbol{y}^0, \boldsymbol{w}^0, 0) = \mathbf{A}(\alpha \cdot \boldsymbol{w}^0)^{-1}(\boldsymbol{\beta} \cdot \boldsymbol{y}^0)$. As time marches on, the last 3 terms on the right hand side of (26) will tend to be fairly close to zero so the elasticities defined by (27) will be approximately equal to the following expressions for $n = 1, \ldots, 6$ and $j = 1, \ldots, 6$:

$$Ex^t_{nj} \approx a_{nj}(w^t_j/\alpha \cdot \boldsymbol{w}^t)/(x^t_n/\boldsymbol{\beta} \cdot \boldsymbol{y}^t) \quad (28)$$

where a_{nj} is the component of the estimated $\mathbf{A}$ matrix in row n and column j. If there are divergent trends in either the input prices $\boldsymbol{w}^t$ or in the output quantities $\boldsymbol{y}^t$ over time, then it can be seen that the input substitution elasticities will have trends over time.

The problem of trending elasticities also arises with respect to output substitution elasticities. Define the vector of (inverse) output supply functions $\boldsymbol{p}(\boldsymbol{y}^t, \boldsymbol{w}^t, t)$ as the right hand side of equations (21). The 4 by 4 matrix of derivatives of the output supply functions with respect to the components of the output quantity vector $\boldsymbol{y}$ evaluated at the year t values for $\boldsymbol{y}$ and $\boldsymbol{x}$, $\nabla_y \boldsymbol{p}(\boldsymbol{y}^t, \boldsymbol{w}^t, t)$, can be obtained by differentiating the right hand side of (21) with respect to the components of $\boldsymbol{y}$:

$$\begin{aligned}\nabla_y \boldsymbol{p}(\boldsymbol{y}^t, \boldsymbol{w}^t, t) = {} & \mathbf{B}(\alpha \cdot \boldsymbol{w}^t)(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)^{-1} - (\mathbf{B}\boldsymbol{y}^t)\boldsymbol{\beta}^T(\alpha \cdot \boldsymbol{w}^t)(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)^{-2} \\ & - \boldsymbol{\beta}(\mathbf{B}\boldsymbol{y}^t)^T(\alpha \cdot \boldsymbol{w}^t)(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)^{-2} + \boldsymbol{\beta}\boldsymbol{\beta}^T(\boldsymbol{y}^t \cdot \mathbf{B}\boldsymbol{y}^t)(\alpha \cdot \boldsymbol{w}^t)(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)^{-3}. \end{aligned} \quad (29)$$

Define the *year t elasticity of the (inverse) supply for output m* with respect to a change in the price of output k, Ep^t_{mk}, as follows for $m = 1, \ldots, 4$ and $k = 1, \ldots, 4$:

$$Ep^t_{mk} \equiv \partial \ln p_m(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial \ln y_k \equiv [\partial p_m(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial y_k][y^t_k/p^t_m] \quad (30)$$

where $p^t_m \equiv p_m(\boldsymbol{y}^t, \boldsymbol{w}^t, t)$ is the fitted price for output m in year t defined by the mth component on the right hand side of equation (21) for year t and $\partial p_m(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial y_k$ is defined in equations (29).

Recall that when $t = 0$, $\boldsymbol{y}^0$ and $\mathbf{B}$ satisfy the equation $\mathbf{B}\boldsymbol{y}^0 = \mathbf{0}_4$. Thus when $t = 0$, the right hand side of (29) when $t = 0$ collapses to $\nabla_y \boldsymbol{p}(\boldsymbol{y}^0, \boldsymbol{w}^0, 0) = \mathbf{B}(\alpha \cdot \boldsymbol{w}^t)(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)^{-1}$. The last 3 terms on the right hand side of (29) will tend to be fairly close to zero so the elasticities defined by (30) will be approximately equal to the following expressions for $m = 1, \ldots, 4$ and $k = 1, \ldots, 4$:

$$Ep^t_{mk} \approx b_{mk}(y^t_k/\boldsymbol{\beta} \cdot \boldsymbol{y}^t)/(p^t_m/\alpha \cdot \boldsymbol{w}^t) \quad (31)$$

where b_{mk} is the component of the estimated $\mathbf{B}$ matrix in row m and column k. If there are divergent trends in either the input prices $\boldsymbol{w}^t$ or in the output quantities $\boldsymbol{y}^t$ over time, then it can be seen that the output substitution elasticities will have trends over time.

Diewert and Lawrence (2002; 149-151)[21] noticed the above trending elasticities problem in the context of estimating a Normalized Quadratic GDP function and they suggested a method for dealing with this problem: let the components of the $\mathbf{A}$ and $\mathbf{B}$ (or $\mathbf{U}$ and $\mathbf{V}$) matrices have linear time trends over the sample period. We implemented their method for our US data. Thus the matrices $\mathbf{A}$ and $\mathbf{B}$ in the estimating equations (20) and (21) become functions of time t, $\mathbf{A}(t)$ and $\mathbf{B}(t)$.[*16] This new model has a total of 72 parameters. The log likelihood for this model increased by 78.021 points for adding 21 new parameters. The new R^2 were as follows:

0.9904 0.9914 0.9829 0.9875 0.9980 0.9383 0.9988 0.9982 0.9925 0.9745.

The lowest R^2 was for the demand for land services equation which was 0.9383; the R^2 for the remaining equations were all above 0.97. Ten out of the 42 substitution matrix parameters converged to 0. Thus the ranks of the beginning and end of period input substitution matrices were equal to 3 (maximum possible rank is 5) and the initial inverse output supply substitution matrix had rank 1 and the final output substitution matrix had rank 2 (maximum possible rank is 3).

The use of linear time trends to model technical progress will lead to smooth estimates of overall technical progress. But we know from index number estimates of technical progress that the rate varies substantially from year to year and the trends also vary substantially from decade to decade.[*17] Thus we will follow the example of Diewert and Wales (1992)[25] by replacing the linear trends in equations 1-6 by piece-wise linear spline functions. In order to determine the break points (or knots) for the spline functions, we looked at the plots of the regression residuals. These residual plots showed systematic trends that changed abruptly from time to time. We used the observations where breaks in the residual trends occurred in equations 1-6 as our break points for the piece-wise linear functions that we use to model technical progress. The break points occurred at the following observations where we now number the years 1970-2022 as 1-53:

Equation 1: 8 18 23 37 51
Equation 2: 8 13 17 22 36 51
Equation 3: 10 17 26 37 40 51
Equation 4: 13 27 31 34 38 50
Equation 5: 5 17 22 31 35 51
Equation 6: 19 22 31 37.

The terms tb_n for $n = 1, \ldots, 6$ appear in the estimating equations (20) and (21). We need to replace these linear functions of time t by piece-wise linear functions of time where the line segments are linked together to form continuous functions of time. Thus the linear function of time, tb_1, is replaced by the function $\sum_{i=1}^{6} b_{1i} f_{1i}(t)$ where the b_{1i} are constants to be estimated and the functions $f_{1i}(t)$ of time t are defined below:

$f_{11}(t) \equiv t$ for $t = 0, 1, \ldots, 7$; $f_{11}(t) \equiv 7$ for $t = 8, 9, \ldots, 52$;

$f_{12}(t) \equiv 0$ for $t = 0, 1, \ldots, 7$; $f_{12}(t) \equiv t - 7$ for $t = 8, 9, \ldots, 17$;
$f_{12}(t) \equiv f_{12}(17)$ for $t = 18, 19, \ldots, 52$;

[*16] Thus $\mathbf{A}(t) \equiv [1 - (t/52)]\mathbf{U}^0\mathbf{U}^{0T} + [t/52]\mathbf{U}^1\mathbf{U}^{1T}$ where $\mathbf{U}^0$ and $\mathbf{U}^1$ are lower triangular matrices that satisfy $\mathbf{U}^{0T}\mathbf{1}_6 = \mathbf{0}_6$ and $\mathbf{U}^{1T}\mathbf{1}_6 = \mathbf{0}_6$ and $\mathbf{B}(t) \equiv [1 - (t/52)]\mathbf{V}^0\mathbf{V}^{0T} + [t/52]\mathbf{V}^1\mathbf{V}^{1T}$ where $\mathbf{V}^0$ and $\mathbf{V}^1$ are lower triangular matrices that satisfy $\mathbf{V}^{0T}\mathbf{1}_4 = \mathbf{0}_4$ and $\mathbf{V}^{1T}\mathbf{1}_4 = \mathbf{0}_4$.

[*17] See Appendix C.

$f_{13}(t) \equiv 0$ for $t = 0, 1, \ldots, 17$; $f_{13}(t) \equiv t - 17$ for $t = 18, \ldots, 22$;
$f_{13}(t) \equiv f_{13}(22)$ for $t = 23, \ldots, 52$;
$f_{14}(t) \equiv 0$ for $t = 0, 1, \ldots, 22$; $f_{14}(t) \equiv t - 22$ for $t = 23, \ldots, 36$;
$f_{14}(t) \equiv f_{14}(36)$ for $t = 37, \ldots, 52$;
$f_{15}(t) \equiv 0$ for $t = 0, 1, \ldots, 36$; $f_{15}(t) \equiv t - 36$ for $t = 37, \ldots, 50$;
$f_{15}(t) \equiv f_{15}(50)$ for $t = 51, 52$;
$f_{16}(t) \equiv 0$ for $t = 0, 1, \ldots, 50$; $f_{16}(t) \equiv t - 50$ for $t = 51, 52$.

Note how the break points for equation 1, 8 18 23 37 51 appear in the above definitions for the $f_{1i}(t)$. Note also that $\sum_{i=1}^{6} f_{1i}(t) = t$ for $t = 0, 1, \ldots, 52$.

The linear function of time tb_2 is replaced by the function $\sum_{i=1}^{7} b_{2i} f_{2i}(t)$ where the b_{2i} are constants to be estimated and the functions $f_{2i}(t)$ of time t are defined below using the break points for equation 2:

$f_{21}(t) \equiv t$ for $t = 0, 1, \ldots, 7$; $f_{21}(t) \equiv 7$ for $t = 8, 9, \ldots, 52$;
$f_{22}(t) \equiv 0$ for $t = 0, 1, \ldots, 7$; $f_{22}(t) \equiv t - 7$ for $t = 8, 9, \ldots, 12$;
$f_{22}(t) \equiv f_{22}(12)$ for $t = 13, \ldots, 52$;
$f_{23}(t) \equiv 0$ for $t = 0, 1, \ldots, 12$; $f_{23}(t) \equiv t - 12$ for $t = 13, \ldots, 16$;
$f_{23}(t) \equiv f_{23}(16)$ for $t = 17, \ldots, 52$;
$f_{24}(t) \equiv 0$ for $t = 0, 1, \ldots, 16$; $f_{24}(t) \equiv t - 16$ for $t = 17, \ldots, 21$;
$f_{24}(t) \equiv f_{24}(21)$ for $t = 22, \ldots, 52$;
$f_{25}(t) \equiv 0$ for $t = 0, 1, \ldots, 21$; $f_{25}(t) \equiv t - 21$ for $t = 22, \ldots, 35$;
$f_{25}(t) \equiv f_{25}(35)$ for $t = 36, \ldots, 52$
$f_{26}(t) \equiv 0$ for $t = 0, 1, \ldots, 35$; $f_{26}(t) \equiv t - 35$ for $t = 36, \ldots, 50$;
$f_{26}(t) \equiv f_{26}(36)$ for $t = 51, 52$;
$f_{27}(t) \equiv 0$ for $t = 0, 1, \ldots, 50$; $f_{27}(t) \equiv t - 50$ for $t = 51, 52$.

The spline functions $f_{3i}(t), f_{4i}(t)$ and $f_{5i}(t)$ for $i = 1, \ldots, 7$ and $f_{6i}(t)$ for $i = 1, \ldots, 5$ were defined in a similar fashion using the break points listed above. Thus replace each tb_n term in equations (20) by the linear spline function $\sum_i b_{ni} f_{ni}(t)$ for $n = 1, \ldots, 6$. This adds 33 additional parameters to the previous model for a total of 105 parameters. In equations (21), the scalar term $(\boldsymbol{b} \cdot \boldsymbol{w}^t)t$ appears on the right hand side. It needs to be replaced with the following terms: $w_1^t \sum_{i=1}^{6} b_{1i} f_{1i}(t) + w_2^t \sum_{i=1}^{7} b_{2i} f_{2i}(t) + w_3^t \sum_{i=1}^{7} b_{3i} f_{3i}(t) + w_4^t \sum_{i=1}^{7} b_{4i} f_{4i}(t) + w_5^t \sum_{i=1}^{7} b_{5i} f_{5i}(t) + w_6^t \sum_{i=1}^{5} b_{6i} f_{6i}(t)$.

We used the final coefficients in the previous nonlinear regression as starting values in the new regression with 105 unknown parameters. The starting coefficient for each b_{ni} was set equal to the final coefficient for b_n in the previous regression for $n = 1, \ldots, 6$. Somewhat surprisingly, Shazam had no difficulty in converging to new estimates; the nonlinear option took only 2.8 seconds with 700 iterations. The log likelihood increased by 273.11 points for adding 33 technical progress parameters. The R^2 for the 10 equations were as follows:

0.9991 0.9978 0.9962 0.9984 0.9987 0.9901 0.9996 0.9994 0.9981 0.9791.

Our final Joint Cost Function model fits the data very well indeed. However, there are a large number of parameters in our final model. In the following section, we will see if the elasticities and the estimates for technical progress that the model generates are "reasonable".[*18]

[*18] Standard errors for our estimated coefficients and our Shazam codes are available on demand.

5 Estimates for Elasticities and Biases in Technical Progress

In the previous section, we derived formula (27) (which drew on (26)) for the year t *elasticity of demand* for input n with respect to a change in the price of input j, $Ex^t_{nj} \equiv [\partial x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial w_j][w^t_j/x^t_n]$. This formula was derived using linear time trends to describe technical progress but the same formula is also valid for year t using our spline model for technical progress provided that we replace the constant matrix $\mathbf{A}$ by the weighted average matrix for year t, $\mathbf{A}(t)$. Thus we use our estimated coefficients to calculate these input elasticities of input demand for each year. The resulting averages of these elasticities are listed in Table 1 below. The main diagonal elasticities for each year are listed in Appendix D.

Table 1 Average Price Elasticities of Input Demand over the Period 1970-2022

Ex_{nj}		w_1 M	w_2 L	w_3 $K_{M\&E}$	w_4 K_S	w_5 K_O	w_6 K_L
x_1	M	-0.4352	0.4203	-0.2333	0.0513	0.1564	0.0404
x_2	L	0.0736	-0.2872	0.1614	0.0584	-0.0233	0.0172
x_3	$K_{M\&E}$	-0.2541	1.0037	-0.8031	-0.1320	0.2257	-0.0403
x_4	K_S	0.0366	0.1889	-0.0683	-0.0997	-0.0298	-0.0277
x_5	K_O	0.3426	-0.2464	0.4017	-0.0912	-0.3389	-0.0679
x_6	K_L	0.0667	0.1337	-0.0458	-0.0679	-0.0486	-0.0381

Note: Imports (M), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

The main diagonal elements of the above matrix are negative, which is consistent with the concavity in input prices property of a joint cost function. A pair of inputs n and j are *substitutes* if $\partial x_n(\boldsymbol{y}, \boldsymbol{w}, t)/\partial w_j = \partial x_j(\boldsymbol{y}, \boldsymbol{w}, t)/\partial w_n$ is positive and are *complements* if these partial derivatives are negative. From Table 1, we see that there are 8 pairs of substitute inputs and 7 pairs of complementary inputs on average. This means that it is unlikely that Cobb-Douglas or CES cost functions could provide a satisfactory approximation to the US aggregate technology (because all inputs are substitutes using these functional forms). Another point to note that follows from an examination of the above Table 1 is that most of the elasticities are small in magnitude.

In the previous section, we derived a formula (30) (which drew on (29)) for the year t *elasticity of marginal cost* for output m with respect to a change in the quantity of output k, $Ep^t_{mk} \equiv [\partial p_m(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial y_k][y^t_k/p^t_m]$. This formula was derived using linear time trends to describe technical progress but the same formula is also valid using our spline model for technical progress, provided that for year t elasticities, we replace the constant matrix $\mathbf{B}$ in (29) by the weighted average matrix $\mathbf{B}(t)$. Thus we use our estimated coefficients to calculate these marginal cost elasticities, which can also be interpreted as (inverse) output supply elasticities. The resulting averages of these elasticities are listed in Table 2 below. The main diagonal elasticities for each year are listed in Appendix A.[*19]

The main diagonal elasticities are all positive and this follows from the fact that we have imposed convexity in $\boldsymbol{y}$ on our estimated joint cost function. However, the elasticities in Table 2 are all quite small. This indicates that the US aggregate joint cost function is close to being

[*19] We did not attempt to compute standard errors for these elasticities. The standard errors are likely to be large so our tables of elasticities are only very rough approximations to the "true" elasticities.

Table 2 Average Elasticities of Inverse Output Supply over the Period 1970-2022

Ep_{mk}		y_1 C	y_2 G	y_3 I	y_4 X
p_1	C	0.0787	-0.0445	-0.0474	0.0131
p_2	G	-0.1683	0.1195	0.0628	-0.0140
p_3	I	-0.1279	0.0449	0.1225	-0.0396
p_4	X	0.0914	-0.0312	-0.0892	0.0914

Note: Consumption (C), Government (G), Investment (I), Export (X).

equal to the sum of 4 sectoral cost functions that produce non overlapping products; i.e., the assumption of no joint production almost holds.

In order to define our next set of elasticities, we need to list the functional form for the joint cost function in year t:

$$\begin{aligned} C(\boldsymbol{y}^t, \boldsymbol{w}^t, t) \equiv\ & (1/2)(\boldsymbol{w}^t \cdot \mathbf{A}(t)\boldsymbol{w}^t)(\alpha \cdot \boldsymbol{w}^t)^{-1}(\boldsymbol{\beta} \cdot \boldsymbol{y}^t) + (1/2)(\boldsymbol{y}^t \cdot \mathbf{B}(t)\boldsymbol{y}^t)(\alpha \cdot \boldsymbol{w}^t)(\boldsymbol{\beta} \cdot \boldsymbol{y}^t)^{-1} \\ & + \boldsymbol{w}^t \cdot \mathbf{D}\boldsymbol{y}^t + (\boldsymbol{\beta} \cdot \boldsymbol{y}^t)\Big[w_1^t \textstyle\sum_{i=1}^{6} b_{1i} f_{1i}(t) + w_2^t \sum_{i=1}^{7} b_{2i} f_{2i}(t) + w_3^t \sum_{i=1}^{7} b_{3i} f_{3i}(t) \\ & + w_4^t \textstyle\sum_{i=1}^{7} b_{4i} f_{4i}(t) + w_5^t \sum_{i=1}^{7} b_{5i} f_{5i}(t) + w_6^t \sum_{i=1}^{5} b_{6i} f_{6i}(t)\Big] \end{aligned} \quad (32)$$

where the year t matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ and the piece-wise linear in t spline functions f_{ni} were defined in the previous section. The above definition for $C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)$ along with Shephard's Lemma, $x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t) = \partial C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial w_n$, are used to define the year t *elasticity of input demand n with respect to an increase in output m*, Ex_{nm}^t, as follows for $n = 1, \ldots, 6, m = 1, \ldots, 4$ and $t = 0, 1, \ldots, 52$:

$$Ex_{nm}^t \equiv [\partial x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial y_m][y_m^t/x_n^t] = [\partial^2 C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial w_n \partial y_m][y_m^t/x_n^t]. \quad (33)$$

The averages of these elasticities are listed in Table 3 below.

Table 3 Average Elasticities of Input Demand with Respect to Output Quantities

Ex_{nm}		y_1 C	y_2 G	y_3 I	y_4 X
x_1	M	0.0864	0.0600	0.4385	0.4152
x_2	L	0.6551	0.1717	0.1771	-0.0038
x_3	$K_{M\&E}$	0.6367	-0.4275	0.5149	0.2760
x_4	K_S	0.4815	0.2818	0.0655	0.1712
x_5	K_O	0.8214	0.5049	-0.0554	-0.2709
x_6	K_L	0.5110	0.2280	0.1532	0.1079

Note: Consumption (C), Government (G), Investment (I), Export (X), Imports (M), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

On average, an increase in consumption y_1 leads to substantial increases in the demand for all inputs except imports where the elasticity was only 0.0864. It is somewhat puzzling that an increase in government output y_2 should lead to a substantial drop in the demand for input 3, Machinery and Equipment.*20

*20 In retrospect, it may have been wiser to impose nonnegativity constraints on the components of the

Definition (32) for the year t cost function, $C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)$, along with the price equals marginal cost equations, $p_m(\boldsymbol{y}^t, \boldsymbol{w}^t, t) = \partial C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial y_m$, can be used to define the year t *elasticity of marginal cost m with respect to an increase in the price of input n*, Ep_{mn}^t, as follows for $m = 1, \ldots, 4, n = 1, \ldots, 6$ and $t = 0, 1, \ldots, 52$:

$$Ep_{mn}^t \equiv [\partial p_m(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial w_n][w_n^t/p_m^t] = [\partial^2 C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial y_m \partial w_n][w_n^t/p_m^t]. \quad (34)$$

The averages of these elasticities over the years 1970-2022 are listed in Table 4 below.

Table 4 Average Elasticities of Output Prices with Respect to Input Prices

Ep_{mn}	w_1 M	w_2 L	w_3 $K_{M\&E}$	w_4 K_S	w_5 K_O	w_6 K_L
p_1 C	0.0222	0.6119	0.0986	0.1328	0.2931	-0.1554
p_2 G	0.0476	0.6350	-0.2583	0.3060	0.0750	0.0596
p_3 I	0.2269	0.4618	0.2183	0.0492	0.1653	0.1043
p_4 X	0.5175	-0.003	0.2711	0.2931	-0.0056	0.0494

Note: Consumption (C), Government (G), Investment (I), Export (X), Imports (M), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

For the most part, an increase in an input price flows through to output prices and increases the price of each of our four outputs. On average, a one percent increase in the price of labour, w_2, substantially increases the prices of consumption, government and investment, p_1, p_2 and p_3, but has a tiny negative effect on the price of exports. There are puzzles that an increase in the input price of Machinery and Equipment services w_3 leads to a substantial drop in the price of government output p_2 and that an increase in the user cost of land w_6 leads to a decline in the price of consumption p_1.

We turn now to the determination of measures of technical progress and the biases in technical progress. The use of a joint cost function to measure technical progress can be traced back to Salter (1960)[52] at least. The most comprehensive measure of technical change going from year t to year $t+1$ using a joint cost function that depends on time, say $C^t(\boldsymbol{y}, \boldsymbol{w})$ where $\boldsymbol{y}$ and $\boldsymbol{w}$ are arbitrary reference output quantity and input price vectors, is $C^t(\boldsymbol{y}, \boldsymbol{w})/C^{t+1}(\boldsymbol{y}, \boldsymbol{w})$.[*21] Holding constant output levels $\boldsymbol{y}$ and input prices $\boldsymbol{w}$, technical progress should reduce the cost of producing $\boldsymbol{y}$ with input prices held constant so that $C^{t+1}(\boldsymbol{y}, \boldsymbol{w}) < C^t(\boldsymbol{y}, \boldsymbol{w})$ and hence technical progress is greater than 1. We will not use this measure of technical progress but we will use a related measure which has the advantage of giving us a decomposition of technical progress into a sum of explanatory factors.

We define year t *cost saving technical progress* κ^t as the derivative of period t cost with respect to time t divided by year t fitted cost:

$$\kappa^t \equiv \partial \ln C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial t = [\partial C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial t]/C(\boldsymbol{y}^t, \boldsymbol{w}^t, t). \quad (35)$$

The year t fitted input n demand function is $x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t) = \partial C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial w_n$ for $n = 1, \ldots, 6$. Define the year t *input n cost saving bias*, κ_n^t, for $n = 1, \ldots, 6$ as follows:

$$\kappa_n^t \equiv w_n^t[\partial x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial t]/C(\boldsymbol{y}^t, \boldsymbol{w}^t, t) = w_n^t[\partial^2 C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial w_n \partial t]/C(\boldsymbol{y}^t, \boldsymbol{w}^t, t). \quad (36)$$

matrix **D** in our model. The imposition of these nonnegativity constraints would lead to nonnegative elasticities for Ex_{nm}^t and Ep_{mn}^t for $t = 0$.

*21 This measure is due to Balk (1998; 58)[2]. For specializations and applications of this definition, see Balk (2001)[3] (2003)[4] and Diewert (2014)[16].

Thus κ_n^t is the derivative of the year t cost of input n with respect to time, $w_n^t \partial x_n(\boldsymbol{y}^t, \boldsymbol{w}^t, t)/\partial t$, divided by total year t fitted input cost, $C(\boldsymbol{y}^t, \boldsymbol{w}^t, t)$. Using the estimated cost function defined by (32), we can calculate κ^t and the bias terms κ_n^t.*22 Euler's Theorem on functions and the linear homogeneity of our estimated $C(\boldsymbol{y}, \boldsymbol{w}, t)$ in $\boldsymbol{w}$ implies that the following decomposition of κ^t into explanatory input bias terms will hold for each year t:

$$\kappa^t = \textstyle\sum_{n=1}^{6} \kappa_n^t. \quad (37)$$

Table 5 below lists the technical progress estimates κ^t and the corresponding bias components.

From Table 5, we see that on average, technical progress reduced economy wide costs by 0.58 percentage points per year. For the years 1971-2022, the average cost reduction was 0.57 percentage points per year. In Appendix A, we show that the corresponding annual average Total Factor Productivity (TFP) Growth estimates using the Jorgenson and Griliches (1967)[31], Diewert and Morrison (1986)[22] and Kohli (1990)[39] (2003)[43] index number methodology was 0.59 percentage points per year. Using the Diewert and Fox (2018)[19] nonparametric methodology for measuring TFP growth explained in Appendix C led to an estimate of 0.59 percentage points per year over the period 1971-2022. Thus our econometric method for measuring technical progress is consistent with alternative methods for measuring technical progress and TFP growth, at least over longer periods. However, the econometric method smooths the year to year fluctuations technical progress and so it does not capture the annual fluctuations in technical progress that are captured by the index number and nonparametric methods, as shown in Figure 1.*23

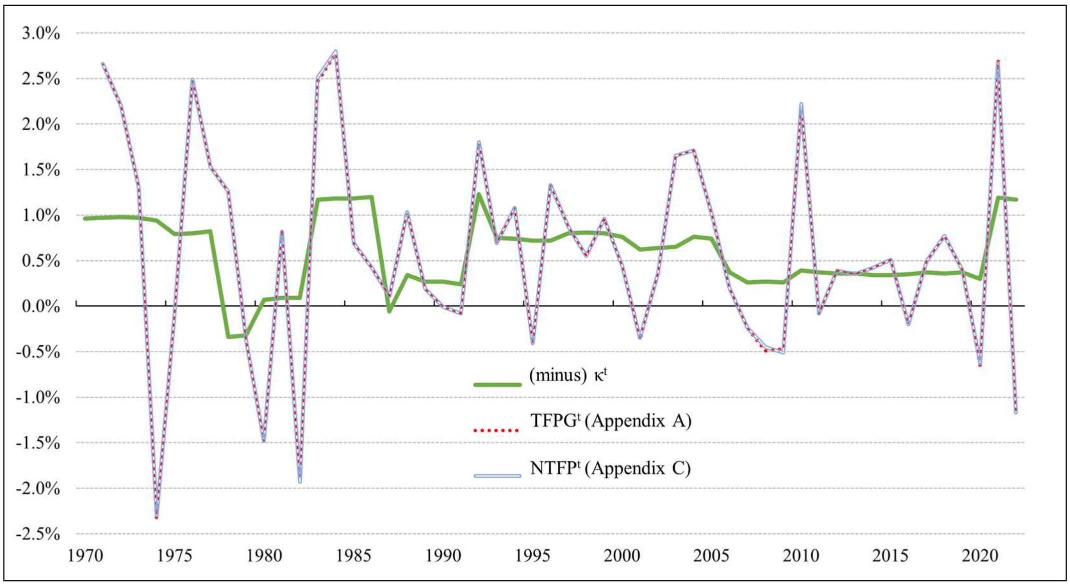

Figure 1 Comparison of Technical Progress Estimates based on Different Approaches

Since $\kappa_2^t, \kappa_3^t, \kappa_4^t$ and κ_6^t were negative on average, technical progress was on average Labour saving, Machinery and Equipment saving, Structure saving and Land saving. Thus technical

*22 The spline functions $f_{ni}(t)$ that were defined in the previous section are not always differentiable with respect to time t. However, the one sided right hand side directional derivatives exist and are straightforward to calculate. These one sided derivatives were used to calculate the various derivatives of the estimated cost function with respect to time.

*23 Note that TFPG^t and NTFP^t are virtually identical in Figure 1.

Table 5 Cost Saving Technical Progress and Input Bias Components for the US

Year t	κ^t	κ_1^t	κ_2^t	κ_3^t	κ_4^t	κ_5^t	κ_6^t
1970	-0.0096	0.0008	-0.0090	-0.0010	-0.0001	0.0007	-0.0010
1971	-0.0097	0.0008	-0.0091	-0.0009	-0.0001	0.0006	-0.0011
1972	-0.0098	0.0008	-0.0093	-0.0006	-0.0001	0.0005	-0.0012
1973	-0.0097	0.0008	-0.0095	-0.0002	-0.0001	0.0005	-0.0011
1974	-0.0094	0.0007	-0.0100	0.0001	0.0000	0.0006	-0.0009
1975	-0.0079	0.0008	-0.0101	0.0000	0.0000	0.0022	-0.0009
1976	-0.0080	0.0008	-0.0099	-0.0001	0.0000	0.0022	-0.0010
1977	-0.0082	0.0008	-0.0099	-0.0001	0.0000	0.0022	-0.0011
1978	0.0034	0.0013	0.0014	0.0000	0.0000	0.0021	-0.0013
1979	0.0032	0.0012	0.0013	0.0001	0.0000	0.0020	-0.0014
1980	-0.0007	0.0011	0.0011	-0.0036	0.0001	0.0019	-0.0014
1981	-0.0009	0.0011	0.0012	-0.0035	0.0001	0.0018	-0.0016
1982	-0.0009	0.0010	0.0012	-0.0033	0.0001	0.0017	-0.0016
1983	-0.0117	0.0011	-0.0078	-0.0035	-0.0016	0.0018	-0.0018
1984	-0.0118	0.0011	-0.0077	-0.0036	-0.0018	0.0019	-0.0017
1985	-0.0118	0.0011	-0.0079	-0.0035	-0.0019	0.0019	-0.0015
1986	-0.0120	0.0011	-0.0082	-0.0034	-0.0020	0.0019	-0.0014
1987	0.0006	0.0011	0.0041	-0.0049	-0.0019	0.0037	-0.0014
1988	-0.0034	-0.0031	0.0040	-0.0045	-0.0019	0.0036	-0.0016
1989	-0.0027	-0.0030	0.0039	-0.0043	-0.0019	0.0035	-0.0009
1990	-0.0027	-0.0031	0.0040	-0.0040	-0.0017	0.0032	-0.0010
1991	-0.0024	-0.0030	0.0040	-0.0038	-0.0016	0.0031	-0.0011
1992	-0.0123	-0.0029	-0.0044	-0.0038	-0.0016	0.0026	-0.0021
1993	-0.0075	0.0019	-0.0044	-0.0039	-0.0018	0.0027	-0.0021
1994	-0.0074	0.0020	-0.0043	-0.0041	-0.0020	0.0029	-0.0019
1995	-0.0072	0.0020	-0.0043	-0.0042	-0.0021	0.0030	-0.0016
1996	-0.0072	0.0020	-0.0044	-0.0041	-0.0023	0.0031	-0.0014
1997	-0.0080	0.0019	-0.0045	-0.0040	-0.0033	0.0031	-0.0012
1998	-0.0081	0.0018	-0.0047	-0.0037	-0.0034	0.0030	-0.0011
1999	-0.0080	0.0018	-0.0049	-0.0034	-0.0035	0.0030	-0.0010
2000	-0.0076	0.0018	-0.0052	-0.0031	-0.0035	0.0031	-0.0008
2001	-0.0062	0.0017	-0.0054	-0.0028	-0.0016	0.0025	-0.0005
2002	-0.0064	0.0017	-0.0055	-0.0027	-0.0016	0.0024	-0.0007
2003	-0.0065	0.0017	-0.0055	-0.0025	-0.0016	0.0023	-0.0009
2004	-0.0076	0.0017	-0.0056	-0.0023	-0.0021	0.0021	-0.0014
2005	-0.0074	0.0017	-0.0056	-0.0022	-0.0020	0.0026	-0.0019
2006	-0.0037	0.0016	-0.0015	-0.0019	-0.0017	0.0024	-0.0026
2007	-0.0026	-0.0011	-0.0016	0.0010	-0.0015	0.0021	-0.0015
2008	-0.0027	-0.0012	-0.0017	0.0009	-0.0012	0.0020	-0.0015
2009	-0.0026	-0.0011	-0.0017	0.0010	-0.0014	0.0021	-0.0014
2010	-0.0039	-0.0010	-0.0017	-0.0003	-0.0017	0.0021	-0.0012
2011	-0.0037	-0.0011	-0.0018	-0.0003	-0.0019	0.0022	-0.0010
2012	-0.0036	-0.0010	-0.0017	-0.0003	-0.0021	0.0023	-0.0008
2013	-0.0036	-0.0010	-0.0017	-0.0003	-0.0023	0.0024	-0.0007
2014	-0.0034	-0.0009	-0.0018	-0.0002	-0.0024	0.0025	-0.0005
2015	-0.0034	-0.0008	-0.0017	-0.0002	-0.0025	0.0024	-0.0005
2016	-0.0035	-0.0008	-0.0018	-0.0002	-0.0025	0.0024	-0.0006
2017	-0.0037	-0.0008	-0.0018	-0.0002	-0.0026	0.0023	-0.0006
2018	-0.0036	-0.0008	-0.0018	-0.0002	-0.0025	0.0023	-0.0007
2019	-0.0037	-0.0008	-0.0018	-0.0002	-0.0025	0.0022	-0.0008
2020	-0.0030	-0.0008	-0.0019	-0.0001	-0.0017	0.0021	-0.0006
2021	-0.0119	0.0065	-0.0167	-0.0004	-0.0018	0.0014	-0.0007
2022	-0.0117	0.0067	-0.0165	-0.0005	-0.0018	0.0014	-0.0009
Mean	-0.0058	0.0005	-0.0040	-0.0018	-0.0016	0.0022	-0.0012

progress reduced the demand for these inputs over time. By far the biggest contribution to overall cost reduction due to technical progress was made by Labour. Since on average, κ_1^t and κ_5^t were positive, technical progress was Import augmenting and Other Capital (R&D and Inventories) augmenting.

Table 5 shows a limitation of our econometric specification. We used piece-wise linear splines to model technical progress and so as time t moves through a break point, the derivatives with respect to time of the spline functions $f_{ni}(t)$ change in a discontinuous manner and this leads to discontinuous changes in the κ_n^t. Thus κ_2^t changes in a discontinuous manner as the spline functions $f_{2i}(t)$ move through break points. The discontinuous movements in κ_2^t largely determine the discontinuous movements in κ^t.

For additional information on the index number and nonparametric methods for measuring technical progress, see Appendices A and C.

For some purposes, it is useful to switch to estimating a GDP function or a Gross Output function or a Variable Profit function where only a few inputs are fixed and all outputs and variable inputs are free to vary. However, as we indicated in Section 3, better global fits can be obtained by estimating joint cost functions if we are using macroeconomic data. It turns out to be possible to get elasticity estimates for these alternative functions using our estimated joint cost function. We show how this can be done in Appendix D.

6 Extensions and Areas for Future Research

It would be of interest to estimate models that used more disaggregated data.[*24] As is explained in Appendix A, the Asian Productivity Data Base has data on more than 20 types of capital for the US for the years 1970-2022. This data base also has information on hours worked by workers classified by: education (2 classes), sex (2 classes), age (5 classes) and type of worker (2, self employed and employee) or 40 types of labour in all.

However, as the number N of inputs and outputs increases, the number of parameters required for a basic flexible functional form is $N(N-1)/2$ and the number of degrees of freedom for a sample of T periods is only NT so the curse of dimensionality becomes a problem. Diewert and Wales (1988)[24] addressed this problem by limiting the number of columns that are allowed to enter into the $\mathbf{U}$ and $\mathbf{V}$ lower triangular matrices defined in Section 4. The same idea could be applied to the $\mathbf{D}$ matrix defined in Section 4; i.e., set $\mathbf{D}$ equal to a limited sum of rank one matrices.[*25]

The choice of the fixed vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ also needs to be examined: are there better ways of choosing $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$?

Are there more efficient ways for specifying technical progress? Should we use quadratic or cubic splines in place of the linear splines that we used? Should we impose nonnegativity restrictions on the components of the $\mathbf{D}$ matrix? Answers to these questions require additional research.

[*24] Of course, it would be preferable to estimate the technology sets for each sector. This option runs into two problems: (i) typically the price and quantity data on capital stocks and labour input are not available by sector and (ii) when estimating sectoral Joint Cost functions or Gross Output functions using sectoral data, we require data on the price and quantity of intermediate inputs, which is also not readily available. When we work with national data, intermediate input flows between sectors cancel out, except for imports.

[*25] Alternatively, we could regard a small set of inputs as fixed and treat the remaining inputs as negative outputs and estimate a variable profit function. This would fix the size of the $\mathbf{A}$ matrix at N by N where N is small and the $\mathbf{D}$ matrix would have size N by M where M is large. The semi-flexible idea could be applied to the $\mathbf{B}$ matrix; i.e., the rank of $\mathbf{B}$ would be limited.

Finally, an extension of the competitive model that we used is possible. Recall equations (6) which equated output prices p_m^t to the marginal cost $\partial C(\boldsymbol{y}^t, \boldsymbol{w}^t)/\partial y_m$ for $m = 1, \ldots, M$. These equations can be replaced by the following equations for year t and $m = 1, \ldots, 4$:

$$p_m^t = (1 - \mu_m^t)^{-1} \partial C(\boldsymbol{y}^t, \boldsymbol{w}^t)/\partial y_m \tag{38}$$

where μ_m^t is the year t *ad valorem markup* for output m.[*26] If we assume that the markup is constant over time, then this new specification for equations (6) would add an additional 4 markup parameters μ_m to be estimated.[*27] However, the underlying data for this more general model would have to be adjusted; i.e., we set the cost of capital equal to a rate of return that causes cost to equal the value of output. For the markup model, we would need to specify an exogenous cost of capital. This is a tricky business: if the cost of capital is set to a high level, markups could become negative. If the cost of capital is set too low, then the resulting markups will be too large.

There are many other problems with our empirical application of flexible functional form theory:

- The General Government and Owner Occupied Housing Sectors should be treated separately from the Business Sector.
- There are missing assets in our model such as Natural Resource Deposits. Environmental Assets and Monetary Holdings.
- An exogenous cost of capital probably should be used in place of our endogenously determined r^t.
- There is a general problem with how to define expected asset inflation rates and how to treat negative user costs when land and natural resource stocks are included in the list of assets.

A host of additional measurement problems are discussed in the excellent survey paper on productivity measurement by Martin and Riley (2024)[49].

7 Conclusion

Here are some of the important points that emerge from our paper:

- It is important to include land as a factor of production in macroeconomic models of the economy.
- It may be better to estimate joint cost functions rather than gross output functions when working with macroeconomic data.
- The Normalized Quadratic Joint Cost function can be used to model aggregate production for an economy but the estimation is quite complex. The advantage of this flexible functional form is that it can impose curvature conditions globally and it can be used to model technical progress in a way that captures longer terms trends in the more variable index number and nonparametric estimates of TFP growth.
- Elasticities of input substitution are far from being constant for aggregate production functions once we disaggregate capital services. For our 6 input model, we found that 8 pairs of inputs were substitutes and 7 pairs were complements. Macroeconomic models based on Cobb-Douglas or CES production functions are not satisfactory descriptions of reality.

[*26] See Diewert and Fox (2008; 176-177)[18] for an example of how this model could be implemented.

[*27] More elaborate models for the markups could be estimated. For example, we could specify linear time trends for the markups or we could specify more flexible spline models.

- Sample wide linear trends used to model technical progress were not satisfactory. Linear spline functions were used to model technical progress with some success.
- A next step in using the methodology outlined in this paper is to include markups in the model. Instead of setting output price equal to marginal cost, set output price equal to a markup plus marginal cost. However, this extension requires exogenous estimates for the economy wide cost of capital and it is difficult to obtain a definitive estimate for this important variable.

Appendix A: Data Construction

In this paper, we use the 1970-2022 US annual data on the prices and quantities of the macroeconomic aggregates $C + G + I + X - M$ and the data for labour and capital stocks developed by the joint project of the Asian Productivity Organization and Keio University led by Koji Nomura. The Augmented Productivity Database (APDB) for the US was produced on November 15, 2023.[*28] The APDB output data are consistent with the recent data published by the Bureau of Economic Analysis (2023)[5], except that the BEA data allocates government investment into the general government sector whereas the APDB data includes government investment in the APDB investment aggregate. The constant dollar APDB investment, depreciation and reproducible capital stock data are perfectly consistent with the geometric model of depreciation.

The APDB has annual current and constant dollar estimates for 10 reproducible capital stocks plus inventory change plus 6 types of land. These 17 constant dollar capital stocks are beginning of the year estimates but the corresponding prices are midyear prices. We convert these midyear prices into beginning of the year prices by taking the arithmetic average of the current midyear price and the previous year midyear price. The APDB also has detailed information on hourly wage rates and annual hours worked for many types of labour classified by age, sex, education and type of worker (employee or self-employed). We will not make use of the detailed labour information: we simply used the resulting APDB aggregate quality adjusted price and quantity of labour for year t, P_L^t and Q_L^t.

The price indexes for the output aggregates for year t are defined as P_C^t (private consumption)[*29], P_G^t (government consumption), P_I^t (gross investment), P_X^t (exports of goods and services) and P_M^t (imports of goods and services)[*30]. The corresponding quantity or volume indexes are defined as $Q_C^t, Q_G^t, Q_I^t, Q_X^t$ and Q_M^t. The 11 components of the APDB investment series are as follows: (1) IT hardware; (2) Communications equipment; (3) Transport equipment; (4) Other machinery and equipment; (5) Dwelling structures; (6) Non-residential buildings; (7) Other structures; (8) Research and development; (9) Computer software; (10) Other intangible assets and (17) Net increase in inventory stocks.[*31] Denote the year t price index for investment good n by P_{In}^t and the corresponding quantity or volume index by Q_{In}^t for $n = 1, \ldots, 17$ and $t = 1970, \ldots, 2022$. The APDB price series for reproducible investments are listed in Table A1 and the corresponding quantity series are listed in Table A2. The units of measurement are in trillions of 1970 constant dollars.

[*28] The methodology for constructing the labour, capital and productivity accounts for the US and Asian countries is explained in Asian Productivity Organization (2022; 165-188)[1].

[*29] The price of consumption was adjusted downward by removing indirect taxes on outputs in order to obtain an approximation to the producer price of consumption.

[*30] The price of imports was adjusted upwards by adding tariffs to the border price of imports.

[*31] We cumulated the estimates for real inventory change to form beginning of the year inventory stocks. We set the price of inventory change equal to the end of year price of inventory stock. These conventions allowed us to apply user cost theory to the stock of inventories. In Tables A1 and A2, we labelled the year t price and quantity of inventory change as P_{I17}^t and Q_{I17}^t. Assets 11-16 are land stocks. We did not include investments in land stocks in this study since these investments are not recognized in the current international System of National Accounts.

Table A1: Price Indexes for US Reproducible Investments

Year	P_{I1}	P_{I2}	P_{I3}	P_{I4}	P_{I5}	P_{I6}	P_{I7}	P_{I8}	P_{I9}	P_{I10}	P_{I17}
1970	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1971	0.67587	1.01677	1.02154	1.02539	1.05098	1.09624	1.07348	1.04382	1.05603	1.08889	0.93258
1972	0.55998	1.06170	1.04238	1.04236	1.11280	1.17834	1.13004	1.07645	1.22239	1.12294	1.00000
1973	0.50039	1.08037	1.05074	1.07326	1.21141	1.26395	1.22238	1.12088	1.35754	1.17141	1.15217
1974	0.43216	1.18518	1.15617	1.19841	1.34337	1.40873	1.44461	1.22354	1.43624	1.22613	1.35922
1975	0.39915	1.30770	1.30601	1.42353	1.48421	1.56112	1.56420	1.32321	1.70886	1.34098	1.40000
1976	0.33373	1.37345	1.41440	1.53608	1.56882	1.62011	1.61517	1.38775	1.83141	1.42098	1.40164
1977	0.28188	1.32475	1.51750	1.63810	1.71554	1.73241	1.66631	1.46302	2.15563	1.48286	1.45752
1978	0.19059	1.37678	1.63484	1.76659	1.92480	1.90627	1.77852	1.54458	2.14661	1.59709	1.59586
1979	0.15771	1.40083	1.76883	1.91389	2.13733	2.11003	1.94016	1.64948	2.67716	1.66546	1.81351
1980	0.12164	1.49599	1.94683	2.15343	2.37790	2.35708	2.01758	1.80049	2.94414	1.74004	1.91254
1981	0.10657	1.61514	2.13599	2.35925	2.56150	2.55678	2.28504	1.94623	3.96771	1.84254	2.15531
1982	0.09808	1.73120	2.30486	2.53575	2.68465	2.73185	2.48623	2.08271	4.74826	1.93262	2.13523
1983	0.08172	1.78463	2.38076	2.63763	2.74432	2.83119	2.34445	2.17215	6.01327	2.01593	2.12408
1984	0.06584	1.83407	2.44665	2.66571	2.81691	2.92301	2.26529	2.24344	6.32033	2.09859	2.35766
1985	0.05582	1.87439	2.50910	2.69479	2.87259	3.02464	2.37306	2.29201	6.74004	2.19965	2.41811
1986	0.04802	1.88606	2.60075	2.81758	3.03598	3.14442	2.65302	2.33250	6.58133	2.30470	2.20688
1987	0.04034	1.87379	2.60770	2.87636	3.17485	3.24433	2.70729	2.37182	6.43226	2.38810	2.30719
1988	0.03722	1.85554	2.63154	2.96440	3.29146	3.37893	2.71097	2.44625	7.04110	2.48258	2.38453
1989	0.03474	1.85899	2.72511	3.05756	3.41345	3.50560	2.84825	2.47653	7.94874	2.62560	2.55637
1990	0.03136	1.85717	2.79755	3.17531	3.48722	3.62340	2.88063	2.50335	8.00673	2.73804	2.47809
1991	0.02821	1.86775	2.90313	3.30326	3.50771	3.72449	2.95890	2.54501	8.16032	2.90359	2.51844
1992	0.02399	1.84335	2.98052	3.37807	3.58406	3.75027	3.04836	2.58129	8.27947	2.94880	2.57584
1993	0.02027	1.80267	3.03054	3.40098	3.72808	3.85506	2.98750	2.59893	8.98822	2.97831	2.63679
1994	0.01783	1.76728	3.12641	3.43791	3.87175	3.97008	3.05306	2.65155	9.29016	3.05778	2.59610
1995	0.01477	1.69672	3.19077	3.47123	4.01020	4.07819	3.35178	2.75317	9.50958	3.13220	2.73869
1996	0.01115	1.64909	3.21131	3.48756	4.07233	4.12580	3.57472	2.75073	9.15994	3.20100	2.78156
1997	0.00859	1.63449	3.22029	3.50559	4.21007	4.26485	3.47910	2.81676	9.46103	3.25780	2.58470
1998	0.00638	1.53893	3.20878	3.51224	4.33512	4.43467	3.71418	2.82870	9.64210	3.24257	2.50680
1999	0.00495	1.43565	3.23021	3.52466	4.50442	4.62237	3.89519	2.87925	10.06312	3.35064	2.50096
2000	0.00433	1.38248	3.26157	3.53518	4.71390	4.81433	3.96958	2.97622	10.62807	3.45741	2.55675
2001	0.00357	1.30871	3.28972	3.57471	4.94473	4.98888	4.12756	2.99321	10.55350	3.51224	2.57458
2002	0.00310	1.23452	3.28924	3.61413	5.06443	5.12985	4.53281	3.03320	9.92951	3.52655	2.55607
2003	0.00276	1.05489	3.30241	3.65456	5.31084	5.23995	4.69601	3.10133	9.44297	3.57264	2.20047
2004	0.00256	0.95174	3.33990	3.70865	5.68881	5.51466	5.02488	3.16943	9.32880	3.59042	2.41006
2005	0.00226	0.90444	3.41053	3.81705	6.09701	5.98045	5.74836	3.23837	9.39630	3.60486	2.80335
2006	0.00198	0.84512	3.44258	3.88153	6.44831	6.44107	6.60118	3.28987	9.42468	3.63731	2.46026
2007	0.00177	0.78927	3.45684	3.97596	6.53946	6.86055	7.19371	3.36838	9.41765	3.67169	2.77872
2008	0.00161	0.70090	3.48075	4.09216	6.44872	7.10587	7.64303	3.45528	9.60404	3.66600	2.80756
2009	0.00149	0.63941	3.55534	4.14941	6.23955	7.28187	7.56937	3.46005	9.41520	3.67803	2.75002
2010	0.00144	0.57676	3.58793	4.14727	6.19757	7.12688	7.67974	3.51914	9.14362	3.61282	3.07708
2011	0.00139	0.54953	3.68260	4.23631	6.22240	7.28125	8.01027	3.59899	9.28274	3.59310	3.23853
2012	0.00138	0.51086	3.77482	4.32437	6.28879	7.45452	8.45721	3.67948	9.37802	3.62280	3.19538
2013	0.00137	0.47782	3.79586	4.32164	6.59165	7.56251	8.51778	3.65614	9.50706	3.61019	3.16564
2014	0.00138	0.42995	3.84272	4.34573	6.99714	7.79153	8.94102	3.70782	9.52453	3.64910	3.09468
2015	0.00138	0.38385	3.90679	4.36940	7.21251	7.95558	9.12524	3.76212	9.47558	3.72009	3.25672
2016	0.00136	0.34207	3.93791	4.36464	7.47320	8.09039	9.10904	3.69651	9.50454	3.79522	3.63563
2017	0.00136	0.32079	3.95331	4.37696	7.81323	8.27618	9.39472	3.76160	9.50764	3.84454	3.10563
2018	0.00137	0.30203	4.03456	4.43593	8.27303	8.62899	9.26953	3.86586	9.37476	3.90106	3.22573
2019	0.00134	0.28488	4.08707	4.49939	8.51296	9.12736	9.48236	3.93387	9.29080	3.94401	3.14483
2020	0.00131	0.27304	4.13364	4.53565	8.79649	9.46904	9.60616	4.10597	9.14762	4.00237	3.90540
2021	0.00133	0.26205	4.27552	4.68426	9.69699	9.86040	9.99053	4.21436	8.95539	4.09541	2.90687
2022	0.00139	0.25988	4.68266	5.07353	11.07624	11.85630	10.78711	4.37195	8.86819	4.35736	3.81354

Table A2: Quantity Indexes for US Reproducible Investments

Year	Q_{I1}	Q_{I2}	Q_{I3}	Q_{I4}	Q_{I5}	Q_{I6}	Q_{I7}	Q_{I8}	Q_{I9}	Q_{I10}	Q_{I17}
1970	0.00325	0.00782	0.01841	0.05139	0.04397	0.03348	0.03378	0.02844	0.00313	0.00413	0.00200
1971	0.00474	0.00742	0.01948	0.04784	0.05604	0.03290	0.03303	0.02807	0.00306	0.00408	0.00890
1972	0.00707	0.00703	0.02278	0.05272	0.06532	0.03349	0.03306	0.02918	0.00305	0.00439	0.00910
1973	0.00782	0.00838	0.02724	0.06210	0.06472	0.03594	0.03379	0.03022	0.00326	0.00420	0.01380
1974	0.01028	0.00865	0.02483	0.06587	0.05167	0.03501	0.03358	0.02980	0.00378	0.00426	0.01030
1975	0.01043	0.00823	0.02143	0.05923	0.04509	0.03051	0.03508	0.02954	0.00388	0.00412	-0.00450
1976	0.01533	0.00897	0.02400	0.06054	0.05532	0.02946	0.03641	0.03120	0.00396	0.00524	0.01220
1977	0.02321	0.01187	0.02889	0.06585	0.06733	0.02886	0.03835	0.03266	0.00355	0.00585	0.01530
1978	0.04530	0.01386	0.03155	0.07325	0.07188	0.03325	0.04267	0.03483	0.00414	0.00575	0.01617
1979	0.07349	0.01647	0.03308	0.07801	0.06919	0.03838	0.04481	0.03707	0.00425	0.00659	0.00993
1980	0.11827	0.01862	0.02696	0.07412	0.05464	0.03882	0.05055	0.03905	0.00469	0.00597	-0.00329
1981	0.18528	0.01945	0.02600	0.07538	0.05039	0.04106	0.05243	0.04126	0.00422	0.00722	0.01383
1982	0.22937	0.01987	0.02298	0.06863	0.04161	0.04189	0.04802	0.04306	0.00413	0.00727	-0.00698
1983	0.35506	0.02007	0.02590	0.06697	0.05919	0.03895	0.04307	0.04565	0.00383	0.00767	-0.00273
1984	0.57983	0.02240	0.02972	0.07632	0.06777	0.04517	0.04908	0.04996	0.00448	0.00855	0.02774
1985	0.74162	0.02425	0.03025	0.07963	0.06985	0.05127	0.04949	0.05479	0.00488	0.00851	0.00902
1986	0.86994	0.02593	0.02809	0.08089	0.07793	0.04889	0.04074	0.05710	0.00544	0.00940	0.00299
1987	1.11170	0.02620	0.02748	0.08083	0.07897	0.04843	0.04078	0.05950	0.00623	0.00924	0.01175
1988	1.24924	0.02881	0.03008	0.08267	0.07806	0.04905	0.04081	0.06141	0.00654	0.00933	0.00776
1989	1.52323	0.02880	0.02677	0.09044	0.07529	0.05084	0.03906	0.06403	0.00702	0.00976	0.01084
1990	1.52828	0.03022	0.02694	0.08806	0.06900	0.05222	0.04281	0.06691	0.00774	0.01010	0.00585
1991	1.66775	0.02920	0.02785	0.08103	0.06338	0.04556	0.04188	0.06933	0.00817	0.01021	-0.00016
1992	2.24713	0.03098	0.02920	0.08113	0.07175	0.04275	0.03959	0.06999	0.00849	0.01070	0.00633
1993	2.79367	0.03221	0.03391	0.08625	0.07714	0.04180	0.04296	0.06932	0.00848	0.01128	0.00789
1994	3.40045	0.03755	0.03928	0.09064	0.08386	0.04382	0.04169	0.06937	0.00853	0.01188	0.02458
1995	5.10148	0.04306	0.04068	0.09639	0.08084	0.04900	0.04042	0.07160	0.00886	0.01286	0.01139
1996	7.35439	0.04881	0.04245	0.10002	0.08755	0.05400	0.04104	0.07639	0.01020	0.01379	0.01107
1997	10.63263	0.05518	0.04534	0.10340	0.08944	0.05811	0.04291	0.08043	0.01214	0.01420	0.02743
1998	15.36354	0.06365	0.04970	0.11035	0.09686	0.06154	0.04281	0.08509	0.01379	0.01496	0.02541
1999	21.86683	0.07750	0.05874	0.11179	0.10281	0.06204	0.04292	0.09048	0.01615	0.01531	0.02431
2000	26.41311	0.10028	0.05526	0.11930	0.10344	0.06452	0.04804	0.09657	0.01745	0.01622	0.02132
2001	27.25659	0.09471	0.04939	0.11642	0.10448	0.06351	0.05130	0.09934	0.01775	0.01572	-0.01488
2002	29.05439	0.07867	0.04576	0.11571	0.11107	0.05551	0.04676	0.09904	0.01818	0.01651	0.00782
2003	32.66864	0.09070	0.04237	0.12126	0.12109	0.05384	0.04733	0.10118	0.01943	0.01775	0.00641
2004	36.95599	0.10334	0.05071	0.12613	0.13277	0.05430	0.04606	0.10311	0.02097	0.01801	0.02660
2005	41.69228	0.11025	0.05692	0.13738	0.14135	0.05333	0.04642	0.10826	0.02231	0.01952	0.02051
2006	52.24049	0.13074	0.06135	0.14598	0.13063	0.05616	0.04799	0.11440	0.02359	0.01973	0.02805
2007	60.17714	0.15550	0.05961	0.15071	0.10606	0.06145	0.05067	0.12005	0.02563	0.01932	0.01224
2008	66.22915	0.16809	0.04700	0.15146	0.08040	0.06362	0.05306	0.12302	0.02715	0.01820	-0.01040
2009	70.77260	0.16532	0.02259	0.12930	0.06308	0.05258	0.04903	0.12129	0.02800	0.01753	-0.05484
2010	79.00717	0.20634	0.04266	0.13451	0.06094	0.04178	0.04872	0.12206	0.02882	0.01928	0.01752
2011	76.41101	0.21959	0.05443	0.14572	0.06075	0.03990	0.04981	0.12569	0.03111	0.01930	0.01430
2012	82.44852	0.23668	0.06272	0.15519	0.06852	0.04116	0.05230	0.12618	0.03346	0.01954	0.02228
2013	80.66736	0.27031	0.06844	0.15905	0.07709	0.04144	0.05133	0.13260	0.03507	0.01959	0.03333
2014	79.38585	0.31051	0.07519	0.16693	0.08010	0.04449	0.05514	0.13725	0.03729	0.02025	0.02740
2015	79.18422	0.36946	0.08348	0.16287	0.08866	0.05095	0.05012	0.14269	0.03926	0.02101	0.04302
2016	79.01030	0.42905	0.07814	0.16096	0.09502	0.05562	0.04422	0.15348	0.04324	0.02187	0.01075
2017	84.12615	0.49073	0.07777	0.16801	0.09886	0.05575	0.04589	0.15864	0.04765	0.02257	0.01053
2018	95.24785	0.50362	0.08402	0.17350	0.09827	0.05669	0.05029	0.16728	0.05349	0.02333	0.01748
2019	97.26011	0.51003	0.08119	0.17929	0.09750	0.05734	0.05250	0.17942	0.05712	0.02362	0.02296
2020	108.15529	0.51269	0.05890	0.16984	0.10451	0.05509	0.04663	0.18495	0.06204	0.02265	-0.00963
2021	119.20516	0.56846	0.05590	0.17881	0.11672	0.05154	0.04468	0.20001	0.06999	0.02226	0.00402
2022	124.82850	0.64307	0.05377	0.18309	0.10620	0.04859	0.04404	0.21028	0.07921	0.02420	0.04125

The year t overall investment price P_I^t is set equal to the chained Törnqvist price index[*32] of the 11 reproducible investment components listed in the above tables and the year t companion constant dollar aggregate investment is denoted by Q_I^t. The APDB aggregate price and quantity component series of GDP (at producer prices) are listed below in Table A3 along with the APDB quality adjusted price and quantity of labour. The units are in trillions of 1970 constant dollars.

[*32] For definitions of the Törnqvist price and quantity indexes and their connection to the economic approach to index number theory, see Diewert (1976)[13] and Diewert and Morrison (1986)[22].

Table A3: Prices and Quantities for Labour and the Output Components of US GDP

Year	P_C^t	P_G^t	P_I^t	P_X^t	P_M^t	P_L^t	Q_C^t	Q_G^t	Q_I^t	Q_X^t	Q_M^t	Q_L^t
1970	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.6035	0.1928	0.2298	0.0597	0.0583	0.6584
1971	1.0406	1.0831	1.0428	1.0368	1.0659	1.0731	0.6265	0.1930	0.2449	0.0608	0.0615	0.6552
1972	1.0795	1.1646	1.0878	1.0814	1.1307	1.1446	0.6649	0.1936	0.2655	0.0656	0.0683	0.6745
1973	1.1359	1.2456	1.1523	1.2260	1.3220	1.2170	0.6978	0.1920	0.2886	0.0777	0.0714	0.7041
1974	1.2528	1.3523	1.2856	1.5102	1.8775	1.3229	0.6919	0.1961	0.2728	0.0839	0.0698	0.7065
1975	1.3641	1.4769	1.4376	1.6643	2.0714	1.4478	0.7076	0.2009	0.2376	0.0833	0.0621	0.6892
1976	1.4404	1.5655	1.5146	1.7187	2.0975	1.5522	0.7471	0.2009	0.2725	0.0870	0.0742	0.7103
1977	1.5379	1.6688	1.6068	1.7889	2.2806	1.6698	0.7786	0.2045	0.3048	0.0891	0.0824	0.7341
1978	1.6482	1.7819	1.7323	1.8977	2.4507	1.8058	0.8127	0.2081	0.3371	0.0985	0.0895	0.7669
1979	1.7971	1.9300	1.8852	2.1265	2.8592	1.9608	0.8320	0.2097	0.3500	0.1083	0.0910	0.7909
1980	1.9855	2.1318	2.0565	2.3419	3.5426	2.1600	0.8294	0.2128	0.3239	0.1199	0.0850	0.7884
1981	2.1491	2.3446	2.2481	2.5156	3.7429	2.3630	0.8410	0.2162	0.3463	0.1213	0.0872	0.7954
1982	2.2869	2.5059	2.4014	2.5282	3.6227	2.5495	0.8534	0.2213	0.3073	0.1121	0.0860	0.7793
1983	2.3885	2.6118	2.4495	2.5388	3.4842	2.6776	0.9018	0.2280	0.3302	0.1091	0.0970	0.7905
1984	2.4752	2.7430	2.4782	2.5625	3.4598	2.7966	0.9493	0.2314	0.4089	0.1180	0.1205	0.8329
1985	2.5655	2.8477	2.5244	2.4861	3.3458	2.9325	0.9984	0.2425	0.4158	0.1220	0.1283	0.8539
1986	2.6298	2.9097	2.6166	2.4460	3.3468	3.0760	1.0397	0.2536	0.4156	0.1312	0.1394	0.8660
1987	2.7153	3.0042	2.6590	2.5001	3.5485	3.1918	1.0749	0.2586	0.4313	0.1456	0.1477	0.8940
1988	2.8191	3.1111	2.7271	2.6286	3.7151	3.3641	1.1199	0.2637	0.4383	0.1691	0.1535	0.9192
1989	2.9436	3.2424	2.8161	2.6722	3.7963	3.4647	1.1526	0.2718	0.4510	0.1887	0.1603	0.9498
1990	3.0703	3.3950	2.8664	2.6874	3.8988	3.6381	1.1761	0.2793	0.4479	0.2054	0.1660	0.9620
1991	3.1640	3.5363	2.9225	2.7173	3.8627	3.8273	1.1784	0.2842	0.4237	0.2189	0.1658	0.9463
1992	3.2502	3.6655	2.9536	2.7040	3.8665	4.0073	1.2219	0.2860	0.4432	0.2341	0.1775	0.9572
1993	3.3363	3.7536	2.9833	2.7084	3.8369	4.0928	1.2646	0.2859	0.4688	0.2418	0.1928	0.9775
1994	3.3934	3.8564	3.0319	2.7399	3.8682	4.1555	1.3135	0.2873	0.5115	0.2632	0.2158	1.0092
1995	3.4698	3.9625	3.0923	2.8013	3.9572	4.2057	1.3523	0.2881	0.5256	0.2902	0.2331	1.0430
1996	3.5464	4.0525	3.0791	2.7639	3.8802	4.3424	1.3992	0.2894	0.5690	0.3139	0.2534	1.0629
1997	3.6063	4.1409	3.0800	2.7143	3.7399	4.4821	1.4520	0.2947	0.6241	0.3514	0.2876	1.0975
1998	3.6369	4.2216	3.0788	2.6501	3.5350	4.6656	1.5290	0.3003	0.6758	0.3596	0.3212	1.1331
1999	3.6960	4.3802	3.1042	2.6304	3.5476	4.8355	1.6113	0.3085	0.7266	0.3775	0.3585	1.1632
2000	3.7935	4.5890	3.1603	2.6808	3.6990	5.1102	1.6925	0.3133	0.7681	0.4089	0.4051	1.1896
2001	3.8835	4.7413	3.1908	2.6597	3.6050	5.3375	1.7343	0.3244	0.7355	0.3861	0.3951	1.1768
2002	3.9272	4.8850	3.2164	2.6367	3.5590	5.4167	1.7784	0.3369	0.7381	0.3785	0.4096	1.1762
2003	4.0093	5.0961	3.2406	2.6783	3.6659	5.6199	1.8348	0.3427	0.7688	0.3865	0.4306	1.1745
2004	4.1030	5.3227	3.3382	2.7760	3.8371	5.8675	1.9039	0.3479	0.8290	0.4238	0.4779	1.1902
2005	4.2213	5.5929	3.4955	2.8719	4.0612	6.0713	1.9713	0.3508	0.8720	0.4532	0.5089	1.2093
2006	4.3339	5.8389	3.6116	2.9631	4.2160	6.2824	2.0285	0.3550	0.9004	0.4962	0.5416	1.2367
2007	4.4539	6.0955	3.6972	3.0748	4.3640	6.5101	2.0782	0.3607	0.8831	0.5397	0.5555	1.2548
2008	4.5913	6.3694	3.7437	3.2148	4.7921	6.6706	2.0814	0.3695	0.8300	0.5709	0.5437	1.2503
2009	4.5961	6.3175	3.7186	3.0252	4.2681	6.7267	2.0543	0.3852	0.6918	0.5232	0.4745	1.1932
2010	4.6711	6.5186	3.6821	3.1515	4.5128	6.8358	2.0935	0.3852	0.7632	0.5893	0.5358	1.1986
2011	4.7838	6.7262	3.7436	3.3500	4.8567	6.9345	2.1291	0.3734	0.7931	0.6316	0.5616	1.2250
2012	4.8684	6.8369	3.8118	3.3756	4.8717	7.0539	2.1582	0.3679	0.8507	0.6570	0.5753	1.2541
2013	4.9247	7.0135	3.8367	3.3807	4.8111	7.1245	2.1959	0.3608	0.8967	0.6768	0.5824	1.2791
2014	4.9901	7.1605	3.9069	3.3824	4.7756	7.2941	2.2580	0.3578	0.9420	0.7032	0.6125	1.3072
2015	5.0010	7.1527	3.9502	3.2192	4.3964	7.4733	2.3342	0.3641	0.9918	0.7053	0.6444	1.3379
2016	5.0580	7.1628	3.9676	3.1544	4.2468	7.5512	2.3915	0.3704	0.9900	0.7087	0.6537	1.3604
2017	5.1464	7.3365	4.0224	3.2367	4.3388	7.7557	2.4546	0.3702	1.0315	0.7379	0.6845	1.3842
2018	5.2438	7.6194	4.0984	3.3444	4.4715	7.9942	2.5219	0.3753	1.0871	0.7589	0.7122	1.4119
2019	5.3266	7.7279	4.1648	3.3280	4.4333	8.2065	2.5724	0.3901	1.1207	0.7628	0.7206	1.4353
2020	5.4091	7.9177	4.2398	3.2436	4.3373	8.6934	2.5076	0.4014	1.0767	0.6629	0.6560	1.3715
2021	5.6180	8.3575	4.4042	3.6184	4.6577	9.1360	2.7186	0.4028	1.1451	0.7048	0.7509	1.4144
2022	5.9758	8.9418	4.7843	3.9724	4.9893	9.4202	2.7874	0.3993	1.1775	0.7540	0.8154	1.4682

We turn now to the APDB data on US capital stocks. The beginning of the year t capital stocks for the above 10 reproducible investment assets are also available in the APDB. Denote the asset n and year t stock price index by P_{Kn}^t and the corresponding quantity index by Q_{Kn}^t for $n = 1, \ldots, 10$ and $t = 1970, 1971, \ldots, 2022$. The APDB also has current and constant dollar series for 6 types of land used by the US production sector: (11) Agricultural Land; (12) Forest Land; (13) Industrial Land; (14) Commercial Land; (15) Residential Land and (16) Other Use Land.[*33] Denote the beginning of the year t price index for these land assets by P_{Kn}^t and the corresponding quantity index by Q_{Kn}^t for $n = 11, \ldots, 16$ and $t = 1970, \ldots, 2022$. We also constructed a beginning of the year stock of inventories by cumulating the constant dollar inventory investments, Q_{I17}^t. Denote the beginning of the year stock of inventories for year t by Q_{K17}^t and the corresponding beginning of the year price by P_{K17}^t. Thus Asset 17 is Inventory Stocks. The beginning of the year prices and quantities of these 17 capital stocks are listed in Tables A4 and A5 along with the price and quantity of a Törnqvist capital stock aggregate, P_K^t and Q_K^t.

[*33] This is mainly government owned land used for various purposes.

Table A4: Beginning of the Year Prices for US Capital Stocks

Year	P_K^t	P_{K1}^t	P_{K2}^t	P_{K3}^t	P_{K4}^t	P_{K5}^t	P_{K6}^t	P_{K7}^t	P_{K8}^t
1970	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1971	1.03507	0.83828	1.00838	1.01077	1.01270	1.02549	1.04812	1.03674	1.02191
1972	1.10345	0.61853	1.03921	1.03196	1.03388	1.08189	1.13729	1.10176	1.06013
1973	1.18431	0.53068	1.07102	1.04656	1.05781	1.16210	1.22115	1.17621	1.09867
1974	1.30856	0.46660	1.13276	1.10345	1.13583	1.27739	1.33634	1.33350	1.17222
1975	1.48624	0.41584	1.24642	1.23108	1.31097	1.41379	1.48493	1.50441	1.27338
1976	1.63924	0.36663	1.34056	1.36020	1.47981	1.52651	1.59062	1.58969	1.35549
1977	1.75868	0.30796	1.34909	1.46594	1.58709	1.64218	1.67626	1.64074	1.42539
1978	1.92623	0.23635	1.35076	1.57616	1.70235	1.82017	1.81935	1.72242	1.50381
1979	2.15475	0.17425	1.38880	1.70182	1.84024	2.03106	2.00815	1.85934	1.59704
1980	2.38301	0.13976	1.44840	1.85782	2.03366	2.25761	2.23356	1.97887	1.72499
1981	2.57194	0.11417	1.55554	2.04140	2.25634	2.46970	2.45694	2.15131	1.87336
1982	2.74364	0.10239	1.67315	2.22042	2.44750	2.62307	2.64432	2.38563	2.01448
1983	2.83876	0.08995	1.75790	2.34280	2.58669	2.71448	2.78152	2.41534	2.12744
1984	2.85056	0.07382	1.80933	2.41369	2.65167	2.78061	2.87710	2.30488	2.20780
1985	2.89506	0.06086	1.85421	2.47786	2.68025	2.84475	2.97382	2.31918	2.26773
1986	3.01874	0.05195	1.88021	2.55491	2.75619	2.95429	3.08453	2.51304	2.31226
1987	3.18727	0.04421	1.87991	2.60421	2.84697	3.10542	3.19438	2.68016	2.35217
1988	3.36871	0.03880	1.86465	2.61960	2.92038	3.23315	3.31164	2.70913	2.40905
1989	3.53491	0.03600	1.85725	2.67831	3.01098	3.35245	3.44227	2.77961	2.46140
1990	3.65974	0.03307	1.85806	2.76132	3.11644	3.45033	3.56450	2.86444	2.48995
1991	3.68149	0.02980	1.86244	2.85033	3.23929	3.49746	3.67395	2.91977	2.52419
1992	3.65890	0.02611	1.85553	2.94181	3.34067	3.54588	3.73739	3.00363	2.56316
1993	3.63326	0.02214	1.82299	3.00552	3.38953	3.65607	3.80267	3.01793	2.59012
1994	3.64411	0.01906	1.78496	3.07846	3.41945	3.79991	3.91257	3.02028	2.62525
1995	3.72662	0.01631	1.73198	3.15858	3.45457	3.94097	4.02414	3.20242	2.70237
1996	3.83155	0.01297	1.67289	3.20103	3.47940	4.04126	4.10200	3.46325	2.75196
1997	3.92026	0.00987	1.64177	3.21579	3.49658	4.14120	4.19533	3.52691	2.78376
1998	4.10025	0.00749	1.58669	3.21452	3.50892	4.27259	4.34977	3.59664	2.82274
1999	4.35702	0.00567	1.48728	3.21948	3.51845	4.41977	4.52853	3.80469	2.85399
2000	4.59168	0.00464	1.40905	3.24588	3.52992	4.60916	4.71836	3.93239	2.92775
2001	4.85769	0.00395	1.34558	3.27563	3.55495	4.82931	4.90161	4.04858	2.98473
2002	5.10694	0.00334	1.27160	3.28947	3.59442	5.00458	5.05937	4.33019	3.01322
2003	5.37056	0.00293	1.14469	3.29581	3.63435	5.18763	5.18491	4.61441	3.06728
2004	5.72530	0.00266	1.00331	3.32113	3.68161	5.49982	5.37731	4.86045	3.13539
2005	6.33815	0.00241	0.92808	3.37520	3.76285	5.89291	5.74757	5.38663	3.20391
2006	7.00678	0.00212	0.87477	3.42654	3.84929	6.27266	6.21077	6.17477	3.26413
2007	7.31432	0.00188	0.81718	3.44969	3.92875	6.49388	6.65082	6.89744	3.32914
2008	7.32293	0.00169	0.74507	3.46877	4.03406	6.49409	6.98322	7.41837	3.41184
2009	6.97863	0.00155	0.67014	3.51802	4.12079	6.34413	7.19388	7.60620	3.45768
2010	6.45953	0.00147	0.60807	3.57162	4.14834	6.21856	7.20438	7.62456	3.48961
2011	6.25588	0.00142	0.56314	3.63525	4.19179	6.20998	7.20407	7.84501	3.55908
2012	6.40211	0.00139	0.53019	3.72870	4.28034	6.25559	7.36789	8.23375	3.63925
2013	6.68223	0.00138	0.49433	3.78532	4.32301	6.44022	7.50852	8.48750	3.66782
2014	7.06991	0.00138	0.45388	3.81927	4.33369	6.79439	7.67703	8.72941	3.68199
2015	7.43901	0.00138	0.40690	3.87474	4.35757	7.10482	7.87356	9.03314	3.73498
2016	7.72553	0.00137	0.36295	3.92233	4.36703	7.34285	8.02299	9.11715	3.72933
2017	8.04857	0.00136	0.33143	3.94559	4.37081	7.64321	8.18329	9.25189	3.72907
2018	8.33643	0.00137	0.31141	3.99392	4.40645	8.04312	8.45260	9.33213	3.81374
2019	8.59752	0.00136	0.29345	4.06080	4.46766	8.39299	8.87819	9.37595	3.89988
2020	8.94065	0.00133	0.27896	4.11034	4.51752	8.65472	9.29821	9.54427	4.01993
2021	9.62338	0.00132	0.26754	4.20456	4.60996	9.24674	9.66473	9.79835	4.16018
2022	10.55863	0.00136	0.26096	4.47907	4.87890	10.38661	10.85837	10.38883	4.29317
2023		0.00139	0.25987	4.68263	5.07353	11.07624	11.85632	10.78712	4.37197

Year	P^t_{K9}	P^t_{K10}	P^t_{K11}	P^t_{K12}	P^t_{K13}	P^t_{K14}	P^t_{K15}	P^t_{K16}	P^t_{K17}
1970	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1971	1.02806	1.04442	1.04569	1.03841	1.02560	1.02560	1.10341	1.08980	1.00243
1972	1.13928	1.10588	1.16544	1.11461	1.09190	1.09190	1.25098	1.25088	0.92943
1973	1.29004	1.14714	1.40807	1.19651	1.16534	1.16534	1.35133	1.44003	1.00270
1974	1.39695	1.19872	1.68306	1.30862	1.24831	1.24831	1.55640	1.70914	1.15279
1975	1.57257	1.28351	1.91557	1.49106	1.40287	1.40287	1.96862	2.09559	1.36087
1976	1.77016	1.38096	2.25512	1.68524	1.56443	1.56443	2.32907	2.49828	1.39290
1977	1.99358	1.45187	2.61475	1.78678	1.66694	1.66694	2.63734	2.87993	1.40565
1978	2.15117	1.53994	3.02686	1.88809	1.79607	1.79607	3.15812	3.43998	1.46189
1979	2.41192	1.63125	3.59217	1.97162	1.98392	1.98392	3.90813	4.23965	1.59806
1980	2.81071	1.70271	4.10491	2.07364	2.20040	2.20039	4.43091	4.90281	1.81456
1981	3.45597	1.79125	4.30774	2.24286	2.40080	2.40080	4.52062	5.15616	1.92802
1982	4.35807	1.88755	4.17254	2.36509	2.68945	2.68945	4.36832	5.06171	2.15954
1983	5.38089	1.97424	4.05451	2.40745	2.90741	2.90741	4.33089	5.04141	2.14079
1984	6.16690	2.05721	3.76294	2.39445	2.84557	2.84557	4.66055	5.23408	2.11065
1985	6.53028	2.14907	3.21130	2.38952	2.77969	2.77969	5.20344	5.51337	2.36233
1986	6.66081	2.25213	2.81405	2.39676	2.75306	2.75306	6.07430	6.14914	2.42417
1987	6.50696	2.34635	2.73662	2.41339	2.87630	2.80761	7.24808	7.18919	2.20189
1988	6.73684	2.43529	2.86115	2.50361	3.17995	2.96241	8.40756	8.34143	2.31500
1989	7.49507	2.55403	2.97620	2.67445	3.43780	3.05146	9.36191	9.37472	2.38629
1990	7.97792	2.68176	3.09466	2.87956	3.65152	3.10456	9.82555	9.96964	2.55962
1991	8.08371	2.82076	3.17470	3.03667	3.82909	3.03975	9.53577	9.85503	2.49128
1992	8.22006	2.92613	3.25202	3.15246	3.98418	2.75380	9.13590	9.64679	2.46724
1993	8.63403	2.96349	3.43081	3.36844	4.13513	2.35852	8.87841	9.57953	2.58443
1994	9.13938	3.01798	3.62183	3.60974	4.30336	2.09527	8.74019	9.59139	2.64310
1995	9.40005	3.09491	3.78845	3.78642	4.51136	2.08314	8.63247	9.64014	2.59999
1996	9.33494	3.16654	3.96694	3.93643	4.78022	2.14650	8.58164	9.74616	2.74223
1997	9.31067	3.22934	4.16517	4.07082	5.12684	2.24448	8.79453	10.10489	2.78756
1998	9.55177	3.25011	4.37084	4.23043	5.56081	2.72568	9.38040	10.89924	2.59072
1999	9.85281	3.29653	4.60353	4.36673	6.08130	3.26726	10.42696	12.22069	2.51246
2000	10.34583	3.40395	4.93527	4.46973	6.67444	3.44520	11.90367	14.06072	2.50650
2001	10.59102	3.48475	5.28086	4.56644	7.31174	3.71645	13.51222	16.23182	2.56327
2002	10.24173	3.51932	5.45543	4.57454	7.95061	3.76246	15.36732	18.71792	2.57780
2003	9.68646	3.54952	5.51233	4.56731	8.53766	3.75776	17.67116	21.66917	2.55926
2004	9.38608	3.58145	6.03167	4.63239	9.01516	3.96208	20.50270	25.46636	2.20298
2005	9.36275	3.59756	7.18595	4.75009	9.33032	4.45552	24.56166	31.01130	2.41321
2006	9.41070	3.62101	8.36558	4.93447	9.44589	5.33774	27.71376	35.68362	2.80954
2007	9.42137	3.65442	8.87795	5.13866	9.34992	6.14649	27.34450	36.07793	2.46424
2008	9.51105	3.66876	8.94943	5.33659	9.06169	6.88237	23.64786	32.09902	2.77903
2009	9.50982	3.67194	8.98778	5.50437	8.63181	6.52851	19.00617	26.59893	2.81705
2010	9.27960	3.64534	9.37018	5.52552	8.13549	4.53200	16.25553	23.43098	2.75460
2011	9.21338	3.60287	10.27922	5.48306	7.65966	3.61624	14.27961	21.48031	3.08338
2012	9.33059	3.60787	11.69926	5.48505	7.28683	4.19682	13.09658	20.72792	3.24133
2013	9.44274	3.61641	13.18125	5.54368	7.07956	4.48792	14.74880	23.44144	3.20238
2014	9.51600	3.62956	14.11635	5.66080	7.06933	5.05679	17.04571	26.77158	3.17072
2015	9.50026	3.68451	14.45474	5.79841	7.25248	5.87177	18.56820	28.97868	3.10056
2016	9.49027	3.75757	14.55009	5.90648	7.59409	6.47944	20.08394	31.18148	3.26343
2017	9.50629	3.81980	14.91295	5.96136	8.03723	6.97179	21.78121	33.72247	3.64123
2018	9.44140	3.87272	15.23799	6.00715	8.51739	7.47011	23.45716	36.22275	3.10892
2019	9.33298	3.92245	15.42207	6.05945	8.96974	7.68706	24.79480	38.26933	3.23321
2020	9.21941	3.97310	15.80484	6.09480	9.34980	8.36316	26.44699	40.67494	3.14917
2021	9.05170	4.04880	16.95692	6.11076	9.60509	9.71709	29.35484	44.99057	3.90865
2022	8.91198	4.22629	18.65843	6.18476	9.75741	11.21533	33.78415	51.78277	2.90857
2023	8.86838	4.35726	19.49921	6.25259	9.82352	12.07903	36.36039	55.73730	3.82082

Table A5: Beginning of the Year US Capital Stock Quantities

Year	Q^t_K	Q^t_{K1}	Q^t_{K2}	Q^t_{K3}	Q^t_{K4}	Q^t_{K5}	Q^t_{K6}	Q^t_{K7}	Q^t_{K8}
1970	3.19747	0.01349	0.02022	0.07947	0.23245	0.55234	0.32395	0.50843	0.07483
1971	3.26861	0.01229	0.02210	0.07847	0.23785	0.56894	0.33003	0.52885	0.08635
1972	3.35665	0.01271	0.02317	0.07864	0.23906	0.59680	0.33502	0.54801	0.09535
1973	3.45911	0.01498	0.02362	0.08168	0.24439	0.63231	0.34004	0.56644	0.10365
1974	3.57489	0.01724	0.02516	0.08807	0.25743	0.66572	0.34724	0.58539	0.11131
1975	3.65371	0.02092	0.02655	0.09090	0.27156	0.68438	0.35287	0.60358	0.11713
1976	3.70262	0.02365	0.02723	0.09009	0.27715	0.69563	0.35357	0.62285	0.12162
1977	3.78480	0.02976	0.02840	0.09176	0.28298	0.71661	0.35318	0.64301	0.12675
1978	3.88794	0.04078	0.03182	0.09742	0.29259	0.74855	0.35223	0.66458	0.13223
1979	4.01444	0.06739	0.03615	0.10421	0.30725	0.78349	0.35575	0.68992	0.13863
1980	4.14166	0.11016	0.04169	0.11084	0.32352	0.81373	0.36399	0.71650	0.14584
1981	4.21806	0.17852	0.04775	0.11058	0.33338	0.82802	0.37201	0.74820	0.15348
1982	4.30964	0.28388	0.05306	0.10954	0.34269	0.83747	0.38165	0.78107	0.16167
1983	4.35062	0.39579	0.05743	0.10603	0.34416	0.83763	0.39128	0.80861	0.16993
1984	4.42874	0.58179	0.06088	0.10587	0.34378	0.85516	0.39706	0.83029	0.17896
1985	4.57180	0.90467	0.06553	0.10917	0.35205	0.88057	0.40866	0.85757	0.19018
1986	4.70776	1.27020	0.07066	0.11220	0.36181	0.90664	0.42529	0.88439	0.20364
1987	4.82847	1.63784	0.07602	0.11267	0.37110	0.93982	0.43829	0.90207	0.21663
1988	4.93892	2.10333	0.08030	0.11249	0.37867	0.97238	0.44973	0.91931	0.22933
1989	5.05997	2.54907	0.08582	0.11468	0.38658	1.00245	0.46082	0.93615	0.24134
1990	5.16438	3.09502	0.08990	0.11334	0.39984	1.02742	0.47239	0.95002	0.25344
1991	5.25163	3.48668	0.09428	0.11254	0.40888	1.04569	0.48474	0.96804	0.26585
1992	5.30371	3.88137	0.09668	0.11271	0.40985	1.05728	0.48931	0.98451	0.27809
1993	5.36929	4.64767	0.09991	0.11387	0.41011	1.07502	0.48993	0.99673	0.28861
1994	5.44599	5.65727	0.10350	0.11907	0.41535	1.09824	0.49000	1.01292	0.29652
1995	5.55905	6.88216	0.11082	0.12783	0.42338	1.12626	0.49173	1.02669	0.30297
1996	5.66495	9.20003	0.12125	0.13601	0.43554	1.15072	0.49889	1.03962	0.31021
1997	5.80444	12.75841	0.13420	0.14403	0.44902	1.18118	0.51065	1.05329	0.32042
1998	5.97484	18.06154	0.14954	0.15285	0.46313	1.21189	0.52549	1.06836	0.33234
1999	6.15790	25.83353	0.16852	0.16362	0.48103	1.24845	0.54247	1.08289	0.34621
2000	6.34231	36.85800	0.19498	0.18008	0.49706	1.28913	0.55851	1.09714	0.36233
2001	6.55247	48.52102	0.23495	0.18990	0.51724	1.32886	0.57587	1.11651	0.38090
2002	6.68621	57.45963	0.26016	0.19229	0.53110	1.36740	0.59062	1.13839	0.39844
2003	6.81927	65.30110	0.26511	0.19094	0.54187	1.41070	0.59617	1.15524	0.41239
2004	6.94936	73.89978	0.27934	0.18683	0.55566	1.46157	0.59945	1.17198	0.42562
2005	7.10395	83.54085	0.30085	0.19085	0.57085	1.52006	0.60229	1.18576	0.43808
2006	7.25761	94.16128	0.32236	0.19905	0.59215	1.58040	0.60242	1.19660	0.45284
2007	7.42491	110.96673	0.35770	0.21014	0.61969	1.63288	0.60770	1.21329	0.47036
2008	7.52989	129.58735	0.40606	0.21724	0.64662	1.65817	0.61780	1.23219	0.48965
2009	7.58682	147.76124	0.45317	0.21135	0.66883	1.65498	0.62863	1.25182	0.50797
2010	7.55260	164.57125	0.48662	0.18521	0.66747	1.63610	0.62806	1.26801	0.52124
2011	7.59068	183.40342	0.54767	0.18261	0.67091	1.61555	0.61655	1.28311	0.53269
2012	7.62719	194.40635	0.60510	0.19100	0.68371	1.59527	0.60393	1.29843	0.54526
2013	7.69440	207.23516	0.66312	0.20485	0.70259	1.58317	0.59341	1.31535	0.55588
2014	7.77324	214.96042	0.73695	0.22096	0.72230	1.58182	0.58466	1.33214	0.57029
2015	7.87649	219.34074	0.82794	0.23958	0.74575	1.58371	0.57976	1.35243	0.58617
2016	7.99470	222.23249	0.94814	0.26147	0.76124	1.59388	0.58167	1.36701	0.60396
2017	8.08877	224.06239	1.09075	0.27375	0.77201	1.60945	0.58792	1.37492	0.62813
2018	8.19136	229.26620	1.25013	0.28246	0.78574	1.62488	0.59260	1.38155	0.65237
2019	8.31064	242.85332	1.38366	0.29536	0.80353	1.64210	0.59897	1.39510	0.67984
2020	8.42130	254.21506	1.49027	0.30299	0.82364	1.65813	0.60560	1.41086	0.71308
2021	8.45731	271.41805	1.57238	0.28898	0.83123	1.67974	0.60920	1.41980	0.74500
2022	8.59638	292.82907	1.68237	0.27522	0.84523	1.71163	0.60864	1.42583	0.78449

Year	Q^t_{K9}	Q^t_{K10}	Q^t_{K11}	Q^t_{K12}	Q^t_{K13}	Q^t_{K14}	Q^t_{K15}	Q^t_{K16}	Q^t_{K17}
1970	0.00657	0.01482	0.16902	0.12774	0.22781	0.53839	0.22792	0.03999	0.04002
1971	0.00701	0.01439	0.16618	0.12802	0.22700	0.54546	0.23201	0.04158	0.04202
1972	0.00726	0.01404	0.16369	0.12831	0.22593	0.55832	0.23762	0.04287	0.05090
1973	0.00741	0.01404	0.16186	0.12858	0.22416	0.57497	0.24226	0.04378	0.05998
1974	0.00768	0.01389	0.16097	0.12887	0.22250	0.59037	0.24698	0.04404	0.07376
1975	0.00830	0.01382	0.16054	0.12915	0.22042	0.59279	0.25066	0.04405	0.08404
1976	0.00880	0.01365	0.16071	0.12915	0.21864	0.60084	0.25575	0.04378	0.07954
1977	0.00920	0.01450	0.16093	0.12915	0.21732	0.61725	0.25945	0.04352	0.09172
1978	0.00913	0.01564	0.16061	0.12896	0.21586	0.63118	0.26491	0.04367	0.10699
1979	0.00957	0.01639	0.16013	0.12940	0.21361	0.65524	0.26865	0.04361	0.12313
1980	0.00996	0.01766	0.15898	0.12926	0.21092	0.67062	0.27853	0.04415	0.13304
1981	0.01060	0.01806	0.15754	0.12912	0.20767	0.67361	0.28416	0.04503	0.12976
1982	0.01062	0.01942	0.15619	0.12898	0.20422	0.68327	0.28857	0.04588	0.14356
1983	0.01056	0.02046	0.15518	0.12798	0.20054	0.68504	0.29080	0.04708	0.13659
1984	0.01028	0.02157	0.15448	0.12784	0.19792	0.70525	0.29585	0.04748	0.13387
1985	0.01063	0.02314	0.15392	0.12770	0.19619	0.72955	0.30129	0.04776	0.16156
1986	0.01120	0.02425	0.15336	0.12756	0.19473	0.75098	0.30724	0.04803	0.17056
1987	0.01205	0.02583	0.15265	0.12742	0.19287	0.77378	0.31134	0.04845	0.17354
1988	0.01328	0.02684	0.15192	0.12853	0.19081	0.78121	0.31726	0.04808	0.18527
1989	0.01436	0.02766	0.15108	0.12848	0.18779	0.80666	0.32338	0.04849	0.19302
1990	0.01548	0.02863	0.15029	0.12842	0.18256	0.82944	0.32608	0.04898	0.20383
1991	0.01683	0.02963	0.14975	0.12836	0.17562	0.84234	0.33009	0.04927	0.20967
1992	0.01810	0.03046	0.14965	0.12835	0.16739	0.84467	0.33526	0.04921	0.20951
1993	0.01922	0.03149	0.14968	0.12870	0.16026	0.86473	0.33870	0.04889	0.21583
1994	0.01996	0.03274	0.15008	0.12906	0.15443	0.87799	0.34193	0.04833	0.22371
1995	0.02050	0.03417	0.15059	0.12942	0.15121	0.89615	0.34868	0.04757	0.24824
1996	0.02113	0.03606	0.15100	0.12978	0.15041	0.90954	0.35182	0.04699	0.25961
1997	0.02267	0.03825	0.15111	0.13014	0.15174	0.93449	0.35717	0.04651	0.27066
1998	0.02533	0.04020	0.15122	0.13017	0.15258	0.96423	0.36557	0.04614	0.29803
1999	0.02849	0.04228	0.15089	0.13020	0.15276	0.99416	0.37357	0.04607	0.32339
2000	0.03257	0.04410	0.15024	0.13024	0.15232	1.01901	0.38009	0.04627	0.34766
2001	0.03639	0.04621	0.14939	0.13027	0.15135	1.04771	0.39476	0.04641	0.36894
2002	0.03920	0.04733	0.14851	0.13048	0.14987	1.05961	0.40224	0.04664	0.35409
2003	0.04145	0.04882	0.14795	0.13056	0.14850	1.07610	0.41253	0.04663	0.36189
2004	0.04399	0.05099	0.14746	0.13064	0.14746	1.09532	0.41904	0.04670	0.36829
2005	0.04698	0.05279	0.14699	0.13072	0.14645	1.11509	0.42752	0.04669	0.39484
2006	0.05011	0.05541	0.14649	0.13080	0.14487	1.13964	0.43513	0.04673	0.41531
2007	0.05327	0.05751	0.14595	0.13088	0.14350	1.15652	0.44466	0.04674	0.44330
2008	0.05709	0.05868	0.14526	0.13142	0.14080	1.16522	0.44563	0.04685	0.45552
2009	0.06092	0.05857	0.14492	0.13196	0.13906	1.16973	0.44541	0.04638	0.44514
2010	0.06420	0.05791	0.14480	0.13251	0.13768	1.15305	0.44504	0.04615	0.39041
2011	0.06708	0.05894	0.14478	0.13305	0.13717	1.16927	0.45101	0.04564	0.40789
2012	0.07092	0.05971	0.14475	0.13349	0.13739	1.17963	0.45308	0.04533	0.42216
2013	0.07545	0.06048	0.14458	0.13361	0.13939	1.19521	0.45580	0.04527	0.44440
2014	0.07984	0.06108	0.14434	0.13373	0.14062	1.20240	0.45817	0.04528	0.47767
2015	0.08463	0.06210	0.14400	0.13385	0.14112	1.22209	0.46235	0.04529	0.50501
2016	0.08948	0.06350	0.14324	0.13397	0.14121	1.24569	0.46613	0.04559	0.54795
2017	0.09606	0.06526	0.14248	0.13397	0.14153	1.26444	0.46738	0.04605	0.55868
2018	0.10414	0.06715	0.14231	0.13384	0.14158	1.28670	0.47155	0.04610	0.56919
2019	0.11443	0.06919	0.14201	0.13384	0.14071	1.30863	0.47458	0.04620	0.58664
2020	0.12437	0.07093	0.14201	0.13384	0.14140	1.32817	0.47419	0.04621	0.60955
2021	0.13513	0.07136	0.14201	0.13384	0.14210	1.29850	0.47656	0.04614	0.59994
2022	0.14898	0.07133	0.14201	0.13384	0.14197	1.34401	0.48259	0.04593	0.60395

Year t nominal Gross Domestic Product (GDP), V_Y^t, is defined as follows:

$$V_Y^t \equiv P_C^t Q_C^t + P_G^t Q_G^t + P_I^t Q_I^t + P_X^t Q_X^t - P_M^t Q_M^t; \qquad t = 1970, \ldots, 2022. \qquad \text{(A1)}$$

The year t price index for GDP is defined as a Törnqvist price index of the 5 component price and quantity series for consumption, government output, investment, exports and (minus) imports, which we denote by P_Y^t. The companion quantity index for GDP is denoted by Q_Y^t. V_Y^t, P_Y^t and Q_Y^t are listed in Table A6 below.

Now that we have estimates for the US capital stock, we can calculate real and nominal capital output ratios for the US economy, Q_{KY}^t and V_{KY}^t, for each year t. The real capital output ratio, $Q_{KY}^t \equiv Q_K^t/Q_Y^t$, is an indicator of partial *efficiency* while the nominal capital output ratio, $V_{KY}^t \equiv V_K^t/V_Y^t$, is a partial indicator of the international *competitiveness* of the US economy: it indicates how many dollars are required to purchase a sufficient amount of capital in order to produce one dollar of output in year t (on average). The lower V_{KY}^t is, the easier it is to start a business in the US in year t. V_{KY}^t and Q_{KY}^t are listed in Table A6 below.

In order to indicate the importance of land as a component of the total capital stock, we construct a year t Törnqvist capital stock aggregate price P_{KR}^t that excludes the 6 land stocks. Denote the companion year t reproducible capital stock quantity by Q_{KR}^t and define the corresponding year t value by $V_{KR}^t \equiv P_{KR}^t Q_{KR}^t$. Define the year t reproducible real and nominal capital output ratios as $Q_{KRY}^t \equiv Q_{KR}^t/Q_Y^t$ and $V_{KRY}^t \equiv V_{KR}^t/V_Y^t$. These variables are also listed in Table A6 below.

Table A6: US GDP Aggregates and Nominal and Real Capital Output Ratios

Year	V^t_{KY}	Q^t_{KY}	V^t_{KRY}	Q^t_{KRY}	P^t_Y	P^t_K	Q^t_Y	Q^t_K	V^t_Y	V^t_K	V^t_{KR}
1970	3.1120	3.1120	1.8167	1.8167	1.0000	1.0000	1.0275	3.1975	1.0275	3.1975	1.8666
1971	3.0375	3.0733	1.7804	1.8132	1.0473	1.0351	1.0635	3.2686	1.1138	3.3833	1.9830
1972	3.0217	2.9958	1.7645	1.7854	1.0940	1.1035	1.1205	3.3567	1.2258	3.7039	2.1629
1973	3.0010	2.9236	1.7532	1.7630	1.1538	1.1843	1.1832	3.4591	1.3651	4.0967	2.3932
1974	3.1643	3.0423	1.8696	1.8609	1.2581	1.3086	1.1751	3.5749	1.4784	4.6780	2.7640
1975	3.3652	3.1219	1.9853	1.9344	1.3788	1.4862	1.1704	3.6537	1.6137	5.4303	3.2036
1976	3.3773	3.0049	1.9489	1.8685	1.4585	1.6392	1.2322	3.7026	1.7971	6.0695	3.5024
1977	3.3282	2.9307	1.9020	1.8354	1.5487	1.7587	1.2914	3.7848	2.0000	6.6563	3.8040
1978	3.3110	2.8523	1.8795	1.8065	1.6593	1.9262	1.3631	3.8879	2.2619	7.4891	4.2512
1979	3.4194	2.8541	1.9216	1.8312	1.7985	2.1548	1.4066	4.0144	2.5297	8.6501	4.8612
1980	3.5940	2.9501	2.0241	1.9150	1.9561	2.3830	1.4039	4.1417	2.7462	9.8696	5.5586
1981	3.5318	2.9276	2.0340	1.9195	2.1319	2.5719	1.4408	4.2181	3.0717	10.8486	6.2479
1982	3.6772	3.0587	2.1731	2.0260	2.2822	2.7436	1.4090	4.3096	3.2155	11.8241	6.9877
1983	3.5317	2.9621	2.0994	1.9737	2.3809	2.8388	1.4688	4.3506	3.4970	12.3503	7.3415
1984	3.2512	2.8102	1.9417	1.8769	2.4640	2.8506	1.5759	4.4287	3.8830	12.6244	7.5396
1985	3.1700	2.7911	1.9236	1.8819	2.5491	2.8951	1.6380	4.5718	4.1753	13.2356	8.0319
1986	3.2197	2.7920	1.9731	1.8979	2.6177	3.0187	1.6862	4.7078	4.4139	14.2115	8.7090
1987	3.2869	2.7662	1.9931	1.8931	2.6824	3.1873	1.7455	4.8285	4.6821	15.3896	9.3322
1988	3.2965	2.7164	1.9599	1.8755	2.7759	3.3687	1.8182	4.9389	5.0471	16.6378	9.8918
1989	3.2878	2.6881	1.9309	1.8649	2.8900	3.5349	1.8824	5.0600	5.4402	17.8865	10.5042
1990	3.2884	2.6906	1.9368	1.8797	2.9944	3.6597	1.9194	5.1644	5.7475	18.9003	11.1315
1991	3.2624	2.7399	1.9578	1.9271	3.0919	3.6815	1.9167	5.2516	5.9262	19.3339	11.6025
1992	3.0921	2.6780	1.9095	1.8919	3.1689	3.6589	1.9805	5.3037	6.2759	19.4057	11.9839
1993	2.9533	2.6375	1.8747	1.8690	3.2448	3.6333	2.0358	5.3693	6.6056	19.5080	12.3833
1994	2.8341	2.5717	1.8394	1.8324	3.3066	3.6441	2.1177	5.4460	7.0024	19.8458	12.8801
1995	2.8187	2.5576	1.8556	1.8315	3.3815	3.7266	2.1735	5.5590	7.3497	20.7165	13.6384
1996	2.7929	2.5081	1.8589	1.8071	3.4409	3.8316	2.2587	5.6649	7.7718	21.7055	14.4473
1997	2.7557	2.4593	1.8298	1.7811	3.4986	3.9203	2.3602	5.8044	8.2573	22.7549	15.1093
1998	2.8072	2.4260	1.8194	1.7666	3.5435	4.1002	2.4628	5.9748	8.7270	24.4983	15.8779
1999	2.8901	2.3892	1.8238	1.7517	3.6019	4.3570	2.5774	6.1579	9.2835	26.8301	16.9316
2000	2.9467	2.3676	1.8310	1.7513	3.6894	4.5917	2.6788	6.3423	9.8830	29.1218	18.0958
2001	3.1136	2.4255	1.8936	1.8040	3.7842	4.8577	2.7015	6.5525	10.2228	31.8299	19.3582
2002	3.2383	2.4351	1.9447	1.8223	3.8403	5.1069	2.7458	6.6862	10.5445	34.1461	20.5064
2003	3.3142	2.4162	1.9546	1.8138	3.9155	5.3706	2.8223	6.8193	11.0505	36.6233	21.5995
2004	3.3793	2.3712	1.9401	1.7891	4.0174	5.7253	2.9307	6.9494	11.7738	39.7872	22.8429
2005	3.5831	2.3448	1.9947	1.7798	4.1478	6.3382	3.0296	7.1039	12.5662	45.0259	25.0654
2006	3.8226	2.3298	2.0900	1.7790	4.2705	7.0068	3.1151	7.2576	13.3031	50.8525	27.8036
2007	3.8916	2.3383	2.1529	1.7977	4.3949	7.3143	3.1753	7.4249	13.9551	54.3082	30.0444
2008	3.8704	2.3674	2.2322	1.8363	4.4793	7.3229	3.1806	7.5299	14.2470	55.1409	31.8022
2009	3.7803	2.4498	2.3120	1.9100	4.5224	6.9786	3.0969	7.5868	14.0055	52.9456	32.3805
2010	3.3556	2.3746	2.2034	1.8523	4.5712	6.4595	3.1806	7.5526	14.5389	48.7863	32.0357
2011	3.1544	2.3516	2.1634	1.8306	4.6638	6.2559	3.2279	7.5907	15.0542	47.4864	32.5689
2012	3.1142	2.3113	2.1415	1.7995	4.7516	6.4021	3.3000	7.6272	15.6800	48.8301	33.5791
2013	3.1600	2.2820	2.1286	1.7780	4.8257	6.6822	3.3717	7.6944	16.2709	51.4157	34.6347
2014	3.2395	2.2495	2.1260	1.7578	4.9093	7.0699	3.4555	7.7732	16.9642	54.9561	36.0651
2015	3.3230	2.2168	2.1346	1.7361	4.9627	7.4390	3.5531	7.8765	17.6328	58.5933	37.6390
2016	3.4054	2.2130	2.1481	1.7380	5.0205	7.7255	3.6126	7.9947	18.1370	61.7633	38.9607
2017	3.4418	2.1857	2.1408	1.7217	5.1113	8.0486	3.7007	8.0888	18.9154	65.1030	40.4937
2018	3.4327	2.1503	2.0995	1.6966	5.2221	8.3364	3.8094	8.1914	19.8929	68.2866	41.7653
2019	3.4471	2.1291	2.1031	1.6863	5.3104	8.5975	3.9033	8.3106	20.7281	71.4509	43.5929
2020	3.6529	2.2092	2.2017	1.7587	5.4071	8.9406	3.8119	8.4213	20.6116	75.2918	45.3816
2021	3.5798	2.1003	2.1163	1.6840	5.6460	9.6234	4.0268	8.4573	22.7353	81.3879	48.1146
2022	3.6618	2.0998	2.1038	1.6812	6.0547	10.5586	4.0939	8.5964	24.7871	90.7660	52.1475
Mean	3.2903	2.5802	1.9913	1.8223	3.3758	4.6218	2.3854	5.8789	9.3025	31.238	19.121

The real capital output ratio, Q^t_{KY}, fell from 3.12 in 1975 to 2.10 in 2022. This is probably a reflection of the growth in services relative to goods: services typically require less capital to produce a dollar of output. However, the nominal capital output ratio grew from 3.11 in 1970 to 3.95 in 1980 and then fell to 2.75 in 1997. It rose to 3.89 in 2007 (due to rising land prices) when the great recession hit the US economy. The capital output ratio fell to 3.11 in 2012 and then rose as a new property price bubble occurred, to finish at 3.66. However, overall, the US economy has not varied too much from the average nominal capital output ratio of 3.29 so that the economy has remained competitive over the years. The average share of reproducible capital in the total capital stock was 61.2% so the average share of land was 38.8%. *Thus land is a huge missing input in most macroeconomic production models of the US economy.*

We turn our attention to the construction of user costs for the US economy. User costs require depreciation rates for assets. Assets 11-17 are non-depreciable assets so we set the period t depreciation rate for these assets, δ^t_n, equal to 0 for $n = 11, \ldots, 17$ and $t = 1970, \ldots, 2022$. Since the APDB uses the geometric model for depreciation for reproducible assets 1-10, the APDB depreciation rates satisfy the following equations:

$$Q^{t+1}_{Kn} = (1 - \delta^t_n)Q^t_{Kn} + Q^t_{In}; \qquad n = 1, \ldots, 10; t = 1970, \ldots, 2022. \tag{A2}$$

We can solve equations (A1) for the depreciation rates δ^t_n for the years 1970-2022 for assets 1-10. The resulting nonzero depreciation rates are listed below in Table A7.

Table A7: Geometric Depreciation Rates for Reproducible US Capital Stocks

Year	δ_1^t	δ_2^t	δ_3^t	δ_4^t	δ_5^t	δ_6^t	δ_7^t	δ_8^t	δ_9^t	δ_{10}^t
1970	0.32975	0.29376	0.24427	0.19786	0.04956	0.08459	0.02627	0.22606	0.40849	0.30782
1971	0.35148	0.28745	0.24605	0.19606	0.04952	0.08456	0.02624	0.22084	0.40202	0.30844
1972	0.37737	0.28389	0.25102	0.19823	0.04997	0.08500	0.02668	0.21903	0.39933	0.31242
1973	0.37153	0.28976	0.25535	0.20074	0.04949	0.08453	0.02621	0.21765	0.40258	0.31057
1974	0.38259	0.28848	0.24979	0.20099	0.04958	0.08461	0.02630	0.21539	0.41114	0.31158
1975	0.36808	0.28417	0.24463	0.19751	0.04946	0.08450	0.02618	0.21392	0.40716	0.31043
1976	0.38981	0.28650	0.24788	0.19744	0.04936	0.08440	0.02608	0.21433	0.40421	0.32201
1977	0.40944	0.29739	0.25320	0.19873	0.04939	0.08442	0.02610	0.21443	0.39372	0.32460
1978	0.45837	0.29955	0.25416	0.20027	0.04936	0.08439	0.02607	0.21498	0.40475	0.31981
1979	0.45576	0.30234	0.25381	0.20095	0.04972	0.08475	0.02643	0.21536	0.40329	0.32448
1980	0.45305	0.30112	0.24557	0.19862	0.04959	0.08462	0.02631	0.21540	0.40774	0.31580
1981	0.44766	0.29613	0.24453	0.19819	0.04943	0.08446	0.02615	0.21550	0.39564	0.32414
1982	0.41377	0.29218	0.24183	0.19596	0.04951	0.08454	0.02622	0.21526	0.39419	0.32070
1983	0.42715	0.28933	0.24583	0.19569	0.04974	0.08477	0.02645	0.21548	0.38987	0.32077
1984	0.44165	0.29139	0.24962	0.19793	0.04954	0.08457	0.02625	0.21648	0.40196	0.32368
1985	0.41571	0.29183	0.24940	0.19848	0.04972	0.08475	0.02643	0.21733	0.40580	0.31980
1986	0.39545	0.29111	0.24612	0.19789	0.04936	0.08439	0.02608	0.21659	0.41022	0.32248
1987	0.39455	0.28839	0.24543	0.19739	0.04938	0.08441	0.02609	0.21605	0.41537	0.31845
1988	0.38201	0.29010	0.24798	0.19742	0.04936	0.08439	0.02607	0.21540	0.41128	0.31712
1989	0.38339	0.28808	0.24510	0.19965	0.05020	0.08523	0.02691	0.21516	0.41067	0.31778
1990	0.36724	0.28733	0.24475	0.19761	0.04938	0.08441	0.02609	0.21504	0.41252	0.31781
1991	0.36512	0.28424	0.24598	0.19581	0.04954	0.08457	0.02625	0.21473	0.41007	0.31667
1992	0.38152	0.28711	0.24879	0.19733	0.05108	0.08611	0.02780	0.21387	0.40743	0.31760
1993	0.38386	0.28642	0.25210	0.19750	0.05015	0.08518	0.02687	0.21278	0.40285	0.31853
1994	0.38456	0.29209	0.25631	0.19891	0.05085	0.08588	0.02757	0.21218	0.40057	0.31915
1995	0.40447	0.29446	0.25424	0.19896	0.05006	0.08509	0.02678	0.21241	0.40135	0.32098
1996	0.41261	0.29574	0.25313	0.19869	0.04962	0.08465	0.02633	0.21335	0.40962	0.32180
1997	0.41772	0.29690	0.25352	0.19884	0.04971	0.08474	0.02643	0.21380	0.41837	0.32027
1998	0.42032	0.29872	0.25470	0.19960	0.04976	0.08479	0.02647	0.21428	0.41986	0.32041
1999	0.41970	0.30294	0.25840	0.19908	0.04976	0.08479	0.02648	0.21479	0.42356	0.31905
2000	0.40019	0.30928	0.25236	0.19941	0.04942	0.08445	0.02613	0.21528	0.41840	0.31981
2001	0.37753	0.29582	0.24745	0.19829	0.04962	0.08466	0.02634	0.21473	0.41047	0.31609
2002	0.36918	0.28337	0.24496	0.19758	0.04956	0.08459	0.02628	0.21357	0.40653	0.31725
2003	0.36860	0.28847	0.24342	0.19832	0.04977	0.08481	0.02649	0.21327	0.40735	0.31924
2004	0.36962	0.29293	0.24989	0.19966	0.05082	0.08586	0.02754	0.21297	0.40867	0.31786
2005	0.37194	0.29497	0.25528	0.20334	0.05329	0.08832	0.03000	0.21344	0.40838	0.32008
2006	0.37632	0.29594	0.25253	0.20002	0.04945	0.08448	0.02616	0.21396	0.40767	0.31822
2007	0.37450	0.29954	0.24987	0.19974	0.04946	0.08450	0.02618	0.21420	0.40940	0.31551
2008	0.37083	0.29794	0.24345	0.19989	0.05041	0.08544	0.02713	0.21383	0.40849	0.31204
2009	0.36520	0.29100	0.23059	0.19536	0.04952	0.08455	0.02624	0.21264	0.40584	0.31056
2010	0.36565	0.29857	0.24438	0.19637	0.04980	0.08483	0.02652	0.21220	0.40408	0.31512
2011	0.35664	0.29609	0.25214	0.19813	0.05016	0.08519	0.02688	0.21237	0.40654	0.31436
2012	0.35811	0.29525	0.25582	0.19937	0.05054	0.08557	0.02725	0.21194	0.40785	0.31433
2013	0.35198	0.29629	0.25546	0.19833	0.04954	0.08458	0.02626	0.21262	0.40670	0.31388
2014	0.34893	0.29787	0.25603	0.19864	0.04944	0.08447	0.02616	0.21282	0.40708	0.31492
2015	0.34783	0.30106	0.25705	0.19764	0.04956	0.08460	0.02628	0.21308	0.40654	0.31585
2016	0.34730	0.30211	0.25191	0.19729	0.04984	0.08487	0.02656	0.21410	0.40974	0.31665
2017	0.35223	0.30378	0.25229	0.19984	0.05184	0.08687	0.02855	0.21395	0.41185	0.31686
2018	0.35618	0.29604	0.25179	0.19817	0.04988	0.08491	0.02660	0.21432	0.41475	0.31706
2019	0.35370	0.29156	0.24906	0.19811	0.04961	0.08464	0.02633	0.21503	0.41238	0.31626
2020	0.35778	0.28892	0.24064	0.19699	0.05000	0.08503	0.02672	0.21460	0.41232	0.31326
2021	0.36031	0.29158	0.24103	0.19828	0.05050	0.08553	0.02722	0.21546	0.41547	0.31228
2022	0.36031	0.29158	0.24103	0.19828	0.05050	0.08553	0.02722	0.21546	0.41547	0.31228
Mean	0.38427	0.29356	0.24909	0.19840	0.04988	0.08491	0.02660	0.21479	0.40769	0.31707

We now explain the user cost of capital methodology developed by Jorgenson and his coworkers.*34 Suppose the beginning of the year t price of a new unit of capital stock n is P_{Kn}^t and a producer faces an annual cost of capital at the beginning of year t of r^t. Thus the cost to the producer is the total of the purchase of one unit of capital stock n at the beginning of year t plus the interest cost of financing the purchase, which is equal to $(1+r^t)P_{Kn}^t$. Suppose further that asset n in year t has a geometric depreciation rate equal to δ_n^t. Various levels of government may charge an annual tax on the use of each unit of asset n at the tax rate τ_n^t at the beginning of year t so that the total tax charged on asset n during year t is $\tau_n^t P_{Kn}^t Q_{Kn}^t$. We think of firms buying their capital stocks at the beginning of year t, using the services of these stocks during period t and then selling the used assets at the end of the period at prevailing prices at the end of the period. The year t user cost U_n^t of asset n is the resulting per unit net cost.*35 Using the geometric model of depreciation, we can use the price of a new unit of capital stock n at the beginning of year t, the P_{Kn}^t, listed in Table A4 as the appropriate price of the year t beginning of the year capital stock quantity Q_{Kn}^t for asset n. Define the year t *ex post inflation rate* for asset n, i_n^t, as follows where P_{Kn}^t is the beginning of year t price for a unit of asset n:

$$(1+i_n^t) \equiv P_{Kn}^{t+1}/P_{Kn}^t; \qquad n=1,\ldots,17; t=1970,\ldots,2022. \tag{A3}$$

Thus the year t user cost for asset n, U_n^t, times the initial year t capital stock, Q_{Kn}^t, is equal to the following expression:

$$\begin{aligned} U_n^t Q_{Kn}^t &\equiv \text{Beginning of period cost} - \text{end of period benefit} \\ &\qquad\qquad\qquad\qquad\qquad n=1,\ldots,17; t=1970,\ldots,2022 \\ &= \{(1+r^t)P_{Kn}^t Q_{Kn}^t + \tau_n^t P_{Kn}^t Q_{Kn}^t\} - P_{Kn}^{t+1}(1-\delta_n^t)Q_{Kn}^t \\ &= [1+r^t+\tau_n^t]P_{Kn}^t Q_{Kn}^t - P_{Kn}^t(1+i_n^t)(1-\delta_n^t)Q_{Kn}^t \qquad \text{using (A3)} \\ &= [r^t+\tau_n^t-i_n^t+\delta_n^t(1+i_n^t)]P_{Kn}^t Q_{Kn}^t. \end{aligned} \tag{A4}$$

Thus the year t Jorgensonian user cost for asset n is $U_n^t = [r^t+\tau_n^t-i_n^t+\delta_n^t(1+i_n^t)]P_{Kn}^t$.*36 Note that the end of period benefit part of the user cost formula, $P_{Kn}^t(1+i_n^t)(1-\delta_n^t)Q_{Kn}^t$, *decreases* as the depreciation rate δ_n^t increases and *increases* as the asset inflation rate i_n^t increases. For land assets, where the depreciation rate is zero, the annual asset inflation rate could be so great that the overall user cost becomes negative instead of being positive. In this case, the user cost becomes a *user benefit*; i.e., instead of the purchase of the asset, using it and then selling it at the end of the period being a cost, it becomes an addition to revenues.

In order to calculate user costs, we require information on the cost of capita r^t, asset tax rates τ_n^t, asset inflation rates i_n^t, depreciation rates δ_n^t and beginning of the year asset prices P_{Kn}^t. Alternative methods for estimating the cost of capital will be discussed below, asset tax rates will be listed in Table A8 below, ex post asset inflation rates as well as asset prices can be calculated using the information in Table A4 on beginning of the year asset prices and depreciation rates are listed in Table A7 above.

*34 See Jorgenson (1963)[29], Jorgenson and Griliches (1967)[31], (1972)[32], Christensen and Jorgenson (1969)[7] and Jorgenson (1989)[30].

*35 This is the method used by Diewert (1980, 472–473)[14] (2014)[16] to derive a user cost formula.

*36 This user cost formula was also derived by Christensen and Jorgenson (1969)[7] using a different method of derivation. Jorgenson was a firm advocate of using ex post asset inflation rates in a user cost formula so we term the user costs defined by (A4) as Jorgensonian user costs.

Table A8: Asset Tax Rates (Percentages)

Year	τ_1^t	τ_2^t	τ_3^t	τ_4^t	τ_5^t	τ_6^t	τ_9^t	τ_{10}^t	τ_{11}^t	τ_{13}^t	τ_{14}^t	τ_{15}^t	τ_{16}^t	τ_{17}^t
1970	0.653	0.653	0.493	0.762	2.105	2.151	0.247	0.247	1.085	1.034	1.299	1.875	0.278	0.248
1971	0.529	0.661	0.499	0.774	2.165	2.257	0.255	0.259	1.136	1.064	1.336	2.058	0.302	0.268
1972	0.579	0.653	0.487	0.752	2.118	2.180	0.260	0.246	1.129	1.050	1.318	1.899	0.287	0.260
1973	0.580	0.620	0.465	0.727	2.065	2.095	0.245	0.238	1.143	1.001	1.257	1.837	0.283	0.255
1974	0.521	0.588	0.444	0.692	1.906	1.951	0.219	0.218	0.993	0.926	1.163	1.768	0.260	0.225
1975	0.502	0.548	0.418	0.662	1.768	1.808	0.215	0.207	0.925	0.889	1.116	1.700	0.247	0.190
1976	0.478	0.538	0.412	0.636	1.739	1.760	0.206	0.205	0.955	0.862	1.082	1.569	0.238	0.213
1977	0.469	0.503	0.400	0.616	1.724	1.742	0.210	0.198	0.898	0.833	1.046	1.587	0.234	0.208
1978	0.359	0.454	0.348	0.539	1.519	1.537	0.168	0.175	0.805	0.738	0.927	1.404	0.209	0.180
1979	0.348	0.388	0.301	0.467	1.304	1.330	0.162	0.149	0.692	0.641	0.805	1.228	0.181	0.137
1980	0.317	0.376	0.287	0.449	1.235	1.264	0.144	0.141	0.636	0.605	0.760	1.063	0.161	0.124
1981	0.354	0.393	0.299	0.462	1.267	1.299	0.165	0.148	0.628	0.622	0.781	1.090	0.163	0.131
1982	0.378	0.408	0.309	0.477	1.301	1.341	0.163	0.153	0.636	0.671	0.843	1.090	0.163	0.130
1983	0.373	0.417	0.315	0.489	1.339	1.377	0.174	0.159	0.684	0.654	0.822	1.213	0.179	0.153
1984	0.386	0.439	0.331	0.508	1.414	1.448	0.168	0.167	0.662	0.667	0.837	1.298	0.187	0.169
1985	0.403	0.444	0.336	0.515	1.430	1.472	0.172	0.170	0.672	0.699	0.878	1.343	0.194	0.148
1986	0.404	0.438	0.335	0.521	1.446	1.466	0.163	0.169	0.688	0.681	0.856	1.364	0.199	0.161
1987	0.391	0.427	0.324	0.505	1.413	1.434	0.161	0.165	0.730	0.717	0.881	1.339	0.198	0.158
1988	0.409	0.424	0.323	0.504	1.398	1.431	0.169	0.164	0.722	0.704	0.865	1.299	0.193	0.152
1989	0.420	0.435	0.333	0.515	1.426	1.457	0.175	0.169	0.736	0.712	0.872	1.305	0.195	0.150
1990	0.427	0.450	0.344	0.535	1.466	1.506	0.171	0.174	0.762	0.731	0.902	1.295	0.193	0.161
1991	0.465	0.492	0.377	0.584	1.586	1.638	0.188	0.191	0.820	0.793	0.947	1.361	0.204	0.172
1992	0.474	0.513	0.394	0.609	1.681	1.705	0.197	0.197	0.873	0.832	0.953	1.468	0.220	0.193
1993	0.478	0.515	0.396	0.610	1.714	1.740	0.206	0.198	0.896	0.841	0.950	1.467	0.220	0.190
1994	0.504	0.533	0.412	0.632	1.768	1.799	0.207	0.207	0.912	0.871	1.038	1.553	0.231	0.204
1995	0.478	0.517	0.402	0.619	1.731	1.762	0.202	0.202	0.899	0.858	1.076	1.489	0.224	0.179
1996	0.460	0.527	0.405	0.626	1.738	1.772	0.199	0.205	0.908	0.874	1.070	1.553	0.231	0.181
1997	0.461	0.528	0.400	0.620	1.738	1.775	0.204	0.203	0.905	0.872	1.096	1.542	0.230	0.197
1998	0.421	0.480	0.372	0.577	1.617	1.660	0.189	0.187	0.839	0.817	1.126	1.491	0.222	0.181
1999	0.415	0.458	0.359	0.555	1.560	1.597	0.184	0.183	0.814	0.787	0.987	1.439	0.214	0.171
2000	0.423	0.445	0.344	0.530	1.496	1.525	0.177	0.175	0.783	0.752	0.910	1.401	0.208	0.158
2001	0.390	0.419	0.327	0.506	1.423	1.445	0.163	0.165	0.738	0.713	0.913	1.303	0.195	0.145
2002	0.405	0.423	0.329	0.511	1.422	1.455	0.160	0.165	0.727	0.717	0.822	1.344	0.199	0.159
2003	0.397	0.388	0.318	0.494	1.389	1.401	0.155	0.160	0.704	0.688	0.880	1.288	0.192	0.160
2004	0.386	0.381	0.305	0.472	1.339	1.356	0.151	0.152	0.721	0.650	0.802	1.247	0.186	0.157
2005	0.348	0.362	0.283	0.439	1.238	1.272	0.141	0.141	0.675	0.595	0.818	1.169	0.175	0.132
2006	0.338	0.350	0.274	0.426	1.200	1.236	0.137	0.138	0.638	0.574	0.774	1.068	0.160	0.141
2007	0.331	0.338	0.271	0.411	1.250	1.223	0.139	0.140	0.637	0.593	0.773	1.084	0.163	0.130
2008	0.316	0.312	0.253	0.384	1.367	1.192	0.124	0.123	0.664	0.618	0.781	1.137	0.141	0.114
2009	0.330	0.327	0.267	0.382	1.560	1.342	0.135	0.136	0.752	0.723	0.820	1.358	0.171	0.127
2010	0.356	0.344	0.281	0.391	1.674	1.512	0.144	0.145	0.818	0.835	0.838	1.612	0.153	0.147
2011	0.335	0.332	0.260	0.373	1.782	1.488	0.133	0.132	0.788	0.842	1.222	1.610	0.158	0.124
2012	0.323	0.313	0.249	0.356	1.788	1.485	0.131	0.131	0.734	0.832	1.123	1.768	0.166	0.125
2013	0.325	0.315	0.252	0.354	1.698	1.489	0.137	0.136	0.662	0.808	1.055	1.767	0.184	0.131
2014	0.327	0.309	0.256	0.354	1.656	1.449	0.142	0.142	0.645	0.790	1.047	1.601	0.173	0.135
2015	0.322	0.304	0.254	0.350	1.591	1.431	0.141	0.142	0.628	0.765	0.984	1.536	0.190	0.125
2016	0.324	0.307	0.257	0.338	1.583	1.384	0.143	0.144	0.650	0.763	0.947	1.505	0.148	0.108
2017	0.314	0.304	0.245	0.328	1.571	1.346	0.131	0.132	0.661	0.752	0.924	1.490	0.116	0.118
2018	0.306	0.296	0.242	0.320	1.532	1.327	0.129	0.131	0.650	0.734	0.893	1.426	0.116	0.117
2019	0.307	0.301	0.243	0.326	1.529	1.369	0.128	0.130	0.677	0.743	0.884	1.433	0.124	0.120
2020	0.323	0.319	0.250	0.341	1.485	1.304	0.134	0.136	0.658	0.702	0.883	1.391	0.139	0.096
2021	0.302	0.294	0.232	0.319	1.406	1.231	0.112	0.114	0.617	0.662	0.805	1.295	0.122	0.113
2022	0.293	0.286	0.231	0.312	1.301	1.167	0.114	0.118	0.585	0.595	0.721	1.196	0.102	0.102
Mean	0.401	0.426	0.331	0.501	1.565	1.532	0.169	0.168	0.773	0.764	0.953	1.434	0.193	0.160

In order to save space, the tax rates listed in Table A8 are in percentage terms; thus the actual 1970 tax rate for Asset 1 is 0.00653 instead of 0.653. The tax rates for assets 7 (Other Structures), 8 (Research and Development) and 12 (Forest Land) were zero for all years.

The economy wide ex post rate of return in year t, r_J^t, can be calculated using Jorgensonian user costs: simply set the value of output in year t, V_Y^t, less the value of labour input, V_L^t, equal to the sum of the year t user cost values defined by (A4):

$$V_Y^t = V_L^t + \sum_{n=1}^{17} [r_J^t + \tau_n^t - i_n^t + \delta_n^t (1 + i_n^t)] P_{Kn}^t Q_{Kn}^t; \qquad t = 1970, \ldots, 2022. \tag{A5}$$

The r_J^t are listed in Table A9 below. Once the ex post rates of return are available, Jorgensonian user cost can be calculated. However, the resulting user costs for land assets were extremely volatile and all land assets had negative Jorgensonian user costs.*[37] But when we aggregated the user costs over all 17 assets using a Törnqvist quantity index to do the aggregation, we found that the resulting capital services aggregate was fairly smooth and well behaved. The aggregate user cost of capital was always positive. Thus the problem of negative user costs seems to arise in a visible manner only if we want to estimate production functions that distinguish different types of capital where land is included as an asset.

We would like user costs to approximate rental prices for assets. But rental prices are always positive so Jorgensonian user costs are not good approximations to rental prices when land assets are in scope. Owners of assets who rent them to producers and determine the rental prices at the beginning of the year are not able to accurately forecast the end of year prices for their assets. Thus they form expectations about year end asset prices when determining rental prices. This suggests that we should use *expected inflation rates* in the user cost formulae in place of actual ex post prices. The problem is that there are many methods for forming expectations. Our approach to this problem is to simply smooth the actual ex post asset inflation rates by a more or less arbitrary method and use these smoothed rates in place of the actual rates. We use a centered 7 year moving average of actual asset inflation rates as an approximation to expected asset inflation rates. Denote this smoothed inflation rate for asset n in year t as $i_n^{t^*}$.*[38] These smoothed inflation rates are listed in Table A9 below. Now replace the i_n^t in equations (A5) by their smoothed counterparts $i_n^{t^*}$, replace r_J^t by r^t and solve the resulting equation for r^t. The r^t are also listed in Table A9.

*[37] Here is a list of the land and inventory assets (11 to 17) and the number of negative user costs: 11: 10; 12: 4; 13: 0; 14: 8; 15: 16; 21; 17: 9. Thus there was a total of 68 negative user costs. The user costs for assets 1-10 were all positive.

*[38] For 1970, set $i_n^{1970^*} = (1/2)(i_n^{1970} + i_n^{1971})$, for 1971, use the 3 year centered moving average $i_n^{1971^*} = (1/3)(i_n^{1970} + i_n^{1971} + i_n^{1972})$ and for year 3, use a 5 year centered average. Similar adjustments to the 7 year centered moving average rule were made for the final 3 years in our sample.

Table A9: Alternative Rates of Return and Smoothed Asset Inflation Rates

Year	r^t	r_J^t	$i_1^{t^*}$	$i_2^{t^*}$	$i_3^{t^*}$	$i_4^{t^*}$	$i_5^{t^*}$	$i_6^{t^*}$	$i_7^{t^*}$	$i_8^{t^*}$
1970	0.1002	0.0854	-0.2119	0.0195	0.0159	0.0168	0.0402	0.0666	0.0497	0.0297
1971	0.1138	0.1216	-0.1886	0.0232	0.0153	0.0189	0.0515	0.0690	0.0557	0.0319
1972	0.1396	0.1307	-0.1591	0.0455	0.0432	0.0569	0.0721	0.0825	0.0858	0.0498
1973	0.1436	0.1633	-0.1534	0.0442	0.0569	0.0694	0.0737	0.0768	0.0740	0.0521
1974	0.1450	0.1857	-0.1635	0.0432	0.0661	0.0780	0.0856	0.0821	0.0758	0.0569
1975	0.1504	0.1527	-0.1636	0.0428	0.0745	0.0866	0.0943	0.0848	0.0782	0.0604
1976	0.1569	0.1258	-0.1716	0.0446	0.0856	0.0983	0.0996	0.0903	0.0778	0.0666
1977	0.1559	0.1497	-0.1805	0.0469	0.0920	0.1034	0.0989	0.0911	0.0711	0.0694
1978	0.1488	0.1748	-0.1797	0.0434	0.0880	0.0935	0.0925	0.0861	0.0683	0.0678
1979	0.1366	0.1589	-0.1801	0.0398	0.0808	0.0832	0.0861	0.0833	0.0620	0.0666
1980	0.1198	0.1249	-0.1829	0.0431	0.0741	0.0764	0.0787	0.0806	0.0509	0.0646
1981	0.1104	0.1154	-0.1748	0.0465	0.0671	0.0676	0.0665	0.0732	0.0447	0.0606
1982	0.0924	0.0778	-0.1581	0.0445	0.0602	0.0601	0.0555	0.0637	0.0453	0.0546
1983	0.0915	0.0548	-0.1512	0.0383	0.0498	0.0498	0.0469	0.0527	0.0456	0.0456
1984	0.0980	0.0754	-0.1425	0.0266	0.0366	0.0378	0.0393	0.0437	0.0347	0.0368
1985	0.0983	0.1042	-0.1380	0.0152	0.0272	0.0301	0.0357	0.0384	0.0228	0.0291
1986	0.0971	0.1144	-0.1323	0.0080	0.0238	0.0270	0.0349	0.0361	0.0254	0.0228
1987	0.0949	0.1122	-0.1208	0.0042	0.0241	0.0290	0.0334	0.0355	0.0347	0.0193
1988	0.0914	0.1038	-0.1134	0.0001	0.0249	0.0320	0.0321	0.0332	0.0379	0.0177
1989	0.0855	0.0923	-0.1142	-0.0044	0.0235	0.0300	0.0310	0.0304	0.0267	0.0164
1990	0.0751	0.0620	-0.1128	-0.0073	0.0242	0.0266	0.0293	0.0294	0.0173	0.0158
1991	0.0680	0.0484	-0.1159	-0.0104	0.0271	0.0244	0.0287	0.0283	0.0243	0.0166
1992	0.0695	0.0513	-0.1349	-0.0147	0.0258	0.0210	0.0271	0.0254	0.0323	0.0161
1993	0.0758	0.0675	-0.1574	-0.0175	0.0220	0.0167	0.0265	0.0236	0.0305	0.0161
1994	0.0880	0.0925	-0.1778	-0.0226	0.0174	0.0115	0.0290	0.0244	0.0306	0.0161
1995	0.0986	0.1003	-0.1948	-0.0310	0.0130	0.0074	0.0320	0.0278	0.0348	0.0155
1996	0.1086	0.0982	-0.1989	-0.0360	0.0111	0.0058	0.0337	0.0313	0.0389	0.0177
1997	0.1167	0.1218	-0.2003	-0.0395	0.0089	0.0056	0.0349	0.0328	0.0430	0.0185
1998	0.1164	0.1335	-0.2020	-0.0431	0.0058	0.0057	0.0348	0.0333	0.0443	0.0157
1999	0.1180	0.1219	-0.1900	-0.0525	0.0042	0.0062	0.0363	0.0341	0.0420	0.0156
2000	0.1188	0.1210	-0.1689	-0.0674	0.0046	0.0074	0.0414	0.0361	0.0470	0.0172
2001	0.1213	0.1095	-0.1480	-0.0734	0.0070	0.0100	0.0471	0.0407	0.0597	0.0183
2002	0.1277	0.1098	-0.1307	-0.0726	0.0090	0.0129	0.0514	0.0463	0.0723	0.0194
2003	0.1288	0.1251	-0.1211	-0.0745	0.0087	0.0154	0.0503	0.0505	0.0842	0.0185
2004	0.1219	0.1656	-0.1138	-0.0807	0.0082	0.0182	0.0435	0.0521	0.0908	0.0193
2005	0.1042	0.1615	-0.1036	-0.0872	0.0097	0.0197	0.0350	0.0518	0.0845	0.0199
2006	0.0827	0.0979	-0.0941	-0.0862	0.0116	0.0191	0.0269	0.0484	0.0755	0.0186
2007	0.0643	0.0529	-0.0857	-0.0791	0.0130	0.0187	0.0181	0.0431	0.0720	0.0183
2008	0.0532	0.0036	-0.0751	-0.0767	0.0144	0.0186	0.0090	0.0365	0.0636	0.0184
2009	0.0515	-0.0205	-0.0588	-0.0782	0.0143	0.0167	0.0040	0.0277	0.0471	0.0168
2010	0.0674	0.0337	-0.0426	-0.0805	0.0147	0.0141	0.0068	0.0208	0.0344	0.0145
2011	0.0791	0.0926	-0.0284	-0.0827	0.0159	0.0111	0.0133	0.0174	0.0286	0.0130
2012	0.0901	0.1147	-0.0171	-0.0837	0.0157	0.0083	0.0214	0.0158	0.0263	0.0109
2013	0.1049	0.1301	-0.0101	-0.0829	0.0143	0.0075	0.0301	0.0184	0.0281	0.0096
2014	0.1124	0.1219	-0.0050	-0.0810	0.0135	0.0072	0.0378	0.0231	0.0252	0.0100
2015	0.1105	0.1052	-0.0031	-0.0808	0.0123	0.0061	0.0429	0.0270	0.0188	0.0100
2016	0.1079	0.1064	-0.0051	-0.0782	0.0118	0.0063	0.0432	0.0311	0.0170	0.0132
2017	0.1081	0.0979	-0.0057	-0.0724	0.0138	0.0089	0.0451	0.0335	0.0167	0.0177
2018	0.1155	0.0950	-0.0018	-0.0611	0.0211	0.0164	0.0562	0.0475	0.0203	0.0202
2019	0.1179	0.1027	0.0022	-0.0463	0.0258	0.0218	0.0609	0.0579	0.0245	0.0230
2020	0.1147	0.1294	0.0035	-0.0353	0.0325	0.0287	0.0666	0.0705	0.0296	0.0277
2021	0.1396	0.1549	0.0157	-0.0232	0.0446	0.0396	0.0860	0.0849	0.0417	0.0284
2022	0.1438	0.1257	0.0259	-0.0144	0.0554	0.0491	0.0948	0.1077	0.0493	0.0252
Mean	0.1074	0.1068	-0.1139	-0.0237	0.0298	0.0313	0.0470	0.0487	0.0487	0.0281

Year	$i_9^{t^*}$	$i_{10}^{t^*}$	$i_{11}^{t^*}$	$i_{12}^{t^*}$	$i_{13}^{t^*}$	$i_{14}^{t^*}$	$i_{15}^{t^*}$	$i_{16}^{t^*}$	$i_{17}^{t^*}$
1970	0.0681	0.0516	0.0801	0.0559	0.0451	0.0451	0.1186	0.1188	-0.0352
1971	0.0895	0.0469	0.1228	0.0618	0.0525	0.0525	0.1058	0.1296	0.0028
1972	0.0954	0.0513	0.1404	0.0837	0.0705	0.0705	0.1468	0.1604	0.0677
1973	0.1041	0.0548	0.1484	0.0870	0.0762	0.0762	0.1499	0.1638	0.0530
1974	0.1114	0.0571	0.1644	0.0896	0.0836	0.0836	0.1634	0.1788	0.0584
1975	0.1133	0.0572	0.1747	0.0854	0.0893	0.0893	0.1782	0.1909	0.0821
1976	0.1180	0.0581	0.1653	0.0823	0.0953	0.0953	0.1858	0.1916	0.0902
1977	0.1390	0.0591	0.1445	0.0806	0.0981	0.0981	0.1670	0.1723	0.0778
1978	0.1583	0.0567	0.1203	0.0685	0.0976	0.0976	0.1244	0.1374	0.0691
1979	0.1739	0.0524	0.0909	0.0524	0.0927	0.0927	0.0970	0.1094	0.0645
1980	0.1767	0.0511	0.0578	0.0430	0.0803	0.0803	0.0890	0.0930	0.0612
1981	0.1738	0.0488	0.0144	0.0346	0.0659	0.0659	0.0774	0.0728	0.0725
1982	0.1594	0.0472	-0.0300	0.0288	0.0496	0.0496	0.0674	0.0561	0.0630
1983	0.1325	0.0469	-0.0543	0.0224	0.0404	0.0369	0.0759	0.0579	0.0305
1984	0.1047	0.0449	-0.0548	0.0160	0.0425	0.0317	0.0958	0.0734	0.0289
1985	0.0835	0.0442	-0.0446	0.0180	0.0369	0.0189	0.1169	0.0937	0.0162
1986	0.0592	0.0447	-0.0349	0.0264	0.0342	0.0098	0.1252	0.1034	0.0278
1987	0.0402	0.0461	-0.0209	0.0350	0.0442	0.0098	0.1101	0.0963	0.0260
1988	0.0342	0.0451	0.0035	0.0407	0.0533	-0.0003	0.0875	0.0856	0.0076
1989	0.0385	0.0401	0.0290	0.0501	0.0601	-0.0194	0.0595	0.0682	0.0106
1990	0.0502	0.0367	0.0409	0.0593	0.0595	-0.0382	0.0297	0.0442	0.0270
1991	0.0492	0.0349	0.0410	0.0610	0.0513	-0.0469	0.0051	0.0220	0.0173
1992	0.0321	0.0313	0.0420	0.0569	0.0483	-0.0469	-0.0120	0.0059	0.0207
1993	0.0226	0.0270	0.0434	0.0508	0.0497	-0.0428	-0.0155	0.0021	0.0127
1994	0.0244	0.0205	0.0468	0.0486	0.0549	-0.0092	-0.0018	0.0149	0.0064
1995	0.0265	0.0172	0.0509	0.0478	0.0625	0.0326	0.0201	0.0353	0.0035
1996	0.0264	0.0200	0.0533	0.0413	0.0710	0.0609	0.0444	0.0578	-0.0037
1997	0.0214	0.0208	0.0554	0.0342	0.0788	0.0881	0.0659	0.0797	-0.0037
1998	0.0127	0.0186	0.0535	0.0275	0.0844	0.0907	0.0873	0.1008	-0.0005
1999	0.0059	0.0165	0.0483	0.0216	0.0864	0.0862	0.1096	0.1218	-0.0094
2000	0.0018	0.0149	0.0546	0.0187	0.0841	0.0874	0.1289	0.1416	-0.0316
2001	-0.0022	0.0147	0.0749	0.0168	0.0770	0.0746	0.1477	0.1614	-0.0079
2002	-0.0060	0.0135	0.0908	0.0177	0.0654	0.0745	0.1501	0.1656	0.0199
2003	-0.0130	0.0102	0.0892	0.0202	0.0500	0.0883	0.1279	0.1457	0.0027
2004	-0.0150	0.0074	0.0804	0.0227	0.0319	0.0942	0.0893	0.1079	0.0177
2005	-0.0103	0.0061	0.0763	0.0269	0.0127	0.0851	0.0417	0.0615	0.0188
2006	-0.0060	0.0038	0.0808	0.0277	-0.0061	0.0416	-0.0004	0.0220	0.0167
2007	-0.0026	0.0009	0.0812	0.0245	-0.0224	0.0049	-0.0407	-0.0149	0.0536
2008	-0.0004	0.0004	0.0736	0.0210	-0.0344	0.0101	-0.0808	-0.0510	0.0473
2009	0.0006	-0.0002	0.0683	0.0169	-0.0402	-0.0083	-0.0811	-0.0539	0.0221
2010	0.0015	-0.0010	0.0697	0.0140	-0.0390	-0.0118	-0.0570	-0.0351	0.0383
2011	-0.0001	0.0006	0.0719	0.0120	-0.0309	-0.0059	-0.0249	-0.0076	0.0169
2012	-0.0002	0.0034	0.0723	0.0102	-0.0174	0.0162	0.0148	0.0277	0.0224
2013	0.0035	0.0068	0.0698	0.0110	-0.0008	0.0707	0.0476	0.0564	0.0421
2014	0.0035	0.0104	0.0590	0.0132	0.0161	0.1098	0.0759	0.0789	0.0042
2015	0.0001	0.0120	0.0410	0.0143	0.0306	0.0910	0.0959	0.0919	0.0026
2016	-0.0034	0.0135	0.0264	0.0137	0.0407	0.0937	0.0874	0.0822	0.0006
2017	-0.0071	0.0157	0.0267	0.0110	0.0448	0.0987	0.0809	0.0771	0.0364
2018	-0.0091	0.0198	0.0376	0.0093	0.0434	0.0977	0.0897	0.0869	0.0031
2019	-0.0096	0.0214	0.0431	0.0082	0.0376	0.0939	0.0889	0.0869	0.0404
2020	-0.0124	0.0239	0.0510	0.0080	0.0291	0.1020	0.0922	0.0906	0.0626
2021	-0.0128	0.0313	0.0728	0.0086	0.0166	0.1310	0.1124	0.1111	0.0996
2022	-0.0102	0.0374	0.0727	0.0115	0.0113	0.1156	0.1136	0.1137	0.0289
Mean	0.0441	0.0286	0.0599	0.0353	0.0444	0.0531	0.0750	0.0827	0.0283

The mean ex post rate of return on US assets over the 53 year sample period is 10.68%. The mean rate of return on assets when we used smoothed asset inflation rates in the user cost formulae is 10.74%. Both average rates of return are quite high. Note that the ex post rate of return in 2009 was −2.05%. The smoothed rates of return r^t are probably much closer to a realistic cost of capital for US firms. Note that the average smoothed asset inflation rates for assets 11 (Agricultural Land), 14 (Commercial Land) and 15 (Residential Land) are 5.99%, 5.31% and 7.50% respectively. These rates of return on holding land are substantial but well below the overall average rate of return on US assets.[*39]

The smoothed year t user cost for asset n is defined as $U_n^t \equiv [r^t + \tau_n^t - i_n^{t^*} + \delta_n^t(1 + i_n^t)]P_{Kn}^t$ and is listed in Table A10.

[*39] However, our accounting framework does not include monetary holdings and natural resource stocks as assets and this omission means that our ex post rates of return on assets are too high.

Table A10: User Costs Using Smoothed Asset Inflation Rates

Year	U_1^t	U_2^t	U_3^t	U_4^t	U_5^t	U_6^t	U_7^t	U_8^t	U_9^t
1970	0.57851	0.38672	0.33739	0.29218	0.13254	0.14531	0.07803	0.30329	0.47085
1971	0.49704	0.39464	0.35715	0.30626	0.13948	0.16541	0.08901	0.31661	0.47791
1972	0.38463	0.41305	0.37479	0.30988	0.15392	0.19443	0.09125	0.33902	0.55168
1973	0.32763	0.43720	0.37807	0.31329	0.16700	0.21838	0.11507	0.35213	0.62755
1974	0.29572	0.46290	0.38580	0.33009	0.16903	0.23250	0.13000	0.37017	0.68831
1975	0.26069	0.51028	0.42217	0.37371	0.18089	0.26044	0.15107	0.40350	0.77458
1976	0.24059	0.55899	0.46864	0.41708	0.19689	0.28039	0.17055	0.43228	0.87249
1977	0.20838	0.57385	0.50493	0.44116	0.21113	0.29228	0.18504	0.45023	0.93196
1978	0.16735	0.57067	0.53720	0.47616	0.22829	0.30877	0.18653	0.46701	0.99168
1979	0.12092	0.57643	0.56693	0.50750	0.23888	0.31812	0.19092	0.47871	1.05592
1980	0.09448	0.57141	0.58026	0.53209	0.24131	0.32003	0.19099	0.49071	1.19257
1981	0.07514	0.58765	0.62721	0.58452	0.26991	0.34617	0.20023	0.52148	1.39159
1982	0.06171	0.59762	0.64766	0.59921	0.26801	0.34923	0.17781	0.53347	1.70687
1983	0.05478	0.62902	0.70973	0.65213	0.29900	0.39454	0.17782	0.57707	2.16496
1984	0.04599	0.67838	0.78079	0.71778	0.34565	0.45191	0.20854	0.63069	2.70736
1985	0.03644	0.71159	0.81913	0.74447	0.36514	0.48349	0.23765	0.66402	2.97896
1986	0.02995	0.72736	0.83954	0.76761	0.37728	0.50304	0.24726	0.68402	3.15736
1987	0.02504	0.72302	0.84749	0.78024	0.39349	0.51471	0.23375	0.69583	3.17802
1988	0.02124	0.71902	0.84839	0.78312	0.40161	0.52869	0.21805	0.70561	3.26192
1989	0.01957	0.70765	0.84676	0.80166	0.40395	0.54208	0.24026	0.70839	3.56153
1990	0.01713	0.69145	0.84211	0.80003	0.38386	0.52612	0.24160	0.69144	3.66842
1991	0.01524	0.67906	0.84741	0.80997	0.37100	0.52562	0.20593	0.68079	3.64493
1992	0.01408	0.69063	0.89079	0.85543	0.39587	0.55845	0.19793	0.69382	3.77968
1993	0.01243	0.69235	0.94776	0.90163	0.43111	0.59622	0.22008	0.71458	4.03373
1994	0.01119	0.71640	1.03269	0.97094	0.48989	0.66312	0.25905	0.75459	4.35018
1995	0.01018	0.72759	1.09647	1.02870	0.53433	0.70754	0.29317	0.80747	4.56952
1996	0.00833	0.72771	1.14443	1.07484	0.58046	0.74787	0.33633	0.84780	4.71076
1997	0.00647	0.73320	1.18186	1.10928	0.62373	0.79367	0.35711	0.87940	4.88440
1998	0.00493	0.71427	1.19104	1.11324	0.63811	0.81503	0.35887	0.89874	5.07048
1999	0.00370	0.68731	1.21350	1.11767	0.65794	0.84963	0.39410	0.91485	5.32077
2000	0.00290	0.67515	1.20479	1.12117	0.66304	0.87510	0.38993	0.93879	5.56525
2001	0.00235	0.63648	1.20144	1.12559	0.67823	0.89800	0.36265	0.96024	5.66338
2002	0.00195	0.59428	1.21442	1.15024	0.71393	0.93311	0.36186	0.98240	5.52416
2003	0.00169	0.54277	1.21541	1.16185	0.75052	0.94071	0.33813	1.00449	5.28280
2004	0.00151	0.47722	1.22425	1.14734	0.79646	0.93404	0.29695	1.00220	5.07699
2005	0.00131	0.43090	1.19869	1.11472	0.80603	0.90853	0.28159	0.96774	4.86965
2006	0.00110	0.38738	1.12858	1.04597	0.74379	0.83990	0.21863	0.92068	4.66140
2007	0.00093	0.34532	1.05934	0.99442	0.70768	0.80811	0.14038	0.87919	4.49004
2008	0.00080	0.30409	1.00015	0.97643	0.70630	0.81809	0.13706	0.86178	4.40542
2009	0.00071	0.26889	0.96310	0.97766	0.71605	0.89281	0.24288	0.86767	4.35922
2010	0.00068	0.25895	1.08402	1.06327	0.79273	1.06831	0.46052	0.93577	4.38023
2011	0.00065	0.24594	1.17038	1.14059	0.83506	1.17671	0.61330	1.00100	4.48753
2012	0.00064	0.23724	1.25560	1.22566	0.86442	1.29759	0.75531	1.06793	4.65954
2013	0.00064	0.22870	1.33311	1.30003	0.91970	1.40788	0.88083	1.13689	4.82416
2014	0.00064	0.21340	1.37828	1.33823	0.96803	1.45993	0.99501	1.16845	4.93666
2015	0.00064	0.19170	1.39874	1.33658	0.96053	1.45397	1.07039	1.17932	4.92517
2016	0.00063	0.16975	1.38663	1.32543	0.97342	1.42949	1.07546	1.16203	4.94515
2017	0.00064	0.15421	1.39066	1.32904	1.01564	1.45496	1.11397	1.14899	4.99462
2018	0.00065	0.14248	1.41360	1.33822	1.02425	1.43878	1.14147	1.19735	5.06859
2019	0.00064	0.13065	1.42109	1.34802	1.04851	1.44872	1.12848	1.22772	5.01341
2020	0.00063	0.12050	1.36942	1.31929	1.00694	1.37881	1.07523	1.23634	4.93869
2021	0.00065	0.12056	1.46810	1.42627	1.13291	1.54439	1.23707	1.38457	5.10271
2022	0.00067	0.11701	1.54572	1.49196	1.21777	1.54713	1.27823	1.45752	5.04712
Mean	0.06931	0.48285	0.96780	0.91151	0.57230	0.76582	0.41470	0.80560	3.58490

Year	U_{10}^t	U_{11}^t	U_{12}^t	U_{13}^t	U_{14}^t	U_{15}^t	U_{16}^t	U_{17}^t
1970	0.37474	0.03093	0.04429	0.06540	0.06805	0.00036	-0.01584	0.13786
1971	0.40988	0.00250	0.05407	0.07381	0.07660	0.03158	-0.01391	0.11397
1972	0.46366	0.01231	0.06237	0.08695	0.08988	0.01480	-0.02233	0.06925
1973	0.48044	0.00945	0.06780	0.09029	0.09327	0.01635	-0.02498	0.09340
1974	0.50281	-0.01584	0.07253	0.08825	0.09121	-0.00103	-0.05324	0.10243
1975	0.54355	-0.02876	0.09689	0.09822	0.10141	-0.02119	-0.07959	0.09552
1976	0.60982	0.00259	0.12572	0.10994	0.11339	-0.03075	-0.08069	0.09589
1977	0.64256	0.05337	0.13458	0.11024	0.11380	0.01252	-0.04043	0.11277
1978	0.66493	0.11070	0.15165	0.10521	0.10860	0.12140	0.04643	0.11907
1979	0.69688	0.18914	0.16604	0.09984	0.10309	0.20287	0.12336	0.11742
1980	0.68455	0.28027	0.15910	0.10012	0.10353	0.18356	0.13921	0.10850
1981	0.72202	0.44079	0.16998	0.12174	0.12556	0.19856	0.20228	0.07560
1982	0.72215	0.53705	0.15049	0.13303	0.13765	0.15686	0.19199	0.06636
1983	0.75429	0.61900	0.16657	0.16760	0.18284	0.12035	0.17859	0.13394
1984	0.80849	0.60003	0.19623	0.17683	0.21233	0.07051	0.13837	0.14936
1985	0.83758	0.48044	0.19180	0.18996	0.24513	-0.02694	0.03568	0.19746
1986	0.88042	0.39069	0.16936	0.19175	0.26388	-0.08790	-0.02660	0.17185
1987	0.90003	0.33700	0.14471	0.16651	0.26367	-0.01281	0.00454	0.15530
1988	0.92376	0.27201	0.12679	0.14339	0.29713	0.14196	0.06381	0.19750
1989	0.96443	0.18989	0.09470	0.11173	0.34675	0.36515	0.18050	0.18224
1990	0.99112	0.12931	0.04541	0.08356	0.37972	0.57323	0.32732	0.12730
1991	1.02306	0.11175	0.02134	0.09417	0.37807	0.72969	0.47322	0.13061
1992	1.07589	0.11779	0.03970	0.11765	0.34660	0.87822	0.63455	0.12507
1993	1.11989	0.14171	0.08412	0.14241	0.30211	0.94060	0.72697	0.16793
1994	1.19276	0.18215	0.14206	0.17980	0.22536	0.92000	0.72239	0.22092
1995	1.26867	0.21471	0.19249	0.20167	0.15989	0.80574	0.63182	0.25197
1996	1.32646	0.25530	0.26490	0.22174	0.12544	0.68444	0.51800	0.31287
1997	1.37193	0.29290	0.33574	0.23874	0.08875	0.58181	0.39708	0.34090
1998	1.38488	0.31153	0.37644	0.22362	0.10091	0.41308	0.19429	0.30770
1999	1.40989	0.35843	0.42123	0.24006	0.13642	0.23833	-0.01962	0.32437
2000	1.46448	0.35550	0.44744	0.28230	0.13967	0.04678	-0.29025	0.38105
2001	1.49513	0.28410	0.47756	0.37662	0.20777	-0.17990	-0.61908	0.33497
2002	1.53919	0.24117	0.50321	0.55270	0.23123	-0.13719	-0.67272	0.28203
2003	1.57132	0.25700	0.49580	0.73169	0.18512	0.24271	-0.32488	0.32690
2004	1.56227	0.29378	0.45958	0.86940	0.14142	0.92288	0.40318	0.23297
2005	1.51665	0.24951	0.36736	0.90977	0.12179	1.82375	1.37820	0.20929
2006	1.44738	0.06917	0.27171	0.89318	0.26107	2.60047	2.22434	0.18952
2007	1.39074	-0.09423	0.20414	0.86599	0.41216	3.16591	2.91600	0.02942
2008	1.34342	-0.12346	0.17210	0.84974	0.35062	3.43781	3.39132	0.01954
2009	1.33501	-0.08288	0.19047	0.85446	0.44425	2.77929	2.84916	0.08642
2010	1.40211	0.05546	0.29484	0.93338	0.39713	2.28360	2.43864	0.08426
2011	1.42090	0.15507	0.36809	0.90719	0.35183	1.71546	1.89764	0.19586
2012	1.45553	0.29440	0.43831	0.84364	0.35724	1.21747	1.32720	0.22343
2013	1.50256	0.55024	0.52063	0.80542	0.20051	1.10585	1.18008	0.20515
2014	1.53018	0.84420	0.56161	0.73655	0.06580	0.89409	0.94329	0.34728
2015	1.54587	1.09565	0.55762	0.63493	0.17214	0.55647	0.59346	0.33855
2016	1.56595	1.27985	0.55669	0.56806	0.15341	0.71418	0.84722	0.35375
2017	1.58709	1.31156	0.57858	0.56859	0.12963	0.91707	1.08404	0.26509
2018	1.62778	1.28555	0.63816	0.67656	0.19953	0.94071	1.07927	0.35321
2019	1.65040	1.25690	0.66464	0.78615	0.25186	1.07373	1.23153	0.25445
2020	1.64051	1.11048	0.65015	0.86630	0.18007	0.96465	1.03926	0.16721
2021	1.74721	1.23861	0.80095	1.24490	0.16182	1.18079	1.33675	0.16072
2022	1.82368	1.43530	0.81787	1.35050	0.39685	1.42448	1.61212	0.33714
Mean	1.14940	0.37721	0.29258	0.42042	0.20555	0.71533	0.63055	0.19214

Moving to the use of smoothed asset inflation rates in the user cost formula did not eliminate the earlier negative user costs but it did reduce the number: there are 5, 8 and 14 negative user costs for assets 11 (Agricultural Land), 15 (Residential Land and 16 (Other Land) respectively.

We use the data for the smoothed user costs, $U_1^t, \ldots, U_{17}^t$, along with the data on the beginning of the year asset stocks, $Q_{K1}^t, \ldots, Q_{K17}^t$, to form 4 capital services aggregates. The first subaggregate is a Machinery and Equipment aggregate consisting of assets 1-4, the second is a Structures subaggregate consisting of assets 5-7, the third is an Other subaggregate which consists of assets 8-10 and 17 (Research and Development, Computer Software, Other Intangibles and Inventory Stocks) and the fourth is a Land subaggregate consisting of assets 11-16. Denote the prices and quantities for these aggregate capital services by $P_{UM}^t, P_{US}^t, P_{UO}^t$ and P_{UL}^t and $Q_{UM}^t, Q_{US}^t, Q_{UO}^t$ and Q_{UL}^t. These aggregate user costs are listed in Table A11 (with prices normalized to equal 1 in 1970).[*40] The four subaggregates were then aggregated into an overall capital services aggregate using Törnqvist direct quantity aggregation. The year t value, price and quantity of the capital services aggregate are denoted by V_U^t, P_U^t and Q_U^t.

[*40] These capital services input subaggregates were formed using Törnqvist direct aggregation of input quantities; i.e., bilateral chained quantity indexes were formed first (quantities were always positive) and then the corresponding aggregate price indexes were calculated by deflating total subaggregate value by the quantity index.

Table A11: Prices and Quantities of Four Capital Services Subaggregates

Year	P_U^t	P_{UM}^t	P_{US}^t	P_{UO}^t	P_{UL}^t	Q_U^t	Q_{UM}^t	Q_{US}^t	Q_{UO}^t	Q_{UL}^t
1970	1.00000	1.00000	1.00000	1.00000	1.00000	0.36904	0.11036	0.15996	0.03686	0.06187
1971	1.08313	1.03597	1.09951	1.01687	1.16595	0.37920	0.11168	0.16463	0.04069	0.06229
1972	1.16234	1.04721	1.21456	1.04646	1.30444	0.39040	0.11269	0.17044	0.04430	0.06317
1973	1.25503	1.05477	1.38372	1.12108	1.35412	0.40492	0.11631	0.17708	0.04772	0.06424
1974	1.28506	1.09630	1.46556	1.18792	1.19002	0.42314	0.12377	0.18379	0.05137	0.06528
1975	1.40903	1.21704	1.62842	1.27334	1.23802	0.43705	0.13053	0.18845	0.05437	0.06530
1976	1.56385	1.34589	1.78646	1.36808	1.48236	0.44421	0.13280	0.19162	0.05570	0.06572
1977	1.69764	1.41718	1.90740	1.44961	1.82235	0.45601	0.13666	0.19582	0.05885	0.06676
1978	1.85720	1.49972	2.00490	1.51045	2.40931	0.47220	0.14445	0.20131	0.06220	0.06756
1979	1.98544	1.56750	2.07522	1.54956	2.92053	0.49305	0.15566	0.20820	0.06605	0.06866
1980	2.02606	1.60895	2.08786	1.57443	3.05638	0.51493	0.16824	0.21538	0.06985	0.06967
1981	2.24890	1.72733	2.27049	1.61651	3.81838	0.53012	0.17698	0.22113	0.07277	0.06987
1982	2.25221	1.75311	2.19204	1.66973	3.97121	0.54553	0.18568	0.22656	0.07661	0.07015
1983	2.49123	1.88233	2.38235	1.95414	4.56370	0.55411	0.18992	0.23028	0.07914	0.07004
1984	2.74284	2.03177	2.75739	2.18078	4.65848	0.56647	0.19563	0.23506	0.08168	0.07083
1985	2.83386	2.07426	2.98456	2.37799	4.28176	0.58979	0.20784	0.24222	0.08798	0.07196
1986	2.85164	2.09597	3.09574	2.40715	3.95130	0.61371	0.22012	0.25024	0.09360	0.07308
1987	2.87525	2.08769	3.12696	2.41093	4.06416	0.63603	0.23041	0.25784	0.09922	0.07437
1988	2.97713	2.06138	3.12571	2.52260	4.88096	0.65663	0.23984	0.26514	0.10565	0.07482
1989	3.16907	2.06988	3.23373	2.55036	6.12367	0.67831	0.25007	0.27203	0.11133	0.07649
1990	3.21591	2.03294	3.13708	2.43889	7.17369	0.69893	0.26095	0.27811	0.11741	0.07784
1991	3.21771	2.02263	2.96704	2.42620	7.85437	0.71619	0.26881	0.28381	0.12367	0.07877
1992	3.35499	2.09770	3.08730	2.47619	8.33654	0.72735	0.27288	0.28720	0.12936	0.07927
1993	3.51648	2.16494	3.35742	2.63595	8.34745	0.74080	0.27928	0.29038	0.13477	0.08029
1994	3.71368	2.28640	3.82189	2.86885	7.66913	0.75636	0.29028	0.29445	0.13910	0.08095
1995	3.81287	2.37694	4.17902	3.08175	6.60094	0.77719	0.30559	0.29928	0.14414	0.08211
1996	3.94083	2.41335	4.56684	3.30574	5.86780	0.80098	0.32680	0.30452	0.14872	0.08274
1997	4.01544	2.41650	4.87639	3.45623	5.16699	0.83132	0.35146	0.31133	0.15545	0.08388
1998	3.96370	2.35375	4.97204	3.46774	4.54040	0.86799	0.37957	0.31879	0.16567	0.08535
1999	4.01973	2.29037	5.22207	3.57517	4.25861	0.91021	0.41370	0.32728	0.17700	0.08673
2000	3.97127	2.22925	5.27010	3.77848	3.43361	0.95787	0.45432	0.33610	0.19015	0.08777
2001	3.91420	2.16638	5.27796	3.75926	3.14927	1.00696	0.49576	0.34545	0.20338	0.08878
2002	4.01220	2.13402	5.47148	3.69325	3.85525	1.04022	0.52053	0.35443	0.21139	0.08891
2003	4.17592	2.09130	5.54995	3.77135	5.53649	1.06567	0.53372	0.36220	0.21960	0.08951
2004	4.38270	2.02325	5.58836	3.57261	8.76703	1.09307	0.54923	0.37077	0.22822	0.09034
2005	4.64007	1.93015	5.53727	3.41956	13.08549	1.12583	0.57220	0.38039	0.23846	0.09153
2006	4.77041	1.78397	4.98457	3.24401	18.40643	1.15993	0.60135	0.38982	0.24951	0.09280
2007	4.82905	1.65948	4.54980	2.85301	22.87290	1.19821	0.64211	0.39935	0.26138	0.09435
2008	4.81727	1.57651	4.54928	2.77749	23.51953	1.22614	0.68267	0.40566	0.27406	0.09455
2009	4.80782	1.52587	5.02662	2.88408	21.39923	1.24361	0.71045	0.40789	0.28536	0.09447
2010	5.10751	1.62739	6.28166	3.02047	18.75001	1.24242	0.70563	0.40624	0.29236	0.09399
2011	5.23770	1.70513	7.10030	3.31386	15.45404	1.25238	0.72289	0.40301	0.30112	0.09496
2012	5.40376	1.79647	7.85808	3.51452	13.10178	1.26462	0.74933	0.40001	0.31128	0.09542
2013	5.57944	1.87690	8.65426	3.65634	11.00654	1.28291	0.78392	0.39866	0.32218	0.09617
2014	5.69367	1.90983	9.30205	3.94162	8.61347	1.30478	0.82042	0.39887	0.33485	0.09659
2015	5.73352	1.89269	9.51683	3.94753	8.39752	1.33152	0.86167	0.40053	0.34833	0.09724
2016	5.77818	1.85610	9.54956	3.94952	9.20232	1.36111	0.90148	0.40349	0.36399	0.09788
2017	5.89187	1.84398	9.88348	3.81218	10.05129	1.38832	0.93152	0.40691	0.38118	0.09823
2018	6.07416	1.84720	9.98701	4.04441	11.26479	1.41677	0.96266	0.40992	0.40039	0.09891
2019	6.16574	1.83940	10.05374	3.94408	12.80517	1.45150	1.00020	0.41416	0.42388	0.09950
2020	5.84925	1.78350	9.60846	3.81624	11.13304	1.48542	1.03225	0.41855	0.44916	0.09987
2021	6.50474	1.90310	10.89250	4.10471	12.98331	1.50862	1.03848	0.42233	0.47315	0.09970
2022	7.11141	1.97545	11.37361	4.43145	17.82376	1.54067	1.05234	0.42630	0.50300	0.10098

The price of aggregate capital services P_U^t increased 7.11 fold over the sample period. The price of Machinery and Equipment Services P_{UM}^t increased 1.98 fold, the price of Structures Services P_{US}^t increased 11.37 fold, the price of Other Capital Services P_{UO}^t increased 4.43 fold and the price of Land Services P_{UL}^t increased 17.82 fold.

Denote the values of the 4 subaggregates in year t by $V_{UM}^t, V_{US}^t, V_{UO}^t$ and V_{UL}^t. Denote the corresponding year t value shares of aggregate capital services by $s_{UM}^t, s_{US}^t, s_{UO}^t$ and s_{UL}^t. These variables are listed in Table A12 below.

Table A12: Values and Shares of Four Capital Services Subaggregates

Year	V_U^t	V_{UM}^t	V_{US}^t	V_{UO}^t	V_{UL}^t	s_{UM}^t	s_{US}^t	s_{UO}^t	s_{UL}^t
1970	0.36904	0.11036	0.15996	0.03686	0.06187	0.29903	0.43344	0.09988	0.16764
1971	0.41072	0.11570	0.18102	0.04138	0.07262	0.28171	0.44073	0.10075	0.17682
1972	0.45378	0.11801	0.20700	0.04636	0.08240	0.26007	0.45618	0.10216	0.18159
1973	0.50819	0.12268	0.24503	0.05349	0.08698	0.24141	0.48217	0.10526	0.17116
1974	0.54376	0.13569	0.26936	0.06103	0.07768	0.24955	0.49536	0.11223	0.14286
1975	0.61582	0.15886	0.30688	0.06923	0.08084	0.25797	0.49833	0.11242	0.13128
1976	0.69468	0.17873	0.34232	0.07620	0.09742	0.25728	0.49278	0.10970	0.14024
1977	0.77414	0.19367	0.37351	0.08530	0.12166	0.25018	0.48248	0.11019	0.15715
1978	0.87697	0.21664	0.40360	0.09395	0.16278	0.24703	0.46023	0.10713	0.18562
1979	0.97893	0.24400	0.43205	0.10235	0.20053	0.24925	0.44135	0.10455	0.20485
1980	1.04329	0.27068	0.44969	0.10998	0.21294	0.25945	0.43103	0.10541	0.20410
1981	1.19219	0.30570	0.50208	0.11763	0.26678	0.25642	0.42114	0.09867	0.22377
1982	1.22864	0.32551	0.49662	0.12792	0.27858	0.26494	0.40420	0.10412	0.22674
1983	1.38042	0.35750	0.54862	0.15466	0.31964	0.25898	0.39743	0.11204	0.23156
1984	1.55373	0.39748	0.64817	0.17813	0.32995	0.25582	0.41717	0.11465	0.21236
1985	1.67139	0.43111	0.72292	0.20923	0.30813	0.25794	0.43252	0.12518	0.18436
1986	1.75009	0.46136	0.77468	0.22531	0.28874	0.26362	0.44265	0.12874	0.16499
1987	1.82875	0.48102	0.80626	0.23922	0.30225	0.26303	0.44088	0.13081	0.16528
1988	1.95486	0.49440	0.82874	0.26650	0.36521	0.25291	0.42394	0.13633	0.18682
1989	2.14961	0.51762	0.87967	0.28394	0.46838	0.24080	0.40922	0.13209	0.21789
1990	2.24770	0.53050	0.87244	0.28634	0.55842	0.23602	0.38815	0.12739	0.24844
1991	2.30449	0.54370	0.84209	0.30004	0.61866	0.23593	0.36541	0.13020	0.26846
1992	2.44024	0.57241	0.88667	0.32032	0.66084	0.23457	0.36335	0.13127	0.27081
1993	2.60502	0.60462	0.97492	0.35525	0.67023	0.23210	0.37425	0.13637	0.25728
1994	2.80889	0.66371	1.12535	0.39904	0.62079	0.23629	0.40064	0.14206	0.22101
1995	2.96331	0.72637	1.25071	0.44420	0.54203	0.24512	0.42206	0.14990	0.18291
1996	3.15652	0.78869	1.39071	0.49162	0.48551	0.24986	0.44058	0.15575	0.15381
1997	3.33812	0.84929	1.51817	0.53726	0.43339	0.25442	0.45480	0.16095	0.12983
1998	3.44043	0.89342	1.58501	0.57449	0.38750	0.25968	0.46070	0.16698	0.11263
1999	3.65878	0.94752	1.70908	0.63282	0.36936	0.25897	0.46712	0.17296	0.10095
2000	3.80395	1.01280	1.77130	0.71849	0.30135	0.26625	0.46565	0.18888	0.07922
2001	3.94143	1.07400	1.82330	0.76454	0.27959	0.27249	0.46260	0.19398	0.07094
2002	4.17358	1.11083	1.93928	0.78071	0.34276	0.26616	0.46466	0.18706	0.08213
2003	4.45017	1.11618	2.01020	0.82821	0.49559	0.25082	0.45171	0.18611	0.11136
2004	4.79062	1.11124	2.07202	0.81535	0.79201	0.23196	0.43252	0.17020	0.16533
2005	5.22393	1.10442	2.10629	0.81544	1.19777	0.21142	0.40320	0.15610	0.22929
2006	5.53336	1.07279	1.94307	0.80942	1.70807	0.19388	0.35116	0.14628	0.30869
2007	5.78623	1.06557	1.81697	0.74573	2.15796	0.18416	0.31402	0.12888	0.37295
2008	5.90667	1.07623	1.84548	0.76121	2.22375	0.18221	0.31244	0.12887	0.37648
2009	5.97906	1.08406	2.05033	0.82299	2.02168	0.18131	0.34292	0.13765	0.33813
2010	6.34566	1.14833	2.55189	0.88307	1.76238	0.18096	0.40215	0.13916	0.27773
2011	6.55958	1.23262	2.86152	0.99788	1.46756	0.18791	0.43624	0.15213	0.22373
2012	6.83371	1.34615	3.14335	1.09399	1.25022	0.19699	0.45998	0.16009	0.18295
2013	7.15793	1.47134	3.45008	1.17801	1.05851	0.20555	0.48199	0.16457	0.14788
2014	7.42901	1.56687	3.71031	1.31983	0.83200	0.21091	0.49944	0.17766	0.11199
2015	7.63428	1.63088	3.81180	1.37506	0.81654	0.21363	0.49930	0.18012	0.10696
2016	7.86474	1.67325	3.85317	1.43757	0.90075	0.21275	0.48993	0.18279	0.11453
2017	8.17979	1.71770	4.02164	1.45314	0.98731	0.20999	0.49166	0.17765	0.12070
2018	8.60569	1.77823	4.09391	1.61932	1.11423	0.20663	0.47572	0.18817	0.12948
2019	8.94954	1.83976	4.16382	1.67183	1.27413	0.20557	0.46526	0.18681	0.14237
2020	8.68861	1.84103	4.02165	1.71411	1.11181	0.21189	0.46287	0.19728	0.12796
2021	9.81320	1.97633	4.60023	1.94215	1.29450	0.20139	0.46878	0.19791	0.13191
2022	10.95630	2.07883	4.84857	2.22903	1.79988	0.18974	0.44254	0.20345	0.16428
Mean	3.87830	0.84691	1.70270	0.63580	0.69288	0.23556	0.43617	0.14449	0.18378

There is a considerable amount of variation in the shares of the four capital services subaggregates in total capital services. The average share of Structures was 43.6%, of Machinery and Equipment was 23.6%, of Land was 18.4% and of Other Capital (mostly R&D and Inventory services) was 18.4%. All of the subaggregate user costs were positive and three of them were fairly smooth but the share of Land services was quite volatile.[*41]

We turn now to the construction of alternative indexes of output, input and Total Factor Productivity (TFP) for the US economy.

In Table A6, the GDP output index Q_Y^t and the companion output price index P_Y^t were listed. The GDP price index P_Y^t is a chained Törnqvist price index that uses the prices $P_C^t, P_G^t, P_I^t, P_X^t, P_M^t$ and the quantity indexes $Q_C^t, Q_G^t, Q_I^t, Q_X^t, -Q_M^t$ as inputs into the price index P_Y^t. The quantity index Q_Y^t was defined residually by deflating the year t value of GDP, V_Y^t by the year t GDP price index; i.e., $Q_Y^t \equiv V_Y^t/P_Y^t$. We are now in a position to define a companion index of the year t quantity of *inputs* Q_Z^t as a chained Törnqvist quantity index that uses the prices of labour and the four types of capital services $P_L^t, P_{UM}^t, P_{US}^t, P_{UO}^t, P_{UL}^t$ and the quantity indexes $Q_L^t, Q_{UM}^t, Q_{US}^t, Q_{UO}^t, Q_{UL}^t$ as inputs into the quantity index Q_Z^t and the companion input price index P_Z^t is defined residually by deflating the year t value of GDP, $V_Z^t = V_Y^t$ by the year t GDP input quantity index; i.e., $P_Z^t \equiv V_Z^t/Q_Y^t$. The year t level of Total Factor Productivity is defined as the year t output quantity Q_Y^t divided by the year t input quantity Q_Z^t; i.e.,

$$\mathrm{TFP}^t \equiv Q_Y^t/Q_Z^t; \qquad t = 1970, \ldots, 2022. \tag{A6}$$

TFP growth is defined as $\mathrm{TFPG}^t \equiv \mathrm{TFP}^t/\mathrm{TFP}^{t-1}$ for $t = 1971, \ldots, 2022$.[*42] The series $Q_Y^t, Q_Z^t, \mathrm{TFP}^t$ and TFPG^t are listed in Table A13.

Recall that Jorgensonian balancing rates of return r_J^t were listed in Table A9 above. These rates of return can be used with actual ex post asset inflation rates to construct Jorgensonian user costs. We used these user costs in place of our smoothed user costs to construct Jorgensonian estimates of year t Total Factor Productivity, TFP_J^t and year t estimates of TFP growth, TFPG_J^t. These alternative GDP based estimates of TFP are also listed in Table A13.

In the main text, we did not use GDP as our output concept. Instead, we used Gross Output (GO) as our output concept. Gross Output simply regards imports as an input instead of as a negative output. Thus the value of Gross Output in year t, V_{GY}^t, is defined as follows:

$$V_{GY}^t \equiv P_C^t Q_C^t + P_G^t Q_G^t + P_I^t Q_I^t + P_X^t Q_X^t; \qquad t = 1970, \ldots, 2022. \tag{A7}$$

The GO price index for year t, P_{GY}^t, is a chained Törnqvist price index that uses the output prices $P_C^t, P_G^t, P_I^t, P_X^t$ and the corresponding quantity indexes $Q_C^t, Q_G^t, Q_I^t, Q_X^t$ as inputs into the price index P_{GY}^t. The corresponding year t GO quantity index Q_{GY}^t is defined residually by deflating the year t value of Gross Output, V_{GY}^t by the year t GO price index; i.e., $Q_{GY}^t \equiv V_{GY}^t/P_{GY}^t$. The companion index of the year t quantity of inputs Q_{GZ}^t is a chained Törnqvist quantity index that uses the prices of imports, labour and the four types of capital services $P_M^t, P_L^t, P_{UM}^t, P_{US}^t, P_{UO}^t, P_{UL}^t$ and the quantity indexes $Q_M^t, Q_L^t, Q_{UM}^t, Q_{US}^t, Q_{UO}^t, Q_{UL}^t$ as inputs into the quantity index Q_{GZ}^t and the companion input price index P_{GZ}^t is defined residually by deflating the year t value of GO, $V_{GZ}^t = V_{GY}^t$ by the year t GO input quantity index; i.e., $P_{GZ}^t \equiv V_{GZ}^t/Q_{GZ}^t$. The year t level of GO Total

[*41] When land prices increase, land inflation rates increase even more on a proportional basis, so the user cost of land tends to decrease when there is a property bubble. When the bubble ends, the user cost of land tends to increase.

[*42] This precise methodology for measuring TFP is due to Diewert and Morrison (1986)[22] and Kohli (1990)[39] but it is a variation on the methodology pioneered by Jorgenson and Griliches (1967)[31] and Christensen and Jorgenson (1969)[7].

Factor Productivity is defined as the year t output quantity Q^t_{GY} divided by the year t input quantity Q^t_{GZ}; i.e.,

$$\mathrm{TFP}^t_G \equiv Q^t_{GY}/Q^t_{GZ}; \qquad t = 1970, \ldots, 2022. \tag{A8}$$

Gross Output TFP growth is defined as $\mathrm{TFPG}^t_G \equiv \mathrm{TFP}^t_G/\,\mathrm{TFP}^{t-1}_G$ for $t = 1971, \ldots, 2022$. The series $Q^t_{GY}, Q^t_{GZ}, \mathrm{TFP}^t_G$ and TFPG^t_G are listed in Table A13.

Table A13: Alternative GDP and Gross Output Measures of Input, Output and Productivity

Year	Q_Y^t	Q_{GY}^t	Q_Z^t	Q_{GZ}^t	TFP^t	TFP$_J^t$	TFP$_G^t$	TFPG^t	TFPG$_J^t$	TFPG$_G^t$
1970	1.0275	1.0858	1.0275	1.0858	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1971	1.0635	1.1250	1.0344	1.0959	1.0281	1.0279	1.0266	1.0281	1.0279	1.0266
1972	1.1205	1.1888	1.0649	1.1331	1.0522	1.0514	1.0492	1.0234	1.0228	1.0220
1973	1.1832	1.2546	1.1090	1.1803	1.0669	1.0662	1.0629	1.0140	1.0141	1.0131
1974	1.1751	1.2445	1.1296	1.1987	1.0402	1.0403	1.0382	0.9750	0.9757	0.9767
1975	1.1704	1.2287	1.1259	1.1842	1.0395	1.0395	1.0376	0.9993	0.9992	0.9994
1976	1.2322	1.3063	1.1541	1.2283	1.0677	1.0675	1.0635	1.0271	1.0270	1.0250
1977	1.2914	1.3755	1.1897	1.2739	1.0855	1.0848	1.0798	1.0167	1.0161	1.0153
1978	1.3631	1.4555	1.2386	1.3312	1.1005	1.0989	1.0934	1.0138	1.0131	1.0126
1979	1.4066	1.4999	1.2835	1.3771	1.0959	1.0929	1.0892	0.9958	0.9945	0.9962
1980	1.4039	1.4876	1.3025	1.3864	1.0779	1.0744	1.0730	0.9836	0.9830	0.9851
1981	1.4408	1.5266	1.3243	1.4108	1.0879	1.0842	1.0820	1.0094	1.0092	1.0084
1982	1.4090	1.4942	1.3223	1.4072	1.0655	1.0614	1.0618	0.9794	0.9790	0.9813
1983	1.4688	1.5683	1.3420	1.4414	1.0945	1.0895	1.0881	1.0272	1.0264	1.0247
1984	1.5759	1.7058	1.3971	1.5254	1.1280	1.1221	1.1182	1.0306	1.0299	1.0277
1985	1.6380	1.7771	1.4412	1.5783	1.1366	1.1309	1.1259	1.0076	1.0078	1.0069
1986	1.6862	1.8387	1.4767	1.6261	1.1419	1.1354	1.1307	1.0047	1.0040	1.0043
1987	1.7455	1.9077	1.5267	1.6853	1.1433	1.1358	1.1320	1.0012	1.0004	1.0011
1988	1.8182	1.9867	1.5723	1.7372	1.1564	1.1480	1.1436	1.0115	1.0107	1.0103
1989	1.8824	2.0586	1.6244	1.7967	1.1588	1.1502	1.1458	1.0021	1.0020	1.0019
1990	1.9194	2.1024	1.6565	1.8350	1.1587	1.1504	1.1457	0.9999	1.0001	0.9999
1991	1.9167	2.0994	1.6556	1.8339	1.1577	1.1495	1.1448	0.9991	0.9993	0.9992
1992	1.9805	2.1768	1.6773	1.8679	1.1808	1.1725	1.1653	1.0199	1.0200	1.0179
1993	2.0358	2.2500	1.7110	1.9176	1.1898	1.1810	1.1733	1.0076	1.0073	1.0069
1994	2.1177	2.3583	1.7587	1.9885	1.2041	1.1952	1.1860	1.0120	1.0120	1.0108
1995	2.1735	2.4339	1.8134	2.0607	1.1986	1.1895	1.1811	0.9954	0.9953	0.9959
1996	2.2587	2.5421	1.8564	2.1238	1.2167	1.2071	1.1969	1.0151	1.0148	1.0134
1997	2.3602	2.6814	1.9209	2.2208	1.2287	1.2184	1.2074	1.0099	1.0093	1.0087
1998	2.4628	2.8200	1.9921	2.3229	1.2363	1.2246	1.2140	1.0062	1.0051	1.0055
1999	2.5774	2.9737	2.0623	2.4261	1.2498	1.2367	1.2257	1.0109	1.0099	1.0096
2000	2.6788	3.1234	2.1327	2.5371	1.2561	1.2422	1.2311	1.0050	1.0044	1.0044
2001	2.7015	3.1365	2.1598	2.5571	1.2508	1.2366	1.2266	0.9958	0.9955	0.9963
2002	2.7458	3.1957	2.1866	2.5964	1.2557	1.2415	1.2308	1.0039	1.0040	1.0035
2003	2.8223	3.2939	2.2059	2.6327	1.2794	1.2651	1.2512	1.0189	1.0190	1.0165
2004	2.9307	3.4501	2.2464	2.7111	1.3046	1.2897	1.2726	1.0197	1.0194	1.0171
2005	3.0296	3.5811	2.2953	2.7860	1.3199	1.3043	1.2854	1.0117	1.0113	1.0101
2006	3.1151	3.7004	2.3546	2.8731	1.3230	1.3065	1.2880	1.0023	1.0017	1.0020
2007	3.1753	3.7754	2.4069	2.9383	1.3193	1.3020	1.2849	0.9972	0.9966	0.9976
2008	3.1806	3.7685	2.4249	2.9474	1.3117	1.2958	1.2786	0.9942	0.9952	0.9951
2009	3.0969	3.6137	2.3742	2.8396	1.3044	1.2898	1.2726	0.9945	0.9953	0.9953
2010	3.1806	3.7590	2.3793	2.8917	1.3368	1.3222	1.2999	1.0248	1.0252	1.0214
2011	3.2279	3.8330	2.4171	2.9512	1.3354	1.3208	1.2988	0.9990	0.9989	0.9991
2012	3.3000	3.9199	2.4598	3.0064	1.3416	1.3269	1.3039	1.0046	1.0046	1.0039
2013	3.3717	3.9996	2.5029	3.0568	1.3471	1.3316	1.3085	1.0041	1.0035	1.0035
2014	3.4555	4.1146	2.5525	3.1315	1.3538	1.3371	1.3139	1.0049	1.0042	1.0042
2015	3.5531	4.2447	2.6091	3.2141	1.3618	1.3450	1.3206	1.0059	1.0059	1.0051
2016	3.6126	4.3144	2.6590	3.2736	1.3586	1.3424	1.3180	0.9977	0.9981	0.9980
2017	3.7007	4.4328	2.7085	3.3469	1.3664	1.3498	1.3245	1.0057	1.0055	1.0049
2018	3.8094	4.5695	2.7632	3.4237	1.3786	1.3617	1.3347	1.0090	1.0088	1.0077
2019	3.9033	4.6742	2.8184	3.4883	1.3849	1.3680	1.3400	1.0046	1.0046	1.0040
2020	3.8119	4.5241	2.7731	3.3984	1.3746	1.3580	1.3313	0.9926	0.9926	0.9935
2021	4.0268	4.8282	2.8413	3.5314	1.4172	1.3995	1.3672	1.0310	1.0305	1.0270
2022	4.0939	4.9532	2.9284	3.6659	1.3980	1.3800	1.3512	0.9864	0.9861	0.9883
Mean	2.3854	2.7426	1.9055	2.2279	1.2164	1.2065	1.1946	1.0065	1.0063	1.0059

The average TFP growth rate using GDP as the output concept was 0.65% per year using smoothed user costs and 0.63% per year using Jorgensonian user costs. Smoothing the user costs did not greatly affect our measures of GDP Total Factor Productivity Growth. Using Gross Output as our output concept, the mean TFP growth rate fell to 0.59% per year. This drop was expected because regarding imports as an input rather than a negative output increases the overall input volume measure by about 10% so we could expect a 10% drop in measured TFP growth using Gross Output in place as GDP as the output measure.

We conclude this data construction Appendix by listing the year by year main diagonal elasticities that correspond to the average elasticities that were listed in Tables 2 and 1 in the main text. These elasticities were generated by differentiating our estimated Joint Cost Function over the years 1970-2022.

Table A14: Main Diagonal Elasticities of Inverse Output Supply 1970-2022

Year	Ep_{11}	Ep_{22}	Ep_{33}	Ep_{44}
1970	0.1536	0.1795	0.1001	0.0199
1971	0.1513	0.1715	0.1039	0.0192
1972	0.1466	0.1621	0.1098	0.0191
1973	0.1428	0.1533	0.1176	0.0206
1974	0.1454	0.1673	0.1136	0.0204
1975	0.1345	0.1881	0.1064	0.0185
1976	0.1332	0.1767	0.1135	0.0186
1977	0.1357	0.1659	0.1175	0.0184
1978	0.1329	0.1552	0.1198	0.0193
1979	0.1295	0.1523	0.1196	0.0196
1980	0.1237	0.1636	0.1146	0.0199
1981	0.1238	0.1547	0.1150	0.0200
1982	0.1141	0.1630	0.1081	0.0183
1983	0.1126	0.1569	0.1111	0.0178
1984	0.1179	0.1405	0.1203	0.0194
1985	0.1137	0.1410	0.1210	0.0195
1986	0.1091	0.1446	0.1200	0.0200
1987	0.1048	0.1421	0.1199	0.0208
1988	0.0976	0.1394	0.1207	0.0224
1989	0.0935	0.1365	0.1207	0.0236
1990	0.0888	0.1365	0.1185	0.0244
1991	0.0827	0.1386	0.1146	0.0254
1992	0.0795	0.1366	0.1168	0.0262
1993	0.0774	0.1323	0.1197	0.0263
1994	0.0757	0.1275	0.1239	0.0275
1995	0.0716	0.1264	0.1255	0.0285
1996	0.0695	0.1200	0.1297	0.0304
1997	0.0674	0.1138	0.1340	0.0335
1998	0.0663	0.1087	0.1364	0.0346
1999	0.0649	0.1041	0.1385	0.0354
2000	0.0618	0.1004	0.1398	0.0373
2001	0.0596	0.1026	0.1359	0.0353
2002	0.0587	0.1029	0.1340	0.0338
2003	0.0575	0.1001	0.1344	0.0339
2004	0.0561	0.0961	0.1363	0.0354
2005	0.0543	0.0925	0.1373	0.0358
2006	0.0515	0.0896	0.1372	0.0375
2007	0.0475	0.0890	0.1347	0.0378
2008	0.0440	0.0901	0.1296	0.0364
2009	0.0409	0.0946	0.1215	0.0335
2010	0.0392	0.0897	0.1252	0.0349
2011	0.0364	0.0854	0.1251	0.0351
2012	0.0351	0.0819	0.1267	0.0362
2013	0.0334	0.0785	0.1276	0.0368
2014	0.0315	0.0751	0.1277	0.0379
2015	0.0304	0.0735	0.1279	0.0392
2016	0.0286	0.0728	0.1255	0.0389
2017	0.0271	0.0707	0.1246	0.0392
2018	0.0258	0.0690	0.1232	0.0400
2019	0.0250	0.0695	0.1215	0.0399
2020	0.0246	0.0737	0.1170	0.0385
2021	0.0223	0.0701	0.1157	0.0375
2022	0.0204	0.0675	0.1149	0.0383

The above inverse elasticities are all small.[*43]

[*43] If there were sectoral production functions for C, G and I (the no joint production hypothesis), then all of the elasticities listed in Table A14 would be equal to zero. It appears that production is "almost" Nonjoint.

Table A15: Main Diagonal Elasticities of Input Demand 1970-2022

Year	Ex_{11}	Ex_{22}	Ex_{33}	Ex_{44}	Ex_{55}	Ex_{66}
1970	-0.5103	-0.2418	-0.6548	-0.0396	-0.4416	-0.0117
1971	-0.5010	-0.2464	-0.6595	-0.0419	-0.4438	-0.0129
1972	-0.4858	-0.2501	-0.6325	-0.0450	-0.4705	-0.0137
1973	-0.5021	-0.2555	-0.5879	-0.0487	-0.5352	-0.0134
1974	-0.6814	-0.2676	-0.6274	-0.0486	-0.4758	-0.0114
1975	-0.7562	-0.2737	-0.7422	-0.0492	-0.4095	-0.0110
1976	-0.6791	-0.2795	-0.7488	-0.0509	-0.4265	-0.0124
1977	-0.6695	-0.2848	-0.7356	-0.0520	-0.4255	-0.0143
1978	-0.6306	-0.2850	-0.7171	-0.0520	-0.4269	-0.0178
1979	-0.6639	-0.2869	-0.7049	-0.0515	-0.3993	-0.0206
1980	-0.7697	-0.2915	-0.7380	-0.0501	-0.3463	-0.0210
1981	-0.7262	-0.2908	-0.7714	-0.0510	-0.3278	-0.0244
1982	-0.7238	-0.2883	-0.8464	-0.0489	-0.3067	-0.0248
1983	-0.6539	-0.2874	-0.9063	-0.0504	-0.3475	-0.0263
1984	-0.5651	-0.2920	-0.9123	-0.0558	-0.3979	-0.0253
1985	-0.5287	-0.2942	-0.9525	-0.0594	-0.4208	-0.0225
1986	-0.5140	-0.2997	-0.9951	-0.0620	-0.4083	-0.0208
1987	-0.5197	-0.2987	-1.0084	-0.0634	-0.3843	-0.0216
1988	-0.5238	-0.2937	-0.9642	-0.0628	-0.3870	-0.0253
1989	-0.5201	-0.2901	-0.9848	-0.0634	-0.3734	-0.0308
1990	-0.5290	-0.2869	-0.9783	-0.0621	-0.3353	-0.0362
1991	-0.5225	-0.2833	-0.9935	-0.0595	-0.3162	-0.0382
1992	-0.5109	-0.2864	-1.0264	-0.0612	-0.3094	-0.0398
1993	-0.4815	-0.2889	-1.0786	-0.0657	-0.3190	-0.0391
1994	-0.4517	-0.2950	-1.1649	-0.0728	-0.3401	-0.0351
1995	-0.4375	-0.3015	-1.2608	-0.0786	-0.3575	-0.0301
1996	-0.3998	-0.3037	-1.2401	-0.0853	-0.3835	-0.0265
1997	-0.3588	-0.3067	-1.1910	-0.0921	-0.4102	-0.0233
1998	-0.3321	-0.3050	-1.1179	-0.0986	-0.4073	-0.0212
1999	-0.3209	-0.3022	-1.0472	-0.1058	-0.4107	-0.0202
2000	-0.3182	-0.2982	-0.9467	-0.1103	-0.4259	-0.0168
2001	-0.3133	-0.2917	-0.9349	-0.1130	-0.3950	-0.0158
2002	-0.3074	-0.2914	-0.9637	-0.1170	-0.3690	-0.0193
2003	-0.3024	-0.2884	-0.9187	-0.1173	-0.3622	-0.0274
2004	-0.2939	-0.2888	-0.8488	-0.1163	-0.3329	-0.0426
2005	-0.2915	-0.2878	-0.7819	-0.1129	-0.3032	-0.0619
2006	-0.2850	-0.2822	-0.6898	-0.1014	-0.2802	-0.0865
2007	-0.2872	-0.2798	-0.6089	-0.0922	-0.2351	-0.1068
2008	-0.3118	-0.2783	-0.5638	-0.0915	-0.2155	-0.1107
2009	-0.2955	-0.2702	-0.5666	-0.1016	-0.2133	-0.1019
2010	-0.2955	-0.2819	-0.5928	-0.1227	-0.2202	-0.0870
2011	-0.3101	-0.2876	-0.6063	-0.1379	-0.2365	-0.0720
2012	-0.3045	-0.2931	-0.6287	-0.1524	-0.2467	-0.0607
2013	-0.2974	-0.2986	-0.6548	-0.1677	-0.2525	-0.0508
2014	-0.2911	-0.2987	-0.6418	-0.1814	-0.2686	-0.0399
2015	-0.2694	-0.2962	-0.6228	-0.1898	-0.2651	-0.0393
2016	-0.2633	-0.2941	-0.6128	-0.1930	-0.2585	-0.0433
2017	-0.2638	-0.2958	-0.5960	-0.1996	-0.2401	-0.0471
2018	-0.2654	-0.2932	-0.5750	-0.2012	-0.2452	-0.0522
2019	-0.2616	-0.2927	-0.5703	-0.2032	-0.2284	-0.0590
2020	-0.2667	-0.2772	-0.5382	-0.2036	-0.2061	-0.0538
2021	-0.2551	-0.2916	-0.5523	-0.2159	-0.2057	-0.0583
2022	-0.2435	-0.3061	-0.5588	-0.2126	-0.2121	-0.0750

The trends in the above own elasticities of input demand generated by our estimated joint cost function are quite interesting. The magnitudes of the Import, Machinery and Equipment and Other Capital Services (inputs 1, 3 and 5) own elasticities of demand trended *downward* over the sample period, indicating that the magnitude of substitution possibilities for these inputs declined over time.[*44] On the other hand, the magnitudes of the Structures and Land Services (inputs 4 and 6) own elasticities of demand trended *upward* over the sample period but the size of these elasticities was small over the entire sample period. During land price bubbles, the own elasticity of demand for land tended to increase in magnitude. Finally, the own elasticity of demand for labour remained relatively constant over the entire sample period.

[*44] Perhaps this phenomenon is due to the growth of multinationals and the resulting internal international supply chains. Another possible explanation is the growth of services relative to goods production in most countries: services inputs may be less substitutable than goods inputs.

Appendix B: Proof of the Flexibility of the Normalized Quadratic Joint Cost Function

A *flexible functional form* for a joint cost function $C(\boldsymbol{y}, \boldsymbol{w})$ has enough free parameters so that it can provide a second order Taylor series approximation to an arbitrary twice continuously differentiable joint cost function $C^*(\boldsymbol{y}, \boldsymbol{w})$ at an arbitrary point $(\boldsymbol{y}^*, \boldsymbol{w}^*)$. We assume that both $C(\boldsymbol{y}, \boldsymbol{w})$ and $C^*(\boldsymbol{y}, \boldsymbol{w})$ are linearly homogeneous in the components of $\boldsymbol{y}$ holding $\boldsymbol{w}$ constant[*45] and are linearly homogeneous in the components of $\boldsymbol{w}$ holding $\boldsymbol{y}$ constant.[*46] It turn out that the linear homogeneity assumptions and the assumption of twice continuous differentiability imply that $C(\boldsymbol{y}, \boldsymbol{w})$ will be a flexible functional form if it has enough parameters to satisfy the following conditions:[*47]

$$\nabla^2_{yw} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla^2_{yw} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*); \qquad MN \text{ restrictions;} \tag{B1}$$

$$\nabla^2_{yy} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla^2_{yy} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*); \qquad M(M-1)/2 \text{ independent restrictions;} \tag{B2}$$

$$\nabla^2_{ww} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla^2_{ww} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*); \qquad N(N-1)/2 \text{ independent restrictions.} \tag{B3}$$

There are M^2 restrictions in the matrix equation (B2) but Young's Theorem in calculus implies that the upper triangle of matrix elements in the matrix of second order partial derivatives of $C(\boldsymbol{y}^*, \boldsymbol{w}^*)$ is equal to the lower triangle; i.e., $[\nabla^2_{yy} C(\boldsymbol{y}^*, \boldsymbol{w}^*)]^T = [\nabla^2_{yy} C(\boldsymbol{y}^*, \boldsymbol{w}^*)]$ and similarly, $[\nabla^2_{yy} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)]^T = [\nabla^2_{yy} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)]$. Thus there are only $M(M+1)/2$ independent restrictions on the second order partial derivatives of $C(\boldsymbol{y}^*, \boldsymbol{w}^*)$ in the matrix equation (B2). But due to the linear homogeneity of $C(\boldsymbol{y}, \boldsymbol{w})$ in the components of $\boldsymbol{y}$, Euler's Theorem on homogeneous functions implies the following M restrictions on the second order partial derivatives of $C(\boldsymbol{y}^*, \boldsymbol{w}^*)$ and $C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$:

$$\nabla^2_{yy} C(\boldsymbol{y}^*, \boldsymbol{w}^*)\boldsymbol{y}^* = \mathbf{0}_M; \; \nabla^2_{yy} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)\boldsymbol{y}^* = \mathbf{0}_M. \tag{B4}$$

Since the M by M matrices $\nabla^2_{yy} C(\boldsymbol{y}^*, \boldsymbol{w}^*)$ and $\nabla^2_{yy} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$ are symmetric, equality of the upper diagonal elements in equations (B2) plus the $2M$ equations in (B4) will imply equality of all M^2 elements in the matrix equation $\nabla^2_{yy} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla^2_{yy} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$.

Similarly, due to the linear homogeneity of $C(\boldsymbol{y}, \boldsymbol{w})$ in the components of $\boldsymbol{w}$, Euler's Theorem on homogeneous functions implies the following N restrictions on the second order partial

[*45] This assumption implies that the N partial derivative functions, $\partial C(\boldsymbol{y}^*, \boldsymbol{w}^*)/\partial w_n$ for $n = 1, \ldots, N$, are also linearly homogeneous in the components of $\boldsymbol{y}$. Thus by Euler's Theorem on homogeneous functions, $\partial C(\boldsymbol{y}^*, \boldsymbol{w}^*)/\partial w_n = \sum_{m=1}^{M} y^*_m \partial^2 C(\boldsymbol{y}^*, \boldsymbol{w}^*)/\partial w_n \partial y_m$ for $n = 1, \ldots, N$. These equations imply that $\nabla_w C(\boldsymbol{y}^*, \boldsymbol{w}^*)^T = \boldsymbol{y}^{*T} \nabla^2_{yw} C(\boldsymbol{y}^*, \boldsymbol{w}^*)$. Similarly, $\nabla_w C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)^T = \boldsymbol{y}^{*T} \nabla^2_{yw} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$. Thus if $\nabla^2_{yw} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla^2_{yw} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$, then $\nabla_w C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla_w C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$.

[*46] This assumption implies that the M partial derivative functions, $\partial C(\boldsymbol{y}^*, \boldsymbol{w}^*)/\partial y_m$ for $m = 1, \ldots, M$, are also linearly homogeneous in the components of $\boldsymbol{w}$. Thus by Euler's Theorem on homogeneous functions, $\partial C(\boldsymbol{y}^*, \boldsymbol{w}^*)/\partial y_m = \sum_{n=1}^{N} w^*_n \partial^2 C(\boldsymbol{y}^*, \boldsymbol{w}^*)/\partial y_m \partial w_n$ for $n = 1, \ldots, N$. These equations imply that $\nabla_y C(\boldsymbol{y}^*, \boldsymbol{w}^*)^T = \boldsymbol{w}^{*T} \nabla^2_{wy} C(\boldsymbol{y}^*, \boldsymbol{w}^*)$. Similarly, $\nabla_y C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)^T = \boldsymbol{w}^{*T} \nabla^2_{wy} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$. Thus if $\nabla^2_{yw} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla^2_{yw} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$, then $\nabla^2_{wy} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla^2_{wy} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$ and $\nabla_y C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla_y C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$. This last equality implies that $C(\boldsymbol{y}^*, \boldsymbol{w}^*) = C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$ since Euler's Theorem implies that $C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \boldsymbol{y}^{*T} \nabla_y C(\boldsymbol{y}^*, \boldsymbol{w}^*)$ and $C^*(\boldsymbol{y}^*, \boldsymbol{w}^*) = \boldsymbol{y}^{*T} \nabla_y C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$.

[*47] Note that the total number of free parameters that $C(\boldsymbol{y}, \boldsymbol{w})$ must have in order to be a flexible functional form for a joint cost function is $MN + (1/2)M(M-1) + (1/2)N(N-1) = (M+N)(M+N-1)/2$. This is the same number of free parameters that is required in order for $f(y_2, \ldots, y_M, x_1, \ldots, x_N)$ to be a flexible functional form for a constant returns to scale production function.

derivatives of $C(\boldsymbol{y}^*, \boldsymbol{w}^*)$ and $C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$:

$$\nabla^2_{ww} C(\boldsymbol{y}^*, \boldsymbol{w}^*)\boldsymbol{w}^* = \mathbf{0}_N;\ \nabla^2_{ww} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)\boldsymbol{w}^* = \mathbf{0}_N. \tag{B5}$$

Since the N by N matrices $\nabla^2_{ww} C(\boldsymbol{y}^*, \boldsymbol{w}^*)$ and $\nabla^2_{ww} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$ are symmetric, equality of the upper diagonal elements in equations (B3) plus the N equations in (B5) will imply equality of all N^2 elements in the matrix equation $\nabla^2_{ww} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \nabla^2_{ww} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$.

Let $\boldsymbol{y}^* \equiv [y_1^*, \ldots, y_M^*] \gg \mathbf{0}_M$ be a positive reference output vector and let $\boldsymbol{w}^* \equiv [w_1^*, \ldots, w_N^*] \gg \mathbf{0}_N$ be a positive vector of reference input prices. Let $\boldsymbol{\alpha} \equiv [\alpha_1, \ldots, \alpha_N] \gg \mathbf{0}_N$ and $\boldsymbol{\beta} \equiv [\beta_1, \ldots, \beta_M] \gg \mathbf{0}_M$ be positive vector of predetermined constants and that satisfy the following linear restrictions:

$$\boldsymbol{\alpha}^T \boldsymbol{w}^* = 1;\ \boldsymbol{\beta}^T \boldsymbol{y}^* = 1. \tag{B6}$$

The basic *Normalized Quadratic Joint Cost Function*, $C(\boldsymbol{y}, \boldsymbol{w})$ is defined as follows:[*48]

$$C(\boldsymbol{y}, \boldsymbol{w}) \equiv (1/2)(\boldsymbol{w}^T \mathbf{A} \boldsymbol{w})(\boldsymbol{\alpha}^T \boldsymbol{w})^{-1}(\boldsymbol{\beta}^T \boldsymbol{y}) + (1/2)(\boldsymbol{y}^T \mathbf{B} \boldsymbol{y})(\boldsymbol{\alpha}^T \boldsymbol{w})(\boldsymbol{\beta}^T \boldsymbol{y})^{-1} + \boldsymbol{w}^T \mathbf{D} \boldsymbol{y}. \tag{B7}$$

The M by N matrix $\mathbf{D}$ is unrestricted. Assume that the matrix $\mathbf{A}$ has the following properties:

$$\mathbf{A} \text{ is a negative semidefinite } N \text{ by } N \text{ matrix}; \tag{B8}$$

$$\mathbf{A} \text{ is symmetric so that } \mathbf{A} = \mathbf{A}^T; \tag{B9}$$

$$\mathbf{A}\boldsymbol{w}^* = \mathbf{0}_N. \tag{B10}$$

Assume that the matrix $\mathbf{B}$ has the following properties:

$$\mathbf{B} \text{ is a positive semidefinite } M \text{ by } M \text{ matrix}; \tag{B11}$$

$$\mathbf{B} \text{ is symmetric so that } \mathbf{B} = \mathbf{B}^T; \tag{B12}$$

$$\mathbf{B}\boldsymbol{y}^* = \mathbf{0}_M. \tag{B13}$$

Now compute the first and second order partial derivatives of $C(\boldsymbol{y}, \boldsymbol{w})$ defined by (B7) and evaluate them at the point $(\boldsymbol{y}^*, \boldsymbol{w}^*)$. Using the restrictions (B6), (B10) and (B13), we obtain the following first and second order partial derivatives:

$$\nabla_y C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \mathbf{D}^T \boldsymbol{w}^*; \tag{B14}$$

$$\nabla_w C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \mathbf{D}\boldsymbol{y}^*; \tag{B15}$$

$$\nabla^2_{yy} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \mathbf{B}; \tag{B16}$$

$$\nabla^2_{ww} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \mathbf{A}; \tag{B17}$$

$$\nabla^2_{yw} C(\boldsymbol{y}^*, \boldsymbol{w}^*) = \mathbf{D}^T. \tag{B18}$$

To prove the flexibility of the Normalized Quadratic Joint Cost Function defined by (B7) with the restrictions (B6)-(B13), we need to find matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ that lead to the satisfaction of equations (B1)-(B3). Using equations (B16)-(B18), this is very simple: define $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ as follows:

$$\mathbf{A} \equiv \nabla^2_{ww} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*); \tag{B19}$$

$$\mathbf{B} \equiv \nabla^2_{yy} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*); \tag{B20}$$

$$\mathbf{D} \equiv [\nabla^2_{yw} C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)]^T. \tag{B21}$$

[*48] This functional form is analogous to the Normalized Quadratic Value Added Function $\Pi(\boldsymbol{p}, \boldsymbol{x})$ that was defined in Diewert and Fox (2021)[20].

Under our regularity conditions on the production possibilities set S, it can be shown that $\nabla^2_{ww}C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$ is a symmetric negative semidefinite matrix which satisfies $\nabla^2_{ww}C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)\boldsymbol{w}^* = \mathbf{0}_N$ and hence, the matrix $\mathbf{A}$ will satisfy the restrictions (B8)-(B10). It also can be shown[*49] that $\nabla^2_{yy}C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)$ is a symmetric positive semidefinite matrix which satisfies $\nabla^2_{yy}C^*(\boldsymbol{y}^*, \boldsymbol{w}^*)\boldsymbol{y}^* = \mathbf{0}_M$ and hence, $\mathbf{B}$ will satisfy the restrictions (B11)-(B13). This establishes the flexibility of the basic Normalized Quadratic Joint Cost function.

[*49] See Diewert (2022)[17].

Appendix C: Nonparametric Estimates for Technical Progress

In this section, we will calculate nonparametric estimates for gross output Total Factor Productivity Growth and compare these estimates with our Appendix A index number estimates TFP growth and our main text estimates of technical progress. The analysis in this section is based on Diewert and Fox (2018)[19].[*50] Their nonparametric methodology has the advantage of giving a decomposition of TFP growth into technical progress and inefficiency components. There are two key concepts that their analysis is based on:

- An approximation to the aggregate production possibilities set for an economy can be formed by using linear multiples of past output and input vectors and
- The *Cost Constrained Gross Output Function* can be used to form measures of efficiency, output price change, input price change, input quantity change and technology change.

Denote the year t observed output and input quantity vectors by $\boldsymbol{y}^t$ and $\boldsymbol{x}^t$ and the corresponding price vectors by $\boldsymbol{p}^t$ and $\boldsymbol{w}^t$. Following Diewert and Fox (2018)[19], we assume that the year t national technology set can be approximated by assuming it consists of past observed output and input vectors, $(\boldsymbol{y}^s, \boldsymbol{x}^s)$, and linear multiples of these vectors for past years and the current year t. Let S^t denote the resulting year t production possibilities set. Thus $S^1 \equiv \{(\boldsymbol{y},\boldsymbol{x}) : \boldsymbol{y} = \lambda \boldsymbol{y}^1, \boldsymbol{x} = \lambda \boldsymbol{x}^1; \lambda \geq 0\}$, $S^2 \equiv \{(\boldsymbol{y},\boldsymbol{x}) : \boldsymbol{y} = \lambda_1 \boldsymbol{y}^1, \boldsymbol{x} = \lambda_1 \boldsymbol{x}^1; \lambda_1 \geq 0, \boldsymbol{y} = \lambda_2 \boldsymbol{y}^2, \boldsymbol{x} = \lambda_2 \boldsymbol{x}^2; \lambda_2 \geq 0\}$, ..., $S^t \equiv \{(\boldsymbol{y},\boldsymbol{x}) : \boldsymbol{y} = \lambda_s \boldsymbol{y}^s, \boldsymbol{x} = \lambda_s \boldsymbol{x}^s; \lambda_s \geq 0, s = 1,2,\ldots,t\}$. These definitions for the S^t mean that we are assuming that S^t is a constant return to scale technology set for each period.

The year t *Cost Constrained Gross Output Function* for the US economy, $G^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{x})$, is defined as follows for the positive price vectors $\boldsymbol{p}$ and $\boldsymbol{w}$, positive input vector $\boldsymbol{x}$ and using the period t production possibilities set S^t:[*51]

$$
\begin{aligned}
G^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{x}) &\equiv \max\nolimits_{y,z}\{\boldsymbol{p}\cdot\boldsymbol{y} : (\boldsymbol{y},\boldsymbol{z}) \in S^t; \boldsymbol{w}\cdot\boldsymbol{z} \leq \boldsymbol{w}\cdot\boldsymbol{x}\}; \qquad t = 1970,\ldots,2022. \\
&= \max\nolimits_s\{\boldsymbol{p}\cdot\boldsymbol{y}^s\,\boldsymbol{w}\cdot\boldsymbol{x}/\boldsymbol{w}\cdot\boldsymbol{x}^s : s = 1,2,\ldots,t\}; \\
&= \boldsymbol{w}\cdot\boldsymbol{x}\,\max\nolimits_s\{\boldsymbol{p}\cdot\boldsymbol{y}^s/\boldsymbol{w}\cdot\boldsymbol{x}^s : s = 1,2,\ldots,t\}. \qquad \text{(C1)}
\end{aligned}
$$

Given the period t technology set S^t, output prices $\boldsymbol{p}$, input prices $\boldsymbol{w}$ and the constraint that primary input costs should not exceed cost $\boldsymbol{w}\cdot\boldsymbol{x}$, we assume that producers choose the output vector $\boldsymbol{y}$ and input vector $\boldsymbol{z}$ to maximize national gross output, $\boldsymbol{p}\cdot\boldsymbol{y}$, subject to total input cost $\boldsymbol{w}\cdot\boldsymbol{z}$ to be equal to or less than the given input cost $\boldsymbol{w}\cdot\boldsymbol{x}$.

Due to our assumptions on the year t national production possibilities set S^t, the year t cost constrained value added function $R^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{x})$ can be calculated for any hypothetical $\boldsymbol{p},\boldsymbol{w}$ and $\boldsymbol{x}$ by solving the very simple maximization problem, $\max_s\{\boldsymbol{p}\cdot\boldsymbol{y}^s/\boldsymbol{w}\cdot\boldsymbol{x}^s : s = 1,2,\ldots,t\}$., which involves taking the maximum of t numbers.

The Cost Constrained Gross Output Function defined by equation (C1) can be used to decompose gross output growth from year $t-1$ to year t, $\boldsymbol{p}^t\cdot\boldsymbol{y}^t/\boldsymbol{p}^{t-1}\cdot\boldsymbol{y}^{t-1}$, into various explanatory growth factors. The explanatory factors are as follows:

- efficiency changes,
- changes in real output prices,
- changes in primary inputs,

[*50] The only difference is that we use the Cost Constrained Gross Output Function in place of the Cost Constrained Value Added Function.

[*51] This function is a variant of Diewert's (1983; 1086)[15] *balance of trade restricted value added function.*

- changes in real input prices, and
- technical progress.

We now define the above explanatory factors using the observed data and the function $G^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{x})$ defined by equation (C1). Following the example of Balk (1998; 143)[2], we define the *gross output efficiency* of the production unit for year t, e^t, as follows:

$$e^t \equiv \boldsymbol{p}^t \cdot \boldsymbol{y}^t / G^t(\boldsymbol{p}^t, \boldsymbol{w}^t, \boldsymbol{x}^t) \leq 1; \qquad t = 1970, \ldots, 2022. \tag{C2}$$

where the inequality in equation (C2) follows using definition (C1).[*52] Thus if $e^t = 1$, then production is allocatively efficient in year t and if $e^t < 1$, then production for the sector during year t is allocatively inefficient. Note that the above definition of value added efficiency is a gross output counterpart to Farrell's (1957; 255)[26] cost based measure of *overall efficiency.*

Define an index of the *change in gross output efficiency* ε^t for the production sector over the years $t-1$ and t as follows:

$$\varepsilon^t \equiv e^t / e^{t-1} = [\boldsymbol{p}^t \cdot \boldsymbol{y}^t / G^t(\boldsymbol{p}^t, \boldsymbol{w}^t, \boldsymbol{x}^t)] / [\boldsymbol{p}^{t-1} \cdot \boldsymbol{y}^{t-1} / G^{t-1}(\boldsymbol{p}^{t-1}, \boldsymbol{w}^{t-1}, \boldsymbol{x}^{t-1})]; \quad t = 1971, \ldots, 2022. \tag{C3}$$

Thus if $\varepsilon^t > 1$, then gross output efficiency has *improved* going from year $t-1$ to t whereas it has *fallen* if $\varepsilon^t < 1$.

We turn our attention to defining nonparametric measures of *output price change* going from year $t-1$ to t. Following the example of Konüs (1939)[46] in his analysis of the true cost of living index, it is natural to single out two special cases of a family of real output price indexes: one choice is α_L^t where we use the year $t-1$ technology and set the reference input prices and quantities equal to the year $t-1$ input prices $\boldsymbol{w}^{t-1}$ and primary input quantities $\boldsymbol{x}^{t-1}$ (which gives rise to a *Laspeyres type output price index*) and another choice is α_P^t where we use the year t technology and set the reference input prices and quantities equal to the year t input prices and quantities $\boldsymbol{w}^t$ and $\boldsymbol{x}^t$ (which gives rise to a *Paasche type real output price index*). We then define an overall measure of real output price change α^t by taking the geometric mean of these two indexes. These indexes are defined as follows:

$$\alpha_L^t \equiv G^{t-1}(\boldsymbol{p}^t, \boldsymbol{w}^{t-1}, \boldsymbol{x}^{t-1}) / G^{t-1}(\boldsymbol{p}^{t-1}, \boldsymbol{w}^{t-1}, \boldsymbol{x}^{t-1}); \qquad t = 1971, \ldots, 2022; \tag{C4}$$

$$\alpha_P^t \equiv G^t(\boldsymbol{p}^t, \boldsymbol{w}^t, \boldsymbol{x}^t) / G^t(\boldsymbol{p}^{t-1}, \boldsymbol{w}^t, \boldsymbol{x}^t); \qquad t = 1971, \ldots, 2022; \tag{C5}$$

$$\alpha^t \equiv [\alpha_L^t \alpha_P^t]^{1/2}; \qquad t = 1971, \ldots, 2022. \tag{C6}$$

Two natural measures of *input quantity change* are the Laspeyres and Paasche input quantity indexes. Denote the year t indexes as β_L^t and β_P^t. Again, it is natural to take the geometric average of these two indexes which gives rise to the Fisher ideal input quantity index, β^t. These indexes are defined as follows:

$$\beta_L^t \equiv \boldsymbol{w}^{t-1} \cdot \boldsymbol{x}^t / \boldsymbol{w}^{t-1} \cdot \boldsymbol{x}^{t-1}; \qquad t = 1971, \ldots, 2022; \tag{C7}$$

$$\beta_P^t \equiv \boldsymbol{w}^t \cdot \boldsymbol{x}^t / \boldsymbol{w}^t \cdot \boldsymbol{x}^{t-1} \qquad t = 1971, \ldots, 2022; \tag{C8}$$

$$\beta^t \equiv [\beta_L^t \beta_P^t]^{1/2}; \qquad t = 1971, \ldots, 2022. \tag{C9}$$

We now consider indexes which measure the effects on cost constrained gross output of a change in real input prices going from year $t-1$ to t. Thus we consider measures of

[*52] Use the fact that $(\boldsymbol{y}^s, \boldsymbol{x}^s)$ is a feasible solution for the year t maximization problem defined by (C1).

the change in cost constrained gross output of the form $G^s(\boldsymbol{p}, \boldsymbol{w}^t, \boldsymbol{x})/G^s(\boldsymbol{p}, \boldsymbol{w}^{t-1}, \boldsymbol{x})$. Since $G^s(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{x})$ is homogeneous of degree 0 in the components of $\boldsymbol{w}$, it can be seen that we cannot interpret $G^s(\boldsymbol{p}, \boldsymbol{w}^t, \boldsymbol{x})/G^s(\boldsymbol{p}, \boldsymbol{w}^{t-1}, \boldsymbol{x})$ as an input price index. If there is only one primary input, $G^s(\boldsymbol{p}, \boldsymbol{w}^t, \boldsymbol{x})/G^s(\boldsymbol{p}, \boldsymbol{w}^{t-1}, \boldsymbol{x})$ is equal to $G^s(\boldsymbol{p}, 1, \boldsymbol{x})/G^s(\boldsymbol{p}, 1, \boldsymbol{x}) = 1$ and this measure of input price change will be independent of changes in the price of the single input. In the case where the number of primary inputs is greater than 1, it is best to interpret $G^s(\boldsymbol{p}, \boldsymbol{w}^t, \boldsymbol{x})/G^s(\boldsymbol{p}, \boldsymbol{w}^{t-1}, \boldsymbol{x})$ as measuring the effects on cost constrained gross output of a change in the relative proportions of primary inputs used in production or in the *mix* of inputs used in production that is induced by a change in relative input prices when there is more than one primary input. We consider two special cases of this family of input mix indexes, Case 1 and Case 2. The first case index, γ_1^t, will use the year t Cost Constrained Gross Output function and the year $t-1$ reference vectors $\boldsymbol{p}^{t-1}$ and $\boldsymbol{x}^{t-1}$ while the second case index, γ_2^t, will use the year $t-1$ Cost Constrained Gross Output function and the year t reference vectors $\boldsymbol{p}^t$ and $\boldsymbol{x}^t$. We take the geometric mean of these two indexes γ^t to provide a measure of the overall effects of a change in input prices.*53

$$\gamma_1^t \equiv G^t(\boldsymbol{p}^{t-1}, \boldsymbol{w}^t, \boldsymbol{x}^t)/G^t(\boldsymbol{p}^{t-1}, \boldsymbol{w}^{t-1}, \boldsymbol{x}^t); \qquad t = 1971, \ldots, 2022; \tag{C10}$$

$$\gamma_2^t \equiv G^{t-1}(\boldsymbol{p}^t, \boldsymbol{w}^t, \boldsymbol{x}^{t-1})/G^{t-1}(\boldsymbol{p}^t, \boldsymbol{w}^{t-1}, \boldsymbol{x}^{t-1}); \qquad t = 1971, \ldots, 2022; \tag{C11}$$

$$\gamma^t \equiv [\gamma_1^t \gamma_2^t]^{1/2}; \qquad t = 1971, \ldots, 2022. \tag{C12}$$

Finally, we use the Cost Constrained Gross Output function in order to define measures of *technical progress* going from year $t-1$ to t. These measures hold $\boldsymbol{p}, \boldsymbol{w}$ and $\boldsymbol{x}$ constant and only change the technology from the year $t-1$ technology to the year t technology.*54 Thus, these measures are of the form $G^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{x})/G^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{x})$. If there is positive technical progress going from year $t-1$ to t, then the production possibilities set S^t will be larger than the period $t-1$ set, S^{t-1}, and thus $G^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{x})$ will be equal to or greater than $G^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{x})$ and our measures of technical progress will be equal to or greater than 1. Our measures of technical progress cannot fall below 1.

We consider two measures of technical progress, a Laspeyres measure τ_L^t and a Paasche measure τ_P^t. However, the Laspeyres case τ_L^t will use the year t input vector $\boldsymbol{x}^t$ as the reference input vector and the year $t-1$ reference output price and input price vectors $\boldsymbol{p}^{t-1}$ and $\boldsymbol{w}^{t-1}$ while the Paasche case τ_P^t will use the year $t-1$ input vector $\boldsymbol{x}^{t-1}$ as the reference input and the year t reference output and input price vectors $\boldsymbol{p}^t$ and $\boldsymbol{w}^t$.*55 As usual, we take our overall year t measure of technical change τ^t to be the geometric mean of the Laspeyres and Paasche measures of technical change.

$$\tau_L^t \equiv G^t(\boldsymbol{p}^{t-1}, \boldsymbol{w}^{t-1}, \boldsymbol{x}^t)/G^{t-1}(\boldsymbol{p}^{t-1}, \boldsymbol{w}^{t-1}, \boldsymbol{x}^t); \qquad t = 1971, \ldots, 2022; \tag{C13}$$

$$\tau_P^t \equiv G^t(\boldsymbol{p}^t, \boldsymbol{w}^t, \boldsymbol{x}^{t-1})/G^{t-1}(\boldsymbol{p}^t, \boldsymbol{w}^t, \boldsymbol{x}^{t-1}); \qquad t = 1971, \ldots, 2022; \tag{C14}$$

$$\tau^t \equiv [\tau_L^t \tau_P^t]^{1/2}; \qquad t = 1971, \ldots, 2022. \tag{C15}$$

*53 These choices of the reference vectors will make our decomposition of gross output growth an exact one. Usually, these input mix growth factors are close to one for all periods.

*54 These measures of technical progress measures were defined by Diewert and Morrison (1986; 662)[22] using the country's GDP function. A special case of the family was defined earlier by Diewert (1983; 1063)[15]. Balk (1998; 99)[2] also used this definition and Balk (1998; 58)[2], following the example of Salter (1960)[52], used the joint cost function to define a similar family of technical progress indexes.

*55 In our case where the reference technology is subject to constant returns to scale, τ_L^t turns out to be independent of $\boldsymbol{x}^t$ and τ_P^t turns out to be independent of $\boldsymbol{x}^{t-1}$. These "mixed" indexes of technical progress turn out to be true Laspeyres and Paasche type indexes.

Using the above definitions, it can be shown that the following exact decomposition of year t nominal gross output growth (relative to the 1970 level of gross output) into explanatory growth factors holds:

$$\mathrm{GO}_G^t \equiv \boldsymbol{p}^t \cdot \boldsymbol{y}^t / \boldsymbol{p}^{t-1} \cdot \boldsymbol{y}^{t-1} = \varepsilon^t \alpha^t \beta^t \gamma^t \tau^t; \qquad t = 1971, \ldots, 2022. \tag{C16}$$

Table C1 lists GO_G^t and the growth components on the right hand side of equation (C16).

Table C1: A Nonparametric Decomposition of Gross Output TFP Growth

Year	GO_G^t	ε^t	α^t	β^t	γ^t	τ^t	$NTFP_G^t$	TFP_G^t	κ^t	e^t
1971	1.0862	1.0000	1.0483	1.0093	1.0000	1.0266	1.0266	1.0266	-0.0097	1.0000
1972	1.1048	1.0000	1.0455	1.0340	1.0000	1.0220	1.0220	1.0220	-0.0098	1.0000
1973	1.1201	1.0000	1.0614	1.0417	1.0000	1.0131	1.0131	1.0131	-0.0097	1.0000
1974	1.1028	0.9775	1.1115	1.0156	0.9995	1.0000	0.9770	0.9767	-0.0094	0.9775
1975	1.0825	0.9988	1.0971	0.9879	1.0001	1.0000	0.9989	0.9994	-0.0079	0.9763
1976	1.1209	1.0241	1.0543	1.0373	1.0006	1.0001	1.0249	1.0250	-0.0080	0.9999
1977	1.1203	1.0001	1.0639	1.0371	1.0000	1.0153	1.0153	1.0153	-0.0082	1.0000
1978	1.1341	1.0000	1.0717	1.0450	1.0000	1.0126	1.0126	1.0126	0.0034	1.0000
1979	1.1244	0.9965	1.0911	1.0345	0.9998	1.0000	0.9962	0.9962	0.0032	0.9964
1980	1.0922	0.9866	1.1013	1.0067	0.9986	1.0000	0.9852	0.9851	-0.0007	0.9830
1981	1.1151	1.0085	1.0868	1.0176	0.9997	1.0000	1.0082	1.0084	-0.0009	0.9914
1982	1.0380	0.9815	1.0613	0.9974	0.9991	1.0000	0.9807	0.9813	-0.0009	0.9731
1983	1.0872	1.0237	1.0354	1.0243	1.0014	1.0000	1.0251	1.0247	-0.0117	0.9961
1984	1.1213	1.0039	1.0307	1.0583	1.0009	1.0230	1.0280	1.0277	-0.0118	1.0000
1985	1.0709	1.0000	1.0279	1.0347	1.0000	1.0069	1.0069	1.0069	-0.0118	1.0000
1986	1.0599	1.0000	1.0244	1.0303	1.0000	1.0043	1.0043	1.0043	-0.0120	1.0000
1987	1.0668	1.0000	1.0282	1.0364	1.0000	1.0011	1.0011	1.0011	0.0006	1.0000
1988	1.0790	1.0000	1.0361	1.0308	1.0000	1.0103	1.0103	1.0103	-0.0034	1.0000
1989	1.0768	1.0000	1.0392	1.0343	1.0000	1.0019	1.0019	1.0019	-0.0027	1.0000
1990	1.0572	1.0000	1.0352	1.0214	0.9999	1.0000	0.9999	0.9999	-0.0027	1.0000
1991	1.0269	0.9995	1.0283	0.9994	0.9997	1.0000	0.9992	0.9992	-0.0024	0.9995
1992	1.0602	1.0005	1.0225	1.0186	1.0000	1.0175	1.0180	1.0179	-0.0123	1.0000
1993	1.0551	1.0000	1.0208	1.0266	1.0000	1.0069	1.0069	1.0069	-0.0075	1.0000
1994	1.0670	1.0000	1.0179	1.0370	1.0000	1.0108	1.0108	1.0108	-0.0074	1.0000
1995	1.0555	0.9958	1.0227	1.0363	1.0001	1.0000	0.9959	0.9959	-0.0072	0.9958
1996	1.0584	1.0042	1.0135	1.0306	0.9999	1.0092	1.0133	1.0134	-0.0072	1.0000
1997	1.0660	1.0000	1.0106	1.0457	1.0000	1.0087	1.0087	1.0087	-0.0080	1.0000
1998	1.0568	1.0000	1.0048	1.0460	1.0000	1.0055	1.0055	1.0055	-0.0081	1.0000
1999	1.0703	1.0000	1.0150	1.0445	1.0000	1.0096	1.0096	1.0096	-0.0080	1.0000
2000	1.0783	1.0000	1.0266	1.0457	1.0000	1.0044	1.0044	1.0044	-0.0076	1.0000
2001	1.0234	0.9968	1.0189	1.0079	0.9997	1.0000	0.9965	0.9963	-0.0062	0.9968
2002	1.0305	1.0032	1.0113	1.0154	0.9999	1.0004	1.0036	1.0035	-0.0064	1.0000
2003	1.0522	1.0000	1.0209	1.0140	1.0000	1.0165	1.0165	1.0165	-0.0065	1.0000
2004	1.0775	1.0000	1.0287	1.0298	1.0000	1.0171	1.0171	1.0171	-0.0076	1.0000
2005	1.0754	1.0000	1.0360	1.0276	1.0000	1.0101	1.0101	1.0101	-0.0074	1.0000
2006	1.0652	1.0000	1.0308	1.0313	1.0000	1.0020	1.0020	1.0020	-0.0037	1.0000
2007	1.0509	0.9980	1.0300	1.0227	0.9997	1.0000	0.9976	0.9976	-0.0026	0.9980
2008	1.0289	0.9961	1.0304	1.0031	0.9994	1.0000	0.9955	0.9951	-0.0027	0.9940
2009	0.9512	0.9936	0.9924	0.9635	1.0013	1.0000	0.9949	0.9953	-0.0026	0.9877
2010	1.0578	1.0125	1.0161	1.0184	1.0006	1.0090	1.0222	1.0214	-0.0039	1.0000
2011	1.0486	0.9992	1.0283	1.0206	1.0000	1.0000	0.9992	0.9991	-0.0037	0.9992
2012	1.0395	1.0008	1.0164	1.0187	0.9999	1.0032	1.0039	1.0039	-0.0036	1.0000
2013	1.0319	1.0000	1.0114	1.0167	1.0000	1.0035	1.0035	1.0035	-0.0036	1.0000
2014	1.0428	1.0000	1.0137	1.0244	1.0000	1.0042	1.0042	1.0042	-0.0034	1.0000
2015	1.0290	1.0000	0.9975	1.0264	1.0000	1.0051	1.0051	1.0051	-0.0034	1.0000
2016	1.0219	0.9981	1.0053	1.0185	1.0000	1.0000	0.9980	0.9980	-0.0035	0.9980
2017	1.0465	1.0020	1.0186	1.0224	0.9999	1.0031	1.0049	1.0049	-0.0037	1.0000
2018	1.0545	1.0000	1.0229	1.0230	1.0000	1.0077	1.0077	1.0077	-0.0036	1.0000
2019	1.0366	1.0000	1.0134	1.0189	1.0000	1.0040	1.0040	1.0040	-0.0037	1.0000
2020	0.9805	0.9943	1.0128	0.9742	0.9994	1.0000	0.9937	0.9935	-0.0030	0.9943
2021	1.1183	1.0057	1.0482	1.0391	1.0003	1.0206	1.0267	1.0270	-0.0119	1.0000
2022	1.1000	0.9884	1.0722	1.0381	0.9999	1.0000	0.9883	0.9883	-0.0117	0.9884
Mean	1.0657	0.9998	1.0348	1.0238	0.99998	1.0061	1.0059	1.0059	-0.0057	0.9970

On average, US nominal gross output grew 6.57% per year. The explanatory factors for this average growth rate are as follows: efficiency growth subtracted 0.02% per year, output price inflation contributed 3.48% per year, input quantity growth contributed 2.38% per year, the input mix effect on average was negligible, and technical progress contributed 0.61% per year. The level of efficiency e^t was negative for 17 years: 1974-1976 (first oil shock recession), 1979-1983 (second oil shock recession), 1991, 1995, 2007-2009 (financial crisis), 2011, 2016, 2020 (covid shock) and 2022. For these years, production was in the interior of our estimated production possibilities set.

A new nonparametric measure of TFP growth for the economy going from year $t-1$ to t can be defined (following Jorgenson and Griliches (1967)[31]) as an index of output growth divided by an index of input growth. An appropriate index of output growth is the gross output ratio $\boldsymbol{p}^t \cdot \boldsymbol{y}^t / \boldsymbol{p}^{t-1} \cdot \boldsymbol{y}^{t-1}$ divided by the gross output price index α^t. An appropriate index of input growth is β^t. Thus define the *nonparametric year t Gross Output TFP growth rate*, NTFP_G^t, for the US economy as follows:

$$\mathrm{NTFP}_G^t \equiv \{[\boldsymbol{p}^t \cdot \boldsymbol{y}^t / \boldsymbol{p}^{t-1} \cdot \boldsymbol{y}^{t-1}]/\alpha^t\}/\beta^t = \varepsilon^t \gamma^t \tau^t; \qquad t = 1971, \ldots, 2022. \tag{C17}$$

where the last equality in equation (C17) follows from equation (C16). The nonparametric year t TFP growth, NTFP_G^t, is equal to the product of period t efficiency change ε^t, the period t input mix index γ^t (which typically will be small in magnitude) and the period t measure of technical progress τ^t. The nonparametric measure of TFP growth, NTFP_G^t, and our old index number measure of TFP growth, TFP_G^t from Table A13 in Appendix A, are listed above in Table C1. Note that the index number and nonparametric estimates of TFP growth are almost identical. Note also that on average TFP growth (measured by both methods) is equal to 0.59% per year over the years 1971-2022. The average rate of technical progress is slightly higher at 0.61% per year. On average, inefficiency dragged down technical progress by 0.02% per year. Preventing recessions is important. Using the Diewert and Fox nonparametric methodology, technical progress can never be negative.

Table C1 also lists the Joint Cost Function based estimates of cost saving technical charge for each year t, κ^t. Taking the negative of these measures and adding one should lead to measures that are approximately equal to our measures of TFP growth. From viewing Table C1, it can be seen that this approximation is not very close. As was mentioned in the main text, the econometric estimates of technical progress are not able to capture year to year shocks to TFP.*56 However, as was noted in the main text, the long term trends in cost saving technical change are fairly close to the long term trends in TFP growth.

We follow the example of Kohli (1990)[39] in order to obtain a levels decomposition for the observed level of nominal gross output in year t, $\boldsymbol{p}^t \cdot \boldsymbol{y}^t$, relative to its observed value in year 1 of our sample, $\boldsymbol{p}^{1970} \cdot \boldsymbol{y}^{1970}$. Define the cumulated explanatory variables as follows:

$$E^{1970} \equiv 1; A^{1970} \equiv 1; B^{1970} \equiv 1; C^{1970} \equiv 1; T^{1970} \equiv 1. \tag{C18}$$

For $t = 1971, \ldots, 2022$, define the above variables recursively as follows:

$$E^t \equiv \varepsilon^t E^{t-1}; A^t \equiv \alpha^t A^{t-1}; B^t \equiv \beta^t B^{t-1}; C^t \equiv \gamma^t C^{t-1}; T^t \equiv \tau^t T^{t-1}; \quad t = 1971, \ldots, 2022. \tag{C19}$$

*56 There is another problem with our joint cost function methodology: it assumes optimizing behavior on the part of producers, which is not consistent with observed outputs and inputs being in the interior of the production possibilities set. The Diewert and Fox nonparametric methodology allows for nonoptimizing behavior on the part of producers but it has the disadvantage that the year t reference technology that their methodology utilizes may not be a very close approximation to the actual year t production possibilities set.

Using the above definitions, it can be seen that we have the following *levels decomposition* for the level of year t observed gross output relative to its level in 1970:

$$\mathrm{GO}^t \equiv \boldsymbol{p}^t \cdot \boldsymbol{y}^t / \boldsymbol{p}^{1970} \cdot \boldsymbol{y}^{1970} = A^t B^t C^t E^t T^t; \qquad t = 1970, \ldots, 2022. \tag{C20}$$

The components of the levels decomposition of gross output given by (C20) are listed in Table C2.

Table C2: The Levels Decomposition of Gross Output for the US 1970-2022

Year	GO^t	E^t	A^t	B^t	C^t	T^t	NTFP^t
1970	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1971	1.0862	1.0000	1.0483	1.0093	1.0000	1.0266	1.0266
1972	1.2000	1.0000	1.0960	1.0436	1.0000	1.0492	1.0492
1973	1.3442	1.0000	1.1633	1.0871	1.0000	1.0629	1.0629
1974	1.4824	0.9775	1.2929	1.1040	0.9995	1.0629	1.0385
1975	1.6047	0.9763	1.4184	1.0906	0.9995	1.0629	1.0373
1976	1.7986	0.9999	1.4954	1.1313	1.0002	1.0631	1.0631
1977	2.0150	1.0000	1.5910	1.1733	1.0001	1.0793	1.0794
1978	2.2852	1.0000	1.7052	1.2261	1.0001	1.0929	1.0930
1979	2.5696	0.9964	1.8605	1.2684	0.9999	1.0929	1.0889
1980	2.8066	0.9830	2.0488	1.2769	0.9985	1.0929	1.0728
1981	3.1296	0.9914	2.2268	1.2994	0.9982	1.0929	1.0816
1982	3.2486	0.9731	2.3632	1.2960	0.9973	1.0929	1.0607
1983	3.5319	0.9961	2.4469	1.3275	0.9987	1.0929	1.0873
1984	3.9604	1.0000	2.5220	1.4049	0.9997	1.1181	1.1177
1985	4.2410	1.0000	2.5924	1.4537	0.9997	1.1258	1.1254
1986	4.4950	1.0000	2.6556	1.4977	0.9997	1.1305	1.1302
1987	4.7951	1.0000	2.7304	1.5522	0.9997	1.1318	1.1314
1988	5.1738	1.0000	2.8289	1.6000	0.9997	1.1435	1.1431
1989	5.5709	1.0000	2.9397	1.6548	0.9997	1.1456	1.1452
1990	5.8897	1.0000	3.0431	1.6901	0.9996	1.1456	1.1451
1991	6.0478	0.9995	3.1292	1.6890	0.9993	1.1456	1.1443
1992	6.4121	1.0000	3.1997	1.7204	0.9993	1.1657	1.1648
1993	6.7652	1.0000	3.2661	1.7661	0.9993	1.1737	1.1728
1994	7.2182	1.0000	3.3247	1.8314	0.9993	1.1863	1.1855
1995	7.6186	0.9958	3.4001	1.8980	0.9993	1.1863	1.1806
1996	8.0634	1.0000	3.4459	1.9561	0.9992	1.1972	1.1963
1997	8.5955	1.0000	3.4824	2.0455	0.9992	1.2077	1.2067
1998	9.0833	1.0000	3.4991	2.1395	0.9992	1.2143	1.2133
1999	9.7215	1.0000	3.5515	2.2346	0.9992	1.2260	1.2250
2000	10.4823	1.0000	3.6459	2.3368	0.9992	1.2313	1.2304
2001	10.7270	0.9968	3.7147	2.3552	0.9989	1.2313	1.2261
2002	11.0542	1.0000	3.7567	2.3914	0.9989	1.2318	1.2304
2003	11.6316	1.0000	3.8351	2.4248	0.9989	1.2522	1.2508
2004	12.5328	1.0000	3.9452	2.4970	0.9989	1.2737	1.2722
2005	13.4772	1.0000	4.0872	2.5660	0.9989	1.2865	1.2850
2006	14.3553	1.0000	4.2132	2.6462	0.9989	1.2891	1.2876
2007	15.0854	0.9980	4.3394	2.7063	0.9985	1.2891	1.2846
2008	15.5213	0.9940	4.4711	2.7147	0.9980	1.2891	1.2788
2009	14.7643	0.9877	4.4372	2.6154	0.9992	1.2891	1.2722
2010	15.6176	1.0000	4.5089	2.6634	0.9998	1.3007	1.3005
2011	16.3770	0.9992	4.6364	2.7182	0.9998	1.3007	1.2995
2012	17.0230	1.0000	4.7126	2.7691	0.9997	1.3049	1.3045
2013	17.5663	1.0000	4.7661	2.8154	0.9997	1.3095	1.3091
2014	18.3180	1.0000	4.8312	2.8842	0.9997	1.3150	1.3146
2015	18.8494	1.0000	4.8190	2.9604	0.9997	1.3217	1.3213
2016	19.2614	0.9980	4.8445	3.0151	0.9996	1.3217	1.3186
2017	20.1568	1.0000	4.9344	3.0826	0.9995	1.3258	1.3251
2018	21.2546	1.0000	5.0474	3.1534	0.9995	1.3360	1.3354
2019	22.0333	1.0000	5.1151	3.2129	0.9995	1.3413	1.3407
2020	21.6040	0.9943	5.1806	3.1301	0.9989	1.3413	1.3323
2021	24.1606	1.0000	5.4305	3.2526	0.9993	1.3689	1.3679
2022	26.5764	0.9884	5.8224	3.3764	0.9992	1.3689	1.3519

Define the period t level of Nonparametric Gross Output TFP relative to 1970, NTFP^t, as follows:

$$\mathrm{NTFP}^{1970} \equiv 1; \mathrm{NTFP}^t \equiv (\mathrm{NTFP}_G^t)(\mathrm{NTFP}^{t-1}); \qquad t = 1971, \ldots, 2022 \tag{C21}$$

where NTFP_G^t is defined by equation (C17). Using definitions (C18)–(C21), it can be seen that we have the following *levels decomposition for Nonparametric TFP* using the cumulated explanatory factors defined by definitions (C18) and (C19):

$$\mathrm{NTFP}^t \equiv \mathrm{GO}^t/[A^t B^t] = C^t E^t T^t; \qquad t = 1970, \ldots, 2022. \tag{C22}$$

The series NTFP^t is listed in Table C2 above and the decomposition given by (C22) is plotted on Figure 2.

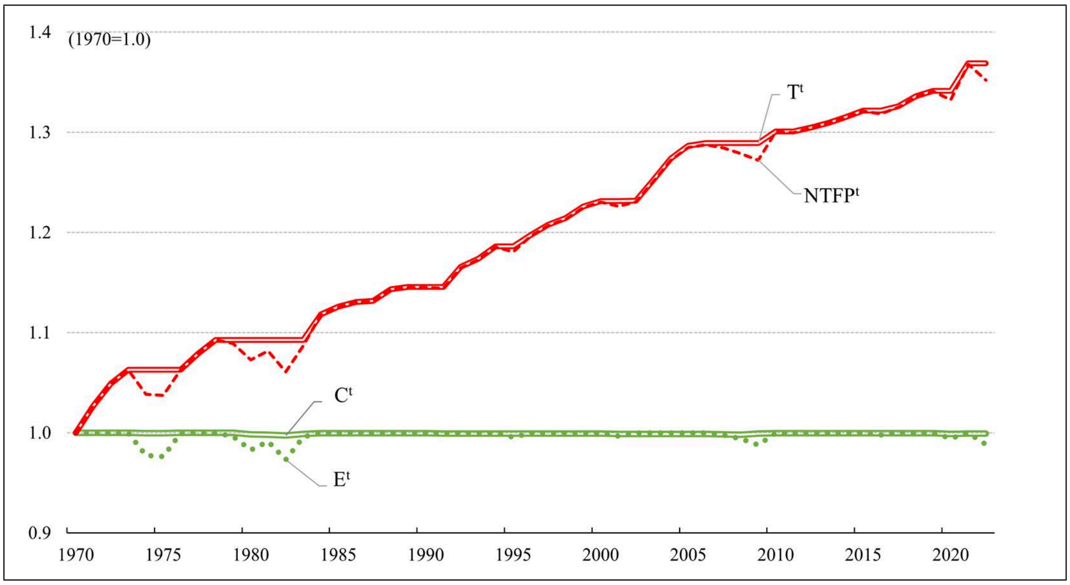

Figure 2 Decomposition of Nonparametric Gross Output TFP into Explanatory Factors

It can be seen that the oil shocks in the 1970s led to large amounts of inefficiency which led to TFP declines. The financial crisis of 2007 and the effects of Covid in 2020 and 2022 also led to inefficiency but the resulting declines in TFP were not as severe.

Appendix D: Converting Cost Function Elasticities into Gross Output Function Elasticities

It is useful to be able to convert the elasticities of inverse output supply and input demand generated by the estimation of a joint cost function into elasticities of output supply and inverse input demand elasticities. We indicate how this can be done in this Appendix.[*57]

Suppose we have estimated a joint cost function $C(\boldsymbol{y},\boldsymbol{w})$ for a particular year where C is twice continuously differentiable with respect to the components of $\boldsymbol{y}$ and $\boldsymbol{w}$ at the year t values for $\boldsymbol{y}$ and $\boldsymbol{w}$. Equations (6) and (7) in section 2 give us an output price vector $\boldsymbol{p}$ and an input demand vector $\boldsymbol{x}$ as functions of the derivatives of the cost function; i.e., $\boldsymbol{p} = \nabla_y C(\boldsymbol{y},\boldsymbol{w})$ and $\boldsymbol{x} = \nabla_w C(\boldsymbol{y},\boldsymbol{w})$. These equations implicitly define $\boldsymbol{y}$ and $\boldsymbol{w}$ as functions of $\boldsymbol{p}$ and $\boldsymbol{x}$, say $\boldsymbol{y} = \boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x})$ and $\boldsymbol{w} = \boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x})$ so we have the following system of 10 equations for our particular example:

$$\boldsymbol{p} = \nabla_y C(\boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x}),\boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x})); \tag{D1}$$

$$\boldsymbol{x} = \nabla_w C(\boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x}),\boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x})). \tag{D2}$$

Denote the 4 by 4 matrix of derivatives of the output supply functions $\boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x})$ with respect to the components of $\boldsymbol{p}$ as $\nabla_p \boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x})$ and the 4 by 6 matrix of derivatives of the functions $\boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x})$ with respect to the components of $\boldsymbol{x}$ as $\nabla_x \boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x})$. Denote the 6 by 4 matrix of derivatives of the inverse input demand functions $\boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x})$ with respect to the components of $\boldsymbol{p}$ as $\nabla_p \boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x})$ and the 6 by 6 matrix of derivatives of the inverse demand functions $\boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x})$ with respect to the components of $\boldsymbol{x}$ as $\nabla_x \boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x})$. If the 10 by 10 matrix of second order partial derivatives of the joint cost function $C(\boldsymbol{y},\boldsymbol{w})$, S_{GO}, has an inverse, then the first order partial derivatives of the functions $\boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x})$ and $\boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x})$ can be defined in terms of the elements of the inverse of the matrix of second order partial derivatives of the joint cost function as follows:[*58]

$$\begin{bmatrix} \nabla_p \boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x}) & \nabla_x \boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x}) \\ \nabla_p \boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x}) & \nabla_x \boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} \nabla^2_{yy} C(\boldsymbol{y},\boldsymbol{w}) & \nabla^2_{yw} C(\boldsymbol{y},\boldsymbol{w}) \\ \nabla^2_{wy} C(\boldsymbol{y},\boldsymbol{w}) & \nabla^2_{ww} C(\boldsymbol{y},\boldsymbol{w}) \end{bmatrix}^{-1} = [S_{GO}]^{-1}. \tag{D3}$$

It can be shown that the 4 by 4 matrix of Gross Output supply functions with respect to output prices, $\nabla_p \boldsymbol{y}^*(\boldsymbol{p},\boldsymbol{x}) = [\partial y^*_m(\boldsymbol{p},\boldsymbol{x})/\partial p_k]$, is a positive semidefinite matrix with rank equal to or less than 3. For our 2022 data, this matrix has rank equal to 2. It can also be shown that the 6 by 6 matrix of derivatives of input prices with respect to input quantities, $\nabla_x \boldsymbol{w}^*(\boldsymbol{p},\boldsymbol{x}) = [\partial w^*_n(\boldsymbol{p},\boldsymbol{x})/\partial x_j]$, is a negative semidefinite matrix with rank equal to or less than 5. For our 2022 data, this matrix has rank equal to 4. These derivatives can be converted into elasticities. Thus define $Ey_m p_k \equiv [\partial y^*_m(\boldsymbol{p},\boldsymbol{x})/\partial p_k][p_k/y_m], Ey_m x_n \equiv [\partial y^*_m(\boldsymbol{p},\boldsymbol{x})/\partial x_n][x_n/y_m], Ew_n x_j \equiv [\partial w^*_n(\boldsymbol{p},\boldsymbol{x})/\partial x_j][x_j/w_n]$ and $Ew_n p_m \equiv [\partial w^*_n(\boldsymbol{p},\boldsymbol{x})/\partial p_m][p_m/w_n]$ for $n = 1,\ldots,6; j = 1,\ldots,6; m = 1,\ldots,4$ and $k = 1,\ldots,4$. These estimated elasticities for the US Gross Output function for 2022 are listed in Tables D1-D4 below.

[*57] This line of research was initiated by Kohli (1981b)[35] (1983b)[38] (1991)[40].

[*58] This follows from the Inverse Function Theorem in advanced calculus; see Rudin (1953; 177)[51].

Table D1: Gross Output Elasticities with Respect to Changes in Gross Output Prices for 2022

$Ey_m p_k$		p_1 C	p_2 G	p_3 I	p_4 X
y_1	C	0.778	-0.416	-0.517	0.154
y_2	G	-1.949	1.962	1.211	-1.224
y_3	I	-1.557	0.779	1.038	-0.260
y_4	X	0.947	-1.602	-0.530	1.185

Note: Consumption (C), Government (G), Investment (I), Export (X).

Note that all main diagonal elasticities are positive; if an output price increases, output supply also increases.

Table D2: Gross Output Elasticities with Respect to Changes in Input Quantities for 2022

$Ey_m x_n$		x_1 M	x_2 L	x_3 $K_{M\&E}$	x_4 K_S	x_5 K_O	x_6 K_L
y_1	C	-0.885	1.722	0.621	1.509	-0.369	-1.598
y_2	G	-0.640	1.239	-2.719	-1.716	-2.433	7.269
y_3	I	3.363	-3.551	0.210	-3.745	3.564	1.158
y_4	X	0.925	0.004	0.106	2.388	-1.011	-1.412

Note: Consumption (C), Government (G), Investment (I), Export (X), Imports (M), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

Table D3: Input Price Elasticities with Respect to Changes in Input Quantities for 2022

$Ew_n x_j$		x_1 M	x_2 L	x_3 $K_{M\&E}$	x_4 K_S	x_5 K_O	x_6 K_L
w_1	M	-5.179	7.258	-0.973	1.831	-8.220	5.283
w_2	L	2.197	-4.035	0.696	-0.734	4.153	-2.278
w_3	$K_{M\&E}$	-1.874	4.432	-2.309	-1.280	-5.556	6.588
w_4	K_S	1.552	-2.056	-0.564	-5.341	0.992	5.417
w_5	K_O	-15.605	26.052	-5.477	2.221	-29.136	21.945
w_6	K_L	11.884	-16.930	7.693	14.368	26.001	-43.015

Note: Imports (M), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

Note that all main diagonal elasticities are negative; if economy wide amount of an input increases, then the demand clearing price for that increased input quantity decreases. Note that the own (inverse) elasticities of demand for Other Capital Services and Land Services (inputs 5 and 6) are *very large* in magnitude. This reflects the fact that the (direct) elasticities of demand for these inputs are *very small* in magnitude.

Table D4: Input Price Elasticities with Respect to Changes in Gross Output Prices for 2022

$Ew_n p_m$		p_1 M	p_2 L	p_3 $K_{M\&E}$	p_4 K_S
w_1	M	-3.612	-0.558	4.555	0.615
w_2	L	2.128	0.327	-1.456	0.001
w_3	$K_{M\&E}$	4.879	-4.565	0.549	0.136
w_4	K_S	5.222	-1.269	-4.302	1.348
w_5	K_O	-2.862	-4.027	9.166	-1.278
w_6	K_L	-14.668	14.252	3.529	-2.114

Note: Imports (M), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

Instead of working with the Gross Output function to model aggregate production in an economy, it is of interest to work with the GDP function. This model of production dates back to Samuelson (1953)[54][*59] and the model assumes that the prices $\boldsymbol{p} \equiv [p_1, p_2, p_3, p_4]$ and w_1 (the price of imports) are fixed for outputs $\boldsymbol{y} \equiv [y_1, y_2, y_3, y_4]$ and imports x_1 and the economy's endowment vector of primary inputs $\boldsymbol{x}^\circ \equiv [x_2, x_3, x_4, x_5, x_6]$ is also viewed as being fixed.[*60] Thus in this model of production, world prices are fixed and the country's endowment of labour and capital are fixed in the short run. Denote the vector of input prices that corresponds to $\boldsymbol{x}^\circ$ (the new vector of fixed factors of production) by $\boldsymbol{w}^\circ \equiv [w_2, w_3, w_4, w_5, w_6]$.

Using our new notation explained in the previous paragraph, the 10 estimating equations for our (old) joint cost function model for the year 2022 can be written as follows:

$$\boldsymbol{p} = \nabla_y C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ); \tag{D4}$$

$$x_1 = \partial C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ)/\partial w_1; \tag{D5}$$

$$\boldsymbol{x}^\circ = \nabla_{w^\circ} C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ) \tag{D6}$$

where $C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ) \equiv C^{2022}(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ) = C^{2022}(\boldsymbol{y}, \boldsymbol{w})$. We now shift imports x_1 from being an input in the Gross Output model of production to imports being a negative output which leads to the *GDP model of production* where $\text{GDP}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ is defined as the maximum of $\boldsymbol{p} \cdot \boldsymbol{y} - w_1 x_1$ (with respect to x_1 and the components of $\boldsymbol{y}$), subject to $(\boldsymbol{y}, x_1, \boldsymbol{x}^\circ) \in S^{2022}$. The solution functions are denoted by $\boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ), x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$, The vector of equilibrium primary input prices is denoted by $\boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) = \nabla_{x^\circ}\text{GDP}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$.[*61]

The first order partial derivatives of the functions $\boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \equiv [y_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ), \ldots, y_4(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)]^T$ and $\boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \equiv [w_2^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ), \ldots, w_6^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)]^T$ can be obtained by differentiating both sides of the following 9 equations[*62] with respect to the 9 components of

[*59] Kohli (1978)[33] (1982)[36] (1991)[40] (1993a)[41] also worked extensively with this model of production.

[*60] In order to define the US GDP function for 2022, the $\boldsymbol{p}$ and $\boldsymbol{y}$ in definition (2) are replaced by $(p_1^{2022f}, \ldots, p_4^{2022f}, w_1^{2022})$ and $(y_1^{2022}, \ldots, y_4^{2022}, -x_1^{2022f})$, the $\boldsymbol{w}$ and $\boldsymbol{x}$ in definition (2) are replaced by $[x_2^{2022f}, \ldots, x_6^{2022f}] \equiv \boldsymbol{x}^\circ$ and $[w_2^{2022}, \ldots, w_6^{2022}] \equiv \boldsymbol{w}^\circ$ and S is the year 2022 production possibilities set for the US economy. The cost function that is used in this Appendix is the estimated US Joint Cost Function for the US for the year 2022, $C(\boldsymbol{y}, \boldsymbol{w}) \equiv C^t(\boldsymbol{y}, \boldsymbol{w})$ where $t = 2022$. The partial derivatives of C are evaluated at $(\boldsymbol{y}, \boldsymbol{w}) = (y^{2022}, w^{2022})$. When we form elasticities, we use predicted values for prices and quantities that are used to convert estimated derivatives into elasticities.

[*61] For the properties of joint cost functions and the relationship between the joint cost function and the GDP function, see the section on joint cost functions in Diewert (2022)[17].

[*62] These equations follow from equations (D4) and (D6). The price of imports w_1° is held constant and so equation (D5) is dropped for now.

$\boldsymbol{p}$ and $\boldsymbol{x}^\circ$:

$$\boldsymbol{p} = \nabla_y C(\boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ), w_1, \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)); \tag{D7}$$

$$\boldsymbol{x}^\circ = \nabla_{w^\circ} C(\boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ), w_1, \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)). \tag{D8}$$

If the 9 by 9 matrix $\mathbf{S}$ defined below has an inverse, the matrices of first order derivatives of the functions $\boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ and $\boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ with respect to the 9 components of $\boldsymbol{p}$ and $\boldsymbol{x}^\circ$ are defined as follows:[*63]

$$\begin{bmatrix} \nabla_p \boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) & \nabla_{x^\circ} \boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \\ \nabla_p \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) & \nabla_{x^\circ} \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \end{bmatrix} = \begin{bmatrix} \nabla^2_{yy} C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ) & \nabla^2_{yw^\circ} C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ) \\ \nabla^2_{w^\circ y} C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ) & \nabla^2_{w^\circ w^\circ} C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ) \end{bmatrix}^{-1} \equiv [\mathbf{S}]^{-1}. \tag{D9}$$

In order to determine the first order partial derivatives of $\boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ and $\boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ with respect to w_1, differentiate both sides of (D7) and (D8) with respect to w_1 and obtain the following matrix equation:

$$\begin{bmatrix} \nabla_{w_1} \boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \\ \nabla_{w_1} \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \end{bmatrix} = -[\mathbf{S}]^{-1} \begin{bmatrix} \nabla^2_{yw_1} C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ) \\ \nabla^2_{w^\circ w_1} C(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ) \end{bmatrix}. \tag{D10}$$

Finally, we need to calculate the first order partial derivatives of $x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ with respect to w_1 and the components of $\boldsymbol{p}$ and $\boldsymbol{x}^\circ$, Define $x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ as follows:

$$x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \equiv \partial C(\boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ), w_1, \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ))/\partial w_1. \tag{D11}$$

Differentiate (D11) with respect to w_1 where all cost function partial derivatives are evaluated at $(\boldsymbol{y}, w_1, \boldsymbol{w}^\circ)$:

$$\begin{aligned} \partial x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)/\partial w_1 &= \nabla^2_{w_1 w_1} C + \begin{bmatrix} \nabla^2_{w_1 y} C & \nabla^2_{w_1 w^\circ} C \end{bmatrix} \begin{bmatrix} \nabla_{w_1} \boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \\ \nabla_{w_1} \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \end{bmatrix} \\ &= \nabla^2_{w_1 w_1} C - \begin{bmatrix} \nabla^2_{w_1 y} C & \nabla^2_{w_1 w^\circ} C \end{bmatrix} \mathbf{S}^{-1} \begin{bmatrix} \nabla^2_{yw_1} C \\ \nabla^2_{w^\circ w_1} C \end{bmatrix} \end{aligned} \tag{D12}$$

where the second equality follows using (D10). Now differentiate (D11) with respect to the components of $\boldsymbol{p}$ and $\boldsymbol{x}^\circ$. We obtain the following formula for the partial derivatives of $x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ where $\nabla^T_p x_1$ denotes the transpose of the column vector $\nabla_p x_1$:

$$\begin{aligned} \begin{bmatrix} \nabla^T_p x_1 & \nabla^T_{x^\circ} x_1 \end{bmatrix} &= \begin{bmatrix} \nabla^2_{w_1 y} C & \nabla^2_{w_1 w^\circ} C \end{bmatrix} \begin{bmatrix} \nabla_p \boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) & \nabla_{x^\circ} \boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \\ \nabla_p \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) & \nabla_{x^\circ} \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \end{bmatrix} \\ &= \begin{bmatrix} \nabla^2_{w_1 y} C & \nabla^2_{w_1 w^\circ} C \end{bmatrix} \mathbf{S}^{-1} \end{aligned} \tag{D13}$$

where the second equality follows using (D9). Comparing (D13) to the partial derivatives defined in (D10) and using the symmetry of the second order derivatives of the joint cost function, we see that:

$$-\nabla_p x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) = \nabla_{w_1} \boldsymbol{y}(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) \text{ and } -\nabla_{x^\circ} x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ) = \nabla_{w_1} \boldsymbol{w}^\circ(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ). \tag{D14}$$

[*63] Apply the Inverse Function Theorem to (D7) and (D8) to get this result.

The above derivatives can be converted into elasticities. Equations (D9) are used to define the partial derivatives for the elasticities listed in Tables D5-D8. Define $Ey_mp_k \equiv [\partial y_m(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)/\partial p_k][p_k/y_m]$, These estimated output price elasticities for the US GDP function for 2022 are listed in Table D5 below.

Table D5: GDP Output Elasticities with Respect to Changes in Output Prices for 2022

Ey_mp_k		p_1 C	p_2 G	p_3 I	p_4 X
y_1	C	1.396	-0.321	-1.295	0.049
y_2	G	-1.503	2.031	0.648	-1.300
y_3	I	-3.902	0.417	3.996	0.139
y_4	X	0.302	-1.702	0.283	1.295

Note: Consumption (C), Government (G), Investment (I), Export (X).

Comparing the own elasticities in Table D5 with the corresponding own elasticities in Table D1, it can seen that the new own elasticities, Ey_mp_m for $m = 1, 2.3.4$, are all bigger than the corresponding Table D1 entries. This follows from Samuelson's Le Chatelier Principle:[*64] the quantity of imports is no longer fixed when we switch from the Gross Output Function to the GDP Function so the economy has more flexibility to respond to an increase in the price of an output and thus the output supply response to an increase in price will tend to be bigger.

Define the elasticity of output m with respect to an increase in the economy's endowment of inputs 2-6 as $Ey_mx_n \equiv [\partial y_m(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)/\partial x_n][x_n/y_m]$ for $m = 1, \ldots, 4$ and $n = 2, \ldots, 6$. These elasticities are listed in Table D6.

Table D6: GDP Output Elasticities with Respect to Changes in Input Quantities for 2022

Ey_mx_n		x_2 L	x_3 $K_{M\&E}$	x_4 K_S	x_5 K_O	x_6 K_L
y_1	C	0.482	0.787	1.196	1.035	-2.500
y_2	G	0.342	-2.599	-1.943	-1.417	6.616
y_3	I	1.162	-0.421	-2.556	-1.773	4.589
y_4	X	1.299	-0.067	2.715	-2.478	-0.469

Note: Consumption (C), Government (G), Investment (I), Export (X), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

Define the elasticity of input price n with respect to an increase in the endowment of input j as $Ew_nx_j \equiv [\partial w_n(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)/\partial x_j][x_j/w_n]$ for $n = 2, \ldots, 6$ and $j = 2, \ldots, 6$. These (inverse) input demand elasticities are listed in Table D7.

[*64] See Samuelson (1947; 36-38)[53] (1960)[55], Diewert (1974a; 146-150)[11] and Kohli (1983b)[38].

Table D7: GDP Inverse Input Demand Elasticities with Respect to Changes in Input Quantities

Ew_nx_j		x_2 L	x_3 $K_{M\&E}$	x_4 K_S	x_5 K_O	x_6 K_L
w_2	L	-0.956	0.205	0.005	12.970	-0.037
w_3	$K_{M\&E}$	1.806	-1.957	-0.183	-10.532	4.676
w_4	K_S	0.120	-0.855	-4.793	-34.581	7.000
w_5	K_O	0.215	-0.624	-0.140	-4.365	0.164
w_6	K_L	-0.276	5.461	18.569	262.729	-30.891

Note: Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

Note that the main diagonal elements in the above 5 by 5 matrix are less negative than the corresponding diagonal elements in the last 5 rows and columns of the matrix defined in Table D3. The GDP model of production allows for more substitution between domestic inputs and imports than the Gross Output model of production, which treats imports as a fixed factor of production. Thus if the quantity of labour or a capital service input *increases* in the GDP model, the corresponding price of that factor of production will *decrease* less than the corresponding change in the input price for the Gross Output model.

Define the elasticity of input price n with respect to an increase in the price of output m as $Ew_np_m \equiv [\partial w_n(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)/\partial p_m][p_m/w_n]$ for $n = 1, \ldots, 6$ and $m = 1, \ldots, 4$.

Table D8: Input Price Elasticities with Respect to Changes in GDP Output Prices for 2022

Ew_np_m		p_1 C	p_2 G	p_3 I	p_4 X
w_2	L	0.596	0.037	0.001	0.296
w_3	$K_{M\&E}$	6.187	-1.699	-0.100	-1.145
w_4	K_S	4.139	-0.129	-0.454	-0.653
w_5	K_O	8.023	-0.241	-0.034	-0.211
w_6	K_L	-0.056	0.525	1.122	3.165

Note: Consumption (C), Government (G), Investment (I), Export (X), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

We use (D13) to calculate the derivatives of the import demand function, $x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ with respect to the components of $\boldsymbol{p}$ and $\boldsymbol{x}^\circ$. These derivatives are then transformed into the elasticities of import demand with respect to output prices, Ex_1p_m for $m = 1, \ldots, 4$ and with respect to input endowments, Ex_1x_n for $n = 2, 3, \ldots, 6$. These elasticities for the year 2022 are listed in Table D9.

Table D9: Elasticities of Import Demand with Respect to Changes in Output Prices and Input Quantities for the GDP Model of Production

Ex_1p_1 C	Ex_1p_2 G	Ex_1p_3 I	Ex_1p_4 X	Ex_1x_2 L	Ex_1x_3 $K_{M\&E}$	Ex_1x_4 K_S	Ex_1x_5 K_O	Ex_1x_6 K_L
-0.698	-0.108	0.880	0.119	1.401	-0.188	0.354	-1.587	1.020

Note: Consumption (C), Government (G), Investment (I), Export (X), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

Using (D14), we can calculate the derivatives of $y_m(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ for $m = 1, 2, 3, 4$ and the derivatives of $w_n(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)$ for $n = 2, \ldots, 6$ with respect to w_1, the price of imports. These derivatives can be turned into elasticities and they are reported in Table D10.

Table D10: Elasticities of Output Supply and Input Price with Respect to the Price of Imports

Ey_1w_1 C	Ey_2w_1 G	Ey_3w_1 I	Ey_4w_1 X	Ew_2w_1 L	Ew_3w_1 $K_{M\&E}$	Ew_4w_1 K_S	Ew_5w_1 K_O	Ew_6w_1 K_L
0.171	0.124	-0.649	-0.179	-0.424	0.362	-0.300	3.013	-2.295

Note: Consumption (C), Government (G), Investment (I), Export (X), Labour (L), Machinery and Equipment ($K_{M\&E}$), Structures (K_S), Other Capital (mainly R&D and Inventories: K_O) and Land (K_L).

Finally, we can use (D12) to calculate the derivative of import demand with respect to an increase in the price of imports for 2022, $\partial x_1(\boldsymbol{p}, w_1, \boldsymbol{x}^\circ)/\partial w_1$. This derivative can be multiplied by w_1^{2022} and divided by the fitted value for imports in 2022 from our estimated cost function model to construct the GDP model elasticity $Ex_1w_1 = -0.119$. This elasticity can be compared to the own price elasticity of import demand which is generated by our estimated cost function for 2022. This cost function based import demand function is $x_1(\boldsymbol{y}^t, \boldsymbol{w}^t) = \partial C^t(\boldsymbol{y}^t, \boldsymbol{w}^t)/\partial w_1$ where $t = 2022$. The corresponding cost function based import demand own price elasticity of demand turned out to equal -0.028, which is less than -0.119 in magnitude. Using the Joint Cost function model of producer behavior, outputs $\boldsymbol{y}$ are held fixed as are domestic input prices $\boldsymbol{w}^\circ$. Using the GDP model of production, output prices $\boldsymbol{p}$ and domestic endowments $\boldsymbol{x}^\circ$ are held fixed. Thus when the price of imports increases, producers are free to substitute other inputs for imports using the Joint Cost function model which evidently leads to a larger reduction in import demand compared to the corresponding response using the GDP function model where the quantities of other inputs are held fixed in response to the increase in the price of imports (but outputs are free to vary). Thus the magnitude of producer responses to changes in prices and quantities can be very different depending on what is being held constant. Different models should be used to answer different questions.

A final note of caution: for this conversion of elasticities method to work successfully, the underlying joint cost function should provide a good fit to the data.

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