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Challenges in Consumer Price Index Construction

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Keywords

Index number theory, multilateral indexes, the chain drift problem, the lack of matching problem, test approach, stochastic approach, economic approach, hedonic regressions, the Commensurability Test.

Journal of Economic Literature Classification Codes

C32, C43, D20, D57, E31.

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1 Introduction

Economists are familiar with the concept of a Consumer Price Index (CPI). But how exactly is a CPI constructed? It turns out that there are many difficult technical problems that are associated with the construction of a CPI. In order to describe these problems, it will be necessary to review several alternative approaches to the construction of a price index. Four main approaches will be described in section 2 of the paper. These approaches are: (i) fixed basket approaches; (ii) stochastic approaches; (iii) axiomatic approaches and (iv) economic approaches.[*1] The object of a price (or quantity) index is to summarize information; i.e., a bilateral price index attempts to provide users with a measure of the general level of prices in the current period with the corresponding level of prices in a base period. Thus the four approaches mentioned above are approaches to *bilateral index number theory*: only the prices (and quantities) pertaining to two periods are considered in these approaches.

Once the various approaches to bilateral index number theory have been explained, two main challenges to these traditional approaches to index number theory can be discussed:

- The chain drift problem and
- The problem of a lack of price matching.

The chain drift problem will be explained in detail below. It arises when prices and quantities of the products under consideration fluctuate in response to sales or seasonal influences. It can be "cured" if we abandon bilateral index number theory and move to multilateral index number theory. Instead of comparing prices (and quantities) over two periods, multilateral index number theory compares prices (and quantities) over multiple periods. Some of the main multilateral index number methods will be explained in section 3. However, although multilateral indexes largely eliminate the chain drift problem, they can suffer from the second problem listed above: the problem of the lack of price matching over time. This problem is caused by disappearing products (due to seasonal influences or obsolescence) or by the appearance of new products or changes in the quality of existing products. It turns out that econometric methods can deal with the lack of matching prices problem and econometric approaches to price index construction will be discussed in section 4.

Section 5.1 offers some thoughts on which method to use to avoid the above two problems and it also describes a third challenge that compilers of a Consumer Price Index face:

- The extension problem.

This problem arises from the use of multilateral methods to solve the above two challenges: the challenge is how to link the results of a new set of multilateral indexes that results from incorporating the data of a new period to a previous set of multilateral indexes.

Section 5.2 mentions other challenges to price index construction and Section 6 offers a conclusion.

[*1] These approaches are described in more detail in IMF, ILO, Eurostat, UNECE, OECD, World Bank, IMF (2025), *Consumer Price Index Manual: Theory*, Washington DC: International Monetary Fund[113]. Subsequent references to this publication will be abbreviated to IMF (2025). This paper draws heavily on this publication as well as on Diewert (2023)[60] and Diewert, Abe, Tonogi and Shimizu (2025)[62]. For alternative surveys of index number theory, see Frisch (1936)[99] and Balk (2008)[11]. We note that approaches (ii), (iii) and (iv) were discussed by Samuelson and Swamy (1974; 567)[146] but the main focus in their paper was on the economic approach.

2 Approaches to Bilateral Index Number Theory

2.1 Setting the Stage: Notation and the Price Levels Approach

In order to further define the index number problem, we introduce some notation. We specify accounting periods, $t = 0, 1, \ldots, T$ for which we have micro price and quantity data for N commodities in scope pertaining to transactions by a definite consumer or a definite group of consumers. Denote the price and quantity of commodity n in period t by p_{tn} and q_{tn} respectively for $n = 1, 2, \ldots, N$ and $t = 0, 1, \ldots, T$. Before proceeding further, we need to discuss the exact meaning of the microeconomic prices and quantities if there are *multiple* transactions for say commodity n within period t. In this case, it is natural to interpret q_{tn} as the *total* amount of commodity n transacted within period t. In order to conserve the value of transactions, it is necessary that p_{tn} be defined as a *unit value*; i.e., p_{tn} must be equal to the value of transactions in commodity n for period t divided by the total quantity transacted, q_{tn}. Fisher[*2] and Hicks[*3] assumed that the length of the accounting period was chosen so that variations in commodity prices within the periods were small compared to their variations between periods. For $t = 0, 1, \ldots, T$, define *the value of transactions in period* t or *expenditures* e^t on the N products in scope as:

$$e^t \equiv \textstyle\sum_{n=1}^{N} p_{tn} q_{tn} \equiv \boldsymbol{p}^t \cdot \boldsymbol{q}^t \tag{1}$$

where $\boldsymbol{p}^t \equiv (p_{t1}, \ldots, p_{tN})$ is the period t price vector, $\boldsymbol{q}^t \equiv (q_{t1}, \ldots, q_{tN})$ is the period t quantity vector and $\boldsymbol{p}^t \cdot \boldsymbol{q}^t$ denotes the inner product of these two vectors.

Using the above notation, we can now state the following *(levels) version of the index number problem*: for $t = 0, 1, \ldots, T$, find scalar numbers P^t and Q^t such that

$$e^t = P^t Q^t. \tag{2}$$

The number P^t is interpreted as an *aggregate period* t *price level* while the number Q^t is interpreted as an *aggregate period* t *quantity level*. The aggregate price level P^t is assumed to be a function of the period t price vector, $\boldsymbol{p}^t$ while the aggregate period t quantity level Q^t is assumed to be a function of the period t quantity vector, $\boldsymbol{q}^t$:

$$P^t = c(\boldsymbol{p}^t) \text{ and } Q^t = f(\boldsymbol{q}^t); \quad t = 0, 1, \ldots, T. \tag{3}$$

[*2] "Throughout this book 'the price' of any commodity or 'the quantity' of it for any one year was assumed given. But what is such a price or quantity? Sometimes it is a single quotation for January 1 or July 1, but usually it is an average of several quotations scattered throughout the year. The question arises: On what principle should this average be constructed? The *practical* answer is *any* kind of average since, ordinarily, the variation during a year, so far, at least, as prices are concerned, are too little to make any perceptible difference in the result, whatever kind of average is used. Otherwise, there would be ground for subdividing the year into quarters or months until we reach a small enough period to be considered practically a point. The quantities sold will, of course, vary widely. What is needed is their sum for the year (which, of course, is the same thing as the simple arithmetic average of the per annum rates for the separate months or other subdivisions). In short, the simple arithmetic average, both of prices and of quantities, may be used. Or, if it is worth while to put any finer point on it, we may take the weighted arithmetic average for the prices, the weights being the quantities sold." Irving Fisher (1922; 318)[91].

[*3] "I shall define a week as that period of time during which variations in prices can be neglected. For theoretical purposes this means that prices will be supposed to change, not continuously, but at short intervals. The calendar length of the week is of course quite arbitrary; by taking it to be very short, our theoretical scheme can be fitted as closely as we like to that ceaseless oscillation which is a characteristic of prices in certain markets." J.R. Hicks (1946; 122)[108].

The functions c and f are to be determined somehow. Basically, they aggregate or summarize the information on all N prices p_{tn} in period t into a single representative price, P^t, and summarize the information on all N quantities q_{tn} in period t into a single representative quantity, Q^t. Note that we are requiring that the functional forms for the price aggregation function c and for the quantity aggregation function f be independent of time. This can be viewed as a reasonable requirement since there is no reason to change the method of aggregation as time changes.

Suppose we follow Eichhorn (1978)[84] and take the *axiomatic* or *test approach to the levels approach* to index number theory. Then if we substitute (3) and (2) into (1), Eichhorn's *test approach to price levels* requires that c and f must satisfy the following functional equation for all strictly positive price and quantity vectors:

$$c(\boldsymbol{p})f(\boldsymbol{q}) = \boldsymbol{p} \cdot \boldsymbol{q} \equiv \textstyle\sum_{n=1}^{N} p_n q_n \quad \text{for all } \boldsymbol{p} \gg \boldsymbol{0}_N \text{ and for all } \boldsymbol{q} \gg \boldsymbol{0}_N. \tag{4}$$

Unfortunately, this test approach to the determination of the aggregate price and quantity *levels*, $f(\boldsymbol{q})$ and $c(\boldsymbol{p})$, is doomed to failure. Assuming that the functions $c(\boldsymbol{p})$ and $f(\boldsymbol{q})$ are required to be positive and $N \geq 2$, Eichhorn (1978; 144)[84] showed that it is not possible to find functions $c(\boldsymbol{p})$ and $f(\boldsymbol{q})$ that satisfy equations (4) for *all* strictly positive price and quantity vectors, $\boldsymbol{p}$ and $\boldsymbol{q}$.

A way of dealing with this negative result is to give up on attempting to determine aggregate price and quantity *levels*. Instead of attempting to determine aggregate price and quantity levels for a single period, try to determine the *ratio* of say period 1's price level P^1 to period 0's price level, P^0. Let this ratio, P^1/P^0 be a function $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ of the price and quantity vectors for periods 0 and 1. Similarly, instead of trying to determine a reasonable aggregator function for the period 0 and 1 quantity levels, Q^0 and Q^1, try to determine a reasonable function of the prices and quantities of both periods, $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$, such that $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = Q^1/Q^0$. The function $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is called a *bilateral price index* and the function $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is called a *bilateral quantity index*. We would like these two functions to satisfy the following *Product Test*:

$$P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = \boldsymbol{p}^1 \cdot \boldsymbol{q}^1/\boldsymbol{p}^0 \cdot \boldsymbol{q}^0 = e^1/e^0. \tag{5}$$

It can be seen that if we can find a suitable price index function $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$, then its companion quantity index function $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ can be defined residually using the Product Test:

$$Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \equiv [\boldsymbol{p}^1 \cdot \boldsymbol{q}^1/\boldsymbol{p}^0 \cdot \boldsymbol{q}^0]/P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1). \tag{6}$$

The following subsections provide alternative approaches for finding "reasonable" bilateral price index functions, $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$.

2.2 Fixed Basket Approaches

The fixed basket approach to the construction of a consumer price index is one of the earliest approaches[*4] to bilateral index number theory.[*5] Lowe (1823; 333-335)[129] was not the first to propose the fixed basket approach, but he laid out the construction of a fixed basket index (and its use) in some detail. Basically, he proposed that the government should form a list

[*4] See Diewert (1993a)[47] and Balk (2008)[11] for reviews of the early history of index number theory.

[*5] Bilateral index number theory compares price levels in two periods. Multilateral index number theory compares price levels across more than two periods.

of articles of general consumption, which is a representative annual basket $\boldsymbol{q} \equiv (q_1, \ldots, q_N)$, and if the prices of year t were $\boldsymbol{p}^t \equiv (p_{t1}, \ldots, p_{tN})$ and the prices of the base year 0 were $\boldsymbol{p}^0 \equiv (p_{01}, \ldots, p_{0N})$, then the *Lowe price index* which gives the aggregate level of prices in year t relative to the level of prices in year 0 is $P_{\text{Lo}} \equiv \boldsymbol{p}^t \cdot \boldsymbol{q}/\boldsymbol{p}^0 \cdot \boldsymbol{q}$. Basically, the Lowe index compares the cost of the same fixed basket of goods and services for the two years in question and defines the price index as the ratio of the resulting costs.*6

As time passed, economists and price statisticians demanded a bit more precision with respect to the specification of the basket vector $\boldsymbol{q}$. There are two natural choices for the reference basket: the period 0 commodity vector $\boldsymbol{q}^0$ or the period t commodity vector $\boldsymbol{q}^t$. These two choices lead to the *Laspeyres* (1871)[127] *price index* $P_{\text{L}} \equiv \boldsymbol{p}^t \cdot \boldsymbol{q}^0/\boldsymbol{p}^0 \cdot \boldsymbol{q}^0$ and the *Paasche* (1874)[133] *price index* $P_{\text{P}} \equiv \boldsymbol{p}^t \cdot \boldsymbol{q}^t/\boldsymbol{p}^0 \cdot \boldsymbol{q}^t$.

The problem with the Laspeyres and Paasche index number formulae is that they are equally plausible but in general, they will give different answers. This suggests that if we require a single estimate for the price change between the two periods, then we should take some sort of evenly weighted average of the two indexes as our final estimate of price change between periods 0 and 1. Examples of such symmetric averages are the arithmetic mean, which leads to the *Sidgwick* (1883; 68)[152] *Bowley* (1901; 227)[18] *index* , $(1/2)P_{\text{L}} + (1/2)P_{\text{P}}$, and the geometric mean, which leads to the *Fisher* (1922)[91] *ideal index*, P_{F}, defined as the square root of the product of the Laspeyres and Paasche indexes, $[P_{\text{L}}P_{\text{P}}]^{1/2}$.

The problem with the arithmetic mean of the Paasche and Laspeyres indexes is that the resulting index does not satisfy the Time Reversal Test. Thus if $P(\boldsymbol{p}^0, \boldsymbol{p}^t, \boldsymbol{q}^0, \boldsymbol{q}^t)$ denotes a generic index number formula between periods 0 and t going forward from period 0, then we would like the formula to satisfy $P(\boldsymbol{p}^t, \boldsymbol{p}^0, \boldsymbol{q}^t, \boldsymbol{q}^0) = 1/P(\boldsymbol{p}^0, \boldsymbol{p}^t, \boldsymbol{q}^0, \boldsymbol{q}^t)$; i.e., if we measure price change going backwards from period t to period 0, then we would like the resulting index, $P(\boldsymbol{p}^t, \boldsymbol{p}^0, \boldsymbol{q}^t, \boldsymbol{q}^0)$, to equal the reciprocal of the index that goes forward from period 0 to t, $P(\boldsymbol{p}^0, \boldsymbol{p}^t, \boldsymbol{q}^0, \boldsymbol{q}^t)$. However, the Fisher index does satisfy the time reversal test. Moreover, it is the *only* index that is a homogeneous symmetric average of the Laspeyres and Paasche price indexes, P_{L} and P_{P}, that also satisfies the Time Reversal Test.*7

Thus the basket approach to consumer price indexes leads to the Fisher ideal index as a possible "best" choice of bilateral index number formula.

2.3 Stochastic Approaches

The stochastic approach to the determination of the price index can be traced back to Jevons (1865)[117] (1884)[118] and Edgeworth (1888)[82] (1925)[83] over a hundred years ago.

The basic idea behind the unweighted (or more accurately, the evenly weighted) stochastic approach*8 is that each price ratio or price relative, p_{tn}/p_{0n} for $n = 1, 2, \ldots, N$, can be regarded as an estimate of a common inflation rate α between periods 0 and t; i.e., it is assumed that

$$p_{tn}/p_{0n} = \alpha + \varepsilon_n; \quad n = 1, 2, \ldots, N \tag{7}$$

where α is the common inflation rate and the ε_n are random variables with mean 0 and variance σ^2. The least squares estimator for α is the *Carli* (1764)[22] *price index* P_{C} defined

*6 Lowe (1823; Appendix page 95)[129] suggested that the commodity basket vector $\boldsymbol{q}$ should be updated every five years.

*7 See Diewert (1997; 138)[50].

*8 Frisch (1936; 3)[99] called the stochastic approach the atomistic approach in his survey of index number theory.

as follows:

$$P_{\mathrm{C}}(\boldsymbol{p}^0, \boldsymbol{p}^t) \equiv \textstyle\sum_{n=1}^{N}(1/N)(p_{tn}/p_{0n}). \tag{8}$$

Unfortunately, P_{C} does not satisfy the Time Reversal Test, i.e., $P_{\mathrm{C}}(\boldsymbol{p}^t, \boldsymbol{p}^0) \neq 1/P_{\mathrm{C}}(\boldsymbol{p}^0, \boldsymbol{p}^t)$*9.

Now assume that the logarithm of each price relative, $\ln(p_{tn}/p_{0n})$, is an unbiased estimate of the logarithm of the inflation rate between periods 0 and t, β say. Thus we have:

$$\ln(p_{tn}/p_{0n}) = \beta + \varepsilon_n; \quad n = 1, 2, \ldots, N \tag{9}$$

where $\beta \equiv \ln \alpha$ and the ε_n are independently distributed random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for β is the logarithm of the geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α is the *Jevons* (1865)[117] *price index* P_{J} defined as:

$$P_{\mathrm{J}}(\boldsymbol{p}^0, \boldsymbol{p}^t) \equiv \textstyle\prod_{n=1}^{N}(p_{tn}/p_{0n})^{1/N}. \tag{10}$$

The Jevons price index P_{J} does satisfy the Time Reversal Test and hence is much more satisfactory than the Carli index P_{C}. However, both the Jevons and Carli price indexes suffer from an important flaw: each price relative p_{tn}/p_{0n} is regarded as being equally important and is given an equal weight in the index number formulae (2) and (4).*10

Theil (1967; 136-137)[157] proposed a solution to the lack of weighting in (8) and (10). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the nth price relative is equal to $s_{0n} \equiv p_{0n}q_{0n}/\boldsymbol{p}^0 \cdot \boldsymbol{q}^0$, the period 0 expenditure share for commodity n. Thus the overall mean (period 0 weighted) logarithmic price change is $\sum_{n=1}^{N} s_{0n} \ln(p_{tn}/p_{0n})$. Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period t has an equal probability of being selected. This leads to the overall mean (period t weighted) logarithmic price change of $\sum_{n=1}^{N} s_{tn} \ln(p_{tn}/p_{0n})$. Each of these measures of overall logarithmic price change seems equally valid so we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change. Theil argued that a nice symmetric index number formula can be obtained if we make the probability of selection for the nth price relative equal to the arithmetic average of the period 0 and t expenditure shares for commodity n. Using these probabilities of selection, Theil's final measure of overall logarithmic price change was

$$\ln P_{\mathrm{T}}(\boldsymbol{p}^0, \boldsymbol{p}^t, \boldsymbol{q}^0, \boldsymbol{q}^t) \equiv \textstyle\sum_{n=1}^{N}(1/2)(s_{0n} + s_{tn}) \ln(p_{tn}/p_{0n}). \tag{11}$$

Taking the exponential of both sides of (11) leads to the *Törnqvist-Theil price index*, $P_{\mathrm{T}}(\boldsymbol{p}^0, \boldsymbol{p}^t, \boldsymbol{q}^0, \boldsymbol{q}^t)$.*11 There are many more stochastic approaches to index number theory that are available to price statisticians but for our purposes, we will regard $P_{\mathrm{T}}(\boldsymbol{p}^0, \boldsymbol{p}^t, \boldsymbol{q}^0, \boldsymbol{q}^t)$ as a "best" index number formula from the viewpoint of the stochastic approach to index number theory.*12

*9 In fact Fisher (1922; 66)[91] noted that $P_{\mathrm{C}}(\boldsymbol{p}^0, \boldsymbol{p}^1)P_{\mathrm{C}}(\boldsymbol{p}^1, \boldsymbol{p}^0) \geq 1$ unless the period 1 price vector $\boldsymbol{p}^1$ is proportional to the period 0 price vector $\boldsymbol{p}^0$; i.e., Fisher showed that the Carli index has a definite upward bias. He urged statistical agencies not to use this formula.

*10 Walsh (1901; 104)[164] (1921a; 82-83)[165], Fisher (1911; 194-196)[90] and Keynes (1930; 71-81)[122] vigorously criticized the lack of weighting in the unweighted stochastic approach to index number theory.

*11 Leo Törnqvist (1936)[158] probably had this formula in mind but it was not explicit in his 1936 paper. It was explicit in Törnqvist and Törnqvist (1937)[159].

*12 For additional materials on the stochastic approach, see Selvanathan and Rao (1994)[149], Clements,

2.4 The Test Approach to Bilateral Index Number Theory

In this section, our goal will be to assume that the bilateral price index formula, $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$, satisfies a sufficient number of "reasonable" tests or properties so that the functional form for P is determined.[*13] The word "bilateral"[*14] refers to the assumption that the function P depends only on the data pertaining to the two situations or periods being compared; i.e., P is regarded as a function of the two sets of price and quantity vectors, $\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1$, that are to be aggregated into a single number that summarizes the overall change in the N price ratios, $p_1^1/p_1^0, \ldots, p_N^1/p_N^0$. Note that the price (and quantity) vectors are assumed to be strictly positive so that price (and quantity) ratios are well defined.

We will take the perspective that was used in section 2.1; i.e., along with the price index $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$, there is a companion quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ such that the product of these two indexes equals the value ratio between the two periods. Thus, throughout this section, we assume that the bilateral price index $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ and its companion bilateral quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ satisfy the following *Product Test*:[*15]

$$\boldsymbol{p}^1 \cdot \boldsymbol{q}^1 / \boldsymbol{p}^0 \cdot \boldsymbol{q}^0 = P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1). \tag{12}$$

Equation (12) means that as soon as the functional form for the price index $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is determined, then (12) can be used to determine the functional form for the quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$. However, a further advantage of assuming that the Product Test holds is that we can assume that the quantity index Q satisfies a "reasonable" property and then use (12) to translate this test on the quantity index into a corresponding test on the price index P.[*16]

If $N = 1$, so that there is only one price and quantity to be aggregated, then a natural candidate for P is p_1^1/p_1^0, the single price ratio, and a natural candidate for Q is q_1^1/q_1^0, the single quantity ratio. When the number of commodities or products to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multi-dimensional analogues to the one product price index formula, p_1^1/p_1^0. Below, we list twenty-one tests that turn out to characterize the Fisher ideal price index.

We assume that every component of each price and quantity vector is positive; i.e., $\boldsymbol{p}^t \gg \boldsymbol{0}_N$ and $\boldsymbol{q}^t \gg \boldsymbol{0}_N$[*17] for $t = 0, 1$. If we want to set $\boldsymbol{q}^0 = \boldsymbol{q}^1$, we call the common quantity vector $\boldsymbol{q}$; if we want to set $\boldsymbol{p}^0 = \boldsymbol{p}^1$, we call the common price vector $\boldsymbol{p}$.

Our first two tests are not very controversial and so we will not discuss them.

Izan and Selvanathan (2006)[25] and Diewert (2010)[56]. However, there are many more stochastic approaches to index number theory. For example, the estimation of hedonic regressions can be regarded as a stochastic approach to index number theory; see Court (1939)[32] who introduced the term "hedonic regression".

[*13] Much of the material in this section is drawn from Diewert (1992)[46] (1993a)[47]. For surveys of the axiomatic approach see Balk (1995)[9] (2008)[11] and Auer (2001)[5].

[*14] Multilateral index number theory refers to the situation where there are more than two situations whose prices and quantities need to be aggregated.

[*15] Frisch (1930; 399)[98] introduced this name for this test but the concept of this test can be traced back to Fisher (1911; 405)[90].

[*16] This observation was first made by Fisher (1911; 400-406)[90]. Vogt (1980)[162] and Diewert (1992)[46] pursued this idea.

[*17] Notation: $\boldsymbol{q} \gg \boldsymbol{0}_N$ means that each component of the vector $\boldsymbol{q}$ is positive; $\boldsymbol{q} \geq \boldsymbol{0}_N$ means each component of $\boldsymbol{q}$ is nonnegative and $\boldsymbol{q} > \boldsymbol{0}_N$ means $\boldsymbol{q} \geq \boldsymbol{0}_N$ and $\boldsymbol{q} \neq \boldsymbol{0}_N$. Finally, $\boldsymbol{p} \cdot \boldsymbol{q} \equiv \sum_{n=1}^N p_n q_n$ denotes the inner product of the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$.

T1: *Positivity*[*18]: $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) > 0$.

T2: *Continuity*[*19]: $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is a continuous function of its arguments.

Our next two tests are somewhat more controversial.

T3: *Strong Identity or Constant Prices Test*[*20]: $P(\boldsymbol{p}, \boldsymbol{p}, \boldsymbol{q}^0, \boldsymbol{q}^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial part of this test is that the two quantity vectors are allowed to be different in the above test.[*21]

T4: *Fixed Basket or Constant Quantities Test*[*22]:

$$P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}, \boldsymbol{q}) = \textstyle\sum_{i=1}^N p_i^1 q_i / \sum_{i=1}^N p_i^0 q_i = \boldsymbol{p}^1 \cdot \boldsymbol{q} / \boldsymbol{p}^0 \cdot \boldsymbol{q}.$$

That is, if quantities are constant during the two periods so that $\boldsymbol{q}^0 = \boldsymbol{q}^1 \equiv \boldsymbol{q}$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^N p_i^1 q_i$, divided by the expenditure on the basket in period 0, $\sum_{i=1}^N p_i^0 q_i$.

It can be shown[*23] that if the price index P satisfies Test T4 and $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ and $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ jointly satisfy the Product Test, (12) above, then $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ must satisfy the identity test $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}, \boldsymbol{q}) = 1$ for all strictly positive vectors $\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}$. This *constant quantities test* for Q is also somewhat controversial since $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ are allowed to be different.

The following four tests restrict the behavior of the price index $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ as the scale of any one of the four vectors $\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1$ changes.

T5: *Proportionality in Current Prices*[*24]: $P(\boldsymbol{p}^0, \lambda\boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = \lambda P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Equivalently, the price index function $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector $\boldsymbol{p}^1$. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

[*18] Eichhorn and Voeller (1976, 23)[85] suggested this test.

[*19] Fisher (1922; 207-215)[91] informally suggested the essence of this test.

[*20] Laspeyres (1871; 308)[127], Walsh (1901; 308)[164] and Eichhorn and Voeller (1976; 24)[85] have all suggested this test. Laspeyres came up with this test or property to discredit the ratio of unit values index of Drobisch (1871)[81], which does not satisfy this test. This test is also a special case of Fisher's (1911; 409-410)[90] Price Proportionality Test. A weaker version of this test is: if the price of every product is identical and the quantity of every product is identical during the two periods, then the price index should equal unity; i.e., $P(\boldsymbol{p}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{q}) = 1$. This Weak Identity Test is weaker because it is easier to satisfy since it is implied by the Strong Identity Test.

[*21] Usually, economists assume that given a price vector $\boldsymbol{p}$, the corresponding quantity vector $\boldsymbol{q}$ is uniquely determined. Here, we have the same price vector but the corresponding quantity vectors are allowed to be different.

[*22] The origins of this test go back at least two hundred years to the Massachusetts legislature which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution; see Willard Fisher (1913)[92]. Other researchers who have suggested the test over the years include Lowe (1823, Appendix, 95)[129], Jevons (1865)[117], Sidgwick (1883, 67-68)[152], Edgeworth (1925, 215)[83] originally published in 1887, Marshall (1887, 363)[130], Pierson (1895, 332)[136], Walsh (1901, 540)[164] (1921b; 544)[166], and Bowley (1901, 227)[18]. Vogt and Barta (1997; 49)[163] correctly observed that this test is a special case of Fisher's (1911; 411)[90] Proportionality Test for quantity indexes which Fisher (1911; 405)[90] translated into a test for the price index using the Product Test (12).

[*23] See Vogt (1980; 70)[162].

[*24] This test was proposed by Walsh (1901, 385)[164], Eichhorn and Voeller (1976, 24)[85] and Vogt (1980, 68)[162].

Walsh (1901)[164] and Fisher (1911; 418)[90] (1922; 420)[91] proposed the related proportionality test $P(\boldsymbol{p}, \lambda\boldsymbol{p}, \boldsymbol{q}^0, \boldsymbol{q}^1) = \lambda$. This last test is a combination of T3 and T5; in fact Walsh (1901, 385)[164] noted that this last test implies the identity test, T3.

In the next test, instead of multiplying all period 1 prices by the same number, we multiply all period 0 prices by the number λ.

T6: *Inverse Proportionality in Base Period Prices*[*25]:

$$P(\lambda\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = \lambda^{-1}P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \text{ for } \lambda > 0.$$

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Thus the price index function $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector $\boldsymbol{p}^0$.

The following two homogeneity tests can also be regarded as invariance tests.

T7: *Invariance to Proportional Changes in Current Quantities*:

$$P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \lambda\boldsymbol{q}^1) = P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \text{ for all } \lambda > 0.$$

Thus the price index function $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is (positively) homogeneous of degree zero in the components of the period 1 quantity vector $\boldsymbol{q}^1$. Vogt (1980, 70)[162] was the first to propose this test[*26] and his derivation of the test is of some interest. Suppose the quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ satisfies the quantity analogue to the price test T5; i.e., suppose $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ satisfies $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \lambda\boldsymbol{q}^1) = \lambda Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ for $\lambda > 0$. Then using the product test (12), we see that $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ must satisfy T7.

T8: *Invariance to Proportional Changes in Base Quantities*[*27]:

$$P(\boldsymbol{p}^0, \boldsymbol{p}^1, \lambda\boldsymbol{q}^0, \boldsymbol{q}^1) = P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \text{ for all } \lambda > 0.$$

Thus the price index function $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is (positively) homogeneous of degree zero in the components of the period 0 quantity vector $\boldsymbol{q}^0$. If the quantity index Q satisfies the following counterpart to T8: $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \lambda\boldsymbol{q}^0, \boldsymbol{q}^1) = \lambda^{-1}Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ for all $\lambda > 0$, then using (12), the corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P.

T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors $\boldsymbol{q}^0$ and $\boldsymbol{q}^1$.

The next five tests are invariance or symmetry tests. Fisher (1922; 62 63, 458-460)[91] and Walsh (1921b; 542)[166] seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, 62-63)[91] spoke of fairness but it is clear that he had symmetry properties in mind. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity Reversal Test* (or Invariance to Changes in the Ordering of Commodities):

$$P(\boldsymbol{p}^{0^*}, \boldsymbol{p}^{1^*}, \boldsymbol{q}^{0^*}, \boldsymbol{q}^{1^*}) = P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$$

where $\boldsymbol{p}^{t^*}$ denotes a permutation of the components of the vector $\boldsymbol{p}^t$ and $\boldsymbol{q}^{t^*}$ denotes the same permutation of the components of $\boldsymbol{q}^t$ for $t = 0, 1$. This test is due to Irving Fisher (1922)[91], and it is one of his three famous Reversal Tests. The other two are the Time Reversal Test and the Factor Reversal Test which will be considered below.

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*25 Eichhorn and Voeller (1976; 28)[85] suggested this test.

*26 Fisher (1911; 405)[90] proposed the related test $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \lambda\boldsymbol{q}^0) = P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^0) = \sum_{i=1}^N p_i^1 q_i^0 / \sum_{i=1}^N p_i^0 q_i^0$.

*27 This test was proposed by Diewert (1992; 216)[46].

T10: *Invariance to Changes in the Units of Measurement* (Commensurability Test):

$P(\alpha_1 p_1^0, \ldots, \alpha_N p_N^0; \alpha_1 p_1^1, \ldots, \alpha_N p_N^1; \alpha_1^{-1} q_1^0, \ldots, \alpha_N^{-1} q_N^0; \alpha_1^{-1} q_1^1, \ldots, \alpha_N^{-1} q_N^1) =$
$P(p_1^0, \ldots, p_N^0; p_1^1, \ldots, p_N^1; q_1^0, \ldots, q_N^0; q_1^1, \ldots, q_N^1)$ for all $\alpha_1 > 0, \ldots, \alpha_N > 0$.

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test was due to Jevons (1884; 23)[118] and the Dutch economist Pierson (1896; 131)[137], who criticized several index number formula for not satisfying this fundamental test. Fisher (1911; 411)[90] first called this test *the Change of Units Test* and later, Fisher (1922; 420)[91] called it the *Commensurability Test*.

T11: *Time Reversal Test*: $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = 1/P(\boldsymbol{p}^1, \boldsymbol{p}^0, \boldsymbol{q}^1, \boldsymbol{q}^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio; this test is satisfied (as are all of the other tests listed in this section). When the number of goods is greater than one, many commonly used price indices fail this test; e.g., the Laspeyres (1871)[127] price index, P_{L} defined earlier in Section 2.2, and the Paasche (1874)[133] price index, P_{P}, both *fail* this fundamental test. This test is due to Pierson (1896; 128)[137], who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test, that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 368)[164] (1921b; 541)[166] and Fisher (1911; 534)[90] (1922; 64)[91].

The next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory. However, these tests are quite consistent with the weighted stochastic approach to index number theory discussed in Section 2.3.

T12: *Quantity Reversal Test* (Quantity Weights Symmetry Test):

$P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^1, \boldsymbol{q}^0)$.

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities $\boldsymbol{q}^0$ and the period 1 quantities $\boldsymbol{q}^1$ must enter the formula in a symmetric or even handed manner. Funke and Voeller (1978; 3)[100] introduced this test; they called it the *Weight Property*.

The next test is the analogue to T12 applied to quantity indexes:

T13: *Price Reversal Test* (Price Weights Symmetry Test)*28:

$\{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0\} / P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = \{\sum_{i=1}^N p_i^0 q_i^1 / \sum_{i=1}^N p_i^1 q_i^0\} / P(\boldsymbol{p}^1, \boldsymbol{p}^0, \boldsymbol{q}^0, \boldsymbol{q}^1)$.

Thus if we use (12) to define the quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ in terms of the price index $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$, then it can be seen that T13 is equivalent to the following property for the associated quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$:

$$Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = Q(\boldsymbol{p}^1, \boldsymbol{p}^0, \boldsymbol{q}^0, \boldsymbol{q}^1). \tag{13}$$

That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

The next three tests are mean value tests.

*28 This test was proposed by Diewert (1992; 218)[46].

T14: *Mean Value Test for Prices*[*29]:

$$\min_i\{p_i^1/p_i^0 : i = 1, \ldots, N\} \leq P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \leq \max_i\{p_i^1/p_i^0 : i = 1, \ldots, N\}.$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is likely to be some kind of an average of the N price ratios, p_i^1/p_i^0, then the price index P should satisfy this test.

The next test is the analogue to T14 applied to quantity indexes:

T15: *Mean Value Test for Quantities*[*30]:

$$\min_i\{q_i^1/q_i^0 : i = 1, \ldots, N\} \leq (e^1/e^0)/P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \leq \max_i\{q_i^1/q_i^0 : i = 1, \ldots, N\}$$

where e^t is the period t value aggregate $e^t \equiv \sum_{n=1}^{N} p_n^t q_n^t$ for $t = 0, 1$. Using (12) to define the quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ in terms of the price index $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$, we see that T15 is equivalent to the following property for the associated quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$:

$$\min_i\{q_i^1/q_i^0 : i = 1, \ldots, N\} \leq Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \leq \max_i\{q_i^1/q_i^0 : i = 1, \ldots, N\}. \quad (14)$$

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth q_i^1/q_i^0 of the individual quantities.

In Section 2.2, we argued that it was very reasonable to take an average of the Laspeyres and Paasche price indexes as a single "best" measure of overall price change. This point of view can be turned into a test:

T16: *Paasche and Laspeyres Bounding Test*[*31]: The price index P lies between the Laspeyres and Paasche indexes, $P_{\mathrm{L}} \equiv \boldsymbol{p}^1 \cdot \boldsymbol{q}^0/\boldsymbol{p}^0 \cdot \boldsymbol{q}^0$ and $P_{\mathrm{P}} \equiv \boldsymbol{p}^1 \cdot \boldsymbol{q}^1/\boldsymbol{p}^0 \cdot \boldsymbol{q}^1$.

It can be shown that if the price index $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ satisfies T16, then the implicit quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ that corresponds to $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ using (12) must lie between the Laspeyres and Paasche quantity indexes, $Q_{\mathrm{L}} \equiv \boldsymbol{p}^0 \cdot \boldsymbol{q}^1/\boldsymbol{p}^0 \cdot \boldsymbol{q}^0$ and $Q_{\mathrm{P}} \equiv \boldsymbol{p}^1 \cdot \boldsymbol{q}^1/\boldsymbol{p}^1 \cdot \boldsymbol{q}^0$.

The next four tests are monotonicity tests; i.e., how should the price index $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ change as any component of the two price vectors $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ increases or as any component of the two quantity vectors $\boldsymbol{q}^0$ and $\boldsymbol{q}^1$ increases.

T17: *Monotonicity in Current Prices*: $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) < P(\boldsymbol{p}^0, \boldsymbol{p}^2, \boldsymbol{q}^0, \boldsymbol{q}^1)$ if $\boldsymbol{p}^1 < \boldsymbol{p}^2$.

That is, if some period 1 price increases, then the price index must increase, so that $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is increasing in the components of $\boldsymbol{p}^1$. This property was proposed by Eichhorn and Voeller (1976; 23)[85] and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in Base Prices*: $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) > P(\boldsymbol{p}^2, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ if $\boldsymbol{p}^0 < \boldsymbol{p}^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is decreasing in the components of $\boldsymbol{p}^0$. Again, this very reasonable property was also proposed by Eichhorn and Voeller (1976; 23)[85].

T19: *Monotonicity in Current Quantities*: if $\boldsymbol{q}^1 < \boldsymbol{q}^2$, then

$$\{\textstyle\sum_{i=1}^{N} p_i^1 q_i^1/\sum_{i=1}^{N} p_i^0 q_i^0\}/P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) < \{\sum_{i=1}^{N} p_i^1 q_i^2/\sum_{i=1}^{N} p_i^0 q_i^0\}/P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^2).$$

T20: *Monotonicity in Base Quantities*: if $\boldsymbol{q}^0 < \boldsymbol{q}^2$, then

[*29] This test was first proposed by Eichhorn and Voeller (1976; 10)[85].
[*30] This test was proposed by Diewert (1992; 219)[46].
[*31] Bowley (1901; 227)[18] and Fisher (1922; 403)[91] both endorsed this property for a price index.

$$\{\sum_{i=1}^{N} p_i^1 q_i^1 / \sum_{i=1}^{N} p_i^0 q_i^0\}/P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) > \{\sum_{i=1}^{N} p_i^1 q_i^1 / \sum_{i=1}^{N} p_i^0 q_i^2\}/P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^2, \boldsymbol{q}^1).$$

If we define the implicit quantity index Q that corresponds to P using (12), we find that T19 translates into the following inequality involving Q:

$$Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) < Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^2) \text{ if } \boldsymbol{q}^1 < \boldsymbol{q}^2. \tag{15}$$

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into:

$$Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) > Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^2, \boldsymbol{q}^1) \text{ if } \boldsymbol{q}^0 < \boldsymbol{q}^2. \tag{16}$$

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 are due to Vogt (1980, 70)[162].

Irving Fisher's (1921; 534)[90] (1922; 72-81)[91] third Reversal Test (the other two being T9 and T11) is the following test:

T21: *Factor Reversal Test* (Functional Form Symmetry Test):
$P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)P(\boldsymbol{q}^0, \boldsymbol{q}^1, \boldsymbol{p}^0, \boldsymbol{p}^1) = \boldsymbol{p}^1 \cdot \boldsymbol{q}^1 / \boldsymbol{p}^0 \cdot \boldsymbol{q}^0$.

A justification for this test is the following one: if $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ is a good functional form for the price index, then if we reverse the roles of prices and quantities, $P(\boldsymbol{q}^0, \boldsymbol{q}^1, \boldsymbol{p}^0, \boldsymbol{p}^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ and the quantity index $Q(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = P(\boldsymbol{q}^0, \boldsymbol{q}^1, \boldsymbol{p}^0, \boldsymbol{p}^1)$ ought to equal the expenditure value ratio, e^1/e^0. The second part of this argument does not seem to be valid and thus many researchers over the years have objected to the factor reversal test.[*32]

Recall that the Fisher (1922)[91] ideal price index, $P_{\mathrm{F}}(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$, was defined as the geometric mean of the Laspeyres and Paasche price indexes:

$$P_{\mathrm{F}}(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \equiv [\boldsymbol{p}^1 \cdot \boldsymbol{q}^0 / \boldsymbol{p}^0 \cdot \boldsymbol{q}^0]^{1/2} [\boldsymbol{p}^1 \cdot \boldsymbol{q}^1 / \boldsymbol{p}^0 \cdot \boldsymbol{q}^1]^{1/2}. \tag{17}$$

This index showed up in section 2.2 above as being a "best" index from the viewpoint of the fixed basket approach to bilateral index number theory. It turns out that it is also a candidate for being a "best" index number formula from the viewpoint of the test approach to index number theory since it satisfies all 21 of the tests listed above. It can also be shown that it is the only bilateral index number formula that satisfies all of the above tests. In fact, the Fisher price index is the only price index which satisfies the Positivity Test T1 and the three reversal tests T11-T13.[*33] To see this, rearrange the terms in the statement of test T13 into the following equation:

$$\begin{aligned}
\{\boldsymbol{p}^1 \cdot \boldsymbol{q}^1 / \boldsymbol{p}^0 \cdot \boldsymbol{q}^0\} / \{\boldsymbol{p}^0 \cdot \boldsymbol{q}^1 / \boldsymbol{p}^1 \cdot \boldsymbol{q}^0\} &= P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) / P(\boldsymbol{p}^1, \boldsymbol{p}^0, \boldsymbol{q}^0, \boldsymbol{q}^1) \\
&= P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) / P(\boldsymbol{p}^1, \boldsymbol{p}^0, \boldsymbol{q}^1, \boldsymbol{q}^0) \\
&\qquad \text{using T12, the quantity reversal test} \\
&= P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \\
&\qquad \text{using T11, the time reversal test.}
\end{aligned} \tag{18}$$

[*32] See for example, Samuelson and Swamy (1974; 575)[146]: "We must stress again that the factor reversal test offers no stumbling block for our definitions of $P(\boldsymbol{p}^1, \boldsymbol{p}^0; \boldsymbol{q}^a)$ and $Q(\boldsymbol{q}^1, \boldsymbol{q}^0; \boldsymbol{p}^a,)$ if, as we should do logically, we drop the *strong* requirement that the *same* formula should apply to $f(\boldsymbol{q})$ as to $c(\boldsymbol{p})$. A man and wife should be properly matched; but that does not mean I should marry my identical twin!" The notation used by Samuelson and Swamy has been changed to align with our notation.

[*33] See Diewert (1992; 221)[46].

Now take positive square roots on both sides of (18) and we see that the left hand side of the equation is the Fisher index $P_F(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ defined by (17) and the right hand side is $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$. Thus if $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ satisfies T1, T11, T12 and T13, it must equal the Fisher ideal index P_F.

If one is willing to embrace T21 as a basic test, Funke and Voeller (1978; 180)[100] showed that the only index number function $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ which satisfies T1 (Positivity), T11 (Time Reversal Test), T12 (Quantity Reversal Test) and T21 (Factor Reversal Test) is the Fisher ideal index P_F defined by (17). Thus the Price Reversal Test T13 can be replaced by the Factor Reversal Test in order to obtain a minimal set of four tests that lead to the Fisher price index.*34

There is one additional test that has emerged in recent years as being very important:

T22: *Circularity Test* (Transitivity over Time Test)*35:

$$P(\boldsymbol{p}^0, \boldsymbol{p}^2, \boldsymbol{q}^0, \boldsymbol{q}^2) = P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2).$$

The index number on the left hand side of T22, compares prices in period 2 directly with prices in period 0 and $P(\boldsymbol{p}^0, \boldsymbol{p}^2, \boldsymbol{q}^0, \boldsymbol{q}^2)$ is called the *fixed base price index* for period 2. The *chained price index* for period 2, $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$, on the right hand side of T22 compares prices in period 2 with those in period 0 by first comparing prices in period 1 with those in period 0 (this is the *chain link index* $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$) and multiplies that index by the chain link index that compares prices in period 2 to those of period 1, $P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$. If the index number formula P satisfies the Circularity Test T22, then it does not matter whether we use the *chained index* (the right hand side of T22) to compare prices in period 2 with those of the base period 0 or if we use the *fixed base index* (the left hand side of T22): *we get the same answer either way.* Obviously, it would be very useful if we could find an index number formula that satisfied the Circularity Test and had satisfactory axiomatic properties with respect to the other tests that we have considered.

It is straightforward to show that the Laspeyres, Paasche, Fisher and Törnqvist indexes do not pass the Circularity Test. The Jevons and Lowe indexes do pass this test.

In the remainder of this section, we will explore some of the implications of the Circularity Test.

Suppose that the bilateral index number formula, $P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ satisfy the following tests: T1 (Positivity), T3 (Strong Identity) and T22 (Circularity). Then $P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ must be equal to the following ratio:

$$P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2) = c(\boldsymbol{p}^2)/c(\boldsymbol{p}^1) \tag{19}$$

where the function of prices $\boldsymbol{p}$, $c(\boldsymbol{p}) > 0$ for $\boldsymbol{p} \gg \boldsymbol{0}_N$.

To establish this result, define the function $g(\boldsymbol{p}, \boldsymbol{q})$ for $\boldsymbol{p} \gg \boldsymbol{0}_N$ and $\boldsymbol{q} \gg \boldsymbol{0}_N$ using the given price index function $P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ as follows:

$$g(\boldsymbol{p}, \boldsymbol{q}) \equiv P(\boldsymbol{1}_N, \boldsymbol{p}, \boldsymbol{1}_N, \boldsymbol{q}) \tag{20}$$

*34 Other characterizations of the Fisher price index can be found in Funke and Voeller (1978)[100] and Balk (1985)[8] (1995)[9].

*35 The test name is due to Fisher (1922; 413)[91] and the concept was originally due to Westergaard (1890; 218-219)[167].

where $\mathbf{1}_N$ is a vector of ones of dimension N. Note that using Test T1, $g(\boldsymbol{p},\boldsymbol{q}) > 0$ for all $\boldsymbol{p} \gg \mathbf{0}_N$ and $\boldsymbol{q} \gg \mathbf{0}_N$. Again using T1, rearrange the equation that defines the Circularity Test as follows:

$$\begin{aligned} P(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2) &= P(\boldsymbol{p}^0,\boldsymbol{p}^2,\boldsymbol{q}^0,\boldsymbol{q}^2)/P(\boldsymbol{p}^0,\boldsymbol{p}^1,\boldsymbol{q}^0,\boldsymbol{q}^1) \\ &= P(\mathbf{1}_N,\boldsymbol{p}^2,\mathbf{1}_N,\boldsymbol{q}^2)/P(\mathbf{1}_N,\boldsymbol{p}^1,\mathbf{1}_N,\boldsymbol{q}^1) \quad \text{setting } \boldsymbol{p}^0 = \mathbf{1}_N \text{ and } \boldsymbol{q}^0 = \mathbf{1}_N \\ &= g(\boldsymbol{p}^2,\boldsymbol{q}^2)/g(\boldsymbol{p}^1,\boldsymbol{q}^1) \quad \text{using definition (20).} \end{aligned} \tag{21}$$

Set $\boldsymbol{p}^1 = \boldsymbol{p}^2 \equiv \boldsymbol{p}$ and use the Strong Identity Test T3 which leads to the following equation:

$$\begin{aligned} 1 &= P(\boldsymbol{p},\boldsymbol{p},\boldsymbol{q}^1,\boldsymbol{q}^2) \\ &= g(\boldsymbol{p},\boldsymbol{q}^2)/g(\boldsymbol{p},\boldsymbol{q}^1) \quad \text{using (21) with } \boldsymbol{p}^1 = \boldsymbol{p}^2 \equiv \boldsymbol{p}. \end{aligned} \tag{22}$$

Thus

$$g(\boldsymbol{p},\boldsymbol{q}^2) = g(\boldsymbol{p},\boldsymbol{q}^1) \quad \text{for all } \boldsymbol{p} \gg \mathbf{0}_N, \boldsymbol{q}^1 \gg \mathbf{0}_N \text{ and } \boldsymbol{q}^2 \gg \mathbf{0}_N. \tag{23}$$

Equations (23) show that $g(\boldsymbol{p},\boldsymbol{q})$ does not depend on $\boldsymbol{q}$. Thus for definiteness, define $c(\boldsymbol{p}) \equiv g(\boldsymbol{p},\mathbf{1}_N) = P(\mathbf{1}_N,\boldsymbol{p},\mathbf{1}_N,\mathbf{1}_N)$ and substitute this equation into (21). We find that:

$$P(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2) = c(\boldsymbol{p}^2)/c(\boldsymbol{p}^1). \tag{24}$$

If we add additional Tests that $P(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2)$ satisfies, then we can deduce additional properties that $c(\boldsymbol{p})$ satisfies in addition to positivity. If $P(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2)$ satisfies T2 (Continuity), T5 (Linear Homogeneity in Current Prices), T17 (Monotonicity in Current Prices), then $c(\boldsymbol{p})$ satisfies continuity, linear homogeneity and monotonicity in $\boldsymbol{p}$ respectively.[*36]

If we add the Commensurability Test T10 to T1, T3 and T22, then it is possible to show that $c(\boldsymbol{p})$ is proportional to the following Cobb-Douglas (1928)[31] functional form:[*37]

$$c(\boldsymbol{p}^t) \equiv \textstyle\prod_{n=1}^{N} p_{tn}^{c(n)} \tag{25}$$

where the $c(n)$ are constants. If we add the Tests T5 (Linear Homogeneity in Current Period Prices) and T10 (Monotonicity in Current Period Prices), then we can deduce that the $c(n)$ are positive numbers that sum to one.[*38] This result seems to support Fisher's view that the Circularity Test does not lead to realistic indexes between pairs of observations. Fisher (1922; 274)[91] claimed that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*...". Fisher(1922; 274-5)[91] went on to assert:

> "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. ... Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."

*36 For related discussions, see Samuelson and Swamy (1974)[146], Eichhorn and Voeller (1976)[85] and Eichhorn (1978; 164-172)[84].

*37 Konüs and Byushgens (1926; 163-166)[125] established this result. For a translation and commentary on their paper (published in Russian), see Diewert and Zelenyuk (2025)[79].

*38 These results are also due to Konüs and Byushgens (1926)[125]. For alternative derivations of this result, see Eichhorn (1978; 167-168)[84] and Balk (1995)[9].

The role of the Commensurability Test when it is assumed along with the Circularity Test is somewhat controversial as noted by Samuelson and Swamy (1974; 571)[146]. Suppose our price index function is the Lowe index, $P_{\text{Lo}}(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) = c(\boldsymbol{p}^1)/c(\boldsymbol{p}^0) = \boldsymbol{p}^1 \cdot \boldsymbol{q}^*/\boldsymbol{p}^0 \cdot \boldsymbol{q}^*$ where $\boldsymbol{q}^*$ is a representative *annual* quantity vector and $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ are *monthly* price vectors for months 0 and 1. This type of Consumer Price index has been used by many national statistical offices around the world. For this case, we have $c(\boldsymbol{p}) \equiv \boldsymbol{p} \cdot \boldsymbol{q}^* = \sum_{n=1}^N p_n q_n^*$. It is easy to see that the Lowe index satisfies the Circularity Test. What happens when we change the units of measurement?

$$\begin{aligned}
&P_{\text{Lo}}(\alpha_1 p_1^0, \ldots, \alpha_N p_N^0; \alpha_1 p_1^1, \ldots, \alpha_N p_N^1; \alpha_1^{-1} q_1^0, \ldots, \alpha_N^{-1} q_N^0; \alpha_1^{-1} q_1^1, \ldots, \alpha_N^{-1} q_N^1) \\
&\quad = \textstyle\sum_{n=1}^N \alpha_n p_n^1 q_n^* / \sum_{n=1}^N \alpha_n p_n^0 q_n^* \\
&\quad \neq \textstyle\sum_{n=1}^N p_n^1 q_n^* / \sum_{n=1}^N p_n^0 q_n^* \quad \text{for all } \alpha_n > 0 \\
&\quad = P_{\text{Lo}}(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1). \qquad (26)
\end{aligned}$$

It appears that the Lowe index does not satisfy T10, the Invariance to Changes in Units of Measurement Test because the Commensurability Test has changed the function $c(\boldsymbol{p})$. But in a practical sense, it *does* satisfy the Invariance to Changes in the Units of Measurement Test because when we change the monthly price units of measurement, we should also change the annual quantity units of measurement in the opposite direction. Thus the term $\sum_{n=1}^N \alpha_n p_n^1 q_n^* / \sum_{n=1}^N \alpha_n p_n^0 q_n^*$ on the right hand side of (26) should be replaced by $\sum_{n=1}^N (\alpha_n p_n^1)(q_n^*/\alpha_n) / \sum_{n=1}^N (\alpha_n p_n^0)(q_n^*/\alpha_n) = \boldsymbol{p}^1 \cdot \boldsymbol{q}^*/\boldsymbol{p}^0 \cdot \boldsymbol{q}^* = P_{\text{Lo}}(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)$ and hence the Lowe index does in fact satisfy the Commensurability Test if we are allowed to change the function $c(\boldsymbol{p})$ in a sensible way when the units of measurement are changed.[*39]

The above example shows that the Lowe index satisfies a Generalized Invariance to Changes in the Units of Measurement Test (but it is not straightforward to precisely define this Generalized Test). The Lowe index also causes us to question the validity of the implications of the Circularity Test. The Lowe index satisfies T1 (Positivity), T3 (Strong Identity) and T22 (Circularity) and the period t price level function for the Lowe index is $c(\boldsymbol{p}^t) \equiv \boldsymbol{p}^t \cdot \boldsymbol{q}^*$ for $t = 0, 1$. However, we could define $\boldsymbol{q}^*$ as $\boldsymbol{q}^0 + \boldsymbol{q}^1$. In this case, the Lowe price level function is equal to $c(\boldsymbol{p}; \boldsymbol{q}^*) \equiv \boldsymbol{p} \cdot \boldsymbol{q}^* = \boldsymbol{p} \cdot (\boldsymbol{q}^0 + \boldsymbol{q}^1)$, which seems to imply that $c(\boldsymbol{p})$ is not independent of $\boldsymbol{q}^0$ and $\boldsymbol{q}^1$. However, when we do the algebra surrounding the Circularity Test which are equations (20)-(24), this $\boldsymbol{q}^*$ is held constant as we work through the definitions that lead up to (24), $P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2) = c(\boldsymbol{p}^2)/c(\boldsymbol{p}^1)$. Thus the implications of the Circularity Test are a bit tricky.

We will return to this problem with the interpretation of the Circularity and Commensurability Tests when we review econometric approaches to the construction of a consumer price index and the corresponding quantity index.

Our candidate for "best" bilateral price index from the viewpoint of the test approach is the Fisher price index. However, a limitation on the above Test Approach to bilateral index number theory is that the above axioms assumed that all prices p_{tn} and q_{tn} were positive.

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*39 Samuelson and Swamy (1974; 571)[146] flagged this problem with the usual interpretation of the Commensurability Test: "The literature, from Fisher on, . . . , is inadequate on the dimensional invariance test. Properly speaking, once one has introduced the appropriate dimensional constants, we impose thereby no restrictions on the functional form of the index number. . . . See Percy W. Bridgman (1922)[19] for proper treatment of dimensional analysis in the natural sciences and logic." Thus Samuelson and Swamy rejected the usual Commensurability Test. The above analysis shows that some care must be used in applying the strict version of the test.

The axiomatic approach to index number theory has in general not dealt adequately with the case where some prices are missing in one of the two periods being compared.*40

2.5 The Economic Approach to Index Number Theory

The economic approach to index number theory*41 works as follows: suppose all purchasers of the N products in scope have the same preference or utility function, $f(\boldsymbol{q})$,*42 where $\boldsymbol{q} \equiv [q_1, \ldots, q_N]$ is a generic vector of aggregate purchases over a time period and $Q^t \equiv f(\boldsymbol{q}^t)$ is the *period t aggregate quantity* that corresponds to the period t vector of purchases, $\boldsymbol{q}^t$.*43 It is further assumed that all purchasers of the N products face the same period t price vector $\boldsymbol{p}^t$ and they collectively choose $\boldsymbol{q}^t$ as a solution to the following period t utility maximization problem:

$$\max_q\{f(\boldsymbol{q}); \boldsymbol{p}^t \cdot \boldsymbol{q} \leq e^t; \boldsymbol{q} \geq \boldsymbol{0}_N\} \equiv Q^t \tag{27}$$

where e^t is observed period t aggregate expenditures on the N products.*44 The unit cost function that corresponds to the utility function $f(\boldsymbol{q})$, $c(\boldsymbol{p})$, is defined as the minimum cost of achieving the utility level 1. If purchasers face the period t vector of prices $\boldsymbol{p}^t$, we have:

$$c(\boldsymbol{p}^t) \equiv \min_q\{\boldsymbol{p}^t \cdot \boldsymbol{q} : f(\boldsymbol{q}) = 1; \boldsymbol{q} \geq \boldsymbol{0}_N\} \equiv P^t. \tag{28}$$

Thus the *period t aggregate price level*, P^t, is equal to the unit cost $c(\boldsymbol{p}^t)$ and we have the following decomposition of period t expenditure e^t into the product of P^t and Q^t:*45

$$e^t = \boldsymbol{p}^t \cdot \boldsymbol{q}^t = c(\boldsymbol{p}^t) f(\boldsymbol{q}^t) = P^t Q^t. \tag{29}$$

Using this approach, we find that the bilateral price index that compares the prices of period t to the prices of period 0 is equal to $c(\boldsymbol{p}^t)/c(\boldsymbol{p}^0) = P^t/P^0$ (this is the Konüs (1924)[124] *true cost of living index* between periods 0 and t when we have linearly homogeneous utility functions) and the corresponding quantity index is equal to $f(\boldsymbol{q}^t)/f(\boldsymbol{q}^0) = Q^t/Q^0$. Note that this Economic Approach to bilateral index number theory has led to the same type of price index that was defined by (24) in the previous section on the Test Approach to bilateral index number theory.

It appears that the decomposition (29) contradicts Eichhorn's result (4) that there is no $f(\boldsymbol{q})$ and $c(\boldsymbol{p})$ solution to the equation $c(\boldsymbol{p})f(\boldsymbol{q}) = \boldsymbol{p} \cdot \boldsymbol{q} \equiv \sum_{n=1}^{N} p_n q_n$ for all $\boldsymbol{p} \gg \boldsymbol{0}_N$ and for all $\boldsymbol{q} \gg \boldsymbol{0}_N$. But there is no contradiction: the Eichhorn result requires that $c(\boldsymbol{p})f(\boldsymbol{q}) = \boldsymbol{p} \cdot \boldsymbol{q}$ for all strictly positive *price and quantity* vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ whereas the equation $\boldsymbol{p}^t \cdot \boldsymbol{q}^t = c(\boldsymbol{p}^t) f(\boldsymbol{q}^t)$

*40 Zhang, Johansen and Nygaard (2019)[169] and Diewert (2023)[60] made a start on the axiomatic approach to index number theory when there are missing prices.

*41 Frisch (1936: 10)[99] called the economic approach the "functional approach". The founder of the economic approach was Konüs (1924)[124]. The fundamentals of this approach taken in this section, which assumes homothetic preferences, were laid out by Pollak (1971)[139] (1989)[140], Afriat (1972)[1] and Samuelson and Swamy (1974)[146]. Following Samuelson and Swamy, we assume that $f(\boldsymbol{q})$ is linearly homogeneous so that the period t price level $P^t \equiv c(\boldsymbol{p}^t)$ is independent of the observed $\boldsymbol{q}^t$ and the aggregate quantity level $Q^t \equiv f(\boldsymbol{q}^t)$ is independent of the observed $\boldsymbol{p}^t$.

*42 It is assumed that $f(\boldsymbol{q})$ is a linearly homogeneous, nondecreasing and concave function of $\boldsymbol{q}$ over the set of nonnegative consumption vectors, $\boldsymbol{q} \geq \boldsymbol{0}_N$ where $\boldsymbol{0}_N$ is a vector of zeros of dimension N.

*43 Note that the preference function $f(\boldsymbol{q})$ can also be viewed as an *aggregator function*: it aggregates the microeconomic vector of purchases $\boldsymbol{q}$ into a single representative number Q.

*44 Thus $e^t = \boldsymbol{p}^t \cdot \boldsymbol{q}^t$. In practice when dealing with scanner data, $\boldsymbol{p}^t$ is the period t vector of *unit value prices* and the corresponding $\boldsymbol{q}^t$ is a vector of *total purchases* of the N products during period t.

*45 For the details of this decomposition , see Samuelson and Swamy (1974)[146] and Diewert (1974a)[40] (1976)[42].

only holds for all strictly positive *price* vectors $\boldsymbol{p}^t$. The companion quantity vector $\boldsymbol{q}^t$ depends on $\boldsymbol{p}^t$ and is a solution to the unit cost minimization problem defined by (28).*46

One way to regard index number theory is that index numbers can be used to *summarize information* in an efficient way. In the economic approach to index number theory, the $2N$ prices and quantities for period t, $\boldsymbol{p}^t$ and $\boldsymbol{q}^t$, are reduced down to the aggregate period t price, $c(\boldsymbol{p}^t)$ and the aggregate period t quantity, $f(\boldsymbol{q}^t)$. Thus index number theory can be considered a part of *descriptive statistics*, which also tries to summarize information in an efficient manner.

The practical question arises at this point: *how exactly are we to determine the functions $c(\boldsymbol{p})$* or $f(\boldsymbol{q})$ in order to come up with an index number formula that National Statistical Offices could use? There are at least two methods that could be used:

- *Econometric methods* could be used to estimate $c(\boldsymbol{p})$ or $f(\boldsymbol{q})$. This approach will be discussed in section 4.
- The *theory of exact index number formulae* could be used. This approach is discussed below.

If we make certain specific assumptions about the functional form for either $f(\boldsymbol{q})$ or $c(\boldsymbol{p})$, it can be shown that $c(\boldsymbol{p}^t)/c(\boldsymbol{p}^0)$ is equal to a known bilateral index number formula. This is the theory of *exact index numbers* that was developed by Konüs and Byushgens (1926)[125]*47, Afriat (1972)[1], Pollak (1971)[139] (1989)[140]*48 and Diewert (1976)[42]. In order to explain this approach, it is necessary to first establish some results from basic microeconomic theory.*49 This material will also be useful when we discuss the econometric approach to index number theory.

Assume that all purchasers maximize the utility function $f(\boldsymbol{q})$ subject to a budget constraint*50 where $\boldsymbol{q} \equiv [q_1, \ldots, q_N]$ which is linearly homogeneous*51, nondecreasing and concave*52 in quantities of N products in scope that are consumed over the nonnegative orthant. Let $\boldsymbol{p} \equiv [p_1, \ldots, p_N]$ be a vector of positive product prices and let u be a positive utility level. Then the total cost or expenditure function, $C(u, \boldsymbol{p})$, is defined as:

$$C(u, \boldsymbol{p}) \equiv \min_q \{\boldsymbol{p} \cdot \boldsymbol{q} : f(\boldsymbol{q}) \geq u\}. \quad (30)$$

If the expenditure function is differentiable with respect to the components of the commodity price vector $\boldsymbol{p}$, then Shephard's (1953; 11)[150] Lemma applies and the consumer's system of commodity demand functions as functions of the chosen utility level u and the commodity

*46 This distinction between the axiomatic approach to index number theory and the economic approach to index number theory was pointed out by Samuelson and Swamy (1974; 566)[146]: "This seeming contradiction with Frisch is possible because the price and quantity variables are not here allowed to be arbitrary independent variables, but rather are constrained to satisfy the observable demand functions which optimize well-being."

*47 For a translation and commentary on this paper, see Diewert and Zelenyuk (2025)[79].

*48 Pollak did the research for most of Pollak (1989)[140] during the 1970s as a consultant for the US Bureau of Labor Statistics.

*49 For additional exposition and references to the literature, see Diewert (1974a)[40].

*50 In order to aggregate over households, we also need the assumption that all households have the same linearly homogeneous utility function and they face the same vector of prices; see Samuelson and Swamy (1974)[146]. These assumptions are rather strong and many price statisticians object to the economic approach on the grounds that the required assumptions are not realistic.

*51 The assumption of linear homogeneity is required so that the resulting price level function, $c(\boldsymbol{p})$, does not depend on quantities; see Gorman (1953)[104] and Samuelson and Swamy (1974)[146] on this point.

*52 This assumption is required because the assumption of utility maximizing behavior implies that $c(\boldsymbol{p})$ must be a concave function of $\boldsymbol{p}$; see Diewert (1974a)[40].

price vector $\boldsymbol{p}$, $\boldsymbol{q}(u, \boldsymbol{p})$, is equal to the vector of first order partial derivatives of the expenditure function $C(u, \boldsymbol{p})$ with respect to the components of $\boldsymbol{p}$:

$$\boldsymbol{q}(u, \boldsymbol{p}) = \nabla_p C(u, \boldsymbol{p}). \tag{31}$$

The demand functions $\boldsymbol{q}(u, \boldsymbol{p}) \equiv [q_1(u, \boldsymbol{p}), \ldots, q_N(u, \boldsymbol{p})]$ defined by (31) are known as Hicksian[*53] demand functions. Define the *unit utility expenditure function* (or unit cost function) as $c(\boldsymbol{p}) \equiv C(1, \boldsymbol{p})$. It turns out that $c(\boldsymbol{p})$ is linearly homogeneous, nondecreasing and concave in $\boldsymbol{p}$ over the positive orthant. It can be used to indirectly define preferences.[*54] Since $f(\boldsymbol{q})$ is linearly homogeneous, it turns out that $C(u, \boldsymbol{p})$ is equal to u times the unit utility expenditure function; i.e., we have:

$$C(u, \boldsymbol{p}) = uc(\boldsymbol{p}). \tag{32}$$

Suppose that we have chosen a differentiable functional form for $c(\boldsymbol{p})$ and we have price and quantity data, $\boldsymbol{p}^t \equiv [p_{t1}, \ldots, p_{tN}]$ and $\boldsymbol{q}^t \equiv [q_{t1}, \ldots, q_{tN}]$, for periods $t = 1, \ldots, T$ that are generated by utility maximizing behavior using the utility function $f(\boldsymbol{q})$ that is dual to $c(\boldsymbol{p})$. Then using (31) and (32), the data should satisfy the following system of *Hicksian demand functions*:

$$\boldsymbol{q}^t = u^t \nabla_p c(\boldsymbol{p}^t); \quad t = 1, \ldots, T. \tag{33}$$

However, we cannot observe the period t utility levels, $u^t \equiv f(\boldsymbol{q}^t)$ for $t = 1, \ldots, T$. The *unobserved utility levels* u^t could be determined by setting *observed period* t *expenditures*, $e^t \equiv \boldsymbol{p}^t \cdot \boldsymbol{q}^t \equiv \sum_{n=1}^{N} p_{tn} q_{tn}$, equal to $C(u^t, \boldsymbol{p}^t)$ which in turn is equal to $u^t c(\boldsymbol{p}^t)$. Solving these equations $e^t = u^t c(\boldsymbol{p}^t)$ for each period t gives us u^t as the following function of e^t and $\boldsymbol{p}^t$:[*55]

$$u^t = e^t / c(\boldsymbol{p}^t); \quad t = 1, \ldots, T. \tag{34}$$

Substitute equations (34) into equations (33) and we obtain the following system of *market demand functions*:

$$\boldsymbol{q}^t = e^t \nabla_p c(\boldsymbol{p}^t) / c(\boldsymbol{p}^t); \quad t = 1, \ldots, T. \tag{35}$$

Equations (35) are the foundations of an econometric model for the case where preferences can be represented by a linearly homogeneous unit cost function. We need to assume an appropriate functional form for the dual unit cost or expenditure function $c(\boldsymbol{p})$, add error terms to equations (35) and use some form of nonlinear estimation to determine the unknown parameters that enter the functional form for $c(\boldsymbol{p})$. Once these parameters have been estimated, the period t aggregate price level $P^t \equiv c(\boldsymbol{p}^t)$ and the corresponding aggregate period t quantity level $u^t \equiv Q^t \equiv e^t / c(\boldsymbol{p}^t)$ can be calculated for $t = 1, \ldots, T$. Bilateral Konüs true cost of living indexes for period t relative to period r can be calculated as P^t / P^r and these indexes are transitive over periods 1 to T.

But it turns out that there is a problem with using equations (35) as estimating equations: *not all of the parameters that are present in* $c(\boldsymbol{p})$ *can be identified.* This is a consequence of the fact that utility is unobservable and so when we attempt to estimate the direct utility function $f(\boldsymbol{q})$ or its dual unit expenditure function, it is necessary to arbitrarily place at least one restriction on the parameters of $f(\boldsymbol{q})$ or $c(\boldsymbol{p})$ in order to determine the *scale of utility*

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*53 See Hicks (1946; 311-331)[108].

*54 See Samuelson and Swamy (1974)[146] and Diewert (1974a; 112)[40] for additional information. Diewert derived the following explicit formula for deriving the utility function $f(\boldsymbol{q})$ from the given unit cost function $c(\boldsymbol{p})$ for $\boldsymbol{q} \gg \boldsymbol{0}_N$: $f(\boldsymbol{q}) = 1/\max_p\{c(\boldsymbol{p}) : \boldsymbol{p} \cdot \boldsymbol{q} \leq 1, \boldsymbol{p} \geq \boldsymbol{0}_N\}$.

*55 This is known as the indirect utility function.

or the scale of the price level, $P = c(\boldsymbol{p})$. Since equations (35) use the unit cost function to represent purchaser preferences, it is usual to place the following restriction on the unknown parameters in $c(\boldsymbol{p})$:

$$c(\boldsymbol{p}^1) = 1. \tag{36}$$

The restriction (36) also sets the price level in period 1, P^1, equal to unity. This restriction is an example of what Samuelson (1974; 1262)[145] termed *money metric utility scaling*: it sets the utility level in period 1, u^1, equal to period 1 expenditure, $e^1 \equiv \boldsymbol{p}^1 \cdot \boldsymbol{q}^1$.

The above material outlines the usual strategy for consumer demand theory when purchaser preferences can be represented by a linearly homogeneous unit cost function, $c(\boldsymbol{p})$. However, a similar econometric approach can be worked out starting with a functional form for the utility function $f(\boldsymbol{q})$. This alternative approach starts with the assumption that the given linearly homogeneous, concave and increasing function $f(\boldsymbol{q})$ is given and $f(\boldsymbol{q})$ is also once differentiable. It is assumed that the observed period t quantity vector, $\boldsymbol{q}^t$, is a solution to the following utility maximization problem for period t:

$$\max_q\{f(\boldsymbol{q}); \boldsymbol{p}^t \cdot \boldsymbol{q} = e^t; \boldsymbol{q} \geq \boldsymbol{0}_N\}; \quad t = 1, \ldots, T. \tag{37}$$

We assume that $\boldsymbol{q}^t \gg \boldsymbol{0}_N$ solves the constrained maximization problem for period t and the observed period t expenditure is $e^t = \boldsymbol{p}^t \cdot \boldsymbol{q}^t$ for $t = 1, \ldots, T$. Since $f(\boldsymbol{q})$ is differentiable and concave, there exists a scalar Lagrange multiplier λ^t for problem t such that λ^t and $\boldsymbol{q}^t$ satisfy the following first order necessary and sufficient conditions:

$$\begin{aligned} \nabla_q f(\boldsymbol{q}^t) &= \lambda^t \boldsymbol{p}^t; \quad t = 1, \ldots, T; \\ e^t &= \boldsymbol{p}^t \cdot \boldsymbol{q}^t. \end{aligned} \tag{38}$$

Take the inner product of both sides of the first set of equations in (38) and use the linear homogeneity of $f(\boldsymbol{q})$ (which implies by Euler's Theorem on homogeneous functions that $f(\boldsymbol{q}^t) = \boldsymbol{q}^t \cdot \nabla_q f(\boldsymbol{q}^t)$ for $t = 1, \ldots, T$) to solve for $\lambda^t = f(\boldsymbol{q}^t)/\boldsymbol{p}^t \cdot \boldsymbol{q}^t = f(\boldsymbol{q}^t)/e^t$ for $t = 1, \ldots, T$. Substitute these solutions for λ^t into the first set of equations in (38) and we obtain the following system of estimating equations that are the $f(\boldsymbol{q})$ counterparts to the $c(\boldsymbol{p})$ estimating equations (35):

$$\boldsymbol{p}^t = e^t \nabla_q f(\boldsymbol{q}^t)/f(\boldsymbol{q}^t); \quad t = 1, \ldots, T. \tag{39}$$

Equations (39) have expenditures e^t and the quantity vector $\boldsymbol{q}^t$ as independent variables and $\boldsymbol{p}^t$ as the prices that would justify purchasing $\boldsymbol{q}^t$ if purchasers had "income" e^t. This is known as the system of *inverse demand functions* and they were first introduced into the economics literature by Konüs and Byushgens (1926)[125].*[56]

Equations (39) are the foundations of an econometric model for the case where preferences can be represented by a linearly homogeneous utility function. We need to add error terms to equations (39) and use nonlinear estimation to determine the unknown parameters that enter the functional form for $f(\boldsymbol{q})$. Once these parameters have been estimated, the period t aggregate quantity level $Q^t \equiv f(\boldsymbol{q}^t)$ and the corresponding aggregate period t price level $P^t \equiv e^t/f(\boldsymbol{q}^t)$ can be calculated for $t = 1, \ldots, T$. Bilateral Konüs true cost of living indexes for period t relative to period r can be calculated as P^t/P^r and these indexes are transitive over periods 1 to T. The corresponding bilateral quantity index is calculated as Q^t/Q^r.

It turns out that there is a problem with using equations (39) as estimating equations for determining $f(\boldsymbol{q})$: *not all of the parameters that are present in $f(\boldsymbol{q})$ can be identified.* Again,

*[56] See also Diewert and Zelenyuk (2025)[79].

this is a consequence of the fact that utility is unobservable and so when we attempt to estimate the direct utility function $f(\boldsymbol{q})$ or its dual unit expenditure function, it is necessary to place at least one restriction on the parameters of $f(\boldsymbol{q})$ or $c(\boldsymbol{p})$ in order to determine the scale of utility or the scale of the price level, $P = c(\boldsymbol{p})$. If we want the price level P^1 to equal 1 in period 1, then we require the following normalization on $f(\boldsymbol{q})$:

$$f(\boldsymbol{q}^1) = e^1. \tag{40}$$

The restriction (40) is an example of what social welfare economists call *ray scaling.*[*57] The utility that corresponds to the period 1 consumption vector $\boldsymbol{q}^1$ is determined by the normalization (40) so that $f(\boldsymbol{q}^1) = e^1$. We scale the level of utility along the ray that joins the origin to $\boldsymbol{q}^1$ as follows:

$$f(\lambda \boldsymbol{q}^1) = \lambda f(\boldsymbol{q}^1); \quad \lambda \geq 0. \tag{41}$$

Thus all consumption vectors $\boldsymbol{q} \geq \boldsymbol{0}_N$ such that they yield the same utility as $\boldsymbol{q}^1$ are assigned the utility level $f(\boldsymbol{q}^1)$; this is the indifference curve or surface $\{\boldsymbol{q} : f(\boldsymbol{q}) = f(\boldsymbol{q}^1)\}$. Then all consumption vectors $\boldsymbol{q}$ that are on the same indifference surface as $2\boldsymbol{q}$ are given the utility level $2f(\boldsymbol{q}^1)$; this is the indifference surface $\{\boldsymbol{q} : f(\boldsymbol{q}) = f(2\boldsymbol{q}^1) = 2f(\boldsymbol{q}^1)\}$, and so on. Thus the ray that starts at the origin and passes through the reference consumption vector $\boldsymbol{q}^1$ is used to scale utility levels.

We now provide some illustrations as to how the above methodology connects with bilateral index number formulae like the Fisher ideal index $P_{\mathrm{F}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2) \equiv [\boldsymbol{p}^2 \cdot \boldsymbol{q}^1\, \boldsymbol{p}^2 \cdot \boldsymbol{q}^2 / \boldsymbol{p}^1 \cdot \boldsymbol{q}^1\, \boldsymbol{p}^1 \cdot \boldsymbol{q}^2]^{1/2}$.

Suppose that purchasers preferences can be represented by the following unit cost function:[*58]

$$c(\boldsymbol{p}) \equiv [\boldsymbol{p}^T \mathbf{B} \boldsymbol{p}]^{1/2}; \quad \mathbf{B} = \mathbf{B}^T; \boldsymbol{p} \in S^* \tag{42}$$

where $\mathbf{B} \equiv [b_{ij}]$ is an N by N symmetric matrix that has one positive eigenvalue (that has a strictly positive eigenvector) and the remaining $N-1$ eigenvalues are zero or negative. The set S^* is the price region of regularity where the function c is positive, concave and increasing and hence for this set of prices, the unit cost function $c(\boldsymbol{p})$ defined by (42) can provide a valid representation of preferences over this region.[*59]

Differentiating the $c(\boldsymbol{p})$ defined by (42) with respect to the components of $\boldsymbol{p}$ leads to the following vector of first order partial derivatives of the unit cost function:

$$\begin{aligned} \nabla_p c(\boldsymbol{p}) &= \mathbf{B}\boldsymbol{p} / [\boldsymbol{p}^T \mathbf{B} \boldsymbol{p}]^{1/2} \\ &= \mathbf{B}\boldsymbol{p} / c(\boldsymbol{p}) \end{aligned} \tag{43}$$

where the second equation in (43) follows using definition (42). We assume that $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ both belong to the regularity region of prices S^*. Evaluate equations (43) at $\boldsymbol{p}^t$ for $t = 1, 2$.

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*57 This is the type of utility scaling recommended by Blackorby (1975)[17] and other welfare economists because this form of scaling does not depend on prices.

*58 In definition (42), $\boldsymbol{p}$ is regarded as a column vector, $\boldsymbol{p}^T$ is the row vector that is the transpose of $\boldsymbol{p}$ and $\mathbf{B}^T$ is the transpose of the N by N matrix $\mathbf{B} \equiv [b_{ik}]$. Thus $c(\boldsymbol{p}) = [\sum_{i=1}^N \sum_{k=1}^N p_i b_{ik} p_k]^{1/2}$ with $b_{ik} = b_{ki}$ for all i, k.

*59 The region of regularity is defined as the set $S^* \equiv \{\boldsymbol{p} : \mathbf{B}\boldsymbol{p} \gg \boldsymbol{0}_N; \boldsymbol{p} \gg \boldsymbol{0}_N\}$. For the details on the restrictions on the matrix $\mathbf{B}$, see Diewert (1976)[42] and Diewert and Hill (2010)[66]. Konüs and Byushgens (1926)[125] introduced the functional forms defined by (42) and (47) into the economics literature but did not discuss the appropriate regularity conditions that apply to the $\mathbf{B}$ and $\mathbf{A}$ matrices in order to ensure that the functions $c(\boldsymbol{p})$ and $f(\boldsymbol{q})$ are concave functions.

Multiply both sides of the resulting period t equation by $u^t/c(\boldsymbol{p}^t)u^t$. Shephard's Lemma, $\boldsymbol{q}^t = \nabla_p c(\boldsymbol{p})u^t$ and equations (34) imply that $e^t = \boldsymbol{p}^t \cdot \boldsymbol{q}^t = c(\boldsymbol{p}^t)u^t$. We obtain the following equations that are equivalent to equations (43) evaluated at $\boldsymbol{p}^t$ for $t = 1, 2.$:

$$\boldsymbol{q}^t/\boldsymbol{p}^t \cdot \boldsymbol{q}^t = \mathbf{B}\boldsymbol{p}^t/c(\boldsymbol{p}^t)^2 \quad t = 1, 2. \tag{44}$$

Recall that the Fisher price index between two periods was defined by (17). The square of the Fisher price index is equal to the following expression:

$$\begin{aligned} P_{\mathrm{F}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)^2 &= \boldsymbol{p}^2 \cdot \boldsymbol{q}^1\, \boldsymbol{p}^2 \cdot \boldsymbol{q}^2/\boldsymbol{p}^1 \cdot \boldsymbol{q}^1\, \boldsymbol{p}^1 \cdot \boldsymbol{q}^2 \\ &= \boldsymbol{p}^2 \cdot [\boldsymbol{q}^1/\boldsymbol{p}^1 \cdot \boldsymbol{q}^1]/\boldsymbol{p}^1 \cdot [\boldsymbol{q}^2/\boldsymbol{p}^2 \cdot \boldsymbol{q}^2] \\ &= \boldsymbol{p}^2 \cdot [\mathbf{B}\boldsymbol{p}^1/c(\boldsymbol{p}^1)^2]/\boldsymbol{p}^1 \cdot [\mathbf{B}\boldsymbol{p}^2/c(\boldsymbol{p}^2)^2] \quad \text{using (44)} \\ &= c(\boldsymbol{p}^2)^2/c(\boldsymbol{p}^1)^2 \quad \text{using } \mathbf{B} = \mathbf{B}^T \\ &= [c(\boldsymbol{p}^2)/c(\boldsymbol{p}^1)]^2. \end{aligned} \tag{45}$$

Take the positive square root of both sides of (45) and we find that $P_{\mathrm{F}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2) = c(\boldsymbol{p}^2)/c(\boldsymbol{p}^1)$, the Konüs true Cost of Living Index between periods 1 and 2. Thus under the assumption that purchasers of the N products engage in cost minimizing behavior during periods 1 and 2 and have preferences over the N commodities that correspond to the unit cost function defined by (42), the Fisher ideal price index P_{F} is *exactly* equal to the true price index, $c(\boldsymbol{p}^2)/c(\boldsymbol{p}^1)$.[*60] Konüs and Byushgens (1926)[125] noticed the connection with the preferences that are generated by the unit cost function $c(\boldsymbol{p})$ defined by (42) to the Fisher ideal price index so this functional form was called the KBF unit cost function by Diewert and Feenstra (2019)[64] (2022)[65].

A special case of the KBF unit cost function occurs when the matrix $\mathbf{B}$ is equal to a rank 1 matrix generated by the positive column vector $\boldsymbol{b}$:

$$\mathbf{B} = \boldsymbol{b}\boldsymbol{b}^T. \tag{46}$$

In this case, the unit cost function becomes $c(\boldsymbol{p}) \equiv [\boldsymbol{p}^T\boldsymbol{b}\boldsymbol{b}^T\boldsymbol{p}]^{1/2} = \boldsymbol{b}\cdot\boldsymbol{p}$. The system of Hicksian demand functions for this functional form for period t data is given by $\boldsymbol{q}^t = \nabla_p c(\boldsymbol{p}^t)u^t = \boldsymbol{b}u^t$. Thus purchasers do not substitute between products as prices change and this class of preferences is known as *no substitution preferences or Leontief preferences.* The Fisher price index is exact for these preferences.

Another justification for the use of Fisher price and quantity indexes can be obtained by assuming that purchasers of the N products have the following functional form for their utility function:

$$f(\boldsymbol{q}) \equiv [\boldsymbol{q}^T\mathbf{A}\boldsymbol{q}]^{1/2}; \quad \mathbf{A} = \mathbf{A}^T; \boldsymbol{q} \in S \tag{47}$$

where $\mathbf{A} \equiv [a_{ij}]$ is an N by N symmetric matrix that has one positive eigenvalue (that has a strictly positive eigenvector) and the remaining $N-1$ eigenvalues are zero or negative. The set S is the quantity region of regularity where the function f is positive, concave and increasing.

Differentiating the $f(\boldsymbol{q})$ defined by (47) with respect to the components of $\boldsymbol{q}$ leads to the following vector of first order partial derivatives of the utility function evaluated at the period

[*60] See Diewert (1976; 133-134)[42] and Konüs and Byushgens (1926)[125] or Diewert and Zelenyuk (2025)[79].

t quantity vector $\boldsymbol{q}^t$:

$$\begin{aligned}\nabla_q f(\boldsymbol{q}^t) &= \mathbf{A}\boldsymbol{q}^t/[\boldsymbol{q}^{tT}\mathbf{A}\boldsymbol{q}^t]^{1/2} \quad t=1,2\\ &= \mathbf{A}\boldsymbol{q}^t/f(\boldsymbol{q}^t)\end{aligned} \tag{48}$$

where the second equation in (48) follows using definition (47). We assume that $\boldsymbol{q}^1$ and $\boldsymbol{q}^2$ both belong to the regularity region of quantities S. Recall the inverse demand system equations (39). Evaluate these equations at $t = 1, 2$. We obtain the following equations:

$$\begin{aligned}\boldsymbol{p}^t &= e^t\nabla_q f(\boldsymbol{q}^t)/f(\boldsymbol{q}^t) \quad t=1,2\\ &= e^t\mathbf{A}\boldsymbol{q}^t/f(\boldsymbol{q}^t)^2 \quad \text{using (48).}\end{aligned} \tag{49}$$

Since e^t equals $\boldsymbol{p}^t \cdot \boldsymbol{q}^t$, equations (49) can be rewritten as:

$$\boldsymbol{p}^t/\boldsymbol{p}^t\cdot\boldsymbol{q}^t = \mathbf{A}\boldsymbol{q}^t/f(\boldsymbol{q}^t)^2; \quad t=1,2. \tag{50}$$

The Fisher (1922)[91] ideal quantity index Q_{F} is defined as follows:

$$\begin{aligned}Q_{\mathrm{F}}(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2) &\equiv [\boldsymbol{p}^1\cdot\boldsymbol{q}^2\,\boldsymbol{p}^2\cdot\boldsymbol{q}^2/\boldsymbol{p}^1\cdot\boldsymbol{q}^1\,\boldsymbol{p}^2\cdot\boldsymbol{q}^1]^{1/2}\\ &= \{\boldsymbol{q}^2\cdot[\boldsymbol{p}^1/\boldsymbol{p}^1\cdot\boldsymbol{q}^1]/\boldsymbol{q}^1\cdot[\boldsymbol{p}^2/\boldsymbol{p}^2\cdot\boldsymbol{q}^2]\}^{1/2}\\ &= \{\boldsymbol{q}^2\cdot[\mathbf{A}\boldsymbol{q}^1/f(\boldsymbol{q}^1)^2]/\boldsymbol{q}^1\cdot[\mathbf{A}\boldsymbol{q}^2/f(\boldsymbol{q}^2)^2]\}^{1/2} \quad \text{using (50)}\\ &= \{f(\boldsymbol{q}^2)^2/f(\boldsymbol{q}^1)^2\}^{1/2} \quad \text{using } \mathbf{A}=\mathbf{A}^T\\ &= f(\boldsymbol{q}^2)/f(\boldsymbol{q}^1).\end{aligned} \tag{51}$$

Thus the Fisher quantity index is exact for the "true" quantity or utility ratio, $f(\boldsymbol{q}^2)/f(\boldsymbol{q}^1)$.

The Implicit Fisher price index that is the companion index to the Fisher quantity index P_{F}^* is:

$$P_{\mathrm{F}}^*(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2) \equiv [\boldsymbol{p}^2\cdot\boldsymbol{q}^2/\boldsymbol{p}^1\cdot\boldsymbol{q}^1]/Q_{\mathrm{F}}(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2) = P_{\mathrm{F}}(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2). \tag{52}$$

Straightforward computations for the terms in (52) show that the Implicit Fisher price index P_{F}^* is equal to the "regular" Fisher price index P_{F}. Thus under the assumption that purchasers of the N products maximize the KBF utility function $f(\boldsymbol{q})$ defined by (47), the Fisher ideal quantity index Q_{F} is *exactly* equal to the true Pollak-Samuelson-Swamy quantity index $f(\boldsymbol{q}^2)/f(\boldsymbol{q}^1)$ and the Fisher price index P_{F} is exactly equal to the Konüs true Cost of Living price index, $c^*(\boldsymbol{p}^2)/c^*(\boldsymbol{p}^1)$, where $c^*(\boldsymbol{p})$ is the unit cost function that is dual to the $f(\boldsymbol{q})$ defined by (47).

Diewert (1976)[42] noted that the above results hold if the $\mathbf{A}$ matrix that appears in definition (47) has the following representation:

$$\mathbf{A} = \boldsymbol{a}\boldsymbol{a}^T \tag{53}$$

where $\boldsymbol{a}$ is column vector of constants and $\boldsymbol{a}^T \equiv [a_1, \ldots, a_N]$ is the transpose of the vector $\boldsymbol{a}$. In this case, the KBF utility is $f(\boldsymbol{q}) \equiv [\boldsymbol{q}^T\mathbf{A}\boldsymbol{q}]^{1/2} \equiv [\boldsymbol{q}^T\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{q}]^{1/2} = \boldsymbol{a}^T\boldsymbol{q} \equiv \sum_{n=1}^N a_n q_n$. Thus the preference function becomes a simple *linear function of quantities* purchased during the period under consideration. Hence if preferences are linear, then the Fisher ideal quantity index $Q_{\mathrm{F}}^*(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2)$ is *exactly* equal to the true Samuelson-Swamy quantity index $f(\boldsymbol{q}^2)/f(\boldsymbol{q}^1) = \boldsymbol{a}\cdot\boldsymbol{q}^2/\boldsymbol{a}\cdot\boldsymbol{q}^1$ and the Fisher price index $P_{\mathrm{F}}^*(\boldsymbol{p}^1,\boldsymbol{p}^2,\boldsymbol{q}^1,\boldsymbol{q}^2)$ is exactly equal to the Konüs true Cost of Living price index, $c^*(\boldsymbol{p}^2)/c^*(\boldsymbol{p}^1)$, where $c^*(\boldsymbol{p})$ is the unit cost function that corresponds to linear preferences.*61

The above results show that the observable Fisher price index $P_{\mathrm{F}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ is *exactly* equal to the economic price index $c(\boldsymbol{p}^2)/c(\boldsymbol{p}^1)$ if consumers maximize utility and have preferences that are dual to the KBF unit cost function defined by (42).*62 Similarly, the above results show that the observable Fisher quantity index $Q_{\mathrm{F}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ is *exactly* equal to the economic quantity index $f(\boldsymbol{q}^2)/f(\boldsymbol{q}^1)$ if consumers maximize the utility function defined by (47).

We need to introduce the concept of a *flexible functional form.**63 A twice continuously differentiable function $f(\boldsymbol{q})$ of N variables $\boldsymbol{q} \equiv (q_1, \ldots, q_N)$ can provide a *second order approximation* to another such function $f^*(\boldsymbol{q})$ around the point $\boldsymbol{q}^*$ if the level and all of the first and second order partial derivatives of the two functions coincide at $\boldsymbol{q}^*$. It can be shown*64 that the function $f(\boldsymbol{q})$ defined by (47) can provide a second order approximation to an arbitrary f^* around any (strictly positive) point $\boldsymbol{q}^*$ in the class of linearly homogeneous functions. Thus the $f(\boldsymbol{q})$ defined by (47) is a *flexible functional form.* Diewert (1976; 117)[42] termed an index number formula $Q_{\mathrm{F}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ that was *exactly* equal to the true quantity index $f(\boldsymbol{q}^2)/f(\boldsymbol{q}^1)$ (where f is a flexible functional form) *a superlative index number formula.**65 Equation (51) and the fact that the homogeneous quadratic function f defined by (47) is a flexible functional form shows that the Fisher ideal quantity index Q_{F} is a superlative index number formula. Similarly, the function $c(\boldsymbol{p})$ defined by (42) can provide a second order approximation to an arbitrary c^* around any (strictly positive) point $\boldsymbol{p}^*$ in the class of linearly homogeneous functions. Thus the $c(\boldsymbol{p})$ defined by (42) is a *flexible functional form.* Equation (45) which shows that $P_{\mathrm{F}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ is exactly equal to the Konüs price index $c(\boldsymbol{p}^2)/c(\boldsymbol{p}^1)$ and the fact that the KBF function $c(\boldsymbol{p})$ defined by (42) is a flexible functional form shows that the Fisher ideal price index P_{F} is a *superlative index number formula.*

Recall the Törnqvist-Theil price index, $P_{\mathrm{T}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$, that was defined by (11) in section 2.3. It can be shown that this index is also a superlative index. Below, we show that this index is exactly equal to a certain Konüs price index that allows for nonhomothetic preferences; i.e., the exactness result which is explained below does not require the assumption that the utility function $f(\boldsymbol{q})$ be linearly homogeneous.

Before we derive the main result, we require some preliminary results. Suppose the function

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*61 The unit cost function $c^*(p_1, p_2, \ldots, p_N)$ that corresponds to linear preferences is defined as the solution to the following linear programming problem: $c^*(p_1, p_2, \ldots, p_N) \equiv \min_q\{\sum_{n=1}^N p_n q_n : \sum_{n=1}^N a_n q_n \geq 1, q_n \geq 0, n = 1, \ldots, N\}$. Assuming that all a_n and p_n are positive, the solution is $c^*(p_1, p_2, \ldots, p_N) = \min_n\{p_n/a_n : n = 1, \ldots, N\}$. Let S be the set of products n which attain this minimum. Then the $q_1, \ldots, q_N$ solution set to this minimization problem is the set of q_n such that $\sum_{n \in S} a_n q_n = 1, q_n \geq 0$ for $n \in S$ and $q_n = 0$ for $n \notin S$. If the period t quantity vector is strictly positive and the assumption that purchasers maximize a linear utility function holds exactly, then it must be the case that all of the ratios p_{tn}/a_n are equal and that prices vary in a proportional manner over time. The assumption of exact utility maximizing behavior is unlikely to hold empirically but it may be the case that it holds approximately. We will return to this point when we discuss the econometric approach to multilateral indexes.

*62 Other examples of exact index number formulae were derived by Pollak (1971)[139] (1989)[140], Afriat (1972)[1], Samuelson and Swamy (1974)[146] and Diewert (1976)[42].

*63 Diewert (1974a; 133)[40] introduced this term to the economics literature.

*64 See Diewert (1976; 130)[42] and let the parameter r equal 2.

*65 Fisher (1922; 247)[91] used the term superlative to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give some precision to Fisher's definition of superlativeness. Fisher defined an index number formula to be superlative if it approximated the corresponding Fisher ideal results using his data set.

of N variables, $f(z_1, \ldots, z_N) \equiv f(\boldsymbol{z})$, is quadratic; i.e.,

$$f(\boldsymbol{z}) \equiv \boldsymbol{a}_0 + \boldsymbol{a}^T \boldsymbol{z} + (1/2)\boldsymbol{z}^T \mathbf{A} \boldsymbol{z}; \quad \mathbf{A} = \mathbf{A}^T \tag{54}$$

where $\boldsymbol{a}$ is a vector of parameters and $\mathbf{A}$ is a symmetric matrix of parameters. It is well known that the second order Taylor series approximation to a quadratic function is *exact*; i.e., if f is defined by (54) above, then for any two points, $\boldsymbol{z}^1$ and $\boldsymbol{z}^2$, we have

$$f(\boldsymbol{z}^2) - f(\boldsymbol{z}^1) = \nabla f(\boldsymbol{z}^1)^T(\boldsymbol{z}^2 - \boldsymbol{z}^1) + (1/2)(\boldsymbol{z}^2 - \boldsymbol{z}^1)^T \nabla^2 f(\boldsymbol{z}^1)(\boldsymbol{z}^2 - \boldsymbol{z}^1).^{*66} \tag{55}$$

It is less well known that *an average of two first order Taylor series approximations* to a quadratic function is also *exact*; i.e., if f is defined by (54) above, then for any two points, $\boldsymbol{z}^1$ and $\boldsymbol{z}^2$, we have[*67]

$$f(\boldsymbol{z}^2) - f(\boldsymbol{z}^1) = (1/2)[\nabla f(\boldsymbol{z}^1) + \nabla f(\boldsymbol{z}^2)]^T[\boldsymbol{z}^2 - \boldsymbol{z}^1]. \tag{56}$$

Diewert (1976; 118)[42] showed that equation (56) characterized a quadratic function and called the equation the *quadratic approximation lemma*. We will refer to (56) as the *quadratic identity*.

We now suppose that the consumer's cost function, $C(u, \boldsymbol{p})$, has the following *translog functional form*:[*68]

$$\begin{aligned} \ln C(u, \boldsymbol{p}) \equiv a_0 &+ \textstyle\sum_{i=1}^N a_i \ln p_i + (1/2)\sum_{i=1}^N \sum_{k=1}^N a_{ik} \ln p_i \ln p_k \\ &+ b_0 \ln u + \textstyle\sum_{i=1}^N b_i \ln p_i \ln u + (1/2) b_{00} [\ln u]^2 \end{aligned} \tag{57}$$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions:

$$a_{ik} = a_{ki}; \quad i, k = 1, \ldots, N; \tag{58}$$

$$\textstyle\sum_{i=1}^N a_i = 1; \tag{59}$$

$$\textstyle\sum_{i=1}^N b_i = 0; \tag{60}$$

$$\textstyle\sum_{k=1}^N a_{ik} = 0; \quad i = 1, \ldots, N. \tag{61}$$

The parameter restrictions (58)-(61) ensure that $C(u, \boldsymbol{p})$ defined by (57) is linearly homogeneous in $\boldsymbol{p}$. It can be shown that the translog cost function defined by (57)-(61) can provide a second order Taylor series approximation to an arbitrary cost function.[*69]

We assume that the representative consumer engages in cost minimizing behavior during periods 1 and 2 and has preferences that are dual to the translog cost function defined by

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*66 $\nabla^2 f(\boldsymbol{z}^1)$ is the N by N matrix of second order partial derivatives of $f(\boldsymbol{z})$ evaluated at the point $\boldsymbol{z}^1$.

*67 To prove that (55) and (56) are true, substitute definition (54) and its derivatives into (55) and (56).

*68 Christensen, Jorgenson and Lau(1971)[29] (1975)[30] introduced this function into the economics literature.

*69 It can also be shown that if $b_0 = 1$ and all of the $b_i = 0$ for $i = 1, \ldots, N$ and $b_{00} = 0$, then $C(u, \boldsymbol{p}) = uC(1, \boldsymbol{p}) \equiv uc(\boldsymbol{p})$; i.e., with these additional restrictions on the parameters of the general translog cost function, we have linearly homogeneous preferences. Note that we also assume that utility u is scaled so that u is always positive. Finally, we assume that for each of our translog results, the regularity region contains the observed price and quantity data. Note that all prices need to be positive in order to define the translog functional form since the logarithm of zero is not well defined.

(57)-(61).[*70] Applying Shephard's Lemma to the translog cost function leads to the following Hicksian demand function equations in expenditure share form:

$$s_n^t = a_n + \textstyle\sum_{k=1}^N a_{nk} \ln p_k^t + b_n \ln u^t; \quad n = 1, \ldots, N; t = 1, 2 \tag{62}$$

where $s_n^t \equiv p_{tn} q_{tn} / \boldsymbol{p}^t \cdot \boldsymbol{q}^t$ is the period t expenditure share on commodity n. Define the geometric average of the period 1 and 2 utility levels as u^*; i.e., define

$$u^* \equiv [u^1 u^2]^{1/2}. \tag{63}$$

Now observe that the right hand side of the equation that defines the natural logarithm of the translog cost function, equation (57), is a quadratic function of the variables $z_n \equiv \ln p_n$ if we hold utility constant at the level u^*. Hence we can apply the quadratic identity, (56), and get the following equation:

$$\begin{aligned}
&\ln C(u^*, \boldsymbol{p}^2) - \ln C(u^*, \boldsymbol{p}^1) \\
&= (1/2)\textstyle\sum_{n=1}^N [\partial \ln C(u^*, \boldsymbol{p}^1)/\partial \ln p_n + \partial \ln C(u^*, \boldsymbol{p}^2)/\partial \ln p_n][\ln p_n^2 - \ln p_n^1] \\
&= (1/2)\textstyle\sum_{n=1}^N [a_n + \sum_{k=1}^N a_{nk} \ln p_k^1 + b_n \ln u^* + a_n + \sum_{k=1}^N a_{nk} \ln p_k^2 + b_n \ln u^*][\ln p_n^2 - \ln p_n^1] \\
&\qquad\qquad \text{differentiating (57) at the points } (u^*, \boldsymbol{p}^1) \text{ and } (u^*, \boldsymbol{p}^2) \\
&= (1/2)\textstyle\sum_{n=1}^N [a_n + \sum_{k=1}^N a_{nk} \ln p_k^1 + b_n \ln[u^1 u^2]^{1/2} + a_n + \sum_{k=1}^N a_{nk} \ln p_k^2 + b_n \ln[u^1 u^2]^{1/2}] \\
&\qquad\qquad \times [\ln p_n^2 - \ln p_n^1] \quad \text{using definition (63) for } u^* \\
&= (1/2)\textstyle\sum_{n=1}^N [a_n + \sum_{k=1}^N a_{nk} \ln p_k^1 + b_n \ln u^1 + a_n + \sum_{k=1}^N a_{nk} \ln p_k^2 + b_n \ln u^2][\ln p_n^2 - \ln p_n^1] \\
&= (1/2)\textstyle\sum_{n=1}^N [\partial \ln C(u^1, \boldsymbol{p}^1)/\partial \ln p_n + \partial \ln C(u^2, \boldsymbol{p}^2)/\partial \ln p_n][\ln p_n^2 - \ln p_n^1] \\
&\qquad\qquad \text{differentiating (57) at the points } (u^1, \boldsymbol{p}^1) \text{ and } (u^2, \boldsymbol{p}^2) \\
&= (1/2)\textstyle\sum_{n=1}^N [s_n^1 + s_n^2][\ln p_n^2 - \ln p_n^1] \quad \text{using (62)} \\
&\equiv \ln P_{\mathrm{T}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2).
\end{aligned} \tag{64}$$

The last equation in (64) defines the logarithm of an observable index number formula, $P_{\mathrm{T}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$, which is the Törnqvist (1936)[158] (1937)[159] Theil (1967)[157] price index. Hence exponentiating both sides of (64) yields the following equality between the true cost of living between periods 1 and 2, evaluated at the intermediate utility level u^* and the observable price index P_{T}:[*71]

$$C(u^*, \boldsymbol{p}^2)/C(u^*, \boldsymbol{p}^1) = P_{\mathrm{T}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2). \tag{65}$$

Since the translog cost function is a flexible functional form, the Törnqvist-Theil price index P_{T} is also a *superlative index.* The importance of (65) as compared to our earlier exact index number results is that we no longer have to assume that preferences are linearly homogeneous. However, we do have to choose a particular utility level on the left hand side of (65) in order to obtain our new exact result, the geometric mean of u^1 and u^2. If we assume linearly homogeneous translog preferences where $b_0 = 1, b_i = 0$ for $i = 1, \ldots, N$ and $b_{00} = 0$, then $C(u, \boldsymbol{p}) = uC(1, \boldsymbol{p}) \equiv uc(\boldsymbol{p})$ for all $u > 0$ and (65) becomes $c(\boldsymbol{p}^2)/c(\boldsymbol{p}^1) = P_{\mathrm{T}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$.

[*70] Aggregating over consumers when preferences are not represented by a linearly homogeneous utility function is problematic. However, we will eventually specialize the more general preferences that are dual to the translog cost function to the linearly homogeneous case.

[*71] This result is due to Diewert (1976; 122)[42]. For extensions of this result to include demographic and environmental conditioning variables, see Caves, Christensen and Diewert (1982b)[24].

Thus in the case of linearly homogeneous translog preferences, the Konüs true cost of living index becomes the ratio of translog unit cost functions and it is equal to the observable Törnqvist-Theil price index P_{T}.

It is somewhat mysterious how a ratio of *unobservable* cost functions of the form appearing on the left hand side of the above equation can be *exactly* estimated by an *observable* index number formula but the key to this mystery is the assumption of cost minimizing behavior and the quadratic identity (56) along with the fact that derivatives of cost functions are equal to optimal quantities, as specified by Shephard's Lemma. In fact, all of the exact index number results derived in this section can be derived using transformations of the quadratic identity along with Shephard's Lemma (or Wold's (1953)[168] identity (39)).[*72]

The results in this section indicate that the Fisher price index $P_{\mathrm{F}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ and the Törnqvist Theil price index are good indexes when taking the economic approach to index number theory. Is it possible to choose which one is best? When working with aggregated data when there are matching prices in the two periods being compared, it usually will not make much difference which index is chosen. It turns out that $P_{\mathrm{F}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ and $P_{\mathrm{T}}(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ approximate each other numerically to the second order around any point where $\boldsymbol{p}^1 = \boldsymbol{p}^2 \gg \mathbf{0}_N$ and $\boldsymbol{q}^1 = \boldsymbol{q}^2 \gg \mathbf{0}_N$.[*73] However, when aggregating over microeconomic data, the Fisher index is likely to be more sensitive to outlying observations than the Törnqvist Theil index.[*74] At higher levels of aggregation or in situations where the fluctuations in prices and quantities are fairly smooth, P_{F} and P_{T} will usually be quite close.

2.6 Two Stage Aggregation

It is not possible to construct a national Consumer Price Index in a single stage of aggregation. National Statistical Offices break up the universe of consumer products into subgroups and construct separate price indexes for each product group in scope. If they have access to scanner data, then they can also construct the companion quantity indexes for each group. This price and quantity information for each group can then be aggregated at the next stage of aggregation. The four approaches to index number theory can be applied to the first stage aggregation problems and to the second stage problem, provided that the first stage aggregations produced sub-aggregate prices and quantities. If either Laspeyres or Paasche indexes are used at both stages of aggregation, then it can be shown that the two stage Laspeyres index is equal to the single stage Laspeyres index and that two stage Paasche index is equal to the single stage Paasche index; see IMF (2025; 27-32)[113]. If a superlative index is used at both stages of aggregation, then it is not the case that the two stage superlative index is equal to the single stage superlative index. However, it can be shown that there is an approximate equality result for two stage aggregation using a superlative index; see Diewert (1978; 889)[43]. The two stage approximation result for superlative indexes will be fairly accurate provided that all micro prices and quantities are positive and that the changes in these prices and quantities are not too large. However, at the first stage of aggregation, the fluctuations in prices and quantities are frequently very large and moreover, there will typically be missing prices.

It is possible to justify two stage aggregation using the economic approach to bilateral index number theory. This approach assumes that consumer preferences are *separable*; i.e., the consumer's utility function has the form $F[f^1(\boldsymbol{q}^1), f^2(\boldsymbol{q}^2), \ldots, f^M(\boldsymbol{q}^M)]$ where the $\boldsymbol{q}^m$ are

[*72] See Diewert (2002)[52].
[*73] See Diewert (1978; 888)[43].
[*74] See Fox, Level and O'Connell (2025)[97], Dikhanov (2025)[80] and Fox (2025)[96] on this point.

consumption vectors (of varying dimension) for $m = 1, 2, \ldots, M$ and the sub-utility functions $f^m(\boldsymbol{q}^m)$ are nondecreasing, concave, linearly homogeneous functions of the vectors $\boldsymbol{q}^m$ and the macro utility function is nondecreasing and quasiconcave. This approach was initiated by Shephard (1953; 61-71)[150] (1970; 145-146)[151].[*75]

Two stage aggregation also occurs when constructing output, input and value added indexes as well as productivity indexes. The four approaches to CPI bilateral index number theory discussed above can be applied to the construction of subindexes for these production oriented indexes; see Fisher and Shell (1972)[93], Archibald (1977)[3], Diewert (1980)[44] (1983)[45] and Diewert and Morrison (1986)[73] for materials on this topic.[*76]

2.7 Summary of Bilateral Index Number Theory and the Chain Drift Problem

In sections 2.2-2.5, we have outlined four main approaches to bilateral index number theory: the fixed basket approach, the stochastic approach, the test approach and the economic approach. These four approaches led to the Fisher index in section 2.2, the Törnqvist Theil index in section 2.3, the Fisher index in section 2.4 and the Fisher and Törnqvist Theil indexes in section 2.5. In section 2.5, it was noted that in situations where the fluctuations in prices and quantities were not large and there were no missing prices in the two periods being compared, then the Fisher and Törnqvist Theil indexes would give much the same answer. This is an encouraging result in that all approaches seem to lead to the same two bilateral indexes which "normally" give much the same answer.

But the above result does not tell us how we should proceed over more than two periods. The problem of defining aggregate prices and quantities over multiple periods showed up in the test approach to index number theory. Test 22 (the Circularity Test) asked that the bilateral index number formula give the same answer in going from period 0 to period 2 directly or by going from 0 to 1 and then 1 to 2; i.e., we would like $P(\boldsymbol{p}^0, \boldsymbol{p}^2, \boldsymbol{q}^0, \boldsymbol{q}^2)$ to equal $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$. However, both the Fisher and Törnqvist index fail this test. The failure of $P(\boldsymbol{p}^0, \boldsymbol{p}^2, \boldsymbol{q}^0, \boldsymbol{q}^2)$ to equal $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)$ is called the *chain drift problem*. We will explain the problem by using the example and discussion in Diewert (2023)[60] on this problem.

The problem occurs when prices and quantities fluctuate substantially either due to discounted prices (sales) or seasonal factors and chained indexes are used.[*77] The extent of the *price bouncing problem* or the problem of *chain drift* can be measured if we make use of the following test due to Walsh (1901; 389)[164], (1921b; 540)[166]:

T23: *Multiperiod Identity Test*: $P(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1)P(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{q}^1, \boldsymbol{q}^2)P(\boldsymbol{p}^2, \boldsymbol{p}^0, \boldsymbol{q}^2, \boldsymbol{q}^0) = 1$.

Thus price change is calculated over consecutive periods but an artificial final period is introduced as the final period where the prices and quantities revert back to the prices and quantities in the very first period. The test asks that the product of all of these price changes

[*75] For additional materials and references to this literature, see Green (1964)[105] and Diewert (1980; 438-442)[44].

[*76] There is one change in the Samuelson Swamy (1974)[146] approach to linearly homogeneous aggregator functions $f(\boldsymbol{q})$ for a subindex of the overall CPI: if we apply this approach in the aggregation of outputs applying the economic approach, $f(\boldsymbol{q})$ should be an convex function of $\boldsymbol{q}$ instead of a concave function. The dual price function $c(\boldsymbol{p})$ is replaced by a dual unit revenue function say $r(\boldsymbol{p})$ which also must be convex in the output price vector $\boldsymbol{p}$.

[*77] The problem was flagged by Fisher (1922; 293)[91], Persons (1921; 110)[134] (1928; 100-105)[135], Frisch (1936; 8-9)[99], Szulc (1983)[155] (1987)[156] and Hill (1988; 136-137)[109]. The enormous magnitude of fluctuations in prices and quantities in scanner data due to product sales was noted by de Haan (2008)[34] and de Haan and van der Grient (2011)[38].

should equal unity. This test can be used to evaluate the amount of chain drift that occurs if chained indexes are used. *Chain drift* occurs when an index does not return to unity when prices in the current period return to their levels in the base period.

Suppose that we are given the price and quantity data for two commodities for four periods. The data are listed in Table 1 below.*78

Table 1 Price and Quantity Data for Two Products for Four Periods

Period t	p_1^t	p_2^t	q_1^t	q_2^t
1	1.0	1.0	10	100
2	0.5	1.0	5000	100
3	1.0	1.0	1	100
4	1.0	1.0	10	100

The first commodity is subject to periodic sales (in period 2), when the price drops to 1/2 of its normal level of 1. In period 1, we have "normal" off sale demand for commodity 1 which is equal to 10 units. In period 2, the sale takes place and demand explodes to 5000 units.*79 In period 3, the commodity is off sale and the price is back to 1 but many shoppers have stocked up in the previous period so demand falls to only 1 unit. Finally in period 4, the commodity is off sale and we are back to the "normal" demand of 10 units. Commodity 2 exhibits no price or quantity change across periods: its price is 1 in all periods and the quantity sold is 100 units in each period. Note that the only thing that has happened going from period 3 to 4 is that the demand for commodity one has picked up from 1 unit to the "normal" level of 10 units. Also note that, conveniently, the period 4 data are exactly equal to the period 1 data so that for Walsh's test to be satisfied, the product of the period to period chain links must equal one.

Table 2 lists the fixed base Fisher, Laspeyres and Paasche price indexes, $P_{\mathrm{F(FB)}}, P_{\mathrm{L(FB)}}$ and $P_{\mathrm{P(FB)}}$ and as expected, they behave perfectly in period 4, returning to the period 1 level of 1. Then the chained Fisher, Törnqvist-Theil, Laspeyres and Paasche price indexes, $P_{\mathrm{F(CH)}}, P_{\mathrm{T(CH)}}, P_{\mathrm{L(CH)}}$ and $P_{\mathrm{P(CH)}}$ are listed. Obviously, the chained Laspeyres and Paasche indexes have chain drift bias that is extraordinary but what is interesting is that the chained Fisher has a 2% downward bias and the chained Törnqvist has a close to 3% downward bias.

Table 2 Fixed Base and Chained Fisher, Törnqvist-Theil, Laspeyres and Paasche Indexes

Period	$P_{\mathrm{F(FB)}}$	$P_{\mathrm{L(FB)}}$	$P_{\mathrm{P(FB)}}$	$P_{\mathrm{F(CH)}}$	$P_{\mathrm{T(CH)}}$	$P_{\mathrm{L(CH)}}$	$P_{\mathrm{P(CH)}}$
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.698	0.955	0.510	0.698	0.694	0.955	0.510
3	1.000	1.000	1.000	0.979	0.972	1.872	0.512
4	1.000	1.000	1.000	0.979	0.972	1.872	0.512

What explains the results in the above table? The problem is this: when commodity one comes off sale and goes back to its regular price in period 3, *the corresponding quantity does*

*78 This example is taken from Diewert (2012)[57].

*79 This example is based on an actual example that used Dutch scanner data. When the price of a detergent product went on sale in the Netherlands at approximately one half of the regular price, the volume sold shot up approximately one thousand fold; see de Haan (2008; 15)[34] and de Haan and van der Grient (2011)[38]. These papers brought home the magnitude of volume fluctuations due to sales and led Ivancic, Diewert and Fox (2009)[115] (2011)[116] to propose the use of Rolling Window Multilateral Indexes to mitigate the chain drift problem.

not return to the level it had in period 1: the period 3 demand is only 1 unit whereas the "normal" period 1 demand for commodity 1 was 10 units. It is only in period 4, that demand for commodity one recovers to the period 1 level. However, since prices are the same in periods 3 and 4, all of the chain links show no change (even though quantities are changing) and this is what causes the difficulties. If demand for commodity one in period 3 had immediately recovered to its "normal" period 1 level of 10, then there would be no chain drift problem.

For a recent example of how big the chain drift problem can be using chained Fisher or Törnqvist indexes that utilize scanner data, see Fox, Levell and O'Connell (2025)[97].[*80]

A possible solution to solving the chain drift problem is to use Multilateral Indexes that satisfy the Circularity Test for all T periods in scope. In the following section, we will present some of the main multilateral methods that have been applied in recent times.

In addition to the chain drift problem, the bilateral indexes that we have studied above suffer from another problem: all of the approaches outlined thus far *assume that prices and quantities are positive in the two periods being compared.* This is a problem when we are attempting to construct indexes at the first stage of aggregation using scanner data: there is generally a lot of product churn; i.e., new products are constantly introduced and older products are retired. Possible solutions to this *lack of matching problem*[*81] will be discussed in the following sections.

3 Multilateral Approaches

In sections 3 and 4, we no longer assume that every product is present in each of the T periods under consideration.[*82] We assume that there is a total of N products that are present in at least one of the T periods. Denote the set of products that are present in period t as $S(t)$. Thus if $n \in S(t)$, then $p_{tn} > 0$ and $q_{tn} > 0$; i.e., the observed price and quantity for product n in period t is positive. If $n \neq S(t)$, then we define $p_{tn} = 0$ and $q_{tn} = 0$. As usual, $\boldsymbol{p}^t \equiv [p_{t1}, \ldots, p_{tN}], \boldsymbol{q}^t \equiv [q_{t1}, \ldots, q_{tN}]$ and $e^t \equiv \boldsymbol{p}^t \cdot \boldsymbol{q}^t \equiv \sum_{n=1}^{N} p_{tn} q_{tn}$ for $t = 1, \ldots, T$.

3.1 The GEKS Multilateral Price Indexes

The multilateral GEKS method is due to Gini (1924)[102] (1931)[103] and was further developed by Eltetö and Köves (1964)[86] and Szulc (1964)[154].

In order to define the GEKS indexes, we first need to define the Laspeyres, Paasche and Fisher (1922)[91] bilateral price indexes when there are missing products. These indexes which compare the prices of period t to the prices of period r, $P_{\mathrm{L}}(t/r), P_{\mathrm{P}}(t/r)$ and $P_{\mathrm{F}}(t/r)$ are defined as follows (using our new notation explained above):

$$P_{\mathrm{L}}(t/r) \equiv \textstyle\sum_{n \in S(t) \cap S(r)} p_{tn} q_{rn} / \sum_{n \in S(t) \cap S(r)} p_{rn} q_{rn}; \quad 1 \leq r, t \leq T; \tag{66}$$

$$P_{\mathrm{P}}(t/r) \equiv \textstyle\sum_{n \in S(t) \cap S(r)} p_{tn} q_{tn} / \sum_{n \in S(t) \cap S(r)} p_{rn} q_{tn}; \quad 1 \leq r, t \leq T; \tag{67}$$

$$P_{\mathrm{F}}(t/r) \equiv [P_{\mathrm{L}}(t/r) P_{\mathrm{P}}(t/r)]^{1/2}; \quad 1 \leq r, t \leq T. \tag{68}$$

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*80 This paper also shows that the problem of product disappearance is huge.

*81 The lack of matching problem is essentially the same problem as the quality adjustment problem: a new higher quality product is introduced into the marketplace and there is no matched product price in previous periods.

*82 The materials presented in sections 3 and 4 follow the exposition of the various methods that is in Diewert, Abe, Tonogi and Shimizu (2025)[62].

Note that the comparison of prices in period t to the prices in period r is restricted to products that were purchased in both periods; i.e., the above bilateral indexes are based on *matched product prices*.

The sequence of Fisher price indexes that compare the prices of period $1, 2, \ldots, T$ with the prices of period r is $[P_{\mathrm{F}}(1/r), P_{\mathrm{F}}(2/r), \ldots, P_{\mathrm{F}}(T/r)]$. This sequence of numbers defines the *Fisher star indexes* for the T periods relative to the fixed base of period r. Obviously, any period r could be chosen as the base so there are T vectors of Fisher star indexes. The GEKS index simply takes the geometric mean of these indexes. Thus the (preliminary) *GEKS price level* for period t, P^t_{GEKSP}, is defined as follows:

$$P^t_{\mathrm{GEKSP}} \equiv [\textstyle\prod_{r=1}^{T} P_{\mathrm{F}}(t/r)]^{1/T}; \quad t = 1, \ldots, T. \tag{69}$$

We normalize the above T price levels so that the price level for period 1 is set equal to 1. Thus the final GEKS price level for period t is:

$$P^t_{\mathrm{GEKS}} \equiv P^t_{\mathrm{GEKSP}}/P^1_{\mathrm{GEKSP}}; \quad t = 1, \ldots, T. \tag{70}$$

Observed expenditure in period t is $e^t \equiv \boldsymbol{p}^t \cdot \boldsymbol{q}^t \equiv \sum_{n=1}^{N} p_{tn}q_{tn} = \sum_{n \in S(t)} p_{tn}q_{tn}$ where the last equality follows using our convention of setting $p_{tn} = 0 = q_{tn}$ if $n \neq S(t)$ so that the price and quantity of product n in period t is set equal to 0 if product n was not purchased in period t. The *aggregate quantity level* for period t that corresponds to the GEKS price level for period t is defined as follows:

$$Q^t_{\mathrm{GEKS}} \equiv e^t/P^t_{\mathrm{GEKS}}; \quad t = 1, \ldots, T. \tag{71}$$

The resulting GEKS price and quantity levels, P^t_{GEKS} and Q^t_{GEKS}, are transitive over the window of observations consisting of periods 1 to T but there can be a problem with these indexes if there is rapid product churn. Consider the Laspeyres index for period t relative to period r: $P_{\mathrm{L}}(t/r) \equiv \sum_{n \in S(t) \cap S(r)} p_{tn}q_{rn} / \sum_{n \in S(t) \cap S(r)} p_{rn}q_{rn}$. If periods r and t are widely separated, then the index $P_{\mathrm{L}}(t/r)$ may be based on a comparison of only a small number of products that are present in both periods and thus this index and the corresponding Paasche and Fisher indexes may be very unreliable. Thus the GEKS indexes may become substantially influenced by bilateral Laspeyres and Paasche indexes that are based on only a few matched prices and hence, the GEKS indexes may become unreliable.

3.2 Geary Khamis Multilateral Indexes

The GK multilateral method was introduced by Geary (1958)[101] in the context of making international comparisons of prices. Khamis (1970)[123] showed that the equations that define the method have a positive solution under certain conditions. A modification of this method has been adapted to the time series context and is being used to construct some components of the Dutch CPI; see Chessa (2016)[26]. The GK index was the multilateral index chosen by the Dutch to avoid the chain drift problem for the segments of their CPI that use scanner data.

Recall that $S(t)$ was the set of products n that were purchased in period t. Define $S^*(n)$ as the set of periods t where product n was sold. Define the vector $\boldsymbol{q}$ as the sum of the T observed quantity vectors $\boldsymbol{q}^t$ for each period t:

$$\boldsymbol{q} \equiv \textstyle\sum_{t=1}^{T} \boldsymbol{q}^t. \tag{72}$$

The equations which determine the *GK price levels* $P^1, \ldots, P^T$ and *quality adjustment factors* $\alpha_1, \ldots, \alpha_N$ (up to a scalar multiple) are the following ones:

$$\alpha_n = \textstyle\sum_{t \in S^*(n)} [q_{tn}/q_n][p_{tn}/P^t] = \sum_{n=1}^{N} [1/q_n][p_{tn} q_{tn}][1/P^t]; \quad n = 1, \ldots, N; \tag{73}$$

$$P^t = \boldsymbol{p}^t \cdot \boldsymbol{q}^t / \boldsymbol{\alpha} \cdot \boldsymbol{q}^t = e^t / \boldsymbol{\alpha} \cdot \boldsymbol{q}^t; \quad t = 1, \ldots, T. \tag{74}$$

where $\boldsymbol{\alpha} \equiv [\alpha_1, \ldots, \alpha_N]$ is the vector of GK quality adjustment factors, $q_n \equiv \sum_{t=1}^{T} q_{tn} = \sum_{t \in S^*(n)} q_{tn}$ for $n = 1, \ldots, N$ and $e^t \equiv \boldsymbol{p}^t \cdot \boldsymbol{q}^t$ is period t expenditure on the N products. Once a solution $\boldsymbol{\alpha}$ and $P^1, \ldots, P^T$ to equations (73) and (74) has been found, the period t quantity levels Q^t are defined as follows:

$$Q^t \equiv \boldsymbol{\alpha} \cdot \boldsymbol{q}^t; \quad t = 1, \ldots, T. \tag{75}$$

It can be seen that if a solution to equations (73) and (74) exists, then if all of the period price levels P^t are multiplied by a positive scalar λ and all of the quality adjustment factors α_n are divided by the same λ, then another solution to (73) and (74) is obtained. Hence, the α_n and P^t are only determined up to a scalar multiple and an additional normalization is required such as $P^1 = 1$ or $\alpha_1 = 1$ is required to determine a unique solution to the system of equations defined by (73) and (74). The GK price and quantity levels have some good axiomatic properties including invariance to changes in the units of measurement.[*83] Furthermore, the GK price and quantity levels can accommodate new and disappearing products.

A traditional method for obtaining a solution to (73) and (74) is to iterate between these equations. Thus set $\boldsymbol{\alpha} = \mathbf{1}_N$, a vector of ones, and use equations (74) to obtain an initial sequence for the P^t. Substitute these P^t estimates into equations (73) and obtain α_n estimates. Substitute these α_n estimates into equations (74) and obtain a new sequence of P^t estimates. Continue iterating between the two systems until convergence is achieved.

Alternative methods are more efficient. Following Diewert (1999; 26)[51] and Diewert and Fox (2022)[70], substitute equations (74) into equations (73) and after some simplification, obtain the following system of equations that will determine the components of the $\boldsymbol{\alpha}$ vector (up to a scalar multiplicative factor):

$$[\mathbf{I}_N - \mathbf{C}]\boldsymbol{\alpha} = \mathbf{0}_N \tag{76}$$

where $\mathbf{I}_N$ is the N by N identity matrix, $\mathbf{0}_N$ is a vector of zeros of dimension N and the $\mathbf{C}$ matrix is defined as follows:

$$\mathbf{C} \equiv \widehat{\mathbf{q}}^{-1} \textstyle\sum_{t=1}^{T} \boldsymbol{s}^t \boldsymbol{q}^{tT} \tag{77}$$

where $\widehat{\mathbf{q}}$ is an N by N diagonal matrix with the elements of the vector of total purchases $\boldsymbol{q}$ running down the main diagonal and $\widehat{\mathbf{q}}^{-1}$ denotes the inverse of this matrix, $\boldsymbol{s}^t$ is the period t expenditure share column vector, $\boldsymbol{q}^t$ is the column vector of quantities purchased during period t and $\boldsymbol{q}^{tT}$ is the transpose of $\boldsymbol{q}^t$.

The matrix $\mathbf{I}_N - \mathbf{C}$ is singular which implies that the N equations in (76) are not all independent. In particular, if the first $N-1$ equations in (76) are satisfied, then the last equation in (76) will also be satisfied. It can also be seen that the N equations in (76) are homogeneous of degree one in the components of the vector $\boldsymbol{\alpha}$. Thus to obtain a unique solution to (76), set α_N equal to 1, drop the last equation in (76) and solve the remaining $N-1$ equations for $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}$. Once the α_n are known, equations (74) can be used to determine the

[*83] See Zhang, Johansen and Nygaard (2019)[169] and Diewert (2023)[60] on the test properties of various multilateral indexes.

GK price levels, $P^t = \boldsymbol{p}^t \cdot \boldsymbol{q}^t / \boldsymbol{\alpha} \cdot \boldsymbol{q}^t$ for $t = 1, \ldots, T$. This is the efficient procedure that was suggested and used by Diewert and Fox (2022)[70].

It is important to note that the GK indexes are based on the fact that the method essentially assumes that *purchaser preferences are linear*: equations (75) tell us that the period t quantity aggregate Q^t is exactly equal to $\boldsymbol{\alpha} \cdot \boldsymbol{q}^t \equiv \sum_{n=1}^{N} \alpha_n q_{tn}$ for each period t.[*84] Thus the GK method for constructing quantity aggregates could be viewed as a special method for estimating the parameters of a linear utility function.

The next multilateral method relies explicitly on econometric estimation.

4 Econometric Methods

4.1 Weighted and Unweighted Time Product Dummy (TPD) Hedonic Regressions

The unweighted Time Product Dummy (TDP) model dates back to Court (1939)[32] and Summers (1973)[153].

These models are based on the price data satisfying (to some degree of approximation) the following equations:

$$p_{tn} \approx P^t \alpha_n; \quad t = 1, \ldots, T; n \in S(t) \tag{78}$$

where P^t is interpreted as the period t price level and α_n is a parameter which reflects the quality (or marginal utility) of product n.

Taking logarithms of both sides of the approximate equalities defined by (78) leads to the following approximate equalities:

$$\begin{aligned} \ln p_{tn} &\approx \ln P^t + \ln \alpha_n; \quad t = 1, \ldots, T; n \in S(t) \\ &= \rho_t + \beta_n \end{aligned} \tag{79}$$

where $\rho_t \equiv \ln P^t$ for $t = 1, \ldots, T$ and $\beta_n \equiv \ln \alpha_n$ for $n = 1, \ldots, N$. The second set of approximate equations in (79) is a linear regression model. However, the ρ_t and β_n parameters in (79) are not uniquely determined; we require a normalization on one of these parameters. Choose the following normalization:

$$\rho_1 = 0 \quad \text{(which corresponds to } P^1 = 1\text{)}. \tag{80}$$

The linear regression that corresponds to (79) and (80) is the following least squares minimization problem:

$$\min_{\rho' s \ \beta' s} \{ \textstyle\sum_{t=1}^{T} \sum_{n \in S(t)} [\ln p_{tn} - \rho_t - \beta_n]^2 \} \tag{81}$$

where ρ_1 is set equal to 0 in the above minimization problem.[*85]

[*84] However, it turns out that the GK coefficients α_n are such that the GK price and quantity levels are also consistent with purchasers having Leontief (no substitution) preferences; recall the discussion around definitions (42) and (46) where the purchaser unit cost function was defined to be $c(\boldsymbol{p}) \equiv \boldsymbol{b} \cdot \boldsymbol{p}$ where $\boldsymbol{b}$ is an N dimensional vector of positive constants b_n. For proofs of this result, see Diewert (1999; 58-60)[51] or Diewert, Abe, Tonogi and Shimizu (2025; 16)[62]. But Leontief preferences are very unrealistic: they imply that purchasers always purchase all products in a proportional manner over time with no substitution between products as prices change. This assumption is particularly unrealistic at the lowest level of aggregation where new products enter and older products vanish.

[*85] Note that the dependent variables are "stacked" into one big regression equation with only one variance parameter, which can lead to heteroskedasticity problems. An important property of the solution to the minimization problem that is defined by (81) is that the estimates for the ρ_t (and hence the $P^t \equiv \exp[\rho_t]$ are invariant to changes in the units of measurement.

Assume for the moment that the approximate equations (78) hold as exact equations. Multiply both sides of equation p_{tn} by q_{tn} for $n \in S(t)$ and sum the resulting equations over n. For each t, we obtain the following equations:

$$\sum_{n\in S(t)} p_{tn}q_{tn} = \boldsymbol{p}^t \cdot \boldsymbol{q}^t \equiv e^t = P^t \sum_{n\in S(t)} \alpha_n q_{tn} = P^t \boldsymbol{\alpha} \cdot \boldsymbol{q}^t = P^t Q^t; \quad t = 1, \ldots, T; \tag{82}$$

where $e^t \equiv \sum_{n\in S(t)} p_{tn}q_{sn} = \boldsymbol{p}^t \cdot \boldsymbol{q}^t$ is period t expenditure on the N products and the period t quantity aggregate is $Q^t \equiv \boldsymbol{\alpha} \cdot \boldsymbol{q}^t \equiv \sum_{n\in S(t)} \alpha_n q_{tn}$ for $t = 1, \ldots, T$. Thus if equations (78) or (79) hold as equalities rather than as approximate equalities, the Time Product Dummy Hedonic Regression model implicitly assumes that purchasers of the N products in scope have the same linear preferences $f(\boldsymbol{q}) \equiv \boldsymbol{\alpha} \cdot \boldsymbol{q}$ over the N products.

Of course, equations (78) and (79) will not hold as exact equalities. Let ρ_t^* and β_n^* for $t = 2, 3, \ldots, T$ and $n = 1, \ldots, N$ denote the solution to the least squares minimization problem (81) and define $\rho_1^* = 0$. Define the T price levels P^{t^*} by definitions (83) and the N quality adjustment parameters α_n^* by definitions (84):

$$P^{t^*} \equiv \exp[\rho_t^*]; \quad t = 1, \ldots, T; \tag{83}$$

$$\alpha_n^* \equiv \exp[\beta_n^*]; \quad n = 1, \ldots, N. \tag{84}$$

There are two ways to finalize estimates for the period t price and quantity levels. *Method 1* chooses the P^{t^*} defined by (83) and defines the corresponding quantity levels Q^{t^*} as deflated expenditures:

$$Q^{t^*} \equiv e^t / P^{t^*}; \quad t = 1, \ldots, T. \tag{85}$$

Method 2 uses the α_n^* defined by (84) to define the period t quantity levels $Q^{t^{**}}$ as follows:

$$Q^{t^{**}} \equiv \sum_{n=1}^{N} \alpha_n^* q_{tn}; \quad t = 1, \ldots, T. \tag{86}$$

The corresponding period t price levels $P^{t^{**}}$ are defined residually as follows:[*86]

$$P^{t^{**}} \equiv e^t / Q^{t^{**}}; \quad t = 1, \ldots, T. \tag{87}$$

If the fit in the regression defined by (81) is perfect, then the two sets of aggregate price and quantity levels will coincide.[*87]

Using the properties of the least squares minimization problem defined by (81), it can be shown that P^{t^*} and Q^{t^*} are invariant to changes in the units of measurement as are $P^{t^{**}}$ and $Q^{t^{**}}$.

[*86] The estimating equations (78) for the parameters P^t and α_n could be written as $p_{tn} = P^t \alpha_n + \varepsilon_{tn}$ for $t = 1, \ldots, T$ and $n \in S(t)$ where the ε_{tn} are error terms. Then the following equations are the counterparts to equations (82) that take the error terms into account: $e^t = P^t(\sum_{n\in S(t)} \alpha_n q_{tn}) + \varepsilon^t = P^t \boldsymbol{\alpha} \cdot \boldsymbol{q}^t + \varepsilon^t = P^t Q^t + \varepsilon^t$ where $\varepsilon^t \equiv (\sum_{n\in S(t)} \varepsilon_{tn} q_{tn})$ for $t = 1, \ldots, T$. We want to decompose period t expenditure e^t into *exactly* $P^t Q^t$ where P^t represents the period t aggregate price level and Q^t represents the aggregate quantity level. Once we have our estimates for the P^t and α_n, we have two ways of imposing the equality $P^t Q^t = e^t$ for $t = 1, \ldots, T$. Method 1 chooses the P^t as estimates for the price levels and defines the corresponding quantity levels residually as $Q^t \equiv e^t / P^t$ for $t = 1, \ldots, T$. Method 2 uses the estimated α_n to define Q^t as $\boldsymbol{\alpha} \cdot \boldsymbol{q}^t$ and define P^t residually as $P^t \equiv e^t / Q^t$ for $t = 1, \ldots, T$. These conventions ensure that no value disappears or is created by the aggregation process; i.e., the resulting P^t and Q^t satisfy the Product Test $P^t Q^t = e^t$ for all periods t.

[*87] Use the estimates α_n^* and P^{t^*} to define the error terms $e_{tn} \equiv p_{tn} - P^{t^*} \alpha_n^*$ for $t = 1, \ldots, T; n \in S(t)$. If the period t error terms e_{tn} sum to zero, so that $\sum_{n\in S(t)} e_{tn} = 0$, then it can be shown that $P^{t^{**}} = P^{t^*}$ and $Q^{t^{**}} = Q^{t^*}$.

An advantage of the unweighted (or more properly, the equally weighted) TPD price indexes defined by (83) and (85) is that they can be constructed using just price information for the N products in scope. But there is a disadvantage associated with the use of the TPD hedonic regression model: products with low volumes of purchases should not get the same weight in the regression as highly popular products.[*88]

Let $s_{tn} \equiv p_{tn}q_{tn}/\boldsymbol{p}^t \cdot \boldsymbol{q}^t$ be the expenditure share of product n in period t. The least squares regression problem that generalizes the unweighted minimization problem defined by (81) that takes into account the economic importance of each product is the following *Weighted Time Product Dummy* least squares minimization problem:[*89]

$$\min_{\rho\ \beta} \textstyle\sum_{t=1}^{T}\sum_{n\in S(t)} s_{tn}[\ln p_{tn} - \rho_t - \beta_n]^2 = \min_{\rho\ \beta} \sum_{n=1}^{N}\sum_{t\in S^*(n)} s_{tn}[\ln p_{tn} - \rho_t - \beta_n]^2. \quad (88)$$

Note that there are two equivalent ways of writing the least squares minimization problem. The first order necessary conditions for $\rho_1, \ldots, \rho_T$ and $\beta_1, \ldots, \beta_N$ to solve (88) are as follows:[*90]

$$\textstyle\sum_{n\in S(t)} s_{tn}[\rho_t^* + \beta_n^*] = \sum_{n\in S(t)} s_{tn} \ln p_{tn}; \quad t = 1, \ldots, T; \quad (89)$$

$$\textstyle\sum_{t\in S^*(n)} s_{tn}[\rho_t^* + \beta_n^*] = \sum_{t\in S^*(n)} s_{tn} \ln p_{tn}; \quad n = 1, \ldots, N. \quad (90)$$

The solution to (89) and (90) is not unique: if $\boldsymbol{\rho}^* \equiv [\rho_1^*, \ldots, \rho_T^*]$ and $\boldsymbol{\beta}^* \equiv [\beta_1^*, \ldots, \beta_N^*]$ solve (89) and (90), then so do $[\rho_1^* + \lambda, \ldots, \rho_T^* + \lambda]$ and $[\beta_1^* - \lambda, \ldots, \beta_N^* - \lambda]$ for all λ. Thus we can set $\rho_1^* = 0$ in the above equations and drop the first equation in (89) and use linear algebra to find a solution for the resulting equations.

A practical method for obtaining a solution to (88) is to set up a linear regression model defined by (91) below where the e_{tn} are error terms:

$$(s_{tn})^{1/2} \ln p_{tn} = (s_{tn})^{1/2}\rho_t + (s_{tn})^{1/2}\beta_n + e_{tn}; \quad t = 1, \ldots, T; n = 1, \ldots, N. \quad (91)$$

To avoid exact collinearity in the regression defined by (91), set $\rho_1 = 0$.[*91] Once a solution to (89), (90) and $\rho_1^* = 0$ has been found, rewrite equations (89) and (90) as follows:

$$\rho_t^* = \textstyle\sum_{n\in S(t)} s_{tn} \ln(p_{tn}/\alpha_n^*); \quad t = 1, \ldots, T; \quad (92)$$

$$\beta_n^* = \textstyle\sum_{t\in S^*(n)} s_{tn} \ln(p_{tn}/P^{t^*})/\sum_{t\in S^*(n)} s_{tn}; \quad n = 1, \ldots, N. \quad (93)$$

Exponentiate the ρ_t^* and β_n^*. Using definitions (83) and (84) for P^{t^*} and α_n^*, we find that equations (92) and (93) give us the following expressions for the P^{t^*} and α_n^*:

$$P^{t^*} \equiv \exp[\textstyle\sum_{n\in S(t)} s_{tn} \ln(p_{tn}/\alpha_n^*)]; \quad t = 1, \ldots, T; \quad (94)$$

$$\alpha_n^* \equiv \exp[\textstyle\sum_{t\in S^*(n)} s_{tn} \ln(p_{tn}/P^{t^*})/\sum_{t\in S^*(n)} s_{tn}]; \quad n = 1, \ldots, N. \quad (95)$$

[*88] For discussions on the benefits and costs of alternative weighting methods, see Diewert (2004)[53] (2023)[60].

[*89] Rao (1995)[142] (2004)[143] (2005; 574)[144] and Diewert (2004)[53] (2005)[54] considered this model when there were no missing products. However, Balk (1980; 70)[7] suggested this class of models much earlier using different weights. Diewert (2023)[60] considered the case when there were missing products.

[*90] Equations (89) and (90) show that the solution to (88) does not depend on any independently determined reservation prices p_{tn} for products n that are missing in period t.

[*91] There must be a certain amount of product overlap across periods in order to avoid exact collinearity even if we make the normalization $\rho_1 = 0$; i.e., the $\mathbf{X}$ matrix of independent variable column vectors associated with (91) must be nonsingular in order to get a unique solution.

Thus the period t price level, P^{t^*}, is a share weighted geometric average of the quality adjusted prices p_{tn}/α_n^* for products that were purchased in period t and the product n quality adjustment factor α_n^* is a share weighted geometric average of the deflated by P^{t^*} prices of product n over all time periods t where product n was purchased. In both cases, the share weights add up to one.

It turns out that the Weighted Time Product Dummy Hedonic Regression Model is exactly consistent with utility maximizing behavior for at least two utility functions. Earlier in this section, we showed that the Time Product Dummy Hedonic Regression model was consistent with purchasers maximizing a linear utility function and this exactness result carries over to the Weighted Time Product Dummy Hedonic Regression Model. In this case, we must have $p_{tn} = P^t\alpha_n$ for $t = 1, \ldots, T$ and $n \in S(t)$. Thus observed prices for products that were actually purchased must vary in a proportional manner over time for this case to hold exactly.

If purchasers are maximizing a Cobb Douglas (1928)[31] utility function, then the corresponding unit cost function has the following functional form:

$$c_{\mathrm{CD}}(p_1, p_2, \ldots, p_N) = \kappa(0)p_1^{\kappa(1)}p_2^{\kappa(2)}\ldots p_N^{\kappa(N)} \tag{96}$$

where the parameters $\kappa(n) > 0$ for $n = 0, 1, 2, \ldots, N$ and $\sum_{n=1}^{N}\kappa(n) = 1$. If all cost minimizing purchasers have identical Cobb-Douglas preferences, then for each n, observed expenditure shares s_{tn} will be equal to the constant $\kappa(n)$ for $t = 1, \ldots, T$, (which is not very realistic empirically). Comparing (94) and (96), we see that the WTPD price levels P^{t^*} defined by (94) will equal a constant times $c_{\mathrm{CD}}(p_{t1}, p_{t2}, \ldots, p_{tN})$ for $t = 1, \ldots, T$. Thus when we normalize the Cobb-Douglas price levels by choosing $\kappa(0)$ so that $P_{\mathrm{CD}}^1 \equiv c_{\mathrm{CD}}(p_{11}, p_{12}, \ldots, p_{1N}) = 1$, then the CD and WTPD price levels will coincide. Thus the WTPD price levels are exactly equal to the (normalized) Cobb-Douglas price levels. But the Cobb-Douglas model is not very useful at lower levels of aggregation when there is a lot of product churn because Cobb-Douglas preferences do not allow any products to disappear. Linear preferences do allow new products to appear and obsolete products to disappear.

For a discussion of the axiomatic properties of the WTPD price and quantity levels, see Diewert (2023)[60]. The WTPD price and quantity levels functions take the economic importance of the products into account and thus are a clear improvement over their unweighted counterparts. If the estimated errors $e_{tn}^* \equiv \ln p_{tn} - \rho_t^* - \beta_n^*$ that implicitly appear in the weighted least squares minimization problem (88) (and explicitly in (91)) turn out to be small, then the underlying exact model, $p_{tn} = P^t\alpha_n$ for $t = 1, \ldots, T, n \in S(t)$, provides a reasonable approximation to reality and thus this Weighted Time Product Dummy regression model can be used with some confidence.

The GK price and quantity levels and the WTPD price and quantity levels are both consistent with linear preferences. However, the WTPD model tells us how well linear preferences fit the data. The GK model gives us a estimate for the vector of marginal utilities $\boldsymbol{\alpha}$ but we do not know how well this estimate for $\boldsymbol{\alpha}$ fits the data.

In the following section, we will use a different econometric specification in order to estimate linear preferences.

4.2 The Econometric Estimation of Linear Preferences

Sections 3.2 and 4.1 described two multilateral methods (the GK and WTPD indexes) that were consistent with consumers maximizing a linear preference function subject to a budget constraint. In this section, we will again assume that consumers of the N products in scope maximize a linear utility function but we will take an econometric approach that is based on the estimation of a system of *direct demand functions* of the type defined by equations (35) in section 2.5, $\boldsymbol{q}^t = e^t\nabla_p c(\boldsymbol{p}^t)/c(\boldsymbol{p}^t)$, or a system of *inverse demand functions* defined by equations (39), $\boldsymbol{p}^t = e^t\nabla_q f(\boldsymbol{q}^t)/f(\boldsymbol{q}^t)$. However, linear preferences do not lead to a differentiable unit cost function so we cannot use the system of direct demand functions (35) to estimate linear preferences. We can use the following system of indirect demand functions to estimate linear preferences.

$$p_{tn} = e^t\alpha_n/\textstyle\sum_{j\in S(t)}\alpha_j q_{tj} = e^t\alpha_n/\boldsymbol{\alpha}\cdot\boldsymbol{q}^t; \quad t = 1,\ldots,T; n \in S(t) \tag{97}$$

where $\boldsymbol{\alpha}$ is the vector $[\alpha_1,\ldots,\alpha_N]$ of quality adjustment parameters. If we added an error term to the right hand side of each equation in (97), we could attempt to estimate the α_n by solving the following nonlinear least squares minimization problem:

$$\min\nolimits_\alpha \textstyle\sum_{t=1}^T\sum_{n\in S(t)}\{p_{tn} - [\alpha_n/\boldsymbol{\alpha}\cdot\boldsymbol{q}^t]\}^2. \tag{98}$$

If $\boldsymbol{\alpha}^* \equiv [\alpha_1^*,\ldots,\alpha_N^*]$ is a solution to (98), then it can be seen that $\lambda\boldsymbol{\alpha}^*$ is also a solution to (98) where λ is any positive number. This non-uniqueness always occurs when we attempt to estimate utility functions or their dual unit cost functions. The scale of utility is arbitrary so we need to impose at least one normalization on the estimated parameters in order to obtain a cardinal measure of utility. The simplest method of normalization is to set $\alpha_{n^*} = 1$ for some product n^* that is purchased in every period. There is another possible problem with the minimization problem defined by (98): it can be the case that there is no solution to (98). For example, suppose that there are only 2 periods and 2 products in scope. Suppose further that product 1 is only available in period 1 and product 2 is only available in period 2. In this case, there is no solution to (98). Thus we need a certain amount of product overlap in order to obtain a solution to (98).

There is another more serious problem with (98): even if a unique solution to (98) and a normalization like $\alpha_{n^*} = 1$ exists, the resulting quantity levels, $Q^{t^*} \equiv \boldsymbol{\alpha}^*\cdot\boldsymbol{q}^t$ for $t = 1,\ldots,T$, may not be invariant to changes in the units of measurement in the sense that the ratios Q^{t^*}/Q^{1^*} may change if the units of measurement are changed. Invariance to changes in the units of measurement is a fundamental requirement for national statistics.[*92] This problem can be cured if we convert the inverse demand system of estimating equations (97) into share equations. Thus multiply both sides of equation t, n for $t = 1,\ldots,T$ and $n \in S(t)$ in (97) by q_{tn} and divide both sides of the resulting equation by e^t. We obtain the following *system of inverse demand share equations*:

$$s_{tn} = \alpha_n q_{tn}/\textstyle\sum_{j\in S(t)}\alpha_j q_{tj} = \alpha_n q_{tn}/\boldsymbol{\alpha}\cdot\boldsymbol{q}^t; \quad t = 1,\ldots,T; n \in S(t). \tag{99}$$

The corresponding nonlinear least squares minimization problem is:[*93]

$$\min\nolimits_\alpha \textstyle\sum_{t=1}^T\sum_{n\in S(t)}\{s_{tn} - [\alpha_n q_{tn}/\boldsymbol{\alpha}\cdot\boldsymbol{q}^t]\}^2. \tag{100}$$

We require a normalization like $\alpha_{n^*} = 1$ as well as a certain amount of product overlap in order to get a unique solution to (100). Once the solution vector $\boldsymbol{\alpha}^*$ has been calculated, as usual,

*92 The GK and WTPD quantity levels both satisfy this invariance property and their companion price indexes satisfy a similar invariance property.

*93 It should be noted that there is only a single variance parameter associated with the nonlinear estimation problems defined by (98) and (100). In most consumer demand studies, a variance covariance matrix between all prices or shares pertaining to a single period is assumed. However, this specification is not practical when N is large (because there are $N(N+1)/2$ variances and covariances that must be estimated and it is difficult to use this framework when there are missing prices.

define the quantity levels as $Q^{t^*} \equiv \boldsymbol{\alpha}^* \cdot \boldsymbol{q}^t$ and the corresponding price levels as $P^{t^*} \equiv e^t/\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^t$ for $t = 1, \ldots, T$. If we want P^{1^*} to equal 1, then multiply the vector $\boldsymbol{\alpha}^*$ by $\lambda^1 \equiv e^1/\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^1$ and define a new $\boldsymbol{\alpha}^{**} \equiv \lambda^1 \boldsymbol{\alpha}^*$. Define new $Q^{t^{**}} \equiv \boldsymbol{\alpha}^{**} \cdot \boldsymbol{q}^t$ and $P^{t^{**}} \equiv e^t/\boldsymbol{\alpha}^{**} \cdot \boldsymbol{q}^t$ for $t = 1, \ldots, T$. The new price and quantity levels will be proportional to the corresponding initial price and quantity levels and the new price levels will have $P^{1^{**}} = 1$.

It is of interest to compare the above econometric method for estimating linear preferences with Methods 1 and 2 in section 4.1 for estimating a linear utility function using the Time Product Dummy econometric model. Recall that the estimating equations for the TPD model were equations (78), $p_{tn} \approx P^t \alpha_n$ for $t = 1, \ldots, T$ and $n \in S(t)$. Thus the TPD model is a *more flexible model* in that it estimates *two sets of parameters*: the P^t (the period t price levels) and the α_n (the quality adjustment factors) whereas the present econometric model estimates only the α_n. The present model derives the period t price level as $P^t \equiv e^t/Q^t$ where the period t quantity level is defined as $Q^t \equiv \boldsymbol{\alpha} \cdot \boldsymbol{q}^t$ where $\boldsymbol{q}^t$ is the period t quantity vector as usual. However, when quantity information is available (in addition to price information), the TPD method has the disadvantage that there are *two ways* to form the period t price level: $P^{t^*} \equiv P^t$ or $P^{t^{**}} \equiv e^t/\boldsymbol{\alpha} \cdot \boldsymbol{q}^t$. Thus National Statistical Offices must choose between two equally plausible estimates.[*94]

We have outlined three different multilateral indexes (GK, WTPD and the econometric estimation of linear preferences using inverse demand functions) that are consistent with purchasers maximizing a linear utility function. Unfortunately, if there is a lot of variation in prices and quantities over time, the three methods can produce very different (normalized) price levels.[*95] Which method is preferred? From the viewpoint of having a method that is most consistent with the observed price and quantity data, an econometric method is preferred. However, an econometric method is also open to criticism: different econometricians will make different assumptions about the distribution of the error terms and the different specifications can lead to very different empirical estimates. Thus the issue of which method is "best" has not been resolved in a definitive manner.

In the following two sections, we will study other functional forms for the utility function that are more flexible than a linear function but at the same time, contain the linear function as a special case.

4.3 The Estimation of CES Preferences

Our second example of the methodology explained in section 2.5 that uses inverse demand functions to estimate preferences is the case where the utility function is a CES (Constant Elasticity of Substitution) function, where $f(\boldsymbol{q})$ is defined as follows in the case of N products:[*96]

$$f(\boldsymbol{q}) \equiv [\textstyle\sum_{n=1}^{N} \alpha_n (q_n)^r]^{1/r} \tag{101}$$

where the α_n are positive parameters and the parameter r satisfies the following inequalities:

$$0 < r \leq 1.^{*97} \tag{102}$$

[*94] Diewert, Abe, Tonogi and Shimizu (2025)[62] found very little difference between Methods 1 and 2 for their particular data set.

[*95] See Diewert, Abe, Tonogi and Shimizu (2025)[62] for an example of large differences.

[*96] In the mathematics literature, if the α_n sum to one, this aggregator function or utility function is known as a power mean or a mean of order r; see Hardy, Littlewood and Pólya (1934; 12-13)[106]. This functional form was popularized by Arrow, Chenery, Minhas and Solow (1961)[4] in the context of production theory. For more on estimating CES utility functions, see Balk (1999)[10], Melser (2006)[131], de Haan and Krsinich (2024)[39], IMF (2025)[113] and Diewert, Abe, Tonogi and Shimizu (2025)[62].

Note that if the parameter r equals 1, then the CES utility function defined by (101) becomes the linear utility function that was discussed in previous sections.

The extension of the system of inverse demand equations (35) to the case where there are missing observations is the following system of estimating equations:

$$p_{tn} = e^t[\partial f(\boldsymbol{q}^t)/\partial q_n]/f(\boldsymbol{q}^t); \quad t = 1, \ldots, T; n \in S(t). \tag{103}$$

When the above equations do not hold exactly, the estimates which result from using the above equations to estimate the quantity levels $Q^{t^*} = f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*)$ for $t = 1, \ldots, T$ will not be invariant to changes in the units of measurement; i.e., the resulting bilateral indexes, Q^{t^*}/Q^{1^*} will in general change when the units of measurement are changed. As noted earlier, this is not an acceptable situation for producers of consumer price and quantity indexes.

As in the previous section, the lack of invariance problem can be solved by moving to the system of inverse demand *share* equations. Thus multiply both sides of equation t, n in equations (103) by q_{tn}/e^t and we obtain the following system of estimating equations, with shares as the dependent variables instead of prices:

$$s_{tn} = q_{tn}[\partial f(\boldsymbol{q}^t)/\partial q_n]/f(\boldsymbol{q}^t); \quad t = 1, \ldots, T; n \in S(t). \tag{104}$$

Add error terms (with means 0 and constant variances) to the right hand side of equations (104) and we obtain the following nonlinear least squares estimation problem:

$$\min_\alpha \textstyle\sum_{t=1}^T \sum_{n \in S(t)} \{s_{tn} - q_{tn}[\partial f(\boldsymbol{q}^t)/\partial q_n]/f(\boldsymbol{q}^t)\}^2. \tag{105}$$

Assume that $f(\boldsymbol{q})$ is the CES utility function defined by (101) and (102). Equations (104) become the following system of estimating equations:

$$s_{tn} \equiv p_{tn}q_{tn}/e^t = q_{tn}[\partial f(\boldsymbol{q}^t)/\partial q_n]/f(\boldsymbol{q}^t) = \alpha_n(q_{tn})^r / \textstyle\sum_{i \in S(t)} \alpha_i (q_{ti})^r; \; t = 1, \ldots, T, n \in S(t). \tag{106}$$

If $q_{tn} = 0$, then $s_{tn} = 0$ and the right hand side of (106) is also equal to 0 if $0 < r \leq 1$. Thus in this case, the equations (106) can be replaced with the following estimating equations:

$$s_{tn} = \alpha_n(q_{tn})^r / \textstyle\sum_{i=1}^N \alpha_i (q_{ti})^r; \quad t = 1, \ldots, T; n = 1, \ldots, N. \tag{107}$$

Equations (107) are much simpler to program when solving the nonlinear least squares problem (105) when $f(\boldsymbol{q})$ is the CES function defined by (101) and (102). However, going from (106) to (107) does require that the estimated r satisfy the inequalities $0 < r \leq 1$.

Looking at equations (106), it can be seen that the expression $\alpha_n(q_{tn})^r/\sum_{i \in S(t)} \alpha_i(q_{ti})^r$ is homogeneous of degree 0 in $\alpha_1, \ldots, \alpha_N$ for any $r \neq 0$. Thus not all of the α_n can be identified empirically. As was the case of linear preferences, a simple way forward is to set $\alpha_{n^*} = 1$ for some product n^* that is purchased in all T periods.

The nonlinear least squares minimization problem for CES preferences using inverse demand share equations is defined by (108) where we also set $\alpha_n^* = 1$:

$$\min_{\alpha, r} \textstyle\sum_{t=1}^T \sum_{n \in S(t)} \{s_{tn} - [\alpha_n(q_{tn})^r / \sum_{i=1}^N \alpha_i (q_{ti})^r]\}^2. \tag{108}$$

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*97 We require that $r \leq 1$ to ensure that the utility function is concave in the components of $\boldsymbol{q}$ and we require that $r > 0$ in order to ensure that the utility function is well defined if any component of the $\boldsymbol{q}^t$ vector happens to be equal to 0. The restrictions $0 < r < 1$ are required in order to apply Feenstra's (1994)[89] methodology for measuring the welfare effects of increased (or decreased) product choice.

Once a solution r^* and $\alpha_1^*,\ldots,\alpha_N^*$ has been found and it turns out that $0 < r^* \leq 1$, define the quantity levels as $Q^{t^*} \equiv [\sum_{n=1}^{N}\alpha_n^*(q_{tn})^{r^*}]^{1/r^*}$ and the corresponding price levels as $P^{t^*} \equiv e^t/Q^{t^*}$ for $t = 1,\ldots,T$.*98 If we want P^1 to equal 1 and Q^1 to equal e^1, then multiply the vector $\boldsymbol{\alpha}^*$ by $\lambda^1 \equiv [e^1/\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^1]^{r^*}$ and define a new $\boldsymbol{\alpha}^{**} \equiv \lambda^1\boldsymbol{\alpha}^*$. Define new $Q^{t^{**}} \equiv [\sum_{n=1}^{N}\alpha_n^{**}(q_{tn})^{r^*}]^{1/r^*}$ and the new corresponding price levels as $P^{t^{**}} \equiv e^t/Q^{t^{**}}$ for $t = 1,\ldots,T$. The new price and quantity levels will be proportional to the corresponding initial price and quantity levels and the new price levels will set $P^{1^{**}} = 1$.

Once a solution to (108) (with a normalization on the α_n) has been found, reservation prices*99 for products that are not purchased can be calculated as follows:

$$p_{tn}^* \equiv \alpha_n^*(q_{tn})^{r^*-1}/\textstyle\sum_{i\in S(t)}\alpha_i^*(q_{ti})^{r^*}; \quad t = 1,\ldots,T; n \notin S(t). \tag{109}$$

Looking at the right hand side of definitions (109), it can be seen that if $0 < r^* < 1$, then $q_{tn} = 0$ for $n \notin S(t)$ and hence $(q_{tn})^{r^*-1}$ is equal to plus infinity and so is the reservation price p_{tn}^*. This is a weakness of the CES functional form: all reservation prices are equal to plus infinity which seems unlikely empirically. Thus it will be difficult for National Statistical Offices to justify constructing indexes based on estimating CES preferences.

Since the CES utility function has the linear utility function as a special case (where $r = 1$), the CES price and quantity levels using the system of inverse demand share functions will fit the observed data better and in many instances, much better.*100 Thus the CES price and quantity levels based on estimating inverse demand functions should be an improvement over other methods that estimate linear preferences. However, the estimation method is complex and different econometricians will make different error specifications which will lead to different estimates for the CES price and quantity levels. The normalization issue can also be problematic; for example, there may not be a product that is purchased in all time periods and this can lead to multicollinearity problems.

In any case, most academic research that estimate CES functional forms estimate CES cost functions or unit cost functions. The unit cost function $c(\boldsymbol{p})$ that corresponds to the CES utility function $f(\boldsymbol{q})$ defined (101) with the $\alpha_n > 0$ and $r < 1$ but $r \neq 0$ turns out to be the following function assuming that $\boldsymbol{p} \gg \boldsymbol{0}_N$:*101

$$c(\boldsymbol{p}) = [\textstyle\sum_{n=1}^{N}\beta_n(p_n)^{\rho}]^{1/\rho} \tag{110}$$

*98 Recall our discussion about the Commensurability Test at the end of section 2.4. When we change units of measurement for the N products, the term $\alpha_n(q_{tn})^r$ for $n \in S(t)$ in the sum of squared terms defined in (108) becomes $\alpha_n(\lambda_n q_{tn})^r$ where the positive constant λ_n reflects the change in the units of measurement for product n for $n = 1,\ldots,N$. If $\alpha_1^* > 0, \alpha_2^* > 0,\ldots,\alpha_N^* > 0$ and $0 < r^* \leq 1$ are a solution to (108) for the original units of measurement, then $\lambda_1^{-r}\alpha_1^*, \lambda_2^{-r}\alpha_2^*,\ldots,\lambda_N^{-r}\alpha_N^*$ and r^* will be a solution to the new (108) that uses the new units of measurement. This new least squares minimization problem is $\min_{\alpha,r}\sum_{t=1}^{T}\sum_{n\in S(t)}\{s_{tn} - [\alpha_n(\lambda_n q_{tn})^r/\sum_{i=1}^{N}\alpha_i(\lambda_n q_{ti})^r]\}^2$. Note that the expenditure shares s_{tn} are not affected by the changes in the units of measurement. It can be seen that the new period t quantity level, $Q^{t^{**}} \equiv [\sum_{n=1}^{N}\lambda_n^{-r}\alpha_n^*(\lambda_n q_{tn})^{r^*}]^{1/r^*} = [\sum_{n=1}^{N}\alpha_n^*(q_{tn})^{r^*}]^{1/r^*}$ is equal to the original units Q^{t^*}. Thus our econometric price and quantity levels are invariant to changes in the units of measurement. These computations support the viewpoint of Samuelson and Swamy (1974)[146] on the application of the Commensurability Test; i.e., care must be taken in the application of this test.

*99 The reservation price for product n that was not purchased in period t, p_{tn}^*, is the lowest price that would induce potential purchasers to purchase 0 units of the product. The concept of a reservation price is due to Hicks (1940; 114)[107].

*100 See for example Diewert, Abe, Tonogi and Shimizu (2025)[62].

*101 See for example Diewert (2020; 37)[59] or IMF (2025)[113].

where the parameters β_n and ρ are defined in terms of the α_n and r as follows:

$$\beta_n \equiv (\alpha_n)^{1/(1-r)} \quad \text{for } n = 1, \ldots, N \text{ and } \rho \equiv -r/(1-r). \tag{111}$$

When $r = 1, \rho = -\infty$; when $r = 0, \rho = 0$; when $r = -\infty, \rho = 1$. Thus the CES unit cost function satisfies the same regularity conditions as the CES utility function.

It is useful to relate r and ρ to the elasticity of substitution σ_{nk} between products n and k. The unit cost function $c(\boldsymbol{p})$ can be used to define the Allen (1938; 504)[2] Uzawa (1962)[160] *elasticity of substitution* $\sigma_{nk}(\boldsymbol{p})$ between products n and k:

$$\begin{aligned}\sigma_{nk}(\boldsymbol{p}) &\equiv \{c(\boldsymbol{p})\partial^2 c(\boldsymbol{p})/\partial p_n \partial p_k\}/\{[\partial c(\boldsymbol{p})/\partial p_n][\partial c(\boldsymbol{p})/\partial p_k]\} \quad 1 \le n, k \le N \\ &= -\rho + 1 \quad \text{for all } n, k \text{ such that } n \neq k \\ &\equiv \sigma \end{aligned} \tag{112}$$

where the second equality in (112) follows by assuming that the unit cost function is the CES unit cost function defined by (110). Thus the CES functional form imposes the restriction that *the elasticity of substitution between every pair of products is the same constant* $\sigma = -\rho + 1$. Since $\rho = -r/(1-r)$, we see that:

$$\begin{aligned}\sigma &= -\rho + 1 \\ &= r/(1-r) + 1 \quad \text{using } \rho = -r/(1-r) \\ &= r/(1-r) + (1-r)/(1-r) \\ &= 1/(1-r) \quad \text{provided that } r \neq 1. \end{aligned} \tag{113}$$

Hardly any studies that estimate CES preferences estimate the CES utility function directly. Instead, the CES unit cost function $c(\boldsymbol{p}) \equiv [\sum_{n=1}^{N} \beta_n (p_n)^\rho]^{1/\rho}$ is estimated, which was defined by (110) above when there are no missing products. The unit cost method uses the methodology explained in section 2.5. If there are no missing products, equations (35) could be used as estimating equations. Following Feenstra (1994)[89], if there are missing products, we set the reservation price for missing product n in period t, $p_{tn} = +\infty$ for $t = 1, \ldots, T$ and $n \notin S(t)$. Again following Feenstra, assume that ρ satisfies the following inequality:

$$\rho < 0. \tag{114}$$

Note that $\rho < 0$ implies that $\sigma = -\rho + 1 > 1$. With this assumptions, the CES unit cost function $c(\boldsymbol{p}^t)$ is defined as follows:

$$\begin{aligned} c(\boldsymbol{p}^t) &\equiv [\textstyle\sum_{n=1}^{N} \beta_n (p_{tn})^\rho]^{1/\rho} \\ &= [\textstyle\sum_{n \in S(t)} \beta_n (p_{tn})^\rho]^{1/\rho} \end{aligned} \tag{115}$$

where the last equality follows since $n \notin S(t)$ and $\rho < 0$ implies $(p_{tn})^\rho = (1/\infty)^{-\rho} = 0$.

When there are missing prices, the estimating equations (35) become the following equations when $c(\boldsymbol{p}^t)$ is defined by (115)

$$q_{tn} \equiv \beta_n p_{tn}^{\rho-1} / \textstyle\sum_{i \in S(t)} \beta_i (p_{ti})^\rho; \quad t = 1, \ldots, T; n \in S(t). \tag{116}$$

As usual, not all of the parameters β_n can be identified using the estimating equations (116) so in order to eliminate exact multicollinearity, we require a normalization such as $\beta_{n^*} = 1$ for a product n^* that is present in all T periods.

As was the case when estimating the CES utility function, using equations (116) to estimate the unknown β_n and ρ will lead to price and quantity levels that are not invariant to changes in the units of measurement. As in the previous section, the lack of invariance problem can be solved by moving to the system of *share* equations. Thus multiply both sides of equation t, n in equations (116) for $n \in S(t)$ by p_{tn}/e^t and we obtain the following system of estimating equations, with shares as the dependent variables instead of quantities:

$$s_{tn} = \beta_n(p_{tn})^\rho / \textstyle\sum_{i\in S(t)}\beta_i(p_{ti})^\rho; \quad t = 1, \ldots, T; n \in S(t). \tag{117}$$

Add error terms (with means 0 and constant variances) to the right hand side of equations (117) and we obtain the following nonlinear least squares estimation problem:

$$\min_{\beta,\rho}\{\textstyle\sum_{t=1}^{T}\sum_{n\in S(t)}[s_{tn} - \beta_n(p_{tn})^\rho/\sum_{i\in S(t)}\beta_i(p_{ti})^\rho]^2\}. \tag{118}$$

As usual, in order to identify the β_n, we need a normalization on the β_n like $\beta_{n^*} = 1$ for some product n^* that is present in all periods. Once a solution ρ^* and $\beta_1^*, \ldots, \beta_N^*$ has been found and it turns out that $\rho^* < 0$, define the price levels as $P^{t^*} \equiv [\sum_{n\in S(t)}\beta_n^*(p_{tn})^{\rho^*}]^{1/\rho^*}$ and the corresponding quantity levels as $Q^{t^*} \equiv e^t/P^{t^*}$ for $t = 1, \ldots, T$. These preliminary price and quantity levels can be normalized so that the normalized price level in period 1 is equal to 1 and the corresponding normalized quantity level is equal to period 1 expenditure, e^1.

The CES price and quantity levels based on estimating market demand functions is subject to the same criticisms that we noted about the CES price and quantity levels that were based on estimating inverse demand systems: the estimation method is complex and different econometricians will make different error specifications which will lead to different estimates for the CES price and quantity levels. Again, the normalization issue can be problematic; there may not be a product that is purchased in all time periods and this can lead to multicollinearity problems.

In addition to the above problems, the estimation of CES preferences using the unit cost function is subject to another problem that did not apply to the estimation of CES preferences by estimating the utility function directly. Suppose that there are no missing products. Under this assumption, when $\rho = 1$, the CES unit cost function collapses to $c(\boldsymbol{p}) = \sum_{n=1}^{N}\beta_n p_n$. In this case, we have Leontief (No Substitution) preferences and the elasticity of substitution is $\sigma = -\rho + 1 = 0$. If $\sigma = 0$, then using (113), $r = -\infty$. On the other hand, when $r = 1$, the elasticity of substitution is $\sigma = 1/(1 - r) = +\infty$. When estimating the CES functional form using inverse demand share equations, $r = 1$ is a reasonable starting value for the nonlinear regression defined by (108). If all goes well, the result of the nonlinear regression defined by (108) will give as value for r that is less than 1 but probably not too far from 1. Thus linear preferences is a good starting point for estimating σ by estimating the CES utility function directly. On the other hand, using $\rho = 1$ as a starting value for the nonlinear regression defined by (118) is not a good starting point since $\rho = 1$ implies that the elasticity of substitution is equal to 0 which is highly unlikely to be close to the "truth". It seems likely that estimating the elasticity of substation by estimating the CES utility function will generate higher estimates for σ (and better fits) than estimating σ by estimating the CES unit cost function.

Some evidence on differences in estimates for the elasticity of substitution by estimating the CES utility function $f(\boldsymbol{q})$ defined by (101) versus estimating the CES unit cost function $c(\boldsymbol{p})$ defined by (110) can be found in the study on rice demand in Japan by Diewert, Abe, Tonogi and Shimizu (2025)[62]. In this study, the authors found that estimating the CES utility function led to an elasticity of substitution estimate equal to 23.09 whereas estimating the CES unit cost function led to an elasticity of substitution estimate equal to 5.04. The dependent variables in both nonlinear regressions were expenditure shares. The regression R^2 for the $f(\boldsymbol{q})$ regression[*102] was 0.9967 (with a sum of absolute errors between actual shares and predicted shares equal to 7.26) whereas the $c(\boldsymbol{p})$ R^2 was only 0.7080 (with a sum of absolute errors equal to 71.3). Using the same data source and estimation methodology that was used for Rice in Japan was used by the present authors for Japanese data on Cereals, Spaghetti and Cheese and led to the results listed in Table 3.

Table 3 Table of Alternative Estimates for the Elasticity of Substitution for Some Japanese Products

Product	$f(\boldsymbol{q})$ or $c(\boldsymbol{p})$	σ	R^2	Sum of Absolute Errors
Rice	$f(\boldsymbol{q})$	23.09	0.9967	7.26
Rice	$c(\boldsymbol{p})$	5.04	0.7080	71.30
Cereal	$f(\boldsymbol{q})$	21.59	0.9982	5.97
Cereal	$c(\boldsymbol{p})$	2.18	0.7811	69.15
Spaghetti	$f(\boldsymbol{q})$	9.97	0.9969	7.68
Spaghetti	$c(\boldsymbol{p})$	5.11	0.7523	69.70
Cheese	$f(\boldsymbol{q})$	18.57	0.9970	6.12
Cheese	$c(\boldsymbol{p})$	2.82	0.7106	67.44

The above Table shows that the Rice results are not an isolated result. Estimating a CES unit cost function leads to a lower estimate of σ and a very poor fit to the data compared to estimating a CES utility function using the same data.

Since most applications of the CES Feenstra (1994)[89] methodology for measuring the gains from new products are based on estimates for σ that estimate a CES unit cost function, it is likely that these estimated gains are overstated since a higher σ estimate means a lower gain from additional product variety. The above results suggest that researchers should look at estimating a CES utility function rather than estimating the dual unit cost function when it is likely that the products in scope are strong substitutes.

As noted above, a problem with the CES functional form is that there is only one parameter, the elasticity of substitution σ, that is used to describe substitution possibilities between every pair of products. In the following section, we describe a functional form for the purchaser's utility function that has a separate parameter to describe substitution possibilities for each product. Moreover, the reservation prices that are generated by the KBF functional form are finite.

4.4 The Estimation of KBF Preferences

Konüs and Byushgens (1926)[125][*103] introduced the following functional form for a linearly homogeneous utility function:

$$f(\boldsymbol{q}) \equiv (\boldsymbol{q} \cdot \mathbf{A}\boldsymbol{q})^{1/2} = (\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} q_i q_j)^{1/2}; a_{ij} = a_{ji}; \quad 1 \leq i \leq j \leq N. \qquad (119)$$

Thus $\mathbf{A}$ is an N by N symmetric matrix that contains $(N+1)N/2$ unknown a_{ij} parameters. The matrix $\mathbf{A}$ satisfies certain restrictions which are spelled out in Diewert (1976)[42] and

[*102] The R^2 measure we used is the square of the correlation coefficient between the observed and predicted expenditure shares.

[*103] See Diewert and Zelenyuk (2025)[79] for a translation and commentary on their paper. The materials in this section are drawn from materials on the Rank 1 Substitution Matrix KBF functional form that are in Diewert and Feenstra (2019)[64] (2022)[65] and Diewert, Abe, Tonogi and Shimizu (2025)[62].

Diewert and Hill (2010)[66]. Konüs and Byushgens and Diewert showed that this utility function is exact for the Fisher (1922)[91] Ideal quantity and price indexes[*104] so we call preferences defined by (119) KBF preferences. Note that there is no problem in defining $f(\boldsymbol{q})$ if some q_n are equal to 0.

In most empirical applications where there are a large number of products in scope, it is not possible to estimate all $(N+1)N/2$ unknown parameters a_{ij} in the KBF utility function defined by (119). In order to reduce the number of parameters in the $\mathbf{A}$ matrix, we define $\mathbf{A}$ as the following matrix which has rank 2:

$$\mathbf{A} \equiv \boldsymbol{\alpha}\boldsymbol{\alpha}^T - \boldsymbol{\beta}\boldsymbol{\beta}^T \tag{120}$$

where the transposes of the column vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined as $\boldsymbol{\alpha}^T \equiv [\alpha_1, \ldots, \alpha_N]$ and $\boldsymbol{\beta}^T \equiv [\beta_1, \ldots, \beta_N]$.

With $\mathbf{A}$ defined by (120), the system of inverse demand *share* equations (104) becomes the following system of estimating equations:

$$s_{tn} = q_{tn}[\alpha_n \boldsymbol{\alpha} \cdot \boldsymbol{q}^t - \beta_n \boldsymbol{\beta} \cdot \boldsymbol{q}^t]/[(\boldsymbol{\alpha} \cdot \boldsymbol{q}^t)^2 - (\boldsymbol{\beta} \cdot \boldsymbol{q}^t)^2] + e_{tn};\ t = 1, \ldots, T; n = 1, \ldots, N. \tag{121}$$

Equations (121) are valid even when there are missing products because when product n is missing in period t, $s_{tn} = q_{tn} = 0$.

The utility function as a function of the unknown parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and the consumption vector $\boldsymbol{q}$ is defined as follows:

$$f(\boldsymbol{q}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \equiv [\boldsymbol{q}^T(\boldsymbol{\alpha}\boldsymbol{\alpha}^T - \boldsymbol{\beta}\boldsymbol{\beta}^T)\boldsymbol{q}]^{1/2} = [(\boldsymbol{\alpha} \cdot \boldsymbol{q}^t)^2 - (\boldsymbol{\beta} \cdot \boldsymbol{q}^t)^2]^{1/2}. \tag{122}$$

Note that if $\boldsymbol{\beta} = \mathbf{0}_N$, then $f(\boldsymbol{q}, \boldsymbol{\alpha}, \mathbf{0}_N) = [(\boldsymbol{\alpha} \cdot \boldsymbol{q}^t)^2]^{1/2} = \boldsymbol{\alpha} \cdot \boldsymbol{q}^t = \sum_{n=1}^{N} \alpha_n q_n$; i.e., the utility function collapses down to the linear utility function that was studied in earlier sections. This is an important point because it implies that starting coefficients α_n and β_n for the nonlinear least squares minimization problem that is defined below can be set equal to the estimates of the α_n^* that result in the estimation of linear preferences with the starting coefficients for the β_n^* set equal to 0.

There are some tricky aspects to the new utility function as compared to the case of a linear utility function. We need to ensure that $(\boldsymbol{\alpha} \cdot \boldsymbol{q}^t)^2 - (\boldsymbol{\beta} \cdot \boldsymbol{q}^t)^2 > 0$ so that we can calculate the positive square root, $[(\boldsymbol{\alpha} \cdot \boldsymbol{q}^t)^2 - (\boldsymbol{\beta} \cdot \boldsymbol{q}^t)^2]^{1/2}$. We also need to set $\beta_n = 0$ if product n is available in only one period.[*105] In order to identify all of the parameters, we impose our usual normalization so that our present model contains our linear utility function model as a special case where n^* is a product that is available in all T periods :

$$\alpha_{n^*} = 1. \tag{123}$$

Recall that $\boldsymbol{q}^1$ is the observed consumption vector for period 1. In order to prevent multicollinearity between the α_n and β_n parameters, impose the following normalization on the β_n parameters:

$$\boldsymbol{\beta} \cdot \boldsymbol{q}^1 = 0.^{*106} \tag{124}$$

[*104] Recall (47)-(53) in section 2.5 above.

[*105] If a product n appears in only one period in the sample of observations, then our KBF model will be able to estimate the parameter α_n but it will not be able to estimate the parameter β_n; see the Appendix in Diewert (2024)[61].

Estimates for the α_n and β_n parameters are obtained by solving the following nonlinear least squares minimization problem subject to the normalizations (123) and (124):

$$\min_{\alpha,\beta} \sum_{t=1}^{T} \sum_{n=1}^{N} \{s_{tn} - q_{tn}[\alpha_n \boldsymbol{\alpha} \cdot \boldsymbol{q}^t - \beta_n \boldsymbol{\beta} \cdot \boldsymbol{q}^t]/[(\boldsymbol{\alpha} \cdot \boldsymbol{q}^t)^2 - (\boldsymbol{\beta} \cdot \boldsymbol{q}^t)^2]\}^2. \quad (125)$$

Taking into account the normalizations (123) and (124), there are $2N - 2$ free parameters to estimate. Typically, the starting coefficient values for the α_n are set equal to the final coefficient estimates for the linear utility function model discussed in section 4.2 and the starting coefficient values for the β_n are set equal to 0.00001 or −0.00001.

Once the solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ to the nonlinear least squares minimization problem (125) has been obtained, the preliminary period t aggregate quantity and price levels, Q^{t^*} and P^{t^*}, are defined as follows:

$$Q^{t^*} \equiv f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) = [(\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^t)^2 - (\boldsymbol{\beta}^* \cdot \boldsymbol{q}^t)^2]^{1/2}; P^{t^*} \equiv e^t/Q^{t^*}; \quad t = 1, \ldots, T. \quad (126)$$

The normalization (124) ensures that the quantity levels Q^{t^*} defined by (126) are invariant to changes in the units of measurement. Suppose $\alpha_1^*, \ldots, \alpha_N^*$ and $\beta_1^*, \ldots, \beta_N^*$ solve (125) with $\boldsymbol{\beta}^* \cdot \boldsymbol{q}^1 = \sum_{n=1}^{N} \beta_n^* \cdot q_{1n} = 0$. Now change the units of measurement for the N products so that q_{tn} is replaced with $\lambda_n q_{tn}$ where $\lambda_n > 0$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$. The prices p_{tn} are replaced by $(\lambda_n)^{-1} p_{tn}$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$. Now consider the new nonlinear least squares minimization problem that is a counterpart to (125) but uses the new units price and quantity data. The expenditure shares in (125) remain unchanged with the change in units. It can be seen that a solution to the new least squares minimization problem is given by $(\lambda_1)^{-1}\alpha_1^*, \ldots, (\lambda_N)^{-1}\alpha_N^*$ and $(\lambda_1)^{-1}\beta_1^*, \ldots, (\lambda_N)^{-1}\beta_N^*$. Define the new $\alpha_n^{**} \equiv (\lambda_n)^{-1}\alpha_n^*$ and the new $\beta_n^{**} \equiv (\lambda_n)^{-1}\beta_n^*$ for $n = 1, \ldots, N$. Note that the new β_n and q_{1n} will satisfy the new constraint (125); i.e., we have:

$$\sum_{n=1}^{N} \beta_n^{**} \lambda_n q_{1n} = \sum_{n=1}^{N} (\lambda_n)^{-1} \beta_n^* \lambda_n q_{1n} = \sum_{n=1}^{N} \beta_n^* q_{1n} = 0. \quad (127)$$

It can be seen that the quantity levels Q^{t^*} defined by (126) are invariant to changes in the units of measurement.

Normalize the sequence price levels P^{t^*} into the series P_{KBF}^t which is such that the normalized sequence of price levels equals 1 for period 1:

$$P_{\text{KBF}}^t \equiv P^{t^*}/P^{1^*}; \quad t = 1, \ldots, T. \quad (128)$$

With the solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ to (125) in hand, we can calculate Hicksian reservation prices p_{tn}^* for the products n that were *not* present in period t using the following equations where $f_n(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) \equiv \partial f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)/\partial q_n$ for $n = 1, \ldots, N$:

$$p_{tn}^* \equiv e^t f_n(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)/f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*); \quad t = 1, \ldots, T; n \notin S(t). \quad (129)$$

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*106 This normalization ensures that $(\boldsymbol{\alpha} \cdot \boldsymbol{q}^t)^2 - (\boldsymbol{\beta} \cdot \boldsymbol{q}^t)^2 > 0$ when $t = 1$ so that we can define $f(\boldsymbol{q}^1)$ as the positive square root of $(\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^1)^2 - (\boldsymbol{\beta}^* \cdot \boldsymbol{q}^1)^2$ which is equal to $\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^1$ where $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$ solve (125) subject to the normalization $\boldsymbol{\beta}^* \cdot \boldsymbol{q}^1 = 0$. This normalization implies that the matrix of second order partial derivatives of $f(\boldsymbol{q}^1, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ with respect to the components of $\boldsymbol{q}^1$ is equal to $\nabla^2 f(\boldsymbol{q}^1, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) = -(\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^1)^{-1} \boldsymbol{\beta}^* \boldsymbol{\beta}^{*T}$. Thus the *period 1 substitution matrix* $\nabla^2 f(\boldsymbol{q}^1, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ is equal to the negative number $-(\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^1)^{-1}$ times the rank 1 N by N matrix $\boldsymbol{\beta}^* \boldsymbol{\beta}^{*T}$. Note that the diagonal elements of this matrix are equal to $-(\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^1)^{-1}(\beta_n^*)^2 < 0$. Thus each *own elasticity of inverse demand* has a separate parameter, β_n^*, that determines this own elasticity (except that the β_n^* are subject to a single linear constraint, $\sum_{n=1}^{N} \beta_n^* \cdot q_{1n} = 0$). Thus the KBF utility function is far more flexible than the CES utility function.

In an empirical example where this KBF utility function was estimated by Diewert, Abe, Tonogi and Shimizu (2025)[62], the average reservation price for their estimated KBF utility function was 0.43135 while the average predicted price for products that were present in each period was equal to 0.32779. Thus on average, reservation prices were 0.43135/0.32779 = 1.316 or 31.6 percent higher than predicted prices. Recall that the CES model generates infinite reservation prices which is a problem with the CES model.[*107]

The N by N matrix of second order partial derivatives of $f(\boldsymbol{q}, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ evaluated at $\boldsymbol{q} = \boldsymbol{q}^t$ is denoted by $\nabla^2 f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ and it is called the *period t inverse substitution matrix*. For a general twice differentiable linearly homogeneous and concave utility $f(\boldsymbol{q})$, it must be a negative semidefinite matrix that satisfies the restrictions $\nabla^2 f(\boldsymbol{q})\boldsymbol{q} = \mathbf{0}_N$. Thus the rank of $\nabla^2 f(\boldsymbol{q})$ is at most $N-1$. For our particular functional form for $f(\boldsymbol{q}, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ defined by (126), the period t inverse substitution matrix for period t is defined as follows:

$$\nabla^2 f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) \equiv -[f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)]^{-3}[\boldsymbol{\alpha}^*(\boldsymbol{\beta}^* \cdot \boldsymbol{q}^t) - \boldsymbol{\beta}^*(\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^t)][\boldsymbol{\alpha}^*(\boldsymbol{\beta}^* \cdot \boldsymbol{q}^t) - \boldsymbol{\beta}^*(\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^t)]^T. \quad (130)$$

If $\boldsymbol{\beta}^* = \mathbf{0}_N$, then $\nabla^2 f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) = \mathbf{0}_N \mathbf{0}_N^T$ which is an N by N matrix of zeros. If $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$ are both nonzero vectors and $\boldsymbol{\alpha}^* \neq \boldsymbol{\beta}^*$, then the period t substitution matrix defined by (130) will have rank equal to one. Diewert and Wales (1988)[78] called a functional form for a cost function defined over N products a *semiflexible functional form of rank k* if its matrix of second order partial derivatives had rank k. Using this terminology, our $f(\boldsymbol{q}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defined by (126) is a *semiflexible functional form of rank 1*.[*108]

4.5 Summary on the Use of Multilateral Indexes

In the introduction to this paper, we mentioned two Challenges in Measuring Consumer Price Change:

- The chain drift problem (or lack of transitivity problem) and
- The lack of matching problem (caused by product churn).

We suggested that using multilateral index numbers could be used to solve the chain drift problem.[*109] We looked at seven different multilateral index number models in the previous sections: GEKS, GK, WTPD, LP (the Linear Preferences Model using inverse demand functions), CESq (CES preferences estimated using the CES utility function $f(\boldsymbol{q})$), CESp (CES preferences estimated using the CES unit cost function $c(\boldsymbol{p})$) and the KBF utility function model with a rank one substitution matrix. All of these multilateral indexes produce *transitive* price and quantity levels and these methods generate price and quantity indexes that are invariant to changes in the units of measurement. Thus there is no chain drift problem using these multilateral indexes over periods 1 to T. The GK, WTPD and LP indexes are consistent with purchasers maximizing a linear utility function. The elasticity of substitution between every pair of products for these models is equal to $+\infty$, which is problematic. However, many

[*107] Diewert, Abe, Tonogi and Shimizu (2025)[62] also calculated the ratio of reservation prices to the average of predicted prices for the linear utility function model that was discussed in section 4.2. This ratio was 0.40690/0.32549 = 1.2501 so reservation prices for the linear utility function model were on average 25.0 percent higher.

[*108] If $f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) > 0$, then $[f(\boldsymbol{q}^t, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)]^{-3} \equiv k > 0$ and $\mathbf{A} = -k\boldsymbol{\gamma}\boldsymbol{\gamma}^T$ where $\boldsymbol{\gamma}$ is the N dimensional column vector $\boldsymbol{\alpha}^*(\boldsymbol{\beta}^* \cdot \boldsymbol{q}^t) - \boldsymbol{\beta}^*(\boldsymbol{\alpha}^* \cdot \boldsymbol{q}^t)$. Thus $\boldsymbol{z}^T\mathbf{A}\boldsymbol{z} = \boldsymbol{z}^T k\boldsymbol{\gamma}\boldsymbol{\gamma}^T\boldsymbol{z} = k(\boldsymbol{z}^T\boldsymbol{\gamma})^2 \leq 0$ for N dimensional vectors $\boldsymbol{z}$. Thus $\mathbf{A}$ is a negative semidefinite symmetric matrix.

[*109] The use of multilateral methods solves the chain drift problem for the window of T observations. However, when we add a new observation to the initial window and the index is not revisable, then a new chain drift problem is introduced; see section 5.1 below. Typically, the use of a Rolling Window multilateral method greatly reduces the chain drift problem.

National Statistical Offices have used GK or WTPD in constructing indexes at the first stage of aggregation where it is known that the products in scope are highly substitutable. An advantage of WTPD and LP is that one can see how well these methods fit the observed data. These methods can work well.

GEKS, CESq, CESp and KBF are consistent with more flexible preferences but the elasticity of substitution for the two CES models is *constant* for all pairs of products (which is also problematic). GEKS is consistent with the most flexible preferences and so it is a preferred method if product turnover is low. However, in situations where there is a lot of product churn, GEKS can be unsatisfactory because it is constructed using bilateral Fisher indexes between all possible pairs of time periods using only the set of products that are present in both periods being compared. The "true" bilateral Fisher index between two periods where some products are unmatched should be based on Laspeyres and Paasche bilateral indexes that use reservation prices for the unmatched products in order to be consistent with the economic approach to index number theory. However, the GEKS indexes are constructed using only matched prices; i.e., reservation prices are not estimated. The KBF model gets around the lack of product matching and reservation prices for missing products are generated as a byproduct of the estimation method. However, the estimated preferences are not fully flexible. But the preferences that are estimated are more flexible than all other models with the exception of GEKS. Tentative conclusions about the merits of the seven methods for transitive index construction at the first stage of aggregation are as follows:

- Use GEKS (or CCDI)[*110] if product turnover is low so that price matching is high.[*111]
- Use WTPD or LP if product turnover is high and it is thought that the products in scope are highly substitutable.
- If product turnover is high and it is not known if the products in scope are highly substitutable, use CESq (direct utility function estimation) or KBF. KBF is more flexible but is fairly complicated and difficult to explain to the public.
- The use of CESp is not recommended since (limited) experience has shown that estimating a CES unit cost function leads to a model that does not fit the data as well as the comparable CESq model.

But the above conclusions are only tentative. In particular, the use of econometric methods for the KBF indexes is somewhat risky. Many econometricians would not agree that our stochastic specification is satisfactory. Other econometricians may object to the estimation of inverse demand functions. There can be difficulties with choosing alternative normalizations on the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ vectors. Moreover, it would be difficult for national statistical offices to explain the KBF indexes to the public. More experience with estimating KBF preferences is required.

[*110] The CCDI multilateral indexes are constructed in the same way as the GEKS indexes except that instead of using the Fisher index as the basic bilateral index, the Törnqvist-Theil index is used instead. The CCDI index can be traced back to Caves, Christensen and Diewert (1982a)[23] and Inklaar and Diewert (2016)[114]. The CCDI indexes behave better when there are outliers in the data; see Fox, Level and O'Connell (2025)[97] and Dikhanov (2025)[80] on this point.

[*111] At higher levels of aggregation, when price and quantity movements are relatively smooth and there are few missing prices, there is no need to use multilateral methods because the chain drift and lack of matching problems will generally be minimal. In this case, chained superlative indexes will work well and in many cases, chained Laspeyres or Paasche indexes will also be satisfactory.

5 Other Challenges in Measuring Consumer Price Change

5.1 The Extension Problem

As we have seen, the seven multilateral methods explained above provide transitive price and quantity levels within the window of T periods. But National Statistical Offices must produce Consumer Price Indexes in real time and typically, official CPIs are not allowed to be revised. Thus in working with the above multilateral methods, we are faced with the *extension problem*: how exactly are the data for period $T+1$ ($\boldsymbol{p}^{T+1}$ and $\boldsymbol{q}^{T+1}$) to be linked to the indexes already constructed for periods 1 to T?

Ivancic, Diewert, and Fox (2009)[115] (2011)[116] suggested using a *rolling window methodology* with period T as the linking period. To explain what this means, let $P^1, P^2, \ldots, P^{T-1}, P^T$ denote the set of initial normalized price levels that was generated by the chosen multilateral method where $P^1 \equiv 1$. Denote the new set of normalized price levels that is generated by the new window of data spanning periods 2 to $T+1$ by $P^{2^*}, P^{3^*}, \ldots, P^{T^*}, P^{T+1^*}$ with $P^{2^*} \equiv 1$. IDF suggested that the new nonrevised series should be $P^1, P^2, \ldots, P^{T-1}, P^T, P^{T+1}$ where P^{T+1} is defined as P^T times (P^{T+1^*}/P^{T^*}) which is the movement in the new set of price levels from period T to period $T+1$; i.e., $P^{T+1} \equiv P^T(P^{T+1^*}/P^{T^*})$. Krsinich (2016; 383)[126] called this the *movement splice method* for linking the two windows. She did not endorse this method; she suggested using instead the period 2 price level from the initial window of price levels as the linking observation and she called this the *window splice method*. Thus Krsinich defined P^{T+1} as $P^2(P^{T+1^*}/P^{2^*})$. On the other hand, de Haan (2015; 26)[35] suggested that the link period should be chosen to be in the middle of the first window time span; that is, choose $t = T/2$ if T is an even integer or $t = (T+1)/2$ if T is an odd integer. Thus de Haan defined $P^{T+1} \equiv P^t(P^{T+1^*}/P^{t^*})$. The Australian Bureau of Statistics (2016; 12)[6] called this the *half splice method* for linking the results of the two windows. Diewert and Fox (2022; 360)[70] defined $P^{T+1} \equiv [\prod_{t=2}^{T} P^t(P^{T+1^*}/P^{t^*})]^{1/(T-1)}$; i.e., they suggested taking the geometric mean of all possible links of the new series to the initial series which is called the *mean splice*. Finally, using an expanding window on an annual basis (instead of using a rolling window within the year) to construct nonrevisable multilateral indexes was suggested (and implemented) by Chessa (2016)[26] (2021)[27]. The idea of using an *ever expanding window* was suggested by Diewert (2023)[60] in the context of his predicted share price similarity linking methodology. Thus his method uses the chosen multilateral method to construct the initial window of normalized price levels $P^1, P^2, \ldots, P^T$ as before with $P^1 = 1$. When the period $T+1$ data becomes available, then the chosen multilateral method uses all of the data to construct new normalized price levels $P^{1^*}, P^{2^*}, \ldots, P^{T^*}, P^{T+1^*}$ with $P^{1^*} = 1$. The expanding window price level for period $T+1$ is defined as $P^{T+1} = P^1(P^{T+1^*}/P^{1^*}) = P^{T+1^*}$. Thus the expanding window method of linking links back to the first period where we know what the price level is. This method of linking can be related to the econometric estimation of the direct utility function $f(\boldsymbol{q})$ when there is product churn. It is more efficient to use the information from all periods to estimate (constant) preferences and this helps to justify the expanding window methodology. For an empirical example that uses the expanding window methodology, see Diewert and Shimizu (2023)[77]. However, there are disadvantages associated with the ever expanding window method even for indexes that can be revised:

- The method becomes computationally burdensome as the number of periods increases and
- The method does not allow for gradually changing preferences.

The "best" method of linking a new window of data to an earlier window (in the context of indexes that cannot be revised) has not been settled. What has been settled is the fact that the use of multilateral methods to mitigate the chain drift problem does reduce the chain drift problem but it does not completely eliminate it.[*112]

5.2 Other Challenges

There are many other challenges that face National Statistical Offices in their attempts to construct more accurate Consumer Price Indexes. We list some of these problems.

Integration of different methods.

How should statistical agencies integrate portions of their CPI that *use very different methods* to construct different parts of their Consumer Price Indexes? At the first stage of aggregation, some statistical offices use multilateral methods as described above to construct parts of their CPI; e.g., see the Australian Bureau of Statistics (2016)[6]. Other parts of the CPI are constructed using traditional price collection methods where price collectors collect a sample of prices from retail outlets or they collect prices from the internet. To make matters more complicated, most national CPIs use a Lowe index to aggregate the lower level price indexes at higher levels of aggregation. These Lowe indexes typically use a national annual basket from a prior year to form the Lowe quantity basket $\boldsymbol{q}$ and this basket is combined with the monthly subaggregate price indexes that have been constructed using entirely different methodologies. The annual basket weights come from Household Consumer Expenditure Surveys. The end result is an aggregate Consumer Price Index that has no clear methodological justification. There is a particular methodological problem with the treatment of *strongly seasonal products* in a Lowe index: for some months of the year, there will be no price quotes for such products when they are out of season but somehow, monthly prices for these products have to be imputed in order to calculate the Lowe index. For a more detailed discussion of this particular problem, see Chapter 9 in IMF (2025)[113].

How should price indexes for Owner Occupied Housing be constructed?

This is a difficult problem that has been with us for a long time with no complete solution at hand. The problem is this: how exactly should we construct a monthly (imputed) price for the services of a housing unit that is owned by the occupant of the property? There are at least six different approaches to this measurement problem:

- *The Acquisitions Approach.* In this approach, we impute a zero price for a housing property that was not purchased during the current period. If a new housing structure is purchased during the reference month, we charge the entire purchase price of the property (or just the value of the structure) as the price for the new structure. This approach of course does not measure the services of other housing properties that are owned but not newly constructed in the reference month. This approach seemingly avoids imputations but it does not actually avoid the zero imputations for previously built housing units.
- *The Rental Equivalence Approach.* Using this approach, the monthly price for the services of an owned housing unit is the fair market rent for the unit. This approach

[*112] "Multilateral price indices are invoked as a panacea against chain drift, especially at lower aggregation levels of Consumer Price Indices. In this paper it is demonstrated that the chain drift problem inevitably returns as soon as the dataset, over which the multilateral indices are calculated, moves forward and the older index numbers must be retained. This is not a matter of numerical inaccuracy, but due to a mathematical impossibility." Bert M. Balk (2024; 25)[12].

runs into troubles when there are no “similar” housing units that are rented so that “objective” measures of the market rent for high end housing units in particular may be difficult to obtain.

- *The User Cost Approach.* This approach is a hypothetical financial opportunity cost approach. The price is what it would cost to purchase the property at the beginning of the month and then sell it at the end of the month. Suppose that the beginning of the period value of the property is V^0 and the end of the period value is V^1. The property owner forgoes the beginning of the period value of the property so if the owner's opportunity cost of capital is the monthly interest rate is r, then the total initial cost (including interest foregone) is $V^0(1+r)$. The benefit at the end of the month is V^1 so the net opportunity cost of living in the dwelling instead of investing the funds is the (end of period) *user cost value* equal to $V^0(1+r) - V^1$. Decomposing this value into constant quality price and quantity components involves some additional assumptions about the form of depreciation of the structure; see Jorgenson and Griliches (1967)[119], Christensen and Jorgenson (1969)[28] and Diewert (1974b)[41] for more details on this approach. There are additional complications due to the fact that the housing unit consists of separate land and structure components and the structure component depreciates while the land component does not. For a more complete discussion of these complications and references to the literature, see Chapter 10 in IMF (2025)[113]. We mention that Jorgenson and his coauthors have argued in favour of the user cost approach in the CPI to the valuation of *all* durable goods that are purchased by households.[*113]

- *The Opportunity Cost Approach.* This approach suggests that the rent foregone and the financial opportunity cost of tying up capital in an owner occupied housing unit are *both* opportunity costs of living in the house for the reference month and so the “true” opportunity cost of living in ones home is the maximum of the rental foregone and the user cost.[*114]

- *The Payments Approach.* This approach to pricing the services of an owner occupied dwelling unit is a kind of cash flow approach: it simply adds up the costs that are associated with living in the owned unit. These cash costs include mortgage interest on any loans that were made to purchase the unit and property taxes on the unit. This approach ignores imputed costs of ownership like depreciation on the structure and possible imputed benefits such as possible capital gains on the property that might have occurred over the reference period.

- *The Deprival Value Approach.* The above five approaches to valuing the services of an owner occupied dwelling unit are well known[*115] but this sixth approach is perhaps a new approach. This approach is based on the amount of money it would take to deprive the owner(s) of the services of the owned dwelling unit for the accounting period. The formal methodology that justifies this approach is exactly the same as the methodology that was developed to measure the money it would take to compensate a user of the services of a free product (like the use of the internet) for the loss (or deprival) of these services for a reference period.[*116] Conceptually this approach is appealing. A problem

[*113] See Jorgenson and Slesnick (1983)[120] (1984)[121].

[*114] This approach was suggested by Diewert (2008)[55] and modified by Diewert, Nakamura and Nakamura (2009)[68].

[*115] The five approaches are discussed (with references to the literature) in Chapter 10 of IMF (2025)[113].

[*116] See Brynjolfsson, Collis, Diewert, Eggers and Fox (2020)[20] (2025)[21] and Diewert, Fox and Schreyer (2022)[72]. In the first set of papers, an experimental economics approach was used to objectively

with the rental equivalence approach is that an owner of a highly valuable property may find that the market rent for the property is much lower[*117] than the owner's valuation of the benefit of living in the property. It is the latter value that measures the utility derived from the use of the housing unit for the reference period. The practical problem with this approach is the difficulty in obtaining objective measures of the deprival value.

What is the right price of household time?

The economic approach to the construction of a CPI is based on the assumption of households maximizing a utility function subject to a budget constraint. But consumers typically do not derive utility from *purchases* of goods and services; they need to combine their purchases with inputs of *time* using the services of the purchased products to derive overall utility. Thus overall utility is not just a function of the vector of purchases $\boldsymbol{q}$; it is a function of $\boldsymbol{q}$ and a vector of times spent on various household leisure and work activities. Thus households maximize utility subject to both a budget constraint and a time constraint. The pioneering work on integrating the time constraint with the budget constraint is Becker (1965)[15]. For additional work on Becker's theory of the allocation of time and household production, see Pollak and Wachter (1975)[141], Hill (2009)[112] and Schreyer and Diewert (2014)[148]. Becker (1965)[15] used household market wage rates to value household time. However, if all members of the household are retired or unemployed; then there is no market opportunity wage rate to value household time. Furthermore, even if there are market wage rates of household members to value time, it is not clear whether this opportunity cost wage rate should be used to value household work. Instead one could use market wage rates for hiring workers to do the various types of household work. These issues are discussed more fully in Schreyer and Diewert (2014)[148] and in Diewert, Fox and Schreyer (2017)[71].[*118] A related issue is whether household production should be included in a Consumer Price Index. This problem has been with us for a long time but it has assumed new importance with households installing solar panels and supplying electricity not only to themselves, but also to the transmission company that supplies electricity at peak times. How exactly should the purchase of solar panels be treated?

How can we measure the preferences of individual households?

The problem with individual household data is that many purchases are infrequent. This leads to a big lack of matching problem. However, for some recent progress on this problem of sparse data for individual households, see Crawford (2025)[33]. A related problem is: how should we aggregate over individual household CPIs to produce an overall Consumer Price Index? This leads us into the problems associated with the construction of indexes of Social Welfare. For an introduction to this literature, see Chapter 5 in IMF (2025)[113].

Unique products.

In different periods, different products are produced and purchased. This prevents the routine matching of prices. Most of the present paper attempts to deal with this problem but the methods that were suggested do not deal adequately with a service or product that changes in its quality every period so there is absolutely no matching of prices. An example of this extreme lack of matching occurs when we attempt to construct residential property price

determine deprival values for giving up the use of the internet for a month. It will be difficult to implement an experimental approach in the owned housing context.

[*117] Lower income renters cannot afford to pay a high rent for a high end property. Rich folks typically do not rent high end properties; they own them!

[*118] This paper also discusses the complex problems associated with the estimation of consumer preferences when there are two constraints.

indexes.[*119] Each property has a unique location but also the property has a unique structure which does not have the same quality over time; i.e., the structure depreciates. In this case, the matched product approaches that we discussed above cannot be applied; we need to apply hedonic regressions with *characteristics of the product* as explanatory variables rather than the product itself as an explanatory variable as in the Time Product Dummy regression approach described above. For explanations of this Time Product Characteristics approach and references to the literature, see Chapters 7 and 8 in IMF (2025)[113].[*120]

Complex products.

Many service products are very complicated; e.g., telephone and internet access service plans. It is probably necessary to use Time Product Characteristics hedonic regressions in this context.

Bundled products and discounts for volume purchases.

At times, two or more products are bundled together and offered at a discounted price as compared to the total price of the products if they were sold separately. Using the economic approach to index number theory, this situation can be modeled if preferences are known. Consider the household's utility maximization problem for a generic period. Divide the quantity vector into two vectors, $\boldsymbol{q}^1 \equiv [q_1, \ldots, q_M]$ and $\boldsymbol{q}^2 \equiv [q_{M+1}, q_{M+2}, \ldots, q_N]$. Let the corresponding price vectors for these two quantity vectors be $\boldsymbol{p}^1 \equiv [p_1, \ldots, p_M]$ and $\boldsymbol{p}^2 \equiv [p_{M+1}, p_{M+2}, \ldots, p_N]$. Then the household's utility maximization problem when there are no bundled products is the following problem, where the household utility function is $f(\boldsymbol{q}^1, \boldsymbol{q}^2)$ and period "income" or expenditure is $e > 0$:

$$\max_{q^1,q^2}\{f(\boldsymbol{q}^1, \boldsymbol{q}^2) : \boldsymbol{p}^1 \cdot \boldsymbol{q}^1 + \boldsymbol{p}^2 \cdot \boldsymbol{q}^2 \leq e\} \equiv u^*. \tag{131}$$

Suppose that the household also has the opportunity to buy a bundle of products 1 to M at a discounted price $p_{N+1} < \boldsymbol{p}^1 \cdot \boldsymbol{\alpha}$ where $\boldsymbol{\alpha} \equiv [\alpha_1, \ldots, \alpha_M]$ is the vector that describes how many units of products 1 to M are in the bundle; i.e., α_m is the number of units of product m that are in the bundle for $m = 1, \ldots, M$. The utility maximization problem where the household buys q_{N+1} units of the bundled product instead of purchasing individual units of products 1-M is the following constrained maximization problem:

$$\max_{q_{N+1},q^2}\{f(\boldsymbol{\alpha} q_{N+1}, \boldsymbol{q}^2) : p_{N+1} \cdot q_{N+1} + \boldsymbol{p}^2 \cdot \boldsymbol{q}^2 \leq e\} \equiv u^{**}. \tag{132}$$

The optimal decision for the household is to choose the option that leads to the highest utility level. Thus the optimal aggregate quantity level for the household is $Q \equiv \max\{u^*, u^{**}\}$ and the corresponding price level is $P \equiv e/Q$. The same methodology can be applied to discounts for volume purchases of an individual product. In this case, $M = 1$ and α_1 is the number of units of the product that must be purchased in order to get the discounted price. It can be seen that this approach to the treatment of bundled products is too complicated to be implemented by National Statistical Offices so simpler methodologies will have to be applied. One such simpler methodology would be to treat the bundle as an additional product and apply the usual multilateral approaches such as the Weighted Time Product Dummy regression method.

[*119] It is necessary to construct Residential Property Price indexes when implementing the user cost approach to the construction of price indexes for Owner Occupied Housing.

[*120] For applications of the Time Product Characteristics Approach to hedonic regressions in the Residential Property Index context, see de Haan and Diewert (2013)[36], Diewert, de Haan and Hendriks (2015)[63], Diewert and Shimizu (2015)[74] (2017)[75] (2022)[76] and Diewert and Huang (2025)[67].

How to value the household production of environmental "bads".

Households produce undesirable outputs (such as air pollution) as a byproduct of their household product of their consumption of various goods. If governments put a price on the production of household "bads", then this price and the associated quantity of the undesirable output could be treated as just another price and quantity that enter a Consumer Price Index. But this treatment of undesirable outputs seems counterintuitive: an increase in the "consumption" of a bad should not increase utility; it should decrease utility. A practical way of dealing with "bads" in a CPI has not been worked out to our knowledge. For an introduction to the treatment of this issue in the production accounts, see the controversy between Pittman (1983)[138] and Färe, Grosskopf, Lovell and Pasurka (1989)[87]. For a more general discussion of how to integrate the environment into the economic accounts, see Fleurbaey and Blanchet (2013)[95].

Heavily subsidized products.

In the limit, subsidized products can be supplied to consumers free of (explicit) charges. Is zero the "right" price for this type of product? The answer is no. We mentioned above the pioneering deprival value methodology*121 that could be used to obtain prices for free products. For free educational and medical services provided to households by governments, market prices for comparable privately supplied services could be used to value these free services.

Financial products.

What is the "correct" price of a household's monetary deposits? This question has not yet been resolved in a definitive manner. For various approaches to the treatment of financial services, see Barnett (1978)[13] (1980)[14], Feenstra (1986)[88], Fixler and Zieschang (1991)[94], Hill (1996b)[111], Diewert (2014)[58] and Diewert, Fixler and Zieschang (2016)[69] and the references in these papers.

Products involving risk and uncertainty.

What is the correct pricing concept for gambling and insurance expenditures? This question has assumed new importance with climate change leading to larger and larger insurance claims on property damage including residential housing. The treatment of insurance in a CPI is not at all settled. The gross insurance approach regards property insurance payments as the correct value for insurance in a CPI while the net insurance approach subtracts claims from gross insurance and regards the resulting value as the correct one. There is also controversy on how to treat changes in risk: does a change affect the price of insurance or the real quantity of insurance? The academic literature on the utility of insurance and gambling payments is not easy to explain to the public*122 and so National Statistical Offices have tended to use very simple approaches to the treatment of these expenditures in a CPI. There is a need for better approaches in this area.

Although the measurement challenges discussed in this paper arise in many national CPIs, their relative importance differs considerably across countries. For example, Statistics Canada continues to face long-standing difficulties with the treatment of owner-occupied housing and the integration of scanner data with traditional survey-based collection. Japan experiences exceptionally high rates of product churn in digital devices and household electronics, which

*121 See Brynjolfsson, Collis, Diewert, Eggers and Fox (2020)[20] (2025)[21] and Diewert, Fox and Schreyer (2022)[72].

*122 See Diewert (1993b)[48] (1995)[49] and the references to the academic literature.

makes price matching and quality adjustment particularly problematic. In the United States, major challenges include the extensive use of hedonic methods for consumer durables and the reliance on rental equivalence for measuring housing services. Australia has pioneered the implementation of multilateral index methods such as GEKS–Tornqvist, yet still confronts practical issues in real-time updating and the linking of rolling-window indexes. Many European statistical offices are rapidly expanding their use of web-scraped and big-data sources, generating new types of missing-price and classification problems. These country-specific examples illustrate how CPI theory and practice are shaped by local data environments and institutional settings, reinforcing the need for context-sensitive research and methodological development.

It can be seen that there are many challenges that need to be addressed.

6 Conclusion

This paper has reviewed the principal theoretical frameworks for constructing Consumer Price Indexes (CPIs) and clarified the challenges that arise when these frameworks are applied to contemporary data environments. The increasing availability of high frequency scanner data and web scraped information has revealed substantial discontinuities and dynamic patterns in consumer purchasing behavior that diverge sharply from the assumptions underlying much of traditional index number theory. Traditional index number theory has ignored the problems raised by missing prices and huge responses in quantities purchased in response to heavily discounted prices.

A central finding is that chain drift is not merely an artifact of a particular index formula, but rather a structural consequence of the dynamic nature of consumer behavior. When households stock up during sales and subsequently reduce purchases, chained indexes fail fundamental consistency conditions; i.e., when prices return to normal levels, a chained index does not revert back to the level that prevailed before the sale. Understanding and modeling this dynamic behavior—both theoretically and empirically—remains a key direction for future research.

The growing prevalence of product entry and exit in retail markets further complicates CPI measurement. Missing prices and unmatched products cannot be handled satisfactorily through simple carry forward procedures. Although Hicks–type reservation prices provide a theoretically coherent treatment of disappearing goods, their practical construction requires the estimation of inverse demand systems or specific parametric utility structures such as the CES or KBF utility functions, both of which pose substantial econometric challenges. Significant work remains to be done before such methods can be implemented reliably in official statistics.

Multilateral index methods such as GEKS and Similarity-Linked approaches offer important advantages in mitigating chain drift, but their real-time implementation presents additional difficulties. Because index levels must be recomputed whenever new periods are added, these methods are silent on how to update existing indexes to incorporate the new information. Although several splicing techniques have been proposed, including mean, movement, and window splices, consensus has not yet emerged on which approach best balances theoretical coherence with operational feasibility.

Further challenges arise for CES-based indexes such as the Lloyd-Moulton and Sato-Vartia formulas.[*123] While they mitigate substitution bias, their empirical implementation hinges

[*123] See Lloyd (1975)[128], Moulton (1996)[132], Sato (1976)[147] and Vartia (1976)[161].

critically on estimates of the elasticity of substitution. Estimates derived from CES unit cost functions often differ sharply from those obtained from CES utility functions, leading to substantial uncertainty for the value of the elasticity and, consequently, in the resulting index levels. The Sato-Vartia index is also theoretically close to the Fisher index in second-order approximation,[*124] limiting its practical distinctiveness. Moreover, applications such as Feenstra's (1994)[89] new goods welfare measurement methodology rely on externally imposed elasticity values, which may generate substantial overestimation of welfare gains. These issues underscore the need for a more rigorous evaluation of CES based indexes before they can be integrated into official statistical practice.

Finally, the integration of hedonic methods, scanner data, web-scraped prices, and machine learning techniques offers important opportunities for improving quality adjustment and the treatment of product turnover. However, the high dimensionality and heterogeneity of these data introduce challenges related to robustness, identifiability, and computational feasibility. Developing coherent frameworks that combine nonlinear hedonic methods, inverse demand estimation, and hybrid econometric structures represents a promising frontier, but one that requires sustained theoretical and empirical effort.

In sum, this paper has identified a series of open problems that must be addressed for CPI methodology to advance. Many of these challenges stem directly from the dynamic, heterogeneous, and rapidly evolving nature of modern consumer markets. By clarifying the theoretical inconsistencies and empirical limitations inherent in current practices, this study provides a foundation for developing next-generation price index methodologies that integrate richer data environments with structurally grounded economic models. Continued research along these lines will be essential for improving the accuracy, coherence, and relevance of CPIs in the years ahead.

References

[1] Afriat, S.N. (1972), "The Theory of International Comparisons of Real Income and Prices" pp. 13-69 in *International Comparisons of Prices and Outputs*, D.J. Daley (ed.), New York: Columbia University Press.

[2] Allen, R.G.C. (1938), *Mathematical Analysis for Economists*, London: Macmillan.

[3] Archibald, R.B. (1977), "On the Theory of Industrial Price Measurement: Output Price Indexes", *Annals of Economic and Social Measurement* 6, 57-72.

[4] Arrow, K.J., H.B. Chenery, B.S. Minhas and R.M. Solow (1961), "Capital-Labor Substitution and Economic Efficiency", *Review of Economics and Statistics* 63, 225-250.

[5] Auer, L. von (2001), "An Axiomatic Checkup for Price Indices", working Paper No. 1/2001, Faculty of Economics and Management, Otto von Guericke University Magdeburg, Postfach 4120, 39016 Magdeburg, Germany.

[6] Australian Bureau of Statistics (2016), "Making Greater Use of Transactions Data to Compile the Consumer Price Index," Information Paper 6401.0.60.003. [358,360,361]

[7] Balk, B.M. (1980), "A Method for Constructing Price Indices for Seasonal Commodities", *Journal of the Royal Statistical Society*, Series A 143, 68-75.

[8] Balk, B.M. (1985), "A Simple Characterization of Fisher's Price Index," *Statistische Hefte* 26, 59-63.

[9] Balk, B.M. (1995), "Axiomatic Price Index Theory: A Survey", *International Statistical Review* 63, 69-93.

[*124] See Diewert (1978; 887)[43].

[10] Balk, B.M. (1999), "On Curing the CPI's Substitution and New Goods Bias", Paper presented at the Fifth Meeting of the Ottawa Group, Reykjavik, Iceland, 1999.

[11] Balk, B.M.(2008), *Price and Quantity Index Numbers*, New York: Cambridge University Press.

[12] Balk, B.M. (2024), "Extension Methods for Multilateral Price Indices and the Return of Chain Drift", Paper presented at the UNSW-ESCoE Conference on Economic Measurement 2024, University of New South Wales Sydney, November 25-26.

[13] Barnett, W. (1978), "The User Cost of Money", *Economics Letters*, 2, 145-149.

[14] Barnett, W.A., (1980), "Economic Monetary Aggregates: An Application of Aggregation and Index Number Theory", *Journal of Econometrics* 14, 11-48.

[15] Becker, G.S. (1965), "A Theory of the Allocation of Time", *The Economic Journal* 75, 493-517.

[16] Bennet, T.L. (1920), "The Theory of Measurement of Changes in Cost of Living", *Journal of the Royal Statistics Society* 83, 455-462.

[17] Blackorby, C. (1975), "Degrees of Cardinality and Aggregate Partial Orderings", *Econometrica* 43, 845-852.

[18] Bowley, A.L. (1901), *Elements of Statistics*, Westminster: P.S. King and Son.

[19] Bridgeman, P.W. (1922), *Dimensional Analysis*, New Haven: Yale University Press.

[20] Brynjolfsson, E., A. Collis, W.E. Diewert, F. Eggers and K.J. Fox (2020), "Measuring the Impact of Free Goods on Real Household Consumption", *American Economic Association Papers and Proceedings* 110, 25-30.

[21] Brynjolfsson, E., A. Collis, W.E. Diewert, F. Eggers and K.J. Fox (2025), "GDP-B: Accounting for the Value of New and Free Goods in the Digital Economy", *American Economic Journal: Macroeconomics* 17, 312-344.

[22] Carli, G.-R., (1804), "Del valore e della proporzione de' metalli monetati", pp. 297 366 in *Scrittori classici italiani di economia politica*, Volume 13, Milano: G.G. Destefanis (originally published in 1764).

[23] Caves D.W., Christensen, L.R. and Diewert, W.E. (1982a), "Multilateral Comparisons of Output, Input, and Productivity using Superlative Index Numbers", *Economic Journal* 92, 73-86.

[24] Caves, D.W., L.R. Christensen and W.E. Diewert (1982b), "The Economic Theory of Index Numbers and the Measurement of Input, Output, and Productivity," *Econometrica* 50, 1392-1414.

[25] Clements, K.W., H.Y. Izan and E.A. Selvanathan (2006), "Stochastic Index Numbers: A Review", *International Statistical Review* 74, 235-270.

[26] Chessa, A.G. (2016), "A New Methodology for Processing Scanner Data in the Dutch CPI", *EURONA* 1, 49–69.

[27] Chessa, A.G. (2021), "Extension of Multilateral Index Series over Time: Analysis and Comparison of Methods, United Nations ECE/ILO Meeting of the Group of Experts on Consumer Price Indices, Geneva, 2-10 June 2021.

[28] Christensen, L.R., and D.W. Jorgenson (1969), "The Measurement of U.S. Real Capital Input, 1929–1967", *Review of Income and Wealth* 15(4): 293–320.

[29] Christensen, L.R., D.W. Jorgenson and L.J. Lau (1971), "Conjugate Duality and the Transcendental Logarithmic Production Function," *Econometrica* 39, 255 256.

[30] Christensen, L.R., D.W. Jorgenson and L.J. Lau (1975), "Transcendental Logarithmic Utility Functions", American Economic Review 65, 367-383.

[31] Cobb, Charles W. and Paul H. Douglas (1928), "A Theory of Production," *American Economic Review* 18(1): 139-165.

[32] Court, A.T. (1939), "Hedonic Price Indexes with Automotive Examples", pp. 99-117 in *The Dynamics of Automobile Demand*, New York: General Motors Corporation.

[33] Crawford, I. (2025), "Inverse Demand Functions and the Missing Prices Problem", paper presented at the 10th Annual Conference of The Society for Economic Measurement, Athens, Greece, July 10-12.

[34] de Haan, J. (2008), "Reducing Drift in Chained Superlative Price Indexes for Highly Disaggregated Data", paper presented at the Economic Measurement Workshop, Centre for Applied Economic Research, University of New South Wales, December 10.

[35] de Haan, J. (2015), "Rolling Year Time Dummy Indexes and the Choice of Splicing Method", paper presented at the 14th Meeting of the Ottawa Group, Tokyo, Japan. website: `https://stats.unece.org/ottawagroup/meeting/14`

[36] de Haan, J. and W.E. Diewert (eds.) (2013), *Eurostat Handbook on Residential Property Price Indices (RPPIs)*, Eurostat European Commission, Luxembourg: Publications Office of the European Union.

[37] de Haan, W.E. Diewert and R. Hendriks (2015), "Hedonic Regressions and the Decomposition of a House Price Index into Land and Structure Components", *Econometric Reviews* 34, 106-126.

[38] de Haan, J. and H. van der Grient (2011), "Eliminating Chain Drift in Price Indexes Based on Scanner Data", *Journal of Econometrics* 161, 36-46.

[39] de Haan, J. and F. Krsinich (2024), "Product Churn and the GEKS-Törnqvist Price Index: The Feenstra Adjustment", paper presented at the 18th Meeting of the Ottawa Group, Ottawa, Canada. website: `https://stats.unece.org/ottawagroup/meeting/18`

[40] Diewert, W.E. (1974a), "Applications of Duality Theory," pp. 106-171 in M.D. Intriligator and D.A. Kendrick (ed.), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam: North-Holland.

[41] Diewert, W.E. (1974b), "Intertemporal Consumer Theory and the Demand for Durables." *Econometrica* 42: 497–516.

[42] Diewert, W.E. (1976), "Exact and Superlative Index Numbers", *Journal of Econometrics* 4, 114-145.

[43] Diewert, W.E. (1978), "Superlative Index Numbers and Consistency in Aggregation", *Econometrica* 46, 883-900.

[44] Diewert, W.E. (1980), "Aggregation Problems in the Measurement of Capital", pp. 433-528 in *The Measurement of Capital*, Dan Usher (ed.), Studies in Income and Wealth, Vol. 45, National Bureau of Economics Research, Chicago: University of Chicago Press.

[45] Diewert, W.E. (1983), "The Theory of the Output Price Index and the Measurement of Real Output Change", pp. 1049-1113 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Statistics Canada, Ottawa Canada.

[46] Diewert, W.E. (1992), "Fisher Ideal Output, Input and Productivity Indexes Revisited", *Journal of Productivity Analysis* 3, 211-248.

[47] Diewert, W.E. (1993a), "The Early History of Price Index Research", pp. 33-65 in *Essays in Index Number Theory*, Volume 1, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

[48] Diewert, W.E. (1993b), "Symmetric Means and Choice under Uncertainty", pp. 355-433 in *Essays in Index Number Theory*, Volume 1, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

[49] Diewert, W.E. (1995), "Functional Form Problems in Modeling Insurance and Gambling", *The Geneva Papers on Risk and Insurance Theory* 20, 135-150.

[50] Diewert, W.E. (1997), "Commentary on Mathew D. Shapiro and David W. Wilcox: Alternative Strategies for Aggregating Prices in the CPI", *The Federal Reserve Bank of St. Louis Review*, Vol. 79:3, 127-137.

[51] Diewert, W.E. (1999), "Axiomatic and Economic Approaches to International Compar-

isons", 13-87 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Chicago: The University of Chicago Press.

[52] Diewert, W. E. (2002), "The Quadratic Approximation Lemma and Decompositions of Superlative Indexes", *Journal of Economic and Social Measurement* 28, 63-88.

[53] Diewert, W.E. (2004), "On the Stochastic Approach to Linking the Regions in the ICP", Discussion Paper 04-16, Department of Economics, The University of British Columbia, Vancouver, Canada.

[54] Diewert, W.E. (2005), "Weighted Country Product Dummy Variable Regressions and Index Number Formulae", *Review of Income and Wealth* 51, 561-570.

[55] Diewert, W.E. (2008), "OECD Workshop on Productivity Analysis and Measurement: Conclusions and Future Directions", pp. 11-36 in *Proceedings from the OECD Workshops on Productivity Measurement and Analysis*, Paris: OECD.

[56] Diewert, W.E. (2010), "On the Stochastic Approach to Index Numbers", pp. 235-262 in *Price and Productivity Measurement*, W.E. Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox and Alice O. Nakamura (eds.), Trafford Press.

[57] Diewert, W.E. (2012), *Consumer Price Statistics in the UK*, Government Buildings, Cardiff Road, Newport, UK, NP10 8XG: Office for National Statistics.

[58] Diewert, W.E. (2014), "The Treatment of Financial Transactions in the SNA: A User Cost Approach", *Eurostat Review of National Accounts and Macroeconomic Indicators* I, 73-89.

[59] Diewert, W.E. (2020), "The Economic Approach to Index Number Theory", Discussion Paper 20-05, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada V6T 1L4.

[60] Diewert, W.E. (2023), "Scanner Data, Elementary Price Indexes and the Chain Drift Problem", pp. 445-606 in *Advances in Economic Measurement*, D. Chotikapanich, A.N. Rambaldi and N. Rhode (eds.), Singapore: Palgrave Macmillan.

[61] Diewert, W.E. (2024), "The Ottawa Group After 30 Years", Discussion Paper 24-02, Vancouver School of Economics, University of British Columbia, Vancouver B.C., Canada V6T 1L4.

[62] Diewert, W.E., N. Abe, A. Tonogi and C. Shimizu (2025), "Scanner Data and the Construction of Inter-Regional Price Indexes", *Review of Income and Wealth* 71:4, accepted September 1, 2025.

[63] Diewert, W.E., J. de Haan and R. Hendriks (2015), "Hedonic Regressions and the Decomposition of a House Price index into Land and Structure Components", *Econometric Reviews* 34, 106-126.

[64] Diewert, W.E. and R. Feenstra (2019), "Estimating the Benefits of New Products: Some Approximations", Discussion Paper 19-02, Vancouver School of Economics, University of British Columbia, Vancouver, B.C., Canada, V6T 1L4.

[65] Diewert, W.E. and R.C. Feenstra (2022), "Estimating the Benefits of New Products", pp. 437-473 in *Big Data For the Twenty-First Century Economic Statistics*, K.G. Abraham, R.S. Jarmin, B.C. Moyer and M.D. Shapiro (eds.), Chicago: University of Chicago Press.

[66] Diewert, W.E. and R.J. Hill (2010), "Alternative Approaches to Index Number Theory", pages 263-278 in *Price and Productivity Measurement*, W.E. Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox and Alice O. Nakamura (eds.), Victoria Canada: Trafford Press.

[67] Diewert, W.E. and N. Huang (2025), "Decomposing Residential Resale House Prices into Structure and Land Components", *Journal of Official Statistics*, published online July 12, 2025.

[68] Diewert, W.E., A.O. Nakamura and L.I Nakamura (2009),"The Housing Bubble and a New Approach to Accounting for Housing in a CPI", *Journal of Housing Economics*

18:3, 156-171.

[69] Diewert, W.E., D. Fixler and K. Zieschang (2016), "Problems with the Measurement of Banking Services in a National Accounting Framework", pp. 117-176 in *National Accounting and Economic Growth*, J.M. Hartwick (ed.), Cheltenham UK: Edward Elgar Publishing.

[70] Diewert, W.E. and K.J. Fox (2022), "Substitution Bias in Multilateral Methods for CPI Construction", *Journal of Business & Economic Statistics* 40:1, 355–369.

[71] Diewert, W.E., K.J. Fox and P. Schreyer (2017), "The Allocation and Valuation of Time", Discussion Paper 17-04, School of Economics, University of British Columbia, Vancouver, B.C., Canada, V6N 1Z1.

[72] Diewert, W.E., K.J. Fox and P. Schreyer (2022), "Experimental Economics and the New Commodities Problem," *Review of Income and Wealth* 68, 895-905.

[73] Diewert, W.E. and C.J. Morrison (1986), "Adjusting Output and Productivity Indexes for Changes in the Terms of Trade", *The Economic Journal* 96, 659-679.

[74] Diewert, W.E. and C. Shimizu (2015), "Residential Property Price Indexes for Tokyo", *Macroeconomic Dynamics*, Volume 19, 1659-1714.

[75] Diewert, W.E. and C. Shimizu (2017), "Hedonic Regression Models for Tokyo Condominium Sales", *Regional Science and Urban Economics* 60, 300-315.

[76] Diewert, W.E. and C. Shimizu (2022), "Residential Property Price Indexes: Spatial Coordinates versus Neighbourhood Dummy Variables", *Review of Income and Wealth* 68:3, 770-796.

[77] Diewert, W.E. and C. Shimizu (2023), "Scanner Data, Product Churn and Quality Adjustment", Discussion Paper 23-06, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada, V6T 1L4.

[78] Diewert, W.E. and T.J. Wales (1988), "A Normalized Quadratic Semiflexible Functional Form", *Journal of Econometrics* 37, 327-42.

[79] Diewert, W.E. and V. Zelenyuk (2025), "On the Problem of the Purchasing Power of Money by A.A. Konüs and S.S. Byushgens: Translation and Commentary", *Journal of Productivity Analysis* 63, 107–132.

[80] Dikhanov, Y. (2025), "Fisher versus Törnqvist when Relative Price Changes are Large", World Bank, unpublished paper.

[81] Drobisch, M. W. (1871), "Ueber die Berechnung der Veränderungen der Waarenpreise und des Geldwerths", *Jahrbücher für Nationalökonomie und Statistik* 16, 143-156.

[82] Edgeworth, F.Y. (1888), "Some New Methods of Measuring Variation in General Prices", *Journal of the Royal Statistical Society* 51, 346-368.

[83] Edgeworth, F.Y. (1925), *Papers Relating to Political Economy*, Volume 1, New York: Burt Franklin.

[84] Eichhorn, W. (1978), *Functional Equations in Economics*, London: Addison-Wesley.

[85] Eichhorn, W. and J. Voeller (1976), *Theory of the Price Index*, Lecture Notes in Economics and Mathematical Systems, Vol. 140, Berlin: Springer-Verlag.

[86] Eltetö, Ö., and Köves, P. (1964), "On a Problem of Index Number Computation Relating to International Comparisons", (in Hungarian), *Statisztikai Szemle* 42, 507-518.

[87] Färe, R., S. Grosskopf, K.A.C. Lovell and C. Pasurka (1989), "Multilateral Productivity Comparisons when Some Outputs are Undesirable: A Nonparametric Approach", *Review of Economics and Statistics* 71, 90-98.

[88] Feenstra, Robert C. (1986), "Functional Equivalence between Liquidity Costs and the Utility of Money", *Journal of Monetary Economics* 17:2, 271-291.

[89] Feenstra, R.C. (1994), "New Product Varieties and the Measurement of International Prices", *American Economic Review* 84:1, 157-177.

[90] Fisher, I. (1911), *The Purchasing Power of Money*, London: Macmillan.

[91] Fisher, I. (1922), *The Making of Index Numbers*, Boston: Houghton-Mifflin.
[92] Fisher, W.C. (1913), "The Tabular Standard in Massachusetts History," *Quarterly Journal of Economics* 27, 417-451.
[93] Fisher, F.M. and K. Shell (1972), "The Pure Theory of the National Output Deflator". pp.49-113 in *The Economic Theory of Price Indexes* by F.M. Fisher and K. Shell, New York: Academic Press.
[94] Fixler, D. and K.D. Zieschang (1991), "Measuring the Nominal Value of Financial Services in the National Income Accounts", *Economic Inquiry*, 29, 53-68.
[95] Fleurbaey, M. and D. Blanchet (2013), *Beyond GDP: Measuring Welfare and Assessing Sustainability*, New York: Oxford University Press.
[96] Fox, K.J. (2025), "The Curious Case of an Exploding GEKS Index", Paper presented at the UNSW-ESCoE Conference on Economic Measurement 2025, University of New South Wales Sydney, December 8-9.
[97] Fox, K.J., P. Level and M. O'Connell (2025), "Multilateral Index Number Methods for Consumer Price Statistics", *Journal of Business & Economic Statistics*, Published online: October 2025.
[98] Frisch, R. (1930), "Necessary and Sufficient Conditions Regarding the Form of an Index Number which Shall Meet Certain of Fisher's Tests", *Journal of the American Statistical Association* 25, 397- 406.
[99] Frisch, R. (1936), "Annual Survey of General Economic Theory: The Problem of Index Numbers", *Econometrica* 4, 1-39.
[100] Funke, H. and J. Voeller (1978), "A Note on the Characterization of Fisher's Ideal Index," pp. 177-181 in *Theory and Applications of Economic Indices*, W. Eichhorn, R. Henn, O. Opitz and R.W. Shephard (eds.), W • zburg: Physica Verlag.
[101] Geary, R.G. (1958), "A Note on Comparisons of Exchange Rates and Purchasing Power between Countries", *Journal of the Royal Statistical Society Series A* 121, 97-99.
[102] Gini, C. (1924), "Quelques considérations au sujet de la construction des nombres indices des prix et des questions analogues", *Metron* 4:1, 3-162.
[103] Gini, C. (1931), "On the Circular Test of Index Numbers", *Metron* 9:9, 3-24.
[104] Gorman, W.M. (1953). "Community Preference Fields", *Econometrica* 21, 63-80.
[105] Green, H.A.J. (1964), *Aggregation in Economic Analysis: An Introductory Survey*, Princeton, N. J.: Princeton University Press.
[106] Hardy, G.H., J.E. Littlewood and G. Polya (1934), *Inequalities*, Cambridge: Cambridge University Press.
[107] Hicks, J.R. (1940), "The Valuation of the Social Income", *Economica* 7, 105-124.
[108] Hicks, J.R. (1946), *Value and Capital*, Second Edition, Oxford: Clarendon Press.
[109] Hill, T.P. (1988), "Recent Developments in Index Number Theory and Practice", *OECD Economic Studies* 10, 123-148.
[110] Hill, P. (1996a), *Inflation Accounting*, Paris: OECD.
[111] Hill, P. (1996b), "The Services of Financial Intermediaries or FISIM Revisited", paper presented at the Joint UNECE/Eurostat/OECD Meeting on National Accounts, Geneva, April 30-May 3.
[112] Hill, P. (2009), "Consumption of Own Production and Cost of Living Indices", pp. 429-444 in W.E. Diewert, J. Greenlees and C. Hulten (eds.), *Price and Productivity Measurement*; Chicago: University of Chicago Press.
[113] IMF, ILO, Eurostat, UNECE, OECD, World Bank, IMF (2025), *Consumer Price Index Manual: Theory 2025*, Washington DC: International Monetary Fund.
[114] Inklaar, R. and W.E. Diewert (2016), "Measuring Industry Productivity and Cross-Country Productivity Convergence", *Journal of Econometrics* 191, 426-433.
[115] Ivancic, L., W.E. Diewert and K.J. Fox (2009), "Scanner Data, Time Aggregation and

the Construction of Price Indexes", Discussion Paper 09-09, Department of Economics, University of British Columbia, Vancouver, Canada.

[116] Ivancic, L., W.E. Diewert and K.J. Fox (2011), "Scanner Data, Time Aggregation and the Construction of Price Indexes", *Journal of Econometrics* 161, 24-35.

[117] Jevons, W.S., (1865), "The Variation of Prices and the Value of the Currency since 1782", *Journal of the Statistical Society of London* 28, 294-320; reprinted in *Investigations in Currency and Finance* (1884), London: Macmillan and Co., 119-150.

[118] Jevons, W.S., (1884), "A Serious Fall in the Value of Gold Ascertained and its Social Effects Set Forth (1863)", pp. 13-118 in *Investigations in Currency and Finance*, London: Macmillan and Co.

[119] Jorgenson, D.W. and Z. Griliches (1967), "The Explanation of Productivity Change", *Review of Economic Studies* 34, 249–283.

[120] Jorgenson, D.W. and D.T. Slesnick (1983), "Individual and Social Cost of Living Indexes" pp. 241-323 in *Price Level Measurement* , W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

[121] Jorgenson, D.W. and D.T. Slesnick (1984). "Inequality in the Distribution of Individual Welfare", pp. 67-130 in *Advances in Econometrics* , R.L. Basmann and G. Rhodes (eds.), Greenwich: JAI Press.

[122] Keynes, J.M. (1930), *Treatise on Money*, Vol. 1, London: Macmillan.

[123] Khamis, S.H. (1970), "Properties and Conditions for the Existence of a New Type of Index Number", *Sankhya* B 32, 81-98.

[124] Konüs, A.A. (1924), "The Problem of the True Index of the Cost of Living", translated in *Econometrica* 7, (1939), 10-29.

[125] Konüs, A.A. and S.S. Byushgens (1926), "K probleme pokupatelnoi cili deneg", *Voprosi Konyunkturi* 2, 151-172.

[126] Krsinich, F. (2016), "The FEWS Index: Fixed Effects with a Window Splice", *Journal of Official Statistics* 32:2, 375-404.

[127] Laspeyres, E. (1871), "Die Berechnung einer mittleren Waarenpreissteigerung", *Jahrbücher für Nationalökonomie und Statistik* 16, 296-314.

[128] Lloyd, P.J. (1975), "Substitution Effects and Biases in Nontrue Price Indices", *American Economic Review* 65, 301-313.

[129] Lowe, J. (1823), *The Present State of England in Regard to Agriculture, Trade and Finance*, second edition, London: Longman, Hurst, Rees, Orme and Brown.

[130] Marshall, A. (1887), "Remedies for Fluctuations of General Prices', *Contemporary Review* 51, 355-375.

[131] Melser, D. (2006), "Accounting for the Effects of New and Disappearing Goods using Scanner Data", *Review of Income and Wealth* 52:4, 547-568.

[132] Moulton, B.R. (1996), "Constant Elasticity Cost-of-Living Index in Share Relative Form", Bureau of Labor Statistics, Washington D.C., December.

[133] Paasche, H. (1874), "Über die Preisentwicklung der letzten Jahre nach den Hamburger Borsennotirungen", *Jahrbücher für Nationalökonomie und Statistik* 12, 168-178.

[134] Persons, W.M. (1921), "Fisher's Formula for Index Numbers", *Review of Economics and Statistics* 3:5, 103-113.

[135] Persons, W.M. (1928), "The Effect of Correlation Between Weights and Relatives in the Construction of Index Numbers", *The Review of Economics and Statistics* 10:2, 80-107.

[136] Pierson, N.G. (1895), "Index Numbers and Appreciation of Gold", *Economic Journal* 5, 329-335.

[137] Pierson, N.G. (1896), "Further Considerations on Index-Numbers," *Economic Journal* 6, 127-131.

[138] Pittman, R.W. (1983), "Multilateral Productivity Comparisons with Undesirable Out-

puts", *Economic Journal* 93, 883-891.
[139] Pollak, R.A. (1971), "The Theory of the Cost of Living Index", Research Discussion Paper No. 11, Office of Prices and Living Conditions, US Bureau of Labor Statistics, Washington D.C.
[140] Pollak, R.A. (1989), *The Theory of the Cost-of-Living Index*, Oxford: Oxford University Press.
[141] Pollak, R.A. and M.L. Wachter (1975), "The Relevance of the Household Production Function and Its Implications for the Allocation of Time", *Journal of Political Economy* 83, 255-277.
[142] Rao, D.S. Prasada (1995), "On the Equivalence of the Generalized Country-Product-Dummy (CPD) Method and the Rao-System for Multilateral Comparisons", Working Paper No. 5, Centre for International Comparisons, University of Pennsylvania, Philadelphia.
[143] Rao, D.S. Prasada (2004), "The Country-Product-Dummy Method: A Stochastic Approach to the Computation of Purchasing Power parities in the ICP", paper presented at the SSHRC Conference on Index Numbers and Productivity Measurement, June 30-July 3, 2004, Vancouver, Canada.
[144] Rao, D.S. Prasada (2005), "On the Equivalence of the Weighted Country Product Dummy (CPD) Method and the Rao System for Multilateral Price Comparisons", *Review of Income and Wealth* 51:4, 571-580.
[145] Samuelson, P.A. (1974), "Complementarity—An Essay on the 40th Anniversary of the Hicks-Allen Revolution in Demand Theory", *The Journal of Economic Literature* 12, 1255-1289.
[146] Samuelson, P.A. and S. Swamy (1974), "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis", *American Economic Review* 64, 566-593.
[147] Sato, K. (1976), "The Ideal Log-Change Index Number", *Review of Economics and Statistics* 58, 223-228.
[148] Schreyer, P. and W.E. Diewert (2014), "Household Production, Leisure, and Living Standards" pp. 89-114 in *Measuring Economic Sustainability and Progress*, D.W. Jorgenson, J.S. Landefeld and P. Schreyer (eds.), Chicago: University of Chicago Press, (2014), pp. 89-114.
[149] Selvanathan, E.A. and D.S. Prasada Rao (1994), *Index Numbers: A Stochastic Approach*, Ann Arbor: The University of Michigan Press.
[150] Shephard, R.W. (1953), *Cost and Production Functions*, Princeton: Princeton University Press.
[151] Shephard, R.W. (1970), Theory of Cost and Production Functions, Princeton: Princeton University Press.
[152] Sidgwick, H. (1883), *The Principles of Political Economy*, London: Macmillan.
[153] Summers, R. (1973), "International Comparisons with Incomplete Data", *Review of Income and Wealth* 29:1, 1-16.
[154] Szulc, B.J. (1964), "Indices for Multiregional Comparisons", (in Polish), *Przeglad Statystyczny* 3, 239-254.
[155] Szulc, B.J. (1983), "Linking Price Index Numbers," pp. 537-566 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.
[156] Szulc, B.J. (1987), "Price Indices below the Basic Aggregation Level", *Bulletin of Labour Statistics* 2, 9-16.
[157] Theil, H. (1967), *Economics and Information Theory*, Amsterdam: North-Holland Publishing.
[158] Törnqvist, L. (1936), "The Bank of Finland's Consumption Price Index", *Bank of Finland Monthly Bulletin* 10, 1-8.

[159] Törnqvist, L. and E. Törnqvist (1937), Vilket är förhållandet mellan finska markens och svenska kronans köpkraft?", *Ekonomiska Samfundets Tidskrift* 39, 1-39 reprinted as pp. 121-160 in *Collected Scientific Papers of Leo Törnqvist*, Helsinki: The Research Institute of the Finnish Economy, 1981.

[160] Uzawa, H. (1962), "Production Functions with Constant Elasticicities of Substitution", *Review of Economic Studies* 29, 291-299.

[161] Vartia, Y. O. (1976), "Ideal Log Change Index Numbers", *Scandinavian Journal of Statistics* 3, 121-126.

[162] Vogt, A. (1980), "Der Zeit und der Faktorumkehrtest als 'Finders of Tests"', Statistische Hefte 21, 66-71.

[163] Vogt, A. and J. Barta (1997), *The Making of Tests for Index Numbers*, Heidelberg: Physica-Verlag.

[164] Walsh, C.M. (1901), *The Measurement of General Exchange Value*, New York: Macmillan and Co.

[165] Walsh, C.M. (1921a), *The Problem of Estimation*, London: P.S. King & Son.

[166] Walsh, C. M. (1921b), "Discussion", *Journal of the American Statistical Association* 17, 537-544.

[167] Westergaard, H. (1890), *Die Grundzüge der Theorie der Statistik*, Jena: Fischer.

[168] Wold, H. (1953), *Demand Analysis*, New York: John Wiley.

[169] Zhang, L.-C., I. Johansen and R. Nygaard (2019), "Tests for Price Indices in a Dynamic Item Universe", *Journal of Official Statistics* 35:3, 683–697.