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Partial Memories, Inductively Derived Views, and their Interactions with Behavior*

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Abstract

We explore the inductively derived views obtained by players with partial temporal (short-term) memories. A player derives his personal view of the objective game situation from his accumulated (long-term) memories, and uses it for decision making in the objective situation. A salient feature that distinguishes this paper from others on inductive game theory is partiality of a memory function of a player. This creates a multiplicity of possibly derived views. Although this is a difficulty for a player in various senses, it is an essential problem of induction. Faced with multiple possible views, a player may try to resolve this multiplicity using further experiences. This is a two-way interaction between behavior and personal views, which is another distinguishing feature of the present paper.

JEL Classification Number: C70, C72, C79

Key Words: Inductive Game Theory, Partial Memory, Inductive Derivation, Mike's Bike

1. Introduction

1.1. Backgrounds

Game theory and economics are experiential sciences about individual decisions and behavior in social contexts. However, these disciplines have by-passed the experiential

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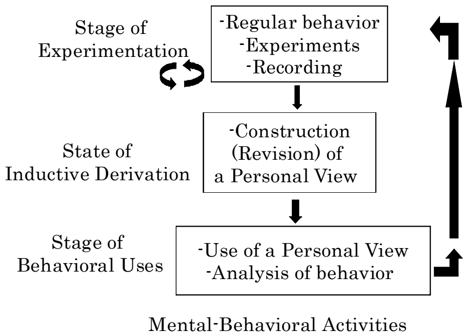

Figure 1.1: From Experimatations to Behavioral Uses

side of the beliefs/knowledge of a player by taking them for granted. As a result, these disciplines are silent about the questions of where basic beliefs come from and how they emerge and change with time[1]. Kaneko-Matsui [12] found this issue and touched it in the context of discrimination and prejudices. Anticipating vast developments, they called the resulting theory *inductive game theory.*

When we dig deeper, many different and untouched aspects are revealed with great potential for further explorations. Kaneko-Kline [9] synthesized these aspects into a skeleton called a *basic scenario.* The basic scenario moves from experimentations to the inductive derivations of personal views, to behavioral uses and further experimentations, and begins the cycle again, as depicted in Fig.1.1. The synthesis reveals a clear-cut skeleton, while sacrificing a lot of details. Kaneko-Kline [8], [10] and Akiyama-Ishikawa-Kaneko-Kline [1] focussed on those details in different parts of the scenario.

In this paper, we continue our exploration from the basic scenario, but now we deal with the case of partial short-term memories and explore reciprocal effects between memories, views, and behavior. We start with the assumption that a player has a weak memory ability, and then define a weakened form of an inductively derived personal view. By doing so, we are able to treat more substantive methods of induction than

[1] In the game theory literature, various approaches appear to be related to ours, e.g., the repeated game approach, the evolutionary game theory approach and behavioral economics. In *ex ante* game theory, behavior results from sophisticated decision-making based on a granted view of the game itself. The repeated game approach (cf. Hart [5]) effectively follows this idea, though the interpretation associated with it may often differ. In evolutionary game theory (cf., Weibull [19]) and behavioral economics (cf. Camerer, [3]), behavior is described by a specified (stochastic or non-stochastic) process within the game itself but without players thinking about the game, and limit behavior is typically analyzed. None of these approaches deals with the origin/emergence of basic beliefs/knowledge.

what we captured in our previous works[2].

With a broader notion of induction, we meet a multiplicity of personal views. We give some uniqueness result, but this should be regarded just as a reference point. In the uniqueness case, the inductive method can be summarized as a mechanical algorithm. In the multiplicity case, we find that a variety of inductive methods may be used with different resulting views. This variety may reflect individual differences in cognitive abilities and propensities. Multiplicity rather than uniqueness has a greater potential to explicate the multitude of different and conflicting views observed in society.

As in Kaneko-Kline [9], this paper covers a long scenario, though this paper discusses more details in each step than in [9]. It would still be inappropriate to only focus on a single theorem in isolation. The contents with theorems need to be taken collectively in order to grasp the full import. For the reader's sake, a summary of important features and results will be given in Sections 1.3 and 9.

1.2. Developing Inductive Game Theory

When a game theorist hears about a development of a new theory, he will likely ask what kind of equilibrium/solution will be proposed and/or justified. Our question does not take such forms, since we do not aim to explore foundations of the extant equilibrium and/or solution concepts. Our primary focus is on the emergence of a player's beliefs/knowledge in a social context, its behavioral consequences, and their reciprocal effects.

The change in focus forces us to rethink or modify even very basic notions such as "information" in game theory. In the standard formulation of an extensive game of von Neumann-Morgenstern [20] and Kuhn [13], information is expressed as a set of possibilities in the form of an "information set". However, at a more basic level, information may be described as a collection of facts or data expressed symbolically. We take the interpretation that information is transmitted and received in symbolic pieces. These pieces and the stored memories of them become the building blocks for the beliefs/knowledge of a player.

Treating information as pieces fits nicely into the context of inductive game theory: Players experience some parts of the game as they play it, and each may perceive and interpret those pieces of information in his own way. To describe individual differences in perception and storing of information, we introduce a memory function for a player.

[2] Induction here is closer to the induction by Bacon [2] than that of Hume [6] based on similarity. Also, biology has a similar aspect of induction. A book review by A. C. Love on Hall [4] describes it as an analogy to a jigsaw puzzle: "The completion of a jigsaw puzzle brings tremendous satisfaction; however, a few missing pieces lead to considerable frustration. Having the intended picture of a puzzle on the container contributes to the satisfaction (or the frustration). How do you know if you have all the pieces? ... Such is the lot of biologists attempting to explain key evolutionary transitions in the history of life" (*Science* 317, 17, Sept.2007).

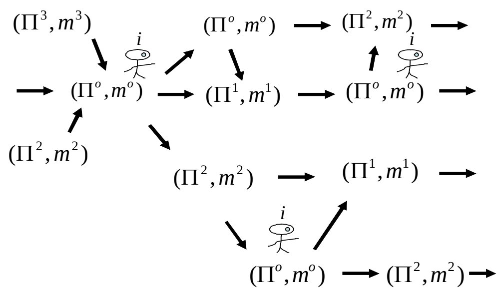

Figure 1.2: Various Social Situations

This additional structure allows us to distinguish between the information to be received in a play and his memories of them. He uses the latter to form his beliefs/knowledge.

For the formation of a player's view based on his memories, we found in Kaneko-Kline [9] that the standard notion of an extensive game needed to be weakened. Since an extensive game consists of hypothetical nodes and branches, it becomes cumbersome for subjective personal views and their derivations. To avoid this, Kaneko-Kline [10] developed the theory of *"information protocols"*, based on information pieces and actions as primitives. It describes more directly a target situation than the theory of extensive games, in that it skips hypothetical nodes and branches. It also takes a simple axiomatic form and can easily distinguish between the objective situation and subjective view. In comparisons of possible views, an advantage of the theory of information protocols is manifested, which will briefly be mentioned in Section 5.1.

In this paper, information protocols are adopted to express target social situations as well as subjective personal views. The entire social system is described in Fig.1.2, where various partial social situations are entangled. We are interested in one particular target situation such as $(\Pi^o, \mathfrak{m}^o)$, which consists of an information protocol Π^o together with a profile of *memory functions* $\mathfrak{m}^o = (\mathfrak{m}_1^o, ..., \mathfrak{m}_n^o)$. Playing this situation from time to time, a player accumulates experiences, and then constructs his personal view from them. This subjective view is also described by an information protocol.

A memory function $\mathfrak{m}_i^o$ for player i is a structure additional to an information protocol Π^o. In the standard literature of game theory, information sets have both roles of information transmission and individual memory. For inductive game theory, we need to separate memories from information transmission. Information pieces play the role of information transmission, and a memory function describes individual local (short-term) memory. This separation will be discussed in Section 3.

A remark related to this separation is on Kuhn's [13] "perfect-recall" condition on information sets. In our context, this can be reformulated as a condition on information pieces and individual histories, which we do not interpret as expressing the memory ability of a player. We call it the *distinguishability condition*, which is shown in Section 6 as a sufficient condition for the existence of a unique smallest view.

1.3. The Steps of this Paper and Some Results

Since this paper has various steps, here we give a small summary of them. In Section 9, the contents and results obtained in this paper will be summarized along these steps.

Step 1: In Section 3, we will give the definitions of an information protocol and memory functions for players. A salient point here is that an individual memory function is allowed to be partial and has a *memory module* as a basic unit of memory. That is, his *short-term* (local) memory is subject to forgetfulness. In particular, the memory module of recall-1 proves to be of importance.

Step 2: The transition process from short-term memories to long-term memories was explained in Kaneko-Kline [9] and more fully in Akiyama *et al.* [1]. Here, the process is only briefly and informally explained in Section 2. Formally, we will take the resulting domain of accumulations of experiences for granted.

Step 3: We will give a generalized definition of an i.d.view, which allows general existence of an i.d.view. However, there are an infinite number of i.d.views. We will focus on minimal/smallest i.d.views. Minimality avoids large redundant views, but there may be still multiple minimal views. When the memory function $\mathfrak{m}_i^o$ is subject to partiality, minimal views may not capture essential structures, since they may be too small.

Step 4: Under Kuhn's distinguishability condition, we show the unique smallest view, which will be discussed in Section 6.

Step 5: As the experienced domain is increased with time, a personal view is evolving, i.e., for some time, it is getting larger but for other time, it gets stuck to a fixed one even if he has more experiences. This will be exemplified with Mike's bike in Sections 2 and 7.

Step 6: The last step is to check an i.d.view with new experiences in the objective situation. He may reach a certain view and it becomes stable in the sense that he does not notice any incoherence between his view and his experiences. However, this takes a long time or he fails to reach it. In Section 8, we consider some difficulties for a player's payoff maximization.

2. Mike's Bike Commuting (1)

One important step of inductive game theory is the transition from a short-term memory to a long-term memory and an accumulation of long-term memories. This step is

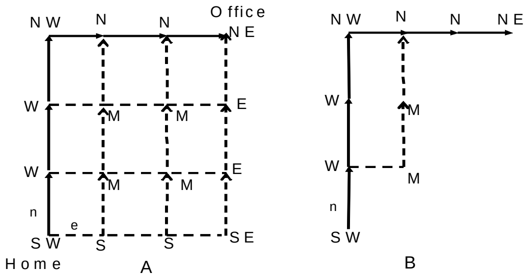

Figure 2.1: Mike's Bike Commuting

elucidated in Kaneko-Kline [9] and Akiyama *et al.* [1]. In this paper, we skip the transition step but adopt certain concepts derived from it such as a domain of accumulation. In this section, we use a variant of "Mike's Bike Commuting" of [1] to illustrate the transition step. This example will be discussed once more in Section 7.

Mike's Bike Commuting: Mike moved to the new town and started commuting from his apartment to his office by bike. The town has the lattice structure depicted in Fig.2.1.A. At each lattice point, he receives an information piece, S, W, N, E, M, SW, SE, NW, or NE. He has two possible actions "e" and "n" at SW, S, W and M. At NW and N, he must choose e, and at SE and E, he must choose n.

Mike regularly takes the route indicated by the bold arrows, directly north from SW to NW and directly east to NE, which his colleague suggested to him. Only occasionally, he deviates to some other behavior and finds some other route. When he deviates from to some new lattice point, like the south M in Fig.2.1.B, he then follows his *default behavior* n when it is available[3].

In the very beginning, he commutes through the *regular* route, which means that the domain of accumulation consists of paths connecting the lattice points within the regular route only. After some time, he might try a deviation from the lower W by taking e there and then following his default behavior up to N. In this case, the *domain of accumulation* consists of the lattice points in the dotted line in addition to those in the bold line in Fig.2.1.B. The deviation to e at the southwest M needs a higher order trial: One deviation to e at the south W and the other deviation to e at that M are required. It is our contention that it takes more time to experience and to learn the results of higher order deviations.

[3]It is assumed in Akiyama *et al.* [1] that Mike is also given a small map of the town. In that paper, local memories involve no partiality, and it was asked how (details of) many routes Mike has learned for a given finite repetitions of commuting. Here, we do not make this assumption, but rather we ask how he can reconstruct a lattice structure of the town with his partial local memories.

The lattice picture is an accurate summary of the town. The player, on the other hand, may not have access to such a full description. Rather, he may receive only the information pieces attached to each lattice point he experiences with the action taken there. At each lattice point he reaches, he receives an information piece and a short-term (local) memory occurs in his mind. One possible form of this memory is to recall only the current and last piece received with the last action taken there. For example at the southwest M, if he comes from the south W, his local memory is just $\langle (W, e), M \rangle$. This is a basic memory module, which we call a *memory module of recall*-1. It plays a fundamental role in this paper.

At each lattice point in the domain of accumulation, the local (short-term) memory is experienced several times and then may be changed to a *long-term memory*. Hence, the set of accumulated long-term memories, which we will call a *memory kit*, is expressed as the set of such local memories over the domain of accumulation. The formal definitions of these concepts will be given in Section 3.3.

Now, Mike's problem of induction is to combine those small modules to one picture. For example, we ask whether or not he can recover the objective picture of Fig.2.1.A from his accumulated memories. We will give some answers in Section 7.

3. Information Protocols, Memory, Views, and Behavior

In Section 3.1, we describe information protocols and the axioms for them introduced in Kaneko-Kline [10]. Section 3.2 introduces the concept of a memory function for a player, which is the interface from the objective world to his mind. Then, we define an objective description $(\Pi^o, \mathfrak{m}^o)$ and a personal view $(\Pi^i, \mathfrak{m}^i)$ of player i. In Section 3.3, we give a definition of a behavior pattern (strategy configuration) for the players, and also describe a domain of accumulation for memories and a memory kit.

3.1. Information Protocols and Axioms

The concept of an information protocol deals with information pieces and actions as primitive concepts, and describes connections between histories to new information pieces and actions. An *information protocol* is given as a quintuple $\Pi = (W, A, \prec, (\pi, N), (h)_{i \in N_*})$, where

IP1: W is a finite nonempty set of *information pieces*;

IP2: A is a finite nonempty set of *actions*;

IP3: $\prec$ is a *causality relation*; formally, it is a finite nonempty subset of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$, where any $w \in W$ and any $a \in A$ occur in some sequence in $\prec$.

The set $(W \times A)^0 \times W$ is stipulated to be W. A sequence in $\prec$ is called a *feasible sequence*. We say that $w \in W$ is a *decision piece* iff w occurs in $[(w_1, a_1), ..., (w_m, a_m)]$ for some feasible sequence $\langle (w_1, a_1), ..., (w_m, a_m), w_{m+1}\rangle$ in $\prec$. We denote the set of all decision pieces by W^D, and define $W^E = W - W^D$, where each piece in W^E is called an *endpiece*. Using those notions, we describe the fourth and fifth components of a protocol.

IP4 (*player assignment*): $N = \{1, ..., n\}$ is a finite set of players, and $\pi : W \to 2^N$ is the player assignment, where $|\pi(w)| = 1$ for all $w \in W^D$ and $\pi(w) = N$ for all $w \in W^E$;

IP5 (*payoff assignment*): $h_i : W^E \to R$ for all $i \in N_*$, where $N_* \subseteq N$.

An information protocol starts with tangible elements in W and A listed in IP1 and IP2. Each $w \in W$ may be interpreted as a pure symbolic expression like a gesture, a sentence in an ordinary language, or a formula in the sense of mathematical logic. In Mike's bike commuting of Fig.2.1.A, $W = \{SE, W, N, E, M, SW, SE, NW, NE\}$ and $A = \{e, n\}$. The set $\prec$ given in IP3 describes the feasible sequences of these elements possibly occurring in the plays of the game. A feasible sequence $\langle (w_1, a_1), ..., (w_m, a_m), w\rangle$ is interpreted as meaning that in one occurrence of the protocol Π, a player first received piece w_1 and took action a_1, then sometime later another player received w_2 and took action a_2, so on, and now, a player receives w. It is not yet assumed that this sequence is an exhaustive history up to w. An exhaustive history will be defined presently.

We sometimes write $[(w_1, a_1), ..., (w_m, a_m)] \prec w$ for $\langle (w_1, a_1), ..., (w_m, a_m), w\rangle \in \prec$. We use $\langle \xi, w\rangle$ to denote a generic element of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. The set $\prec$ is the union of subsets of $(W \times A)^0 \times W = W$, $(W \times A)^1 \times W$, $(W \times A)^2 \times W$,... We are interested only in *finite* information protocols, i.e., W, A and $\prec$ are all finite sets. Throughout the paper, we assume $W \cap A = \emptyset$ to avoid unnecessary complications.

An information protocol is completed by adding the player assignment and the payoff assignment. The player assignment π in IP4 assigns a single player to each decision piece, and the set of all players N to each endpiece. In IP5, the payoff function h_i is specified for each player i in the set $N_* \subseteq N$. We allow N_* to differ from N to describe a personal view where only some players payoffs are known to the player. In the present paper, we consider the case of either $N_* = N$ or $N_* = \{i\}$.

We assume for simplicity that each piece $w \in W$ contains the following information, which player i should be able to read by looking at w:

M1: a full set C_w of available actions at w if w is a decision piece;

M2: the value $\pi(w)$ of the player assignment π if w is a decision piece;

M3: his own payoff $h_i(w)$ (as a numerical value) if w is an endpiece.

In M1, the set C_w of available actions at w is written on the decision piece w. The full set C_w is used for an objective description but may not be used in a subjective protocol Π, which will be defined in Section 3.2. In the subjective case, a player will use only the set of actions at w occurring in his view. We denote this set at w by A_w :

$$A_w := \{a \in A : [(w, a)] \prec u \text{ for some } u \in W\} \subseteq C_w. \tag{3.1}$$

The latter inclusion is the coherence condition with the full set C_w. Condition M2 requires w to include the information of who moves at w. Here, player i may receive (or observe) a decision piece w at which another player j moves. Finally, in M3, each player can read his own payoff from each endpiece.

We use information protocols to describe both the target objective situation and a personal subjective view. The formal distinction between them is made by means of axioms for them. A protocol for the former should satisfy two basic axioms and three non-basic axioms. A protocol for a personal view will be required to satisfy only the two basic axioms. We give the full set of basic and non-basic axioms now.

The first basic axiom is subsequence-closedness. For it, we need a concept of a subsequence of a sequence in $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. We say that a subsequence of $\langle (w_1, a_1), ..., (w_k, a_k),\ w_{k+1}\rangle$ is *legitimate* iff it belongs to $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. For example, $\langle w_t\rangle, \langle (w_1, a_1), w_{k+1}\rangle$ and $\langle (w_2, a_2), ...,\ (w_{k-1}, a_{k-1}), w_{k+1}\rangle$ are legitimate subsequences of $\langle (w_1, a_1), ...,\ (w_k, a_k),\ w_{k+1}\rangle$, but $[(w_2, a_2), (w_3, a_3)]$ is not. A *legitimate supersequence* is defined in the dual manner.

Axiom B1 (Subsequence-Closedness): If $\langle \xi, w\rangle \in \prec$ and $\langle \xi', w'\rangle$ is a legitimate subsequence of $\langle \xi, w\rangle$, then $\langle \xi', w'\rangle \in \prec$.

Since we consider only legitimate subsequences and supersequences throughout this paper, we simply write subsequences and supersequences by abbreviating "legitimate".

The second basic axiom states that the decision pieces can be distinguished from the endpieces.

Axiom B2 (Weak Extension): If $\xi \prec w$ and $w \in W^D$, then there are $a \in A$ and $v \in W$ such that $[\xi, (w, a)] \prec v$.

Any protocol Π that satisfies Axioms B1 and B2 is called a *basic protocol.*

To state the non-basic axioms, we need the notion of an exhaustive history called a position. First, we define an *initial segment* of a sequence $\langle (w_1, a_1), ..., (w_m, a_m), w_{m+1}\rangle$ to be $\langle (w_1, a_1), ..., (w_k, a_k), w_{k+1}\rangle$ for some $k \leq m$. We say that a feasible sequence $\langle \xi, w\rangle$ is *maximal* iff $\prec$ contains no proper feasible supersequence $\langle \eta, v\rangle$ of $\langle \xi, w\rangle$. A *position* $\langle \xi, w\rangle$ is defined to be an initial segment of some maximal feasible sequence $\langle \eta, v\rangle$. Thus, each position is an exhaustive history up to w in Π. We denote the set of all positions by Ξ. Then, we partition Ξ into the sets:

$$\Xi^D = \{\langle \xi, w\rangle \in \Xi : w \in W^D\} \ \text{ and } \ \Xi^E = \{\langle \xi, w\rangle \in \Xi : w \in W^E\}. \tag{3.2}$$

We call $\langle \xi, w\rangle \in \Xi^D$ a *decision position* and $\langle \xi, w\rangle \in \Xi^E$ an *endposition.*

Now, we define the operator Δ by

$$\Delta Y = \{\langle \xi, w\rangle : \langle \xi, w\rangle \text{ is a subsequence of some sequence } \langle \eta, v\rangle \in Y\}. \tag{3.3}$$

for any set $Y \subseteq \bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. Using this, Axiom B1 is stated as $\prec\ = \Delta(\prec)$.

The following lemma states that we can represent a basic protocol in terms of end-positions. It may be much more convenient to describe the set $\prec$.

Lemma 3.1. Let $\Pi = (W, A, \prec, (\pi, N), (h_i)_{i \in N_*})$ be a basic protocol. Then $\prec\; = \Delta\Xi^E$.

Proof. Let $\langle \xi, w \rangle$ be any sequence in $\prec$. Then $\langle \xi, w \rangle$ has a supersequence $\langle \zeta, v \rangle$ which is a maximal feasible sequence in $\prec$. By Axiom B2 (weak extension), $v \in W^E$. Hence, $\langle \zeta, v \rangle \in \Xi^E$, and so $\langle \xi, w \rangle \in \Delta\Xi^E$. For the converse, let $\langle \xi, w \rangle \in \Delta\Xi^E$. Then $\langle \xi, w \rangle$ is a subsequence of some $\langle \zeta, v \rangle \in \Xi^E$. Since $\Xi^E \subseteq \Xi \subseteq \prec$ by Axiom B1 (subsequence-closedness), we have $\langle \zeta, v \rangle \in \prec$, and, again by B1, $\langle \xi, w \rangle \in \prec$. ■

We now list the three non-basic axioms based on the notion of a position.

Axiom N1 (Root): There is a distinguished element $w^0 \in W$ such that $\langle w^0 \rangle$ is an initial segment of every position.

This axiom means that all positions start with w^0. Without this, the protocol may have various starts. The next axiom states that an exhaustive history determines a unique information piece.

Axiom N2 (Determination): Let $\langle \xi, w \rangle$ and $\langle \eta, v \rangle$ be positions. If $\xi = \eta$ and it is nonempty, i.e., $\langle \xi, w \rangle \neq \langle w \rangle$, then $w = v$.

The last axiom states that the set of available actions at an information piece is independent of a history.

Axiom N3 (History-Independent Extension): If $\langle \xi, w \rangle$ is a position and $[(w, a)] \prec v$, then $\langle \xi, (w, a), u \rangle$ is a position for some $u \in W$.

Axiom N3 implies that the set of available actions at any position $\langle \xi, w \rangle$ is the same as A_w given in (3.1). If N3 is violated, the set of available actions differ at two positions ending with the same information piece.

When an information protocol Π satisfies Axioms B1, B2, N1, N2, N3 and $N_* = N$, we call it a *full protocol*. A full protocol will be used to describe a target objective situation, that is, an *objective situation* is a full protocol $\Pi = (W, A, \prec, (\pi, N), (h_i)_{i \in N})$.

For a personal view, we require only Axioms B1, B2, and also, the payoff assignment for only the player in question, that is, a *subjective protocol* is a basic protocol $\Pi = (W, A \prec, (\pi, N), (h_i)_{i \in N_*})$ with $N_* = \{i\}$.

Kaneko-Kline [10] showed that a full protocol is equivalent to an extensive game in Kuhn's [13] sense with the replacement of information sets by information pieces. The equivalence states that from a given full information protocol $\Pi = (W, A, \prec, (\pi, N), (h_i)_{i \in N})$, we can construct an extensive game, and *vice versa*. Also, it is shown that the deletion of each of Axioms N1, N2, N3 corresponds to some weakening of the definitions for an extensive game. It was also shown that such weakenings are arising naturally as inductively derived views. In Sections 4, 6 and 8, we will encounter several examples violating some of Axioms N1-N3.

We now give one example, which will be used in subsequent sections.

Example 3.1. Consider the following 2-person situation in Fig.3.1, in which the endpieces are described as z_1 to z_4, and players 1, 2 move at w_0 and w_1, w_2, respectively.

$$
\begin{array}{ccccccc}
z_1 : \begin{matrix} 3 \\ 3 \end{matrix} & & z_2 : \begin{matrix} 0 \\ 0 \end{matrix} & & z_3 : \begin{matrix} 1 \\ 1 \end{matrix} & & z_4 : \begin{matrix} 5 \\ 2 \end{matrix} \\
& \nwarrow_a \quad \uparrow_b & & & & \uparrow_a \quad \nearrow_b & \\
& w_1 : \mathrm{PL2} & & & & w_2 : \mathrm{PL2} & \\
& & & \nwarrow_a \quad \nearrow_b & & & \\
& & & w_0 : \mathrm{PL1} & & &
\end{array}
$$

Fig.3.1

To describe this as an information protocol, we take $W = \{w_0, w_1, w_2, z_1, ..., z_4\}$ and $A = \{a, b\}$. The set of feasible sequences $\prec$ is quite large, but by Lemma 3.1 it suffices to list only the endpositions $\Xi^E = \{\langle (w_0, a), (w_1, a), z_1 \rangle,\ \langle (w_0, a), (w_1, b), z_2 \rangle,$ $\langle (w_0, b), (w_2, a), z_1 \rangle,\ \langle (w_0, b), (w_2, b), z_4 \rangle\}$. This protocol is full, and can be interpreted as an objective situation.

3.2. Memory Functions and Views

The central part of inductive game theory is the consideration of a derivation of a personal view from memories accumulated in a player's mind. The source for an inductive derivation is his memories from experiences. Therefore, a certain interface from individual experiences to memories is required. Here, we give the concept of a memory function as the description of such an interface.

A memory function describes a personal memory capability within one play of an information protocol. In other words, it describes short-term (local, temporal) memories within one play of the game. Transition from short-term memories to long-term memories needs another structure, which is discussed in Kaneko-Kline [9], Akiyama *et al.* [1]. We will take some resulting concepts for granted.

Now, let Π be a basic information protocol, and let Ξ be the set of positions in Π. In [9], the domain of a memory function for player i is assumed to be the set

$$\Xi_i := \{\langle \xi, w \rangle \in \Xi : i \in \pi(w)\} \tag{3.4}$$

of *player i's positions*. A memory function may give a short-term memory including other players' previous moves. Thus, we extend the domain of a memory function for player i to a superset of Ξ_i. That is, the domain of a memory function is given as a set Y_i with $\Xi_i \subseteq Y_i \subseteq \Xi$.

Definition 3.2 (Memory Functions): A *memory function* $\mathfrak{m}_i$ of player i assigns, to each $\langle \xi, w\rangle \in Y_i$, a finite sequence $\langle \zeta, v\rangle = \langle (v_1, b_1), ..., (v_m, b_m), v\rangle$ satisfying:

$$v = w; \tag{3.5}$$

$$m \geq 0 \text{ and } v_t \in W,\ b_t \in A_{v_t} \text{ for all } t = 1, ..., m. \tag{3.6}$$

Condition (3.5) means that the latest piece is the one received at the current position $\langle \xi, w\rangle$. Except for this requirement, enough flexibility is allowed in (3.6) so as to capture forgetfulness and incorrect memories. Note that the domain Y_i may contain other players' positions, in which case player i receives some other player's information piece.

We call the value $\mathfrak{m}_i\langle \xi, w\rangle = \langle \zeta, v\rangle$ a *memory thread* and each of (v_t, b_t) and v in the thread a *memory knot*. Thus, the most primitive element in memory is a memory knot, and a memory thread is a sequence consisting of several memory knots. When player i reaches a position $\langle \xi, w\rangle$, the memory thread $\langle \zeta, v\rangle = \langle (v_1, b_1), ..., (v_m, b_m), v\rangle$ occurs spontaneously in his mind. Memory knots $v, (v_m, b_m), ..., (v_1, b_1)$ may be recalled in the reverse order. A limitation on a player's short-term memory suggests that these threads should be short. The most basic case is the memory module of *recall*-1, i.e., he receives v and recalls (v_m, b_m) only, which was discussed in Section 2.

We will consider a slightly more general class of memory functions, called "recall-k". By "recall-k", player i can recall back to the k latest memory knots within Y_i; this is a limitation on the length of a memory thread (not a duration of a short-term memory). For this definition, we need the definition of the Y_i-part of a position in Y_i.

Now, let $\langle \xi, w\rangle = \langle (w_1, a_1), ..., (w_m, a_m), w_{m+1}\rangle$ be any position in Y_i. First, we define the index set $\{t : t = 1, ..., m{+}1 \text{ and } \langle (w_1, a_1), ..., (w_{t-1}, a_{t-1}), w_t\rangle \in Y_i\}$,which is denoted by $\{j_1, ..., j_{s+1}\}$. Then, the Y_i*-part* $\langle \xi, w\rangle_i$ of $\langle \xi, w\rangle$ is defined to be $\langle (w_{j_1}, a_{j_1}), ..., (w_{j_s}, a_{j_s}), w_{j_{s+1}}\rangle$. This is the maximal subsequence of $\langle \xi, w\rangle$ with the property that the initial segment of $\langle \xi, w\rangle$ up to each j_t belongs to Y_i. For example, when $\langle \xi, w\rangle = \langle (w_1, a_1), (w_2, a_2), w_3\rangle = \langle (u, a), (u, b), u\rangle$ and $Y_i = \{\langle u\rangle, \langle (u, a), (u, b), u\rangle\}$, the index set is $\{1, 3\}$. Thus, $\langle \xi, w\rangle_i = \langle (w_1, a_1), w_3\rangle = \langle (u, a), u\rangle$.

The recall-k memory function needs the following notation: For $\langle \xi, w\rangle_i = \langle (v_1, b_1), ..., (v_s, b_s), v_{s+1}\rangle$ and a non-negative integer k, we define $\langle \xi, w\rangle_i^k$ by

$$\langle \xi, w\rangle_i^k = \begin{cases} \langle (v_{s-k+1}, b_{s-k+1}), ..., (v_s, b_s), v_{s+1}\rangle & \text{if } k \leq s \\ \langle \xi, w\rangle_i & \text{if } k > s. \end{cases} \tag{3.7}$$

It takes the last k part of $\langle \xi, w\rangle_i$, but when k is larger than s, it takes the entire $\langle \xi, w\rangle_i$. Also, when $k = 0$, we stipulate that $\langle \xi, w\rangle_i^0 = \langle v_{s+1}\rangle = \langle w\rangle$.

The *recall-k memory function* is now formulated as:

$$\mathfrak{m}_i^{Rk}\langle \xi, w\rangle = \langle \xi, w\rangle_i^k \quad \text{for each } \langle \xi, w\rangle \in Y_i. \tag{3.8}$$

When the memory bound k is zero, i.e., player i has no recall ability in short-term memories, it is called the *Markov* memory function $\mathfrak{m}_i^{R0}$. It holds that $\mathfrak{m}_i^{R0}\langle \xi, w\rangle = \langle w\rangle$ for all $\langle \xi, w\rangle \in Y_i$. This is of importance only as a reference point of our analysis.

The recall-k memory functions may include partiality and forgetfulness, but the memories are correct in the sense each memory thread is a subsequence of the true position. This correctness will be used in Theorem 8.1.

When k is longer than the maximum depth of the protocol, we call $\mathfrak{m}_i^{Rk}$ the *perfect-recall memory-function*[4], denoted by $\mathfrak{m}_i^{PR}$. It is given as:

$$\mathfrak{m}_i^{PR}\langle \xi, v\rangle = \langle \xi, v\rangle_i \ \text{ for each } \langle \xi, v\rangle \in Y_i. \tag{3.9}$$

With $\mathfrak{m}_i^{PR}$, player i recalls all the information pieces and actions previously observed by himself. This function will play an important role in Sections 5.2 and 8.

Two extreme cases with respect to Y_i should be emphasized. When Y_i coincides with the set Ξ of all positions, the memory function defined by (3.9) is called the *perfect-information memory function* and is denoted by $\mathfrak{m}_i^{PI}$. In this case, $\mathfrak{m}_i^{PI}\langle \xi, v\rangle = \langle \xi, v\rangle$ for all $\langle \xi, v\rangle \in Y_i = \Xi$. With $\mathfrak{m}_i^{PI}$, player i recalls the complete history within a play of Π including the other players' pieces and actions. The other extreme is given by $Y_i = \Xi_i$, and the memory function $\mathfrak{m}_i^{PR}$ is called the *self-scope perfect-recall memory function,* denoted by $\mathfrak{m}_i^{SPR}$. With this, the player only has memories of his own information pieces and actions. This was exclusively used in Kaneko-Kline [8] and [10].

Having described an information protocol and memory functions, we now have the basic ingredients for objective descriptions and subjective personal views.

The objective description is the target social situation for our study. It exists in the objective world and constitutes one part of the entire social system depicted in Fig.1.2. We regard a full protocol as a complete description up to observables (information pieces and available actions). We require (3.1) to be equality, i.e., M1 is reflected in the objective protocol.

(Objective Situation): A pair $(\Pi^o, \mathfrak{m}^o)$ is called an *objective situation* iff $\Pi^o = (W^o, A^o, \prec^o, (\pi^o, N^o), \{h_j^o\}_{j\in N^o})$ is a full protocol with $A_w^o = C_w$ for all $w \in W^{oD}$ and $\mathfrak{m}^o = (\mathfrak{m}_1^o, ..., \mathfrak{m}_n^o)$ is an n-tuple of memory functions in Π^o.

We use the superscript o to denote the objective situation, and put a superscript i to denote a personal view of player i.

A personal (subjective) view exists in the mind of player i, and it is based on only his observations. Typically such a view is a partial description of the objective situation (description). Thus, we require it to be a basic protocol only. One remark is that the others' payoffs and local memories occur in the others' scopes, and player i cannot

[4] This differs considerably from Kuhn's [13] "perfect-recall" condition on information sets , which will be discussed in Section 6.

directly experience them. Hence, his personal view contains his payoff functions and memory function only.

(**Personal View**): A pair $(\Pi^i, \mathfrak{m}^i)$ is a *personal view* for player i iff $\Pi^i = (W^i, A^i, \prec^i, (\pi^i, N), h^i)$ is a subjective protocol, i.e., it is a basic protocol, with a specification of player i's payoff function h^i, and $\mathfrak{m}^i$ is a memory function for player i in Π^i.

3.3. Behavior Patterns, Closed Domains, and Memory Kits

Suppose that the objective situation $(\Pi^o, \mathfrak{m}^o)$ is played repeatedly. Behavior of each player is described by the concept of a behavior pattern. Let $\Xi_i^{oD} := \Xi^{oD} \cap \Xi_i^o$ be the set of decision positions for player i. A function σ_i on Ξ_i^{oD} is a *behavior pattern* (*strategy*) of player i iff it satisfies: for all $\langle \xi, w \rangle, \langle \eta, v \rangle \in \Xi_i^{oD}$,

$$\sigma_i\langle \xi, w \rangle \in A_w^o; \tag{3.10}$$

$$\mathfrak{m}_i^o\langle \xi, w \rangle = \mathfrak{m}_i^o\langle \eta, v \rangle \text{ implies } \sigma_i\langle \xi, w \rangle = \sigma_i\langle \eta, v \rangle. \tag{3.11}$$

Condition (3.10) means that σ_i prescribes an available action to each decision position, and (3.11) that a strategy depends upon the local memory of the player moving there. We denote, by Σ_i^o, the *set of all behavior patterns* for player i in $(\Pi^o, \mathfrak{m}^o)$. We say that an n-tuple of strategies $\sigma = (\sigma_1, ..., \sigma_n)$ is a *profile of behavior patterns.*

Although a behavior pattern is defined as a complete contingent plan, we do not require that the player be fully aware of it. Rather he should be able to take an action whenever he is called upon to move. Condition M1 ensures that a player can see the available actions, and pick one, maybe, a *default* action, whenever one of his decision pieces is reached. We use the term *behavior pattern* to express the idea that the behavior of a player may initially have no strategic considerations. Once a player has gathered enough information about the game, his behavior may become strategic.

We presume that the players follow some *regular behavior patterns* $\sigma^o = (\sigma_1^o, ..., \sigma_n^o)$. Sometimes, however, some players may deviate from these behavior patterns, which leads to new experiences and short-term memories for them. These short-term memories remain for some periods of time, but after these periods, they would disappear, except when they have occurred frequently enough to reinforce the short-term memories as lasting in his mind. When such a case occurs, a short-term memory becomes a long-term memory, and remains for longer periods.

Since there are many aspects involved in such an evolution process, there would be many possible formulations of the dynamics. Also, since the relevant time structure must be finite, limit theorems are not of interest to us at all. Therefore, we think that a computer simulation is an appropriate method to study the dynamics of accumulation of long-term memories. One simple version is given in Akiyama *et al.* [1]. Here, we do not give a formulation of a dynamics itself. Instead, we give a general definition of possible results of such a dynamic accumulation process, which we call a *memory kit.*

The memory kit is defined over its objective counterpart, a *domain of accumulation* D_i, which is is a subset of Y_i satisfying:

$$D_i \text{ contains at least one endposition } \langle \xi, w \rangle \text{ in } \Xi^o. \tag{3.12}$$

In the beginning of trial-error, this condition may not be satisfied. We consider the inductive process after he reaches a state with (3.12).

For D_i, we start with a basic domain. A subset D_i^{cane} of Y_i is said to be a *cane domain* iff for some endposition $\langle \xi, w \rangle$, D_i^{cane} is given as the set $\{\langle \zeta, v \rangle \in Y_i : \langle \zeta, v \rangle$ is an initial segment of $\langle \xi, w \rangle\}$. Thus, D_i^{cane} is the set of all positions in Y_i continuing to the endposition $\langle \xi, w \rangle$. The regular cane domain is obtained when every player follows his regular behavior pattern σ_i^o with no deviations. A subset D_i of Y_i is said to be a *closed domain of accumulation* iff it is expressed as the union of some cane domains. A closed domain satisfies (3.12). We focus largely on closed domains in this paper.

A domain D_i is still the objective description of experienced positions for player i. However, this gives the *memory kit* T_{D_i} describing the accumulated experiences in the mind of player i :

$$T_{D_i} := \{\mathfrak{m}_i^o \langle \xi, w \rangle : \langle \xi, w \rangle \in D_i\}. \tag{3.13}$$

It is determined by both the domain D_i and the objective memory function $\mathfrak{m}_i^o$ of player i. This is the set of long-term memories changed from short-term memories. See Kaneko-Kline [9] and Akiyama *et al.* [1] for full discussions about such transitions. The memory kit T_{D_i} is the source for an inductive construction of a personal view, i.e., the memory threads in T_{D_i} are used to construct a skeleton of the personal view.

4. Inductive Derivations

We now start the main part of the paper. It is about the inductive construction of a personal view from a memory kit T_{D_i} of a player. The partiality in a player's local memory forces us to consider multiple views for the same memory kit, which opens the theory to new types of induction. In Section 4.3, we discuss the existence of an inductively derived view for each memory kit on a given domain, and conditions for a given set of memory threads to be an inductively view.

4.1. Inductively Derived Views

Suppose that the objective situation $(\Pi^o, \mathfrak{m}^o) = (W^o, A^o, \prec^o, (\pi^o, N^o), \{h_j^o\}_{j \in N^o}, \{\mathfrak{m}_j^o\}_{j \in N^o})$ is fixed. The sets of decision pieces and end pieces in Π^o are denoted by W^{oD}, W^{oE}, and the corresponding sets in a personal view $(\Pi^i, \mathfrak{m}^i)$ are denoted by W^{iD}, W^{iE}. Now, an inductively derived view is defined as follows.

Definition 4.1 (I.D.View). A personal view $(\Pi^i, \mathfrak{m}^i) = (W^i, A^i, \prec^i, (\pi^i, N^i), h^i, \mathfrak{m}^i)$ for player i is an *inductively derived view* from a memory kit T_{D_i} iff

ID1(Information Pieces): $W^i = \{w \in W^o : w$ occurs in some sequence in $T_{D_i}\}$, $W^{iD} \subseteq W^{oD}$ and $W^{iE} \subseteq W^{oE}$;

ID2(Actions): $A^i_w \subseteq A^o_w \ (= C_w)$ for each $w \in W^i$;

ID3(Feasible Sequences): $\Delta T_{D_i} \subseteq \prec^i$;

ID4(Player Assignment): $\pi^i(w) = \pi^o(w)$ if $w \in W^{iD}$ and $\pi^i(w) = N^i$ if $w \in W^{iE}$, where $N^i := \{j \in N^o : j \in \pi^i(w)$ for some $w \in W^{iD}\}$;

ID5(Payoff Assignment): $h^i(w) = h^o_i(w)$ for all $w \in W^{iE}$;

ID6(Memory Function): $\mathfrak{m}^i$ is the perfect-information memory function $\mathfrak{m}^{PI}$ for Π^i.

The above definition is the same as the one in [10] except condition ID3. In [10], the corresponding condition requires equality, i.e., $\Delta T_{D_i} = \prec^i$. The same type of requirement was made in [8] for the extensive game version of an i.d.view. Nevertheless, here we should discuss all of ID1 - ID6. These connect the candidate i.d.view to the objective situation $(\Pi^o, \mathfrak{m}^o)$ by making use of the minimum information conditions stated in M1, M2, and M3. Condition ID3 will be discussed after the other conditions.

Since $(\Pi^i, \mathfrak{m}^i)$ is a personal view, it is required to satisfy Axioms B1 and B2.

Condition ID1 states that player i uses only information pieces he finds in his memory kit, i.e., the set W^i defined from $\prec^i$ by IP3 coincides with the set of pieces occurring in T_{D_i}. It follows from M1 and M3 that he distinguishes between the decision pieces and endpieces; thus, $W^{iD} \subseteq W^{oD}$ and $W^{iE} \subseteq W^{oE}$. Condition ID2 requires that an available action at w in the player's view should be an objectively available one at w, i.e., A^i_w is defined from $\prec^i$ by (3.1), but is not limited to the set of actions occurring in T_{D_i}. Conditions ID4 and ID5 are based on M2 and M3 to connect the player assignment at decision pieces and payoffs at endpieces in Π^i to those found in the objective protocol Π^o. We assume condition ID6 since the view $(\Pi^i, \mathfrak{m}^i)$ is in the mind of player i.

Once a personal view is specified with ID1, ID2 and ID3, the other ID4, ID5 and ID6 uniquely determine the player assignment, payoff and memory function. Hence, all questions about an i.d.view for a given T_{D_i} can be answered by checking ID1 - ID3.

Let us return to ID3. A simple example shows the need for the weaker form, $\Delta T_{D_i} \subseteq \prec^i$, of ID3 when memory is partial.

Example 4.1 (The Absent-minded Driver Game): Consider the 1-player $(\Pi^o, \mathfrak{m}^o_1)$ described as Fig.4.1 with the recall-1 memory function $\mathfrak{m}^o_1 = \mathfrak{m}^{R1}_1$, where payoffs $0, 6, 3$ are regarded as information pieces. Recall-1 gives him the following memories: $\mathfrak{m}^{R1}_1\langle w\rangle = \langle w\rangle$, $\mathfrak{m}^{R1}_1\langle (w,c), w\rangle = \langle (w,c),\ w\rangle$, $\mathfrak{m}^{R1}_1\langle (w,c),(w,c),3\rangle = \langle (w,c),3\rangle$, $\mathfrak{m}^{R1}_1\langle (w,e),0\rangle =$

$\langle (w,e),0\rangle$ and $\mathfrak{m}_1^{R1}\langle (w,c),(w,e),6\rangle = \langle (w,e),6\rangle$. This differs from the interpretation considered in Isbell [7] and Piccione-Rubinstein [15] in that player 1 can distinguish the first $\langle w\rangle$ from the second $\langle (w,c),w\rangle$. Nevertheless, his forgetfulness prevents him from understanding the objective protocol[5].

$$\begin{array}{ccccc} 0 & & 6 & & \\ \uparrow_e & & \uparrow_e & & \\ w & \underset{c}{\rightarrow} & w & \underset{c}{\rightarrow} & 3 \end{array} \qquad\qquad \begin{array}{ccccc} 6 & & 0 & & \\ \uparrow_e & & \uparrow_e & & \\ w & \underset{c}{\rightarrow} & w & \underset{c}{\rightarrow} & 3 \end{array}$$

Fig.4.1 Fig.4.2

Consider the case of the full domain of accumulation $D_1^o = \Xi^o$. His memory kit is $T_{D_1} = \{\langle w\rangle, \langle (w,c),w\rangle, \langle (w,c),3\rangle,\ \langle (w,e),0\rangle, \langle (w,e),6\rangle\}$. It is the first fact that from this T_{D_1}, there is no i.d.view satisfying $\Delta T_{D_1} = \prec^1$: Indeed, if there was an i.d.view with $\Delta T_{D_1} = \prec^1$, then w would be a decision piece in $\prec^1$, but no feasible sequence in ΔT_{D_1} is an extension of $\langle (w,c),w\rangle$, a violation of Axiom B2 (Weak Extension). To avoid this difficulty, we weaken $\Delta T_{D_i} = \prec^i$ into $\Delta T_{D_i} \subseteq \prec^i$ in ID3.

With ID3: $\Delta T_{D_i} \subseteq \prec^i$, it is easy to construct an i.d.view for the above example; both Fig.4.1 and Fig.4.2 satisfies ID3. Thus, we have already multiple i.d.views even if we require them to satisfy Axioms B1,B2 and N1-N3.

4.2. 2-Person Example

In Example 4.1, we have multiple i.d.views caused by partiality in the memory function. With more players, we will have different problems. To see this, we consider the objective situation described in Example 3.1.

Suppose that each i has the self-scope perfect-recall memory function $\mathfrak{m}_i^o = \mathfrak{m}_i^{SPR}$ over the domain $Y_i = \Xi_i^o$. The domain Ξ_1^o of $\mathfrak{m}_1^o$ has five positions ending with $w_0, z_1, ..., z_{4.}$, and Ξ_2^o of $\mathfrak{m}_2^o$ has six positions ending with $w_1, w_2, z_1, ..., z_4$. For example, $\mathfrak{m}_1^{SPR}\langle w_0\rangle = \langle w_0\rangle$ and $\mathfrak{m}_1^{SPR}\langle (w_0,a),(w_1,a),z_1\rangle = \langle (w_0,a),z_1\rangle$.

Let us specify the behavior patterns σ_1 and σ_2 so that they take always actions a. Here, we consider three types of domains of accumulation D_i.

Cane Domains D_i^{cane}: Let D_1, D_2 be the cane domains, e.g., $D_1^{cane} = \{\langle w_0\rangle, \langle (w_0,a), (w_1,a),z_1\rangle\}$; neither player has an experience generated by a deviation. Here, $T_{D_1^{cane}} = \{\langle w_0\rangle, \langle (w_0,a),z_1\rangle\}$ and $T_{D_2^{cane}} = \{\langle w_1\rangle, \langle (w_1,a),z_1\rangle\}$ can be regarded as i.d.views for 1 and 2 respectively, which are represented as Fig.4.3.a and b. Each player i notices the existence of available action b at his decision information piece, i.e., w_0 or w_1, but he

[5] If we follow faithfully the interpretation given in Isbell [7] and Piccione-Rubinstein [15], then the memory function of player 1 is Markov, which gives various strange i.d.views.

does not know where it leads, since he has no experience of b. If each continues choosing only action a, this situation remains stable.

$$
\begin{array}{ccc}
z_1 & & z_1 \\
\uparrow_a & & \uparrow_a \\
w_0 & & w_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
z_1 & & z_3 \\
& \nwarrow_a \; \nearrow_b & \\
& w_0 &
\end{array}
\qquad
\begin{array}{ccc}
z_1 & & z_2 \\
& \nwarrow_a \; \nearrow_b & \\
& w_1 &
\end{array}
$$

Fig.4.3.a and b　　　　　　　　Fig.4.4.a and b

Unilateral Active Domains D_i^{UA}: Now, suppose that player i has the unilateral-active domain, each position of which is obtained by his own deviation:

$$
\begin{aligned}
D_1^{UA} &= \{\langle w_0 \rangle, \langle (w_0, a), (w_1, a), z_1 \rangle, \langle (w_0, b), (w_2, a), z_3 \rangle\}; \\
D_2^{UA} &= \{\langle (w_0, a), w_1 \rangle, \langle (w_0, a), (w_1, a), z_1 \rangle, \langle (w_0, a), (w_1, b), z_2 \rangle\}.
\end{aligned}
$$

In this case, $T_{D_1^{UA}} = \{\langle w_0 \rangle, \langle (w_0, a), z_1 \rangle, \langle (w_0, b), z_1 \rangle\}$ and $T_{D_2^{UA}} = \{\langle w_1 \rangle, \langle (w_1, a), z_1 \rangle, \langle (w_1, b), z_2 \rangle\}$. These form i.d.views, described as Fig.4.4.a and b.

Full Domains $D_i^F = \Xi_i^o$: As stated above, D_1^F and D_2^F have five and six positions, and $T_{D_1^F}$, $T_{D_2^F}$ are given as

$$
\begin{aligned}
T_{D_1^F} &= \{\langle w_0 \rangle, \langle (w_0, a), z_1 \rangle, \langle (w_0, a), z_2 \rangle, \langle (w_0, b), z_3 \rangle, \langle (w_0, b), z_4 \rangle\}; \\
T_{D_2^F} &= \{\langle w_1 \rangle, \langle (w_1, a), z_1 \rangle, \langle (w_1, b), z_2 \rangle, \langle w_2 \rangle, \langle (w_2, a), z_3 \rangle, \langle (w_2, b), z_4 \rangle\}.
\end{aligned}
$$

These are also regarded as i.d.views, described in Fig.4.5 and Fig.4.6. The former violates Axiom N2(Determination), and the latter does Axiom N1(Root).

$$
\begin{array}{ccccc}
z_1 & z_2 & & z_3 & z_4 \\
\nwarrow_a & \uparrow_a & & \uparrow_b & \nearrow_b \\
& & w_0 & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
z_1 & & z_2 \\
& \nwarrow_a \; \nearrow_b & \\
& w_1 &
\end{array}
\qquad
\begin{array}{ccc}
z_3 & & z_4 \\
& \nwarrow_a \; \nearrow_b & \\
& w_2 &
\end{array}
$$

Fig.4.5　　　　　　　　Fig.4.6

We did not yet consider equilibrium: A behavioral use of an i.d.view is rather for decision making/behavior revision before the convergence to an equilibrium point. After having trials and errors many times and having different individual views, the situation may come to equilibrium. Inductive game theory does not start with an equilibrium situation, but may require many repetitions to reach an equilibrium, or even getting stuck in a non-equilibrium situation. To study these problems, we should be careful about each step from trial/error, accumulation of experiences, inductive derivatives of his view, and behavioral uses. In our discourse, we consider each of these steps.

Finally, we consider the full domains with a player's memory described even at the other player's decision pieces : $D_1 = D_2 = \Xi^o$ consisting of all positions in the protocol Π^o. Here, we keep the assumption of each player having the perfect recall memory, but extend his domain of accumulation to the positions of the other player. In this way, a player can accumulate memories about the other player. In this case, the smallest view for each player i is the same as Fig.3.1 except for the other player's payoffs. We treat the other person's payoff personal information. One source for gaining this type of information is considered in Kaneko-Kline [11], where players may switch roles. We show there that the additional structure for role switching may actually facilitate the emergence of cooperation.

4.3. Existence of an I.D.View and the Structure of I.D.Views for a Memory Kit

The existence of an i.d.view is guaranteed with our weakened ID3. Let $(\Pi^o, \mathfrak{m}^o)$ be any objective description.

Theorem 4.1 (Existence of an i.d.view): Let D_i be any domain of accumulation. Then, there exists an i.d.view for the memory kit T_{D_i} obtained from D_i and $\mathfrak{m}_i^o$.

Proof. Define $F = \{\langle \xi, v \rangle \in T_{D_i} : \langle \xi, w \rangle$ is a maximal thread in T_{D_i} and $w \in W^{oD}\}$. Note that F may be empty. For each $\langle \xi, w \rangle \in F$, we choose an action $a_{\langle \xi, w \rangle} \in A_w^o$, and denote the set of those $a_{\langle \xi, w \rangle}$'s by A_F. By (3.12), D_i has at least one endposition $\langle \xi, w^e \rangle$, which implies that $w^e \in W^{oE}$ in some thread in T_{D_i}. We extend T_{D_i} to T'_{D_i} as follows:

$$T'_{D_i} = T_{D_i} \cup \{\langle \xi, (w, a_{\langle \xi, w \rangle}), w^e \rangle : \langle \xi, w \rangle \in F\}. \tag{4.1}$$

This set T'_{D_i} is constructed so that every maximal feasible sequence ends with the endpiece w^e, i.e.,

$$\text{if } \langle \xi, w \rangle \text{ is a maximal feasible sequence in } T'_{D_i}, \text{ then } w \in W^{oE}. \tag{4.2}$$

When $F = \emptyset$, we have $T'_{D_i} = T_{D_i}$ and (4.2) holds.

We define the protocol $\Pi^i = (W^i, A^i, \prec^i, (\pi^i, N^i), h^i)$ as follows: $W^i = \{w \in W^0 : w$ occurs in some sequence in $T_{D_i}\}$; $A^i = \{a \in A^0 : a$ occurs in some sequence in $T_{D_i}\} \cup A_F$; and $\prec^i = \Delta T'_{D_i}$. We use the information pieces occurring in T_{D_i}, the actions in T_{D_i} and newly added actions, and the set of the extended memory threads T'_{D_i}. Observe that by these definitions we ensure that Π^i satisfies ID1, ID2 and ID3: For example, ID3 is shown; since $T_{D_i} \subseteq T'_{D_i}$, we have $\Delta T_{D_i} \subseteq \Delta T'_{D_i} = \prec^i$. As remarked above, the player assignment, payoffs, and memory function are determined by ID4, ID5, and ID6.

It remains to show that this protocol is basic. By using $\prec^i = \Delta T'_{D_i}$, we have B1. Now consider Axiom B2. Let $\langle \xi, w \rangle \in \prec^i$ and $w \in W^{iD} \subseteq W^{oD}$. Then, if $\langle \xi, w \rangle$ is maximal in ΔT_{D_i}, it would be extended by (4.1) since $W^{iD} \subseteq W^{oD}$, so Axiom B2 is satisfied. Suppose that it is not maximal in ΔT_{D_i}. Then, $\langle \xi, w \rangle$ is a proper subsequence of some maximal sequence $\langle \eta, v \rangle$ in ΔT_{D_i} with $v = w$ or $v \neq w$. In the first case, we can extend $\langle \eta, v \rangle = \langle \eta, w \rangle$ by (4.1) to $\langle \eta, (w,a), v \rangle \in T'_{D_i}$, and by B1 for $\prec^i = \Delta T'_{D_i}$, we have $\langle \xi, (w,a), v \rangle \in \prec^i$. In the second case, by B1 for $\prec^i = \Delta T'_{D_i}$, we have some extension $\langle \xi, (w,a), v \rangle$ in $\prec^i = \Delta T'_{D_i}$. ■

As seen in Example 4.1, the multiplicity of i.d.views is an inevitable consequence of our ID3. It comes from various different ways of cutting and extending the memory threads in his memory kit. In fact, for each memory kit, there are a countably infinite number of i.d.views. This can be seen by observing that once we have an i.d.view, we can construct another by adding the same decision piece to the front of each maximal sequence in the view. This implies that great many supersets of ΔT_{D_i} constitute i.d.views. Our next task is to find precisely what shapes they might take.

We say that a superset F of ΔT_{D_i} is *conservative* iff for each $\langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle \in F$, $w_1, ..., w_{m+1}$ occur in ΔT_{D_i} and $a_t \in A^o_{w_t}$ for $t = 1, ..., m$. We note by ID1 and ID2 that if $(\Pi^i, \mathfrak{m}^i)$ is an i.d.view, then $\prec^i$ is a conservative superset of ΔT_{D_i}.

Then, we have the following additional result.

Lemma 4.2. Let F be a conservative superset of ΔT_{D_i}. Then, there is at most one i.d.view from T_{D_i} with $\prec^i = F$.

Proof. Suppose that $(\Pi^i, \mathfrak{m}^i) = (W^i, A^i, \prec^i, (\pi^i, N^i), h^i, \mathfrak{m}^i)$ and $(\Pi'^i, \mathfrak{m}'^i) = (W'^i, A'^i, \prec'^i, (\pi'^i, N'^i), h^{i'}, \mathfrak{m}'^i)$ are both i.d.views from T_{D_i} with $\prec^i = \prec'^i = F$. By IP3, $W^i = W'^i$ and $A^i = A'^i$. Since, $(W'^i, A'^i, \prec'^i) = (W^i, A^i, \prec^i)$, conditions ID4, ID5, and ID6 imply that $(\pi^i, h^i, \mathfrak{m}^i) = (\pi'^i, h'^i, \mathfrak{m}'^i)$. ■

This result is in sharp contrast with Kaneko-Kline [9], where an i.d.view is defined in terms of an extensive game. There we met another type of multiplicity caused by the hypothetical elements of nodes and branches. The use of an information protocol enables us to avoid this problem, which will be mentioned in the end of Section 5.1.

The next theorem gives a necessary and sufficient condition for a conservative superset of ΔT_{D_i} to be an i.d.view. Essentially, condition (i) corresponds to Axiom B1 and condition (ii) to Axiom B2. Thus, we have a direct way to check whether or not a conservative superset F of T_{D_i} will form an i.d.view. Applying this theorem to Example 4.1, we can find more i.d.views.

Theorem 4.3.(Conditions for an i.d.view): Let F be a conservative superset of ΔT_{D_i}. Then, there is an i.d.view $(\Pi^i, \mathfrak{m}^i) = (W^i, A^i, \prec^i, (\pi^i, N^i), h^i, \mathfrak{m}^i)$ from T_{D_i} with $\prec^i = F$ if and only if

(i): $F = \Delta F$;

(ii): $w \in W^{oE}$ for any maximal thread $\langle \xi, w \rangle \in F$.

Proof. (Only-if): Let $(\Pi^i, \mathfrak{m}^i) = (W^i, A^i, \prec^i, (\pi^i, N^i), h^i, \mathfrak{m}^i)$ be an i.d.view from T_{D_i} with $\prec^i = F$. Then (i) holds by Axiom B1. Consider (ii). Let $\langle \xi, w \rangle$ be a maximal thread in $\prec^i = F$. Since this is a basic protocol, $\langle \xi, w \rangle$ must be an endposition in Π^i.

Hence $w \in W^{iE}$ and by ID1, $w \in W^{oE}$.

(If): Suppose that (i) and (ii) hold. Then we define $W^i = \{w \in W^o : w \text{ occurs in } F\}$, $A^i = \{a \in A^o : a \text{ occurs in } F\}$, and $\prec^i = F$.

First, we show Axioms B1 and B2 for $(W^i, A^i, \prec^i)$. By (i), we have Axiom B1.

Consider Axiom B2. Let $\langle \xi, w \rangle \in \prec^i$ and $w \in W^{iD}$. Since F is conservative upon T_{D_i}, we have $w \in W^{oD}$. Thus, by (ii), there can be no maximal sequence in $\prec^i = F$ ending with w. Hence, $\prec^i$ has some feasible sequence $\langle \eta, (w, c), v \rangle$ so that both $\langle \eta, w \rangle$ and $\langle \eta, (w, c), v \rangle$ are supersequences of $\langle \xi, w \rangle$. By Axiom B1, $\langle \xi, (w, c), v \rangle$ is a feasible sequence. Thus, we have Axiom B2 for Π^i.

Next, we show that the conditions ID1 to ID6 are satisfied. The first part of ID1 follows from the supposition that F is conservative upon T_{D_i}. It follows from (ii) and B2 that $W^{iD} \subseteq W^{oD}$ and $W^{iE} \subseteq W^{oE}$. Condition ID2 follows from conservativeness. Condition ID3 follows from $F \supseteq \Delta T_{D_i}$. Since ID1, ID2 and ID3 are satisfied, π^i, h^i, and $\mathfrak{m}^i$ are uniquely determined by ID4, ID5, and ID6. ■

Under a weak additional condition, we can extend Theorem 4.1 to obtain an i.d.view satisfying Axioms N1, N2 and N3.

Theorem 4.4 (Existence of a Full I.D.view): Assume that D_i contains at least one decision piece w with $|A^o_w| \geq 2$. There is an i.d.view from T_{D_i} satisfying Axioms N1, N2 and N3.

$$\begin{array}{ccccccc}
z_1 & & z_2 & & z_3 & & \\
\uparrow_a & & \uparrow_a & & \uparrow_a & & \\
w_0 & \xrightarrow[b]{} & w_0 & \xrightarrow[b]{} & w_0 & \xrightarrow[b]{} & z_4
\end{array}$$

Fig.4.7

This theorem may generate an unnatural view: For example, the memory kit $T_{D_1^F}$ in Section 4.2 has an i.d.view with B1,B2 and N1-N3, described in Fig.4.7. We can, however, prove[6] Theorem 4.4, using this method of extending a given memory kit T_{D_i}.

5. Comparisons of Views

We have the existence of an i.d.view for a given memory kit T_{D_i}. As stated above, Definition 4.1 allows us to have a countably infinite number of i.d.views. A player often has information in addition to T_{D_i} to discriminate between views. One is a criterion to choose a small view. In this section, we consider "smallness" of a view, and also some comparisons of views based on the length of recall-k. In Section 8, we will consider some other sources for discriminating between views.

[6] A proof is found in http://www.sk.tsukuba.ac.jp/SSM/libraries/pdf1201/1207.pdf.

5.1. Small and Minimal Views

Let $(\Pi^i, \mathfrak{m}^i)$, $(\Pi'^i, \mathfrak{m}'^i)$ be two i.d.views from a memory kit T_{D_i}. Then, we say that $(\Pi^i, \mathfrak{m}^i)$ is *smaller than* $(\Pi'^i, \mathfrak{m}'^i)$ iff

$$\prec^i \;\subseteq\; \prec'^i . \tag{5.1}$$

An i.d.view $(\Pi^i, \mathfrak{m}^i)$ is *minimal* iff no i.d.view is strictly smaller than $(\Pi^i, \mathfrak{m}^i)$, and is the *smallest* iff it is smaller than every i.d.view from T_{D_i}. If the smallest view exists, it is unique. Since an i.d.view is finite, it follows from Theorem 4.1 that there exists a minimal i.d.view for any T_{D_i}. On the other hand, when there are more than one minimal views, the smallest view does not exist. In Example 4.1, the protocols of Fig.4.1 and 4.2 are both minimal.

The notion of "smallness" is based on the idea of not using more sequences than what are needed, which is the criterion of the economy of thought (Occam's Razor). We have some other criteria for smallness different from (5.1), e.g., the cardinality $\left|\prec^i\right|$. We can compare any two views by $\left|\prec^i\right|$, but the cardinality ignores the contents of the sequences, while (5.1) captures those contents.

There are some clear-cut cases to have the smallest i.d.view. A simple case is to have an i.d.view $(\Pi^i, \mathfrak{m}^i)$ with $\prec^i = \Delta T_{D_i}$. We state this fact as the next lemma.

Lemma 5.1. Let $(\Pi^i, \mathfrak{m}^i) = (W^i, A^i, \prec^i, (\pi^i, N^i), h^i, \mathfrak{m}^i)$ be an i.d.view from a memory kit T_{D_i}. If $\prec^i = \Delta T_{D_i}$, then $(\Pi^i, \mathfrak{m}^i)$ is the smallest i.d.view for T_{D_i}.

In all the examples in Section 4.2, we have $\prec^i = \Delta T_{D_i}$. Hence, minimal i.d.views often violates at least one of Axioms N1-N3. In Example 4.1, on the other hand, minimal i.d.views with $\Delta T_{D_i} \subsetneq \prec^i$ satisfy N1-N3.

A necessary and sufficient condition for the existence of an i.d.view with $\prec^i = \Delta T_{D_i}$ is given in the following corollary, which is a strengthening of (3.12).

Corollary 5.2. Let T_{D_i} be a memory kit. There is an i.d.view for T_{D_i} with $\prec^i = \Delta T_{D_i}$ if and only if for any maximal thread $\langle \xi, w \rangle$ in T_{D_i}, the piece w appears in W^{oE}.

Proof. This follows from Theorem 4.3. Indeed, if there is an i.d.view for T_{D_i} with $\prec^i$ $= \Delta T_{D_i}$, then condition (ii) of Theorem 4.3 is the latter statement. Conversely, if the latter holds, then by taking $F = \Delta T_{D_i}$ for Theorem 4.3, we have an i.d.view for T_{D_i} with $\prec^i = \Delta T_{D_i}$. ■

Kaneko-Kline [8] and [10] focused on the self-scope perfect-recall memory function $\mathfrak{m}_i^{SPR}$ and used the strict definition $\prec^i = \Delta T_{D_i}$ for an i.d.view. Here, a perfect-recall (not necessarily, self-scope) memory function $\mathfrak{m}_i^{PR}$ determines the smallest i.d.view.

Corollary 5.3. Let T_{D_i} be the memory kit of player i obtained from $\mathfrak{m}_i^o = \mathfrak{m}_i^{PR}$ on a closed domain D_i. There is the smallest i.d.view from T_{D_i} with $\prec^i = \Delta T_{D_i}$.

Proof. It suffices to show the latter part of Corollary 5.2 holds. Let $\langle \xi, w \rangle$ be any maximal thread in T_{D_i}. By the definition of T_{D_i}, $\mathfrak{m}_i^o\langle \eta, v \rangle = \langle \xi, w \rangle$ for some $\langle \eta, v \rangle \in D_i$. Since $\mathfrak{m}_i^o = \mathfrak{m}_i^{PR}$, we have $\mathfrak{m}_i^o\langle \eta, v \rangle = \langle \xi, w \rangle = \langle \eta, v \rangle_i$. Since D_i is closed, we find some endposition $\langle \zeta, u \rangle \in D_i$ such that $\langle \eta, v \rangle$ is an initial segment of $\langle \zeta, u \rangle$. However, since $\langle \xi, w \rangle$ is maximal in T_{D_i}, $\langle \eta, v \rangle$ and $\langle \zeta, u \rangle$ are the same. Hence, $w = v = u \in W^{oE}$. ■

Remark on an advantage of information protocols over extensive games: It is now apt to mention an advantage of the theory of information protocols over extensive games. If we adopt the theory of extensive games, the definition (5.1) takes a quite different form, since its primitives such as nodes and branches are hypothetical additions to observed information pieces. In Kaneko-Kline [9], this comparison is formulated by means of some structure-preserving functions, which is more complicated than (5.1). The theory of information protocols has this advantage in addition to its simpler axiomatic nature.

5.2. Views from Recall-k Memory Functions

Let us explore the recall-k memory function of a player and the associated i.d.views. If his memory ability is very weak, e.g., $k = 0$ or $k = 1$, then we might expect a great multiplicity of minimal i.d.views. However, as his ability gets stronger, the number of minimal i.d.views decreases. For large enough k, we know from Corollary 5.3 that there is a unique smallest view. First, we give some basic results for recall-k memory functions and focus on a particular i.d.view called the PR-view.

We now fix the domain Y_i of player i and also his domain of accumulation D_i. We are interested in how the i.d.views change when the length k of recall-k increases. We have the following result.

Lemma 5.4 (Higher Recall Reduces Possibilities): Let T_{D_i}, T'_{D_i} be the memory kits obtained from the recall-k, recall-k' memory functions. If $k > k'$, then every i.d.view for T_{D_i} is an i.d.view for T'_{D_i}.

Proof. Let $(\Pi^i, \mathfrak{m}^i)$ be an i.d.view for T_{D_i}. We show that $(\Pi^i, \mathfrak{m}^i)$ is also an i.d.view for T'_{D_i}. Since $\prec^i \supseteq \Delta T_{D_i}$ by ID3 for T_{D_i} and $\Delta T_{D_i} \supseteq \Delta T'_{D_i}$ by $k > k'$, we have $\prec^i \supseteq \Delta T'_{D_i}$, i.e., ID3 for T'_{D_i}. Since w is the last piece in $\mathfrak{m}^{Rk}\langle \xi, w \rangle$ and $\mathfrak{m}^{Rk'}\langle \xi, w \rangle$, we have $\{w \in W^o : w \text{ occurs in } T_{D_i}\} = \{w \in W^o : w \text{ occurs in } T'_{D_i}\}$. Thus, ID1, ID2, ID4, ID5, and ID6 for T'_{D_i} follow directly from the corresponding conditions for T_{D_i}. ■

The smallest i.d.view for a perfect-recall memory function $\mathfrak{m}_i^o = \mathfrak{m}_i^{PR}$ was given in Corollary 5.3. By Lemma 5.4, this view is also a view for any level of recall. To state this fact formally, we refer to this i.d.view as the *PR-view for* D_i denoted by $(\Pi^{RR}, \mathfrak{m}^{PI})$, where the set of feasible sequences $\prec^{PR}$ is defined to be $\Delta\{\mathfrak{m}_i^{PR}\langle \xi, w \rangle : \langle \xi, w \rangle \in D_i\}$. The closedness of D_i is sufficient for the PR-view to be an i.d.view.

Corollary 5.5 (PR-View is an i.d.view for any Recall-k Memory Function): Let $\mathfrak{m}_i^o$ be the recall-k memory function $\mathfrak{m}_i^{Rk}$ $(k \geq 0)$ on a closed domain D_i for player i. Then the PR-view $(\Pi^{RP}, \mathfrak{m}^{PI})$ for D_i is an i.d.view for T_{D_i}.

6. Kuhn's Distinguishability Condition

In the theory of extensive games, Kuhn [13] gave a mathematical condition on information sets, which is called "perfect recall" in the game theory literature. In our theory, it is no more than an attribute of information pieces and histories, since a recall ability of a player is expressed by a memory function. Here, it is formulated as follows: An information protocol Π satisfies the *distinguishability condition* for player i iff for any $\langle \xi, w \rangle, \langle \eta, v \rangle \in Y_i$,

$$\langle \xi, w \rangle_i \neq \langle \eta, v \rangle_i \text{ implies } w \neq v. \tag{6.1}$$

That is, when two positions have different personal histories up to Y_i, some difference is in the current pieces[7]. It does not express player i's recall ability. When $Y_i = \Xi^o$, (6.1) is the converse of Axiom N2 (Determination).

The distinguishability condition helps the player avoid unintended concatenations of memory threads in constructing an i.d.view. We have the following theorem, which will be proved in the end of this section.

Theorem 6.1 (Smallest Under Distinguishability): Let (6.1) hold for player i in Π^o, and $\mathfrak{m}_i^o = \mathfrak{m}_i^{Rk}$ for $k \geq 1$. Let D_i be a closed domain. The PR-view $(\Pi^{PR}, \mathfrak{m}^{PI})$ is the smallest i.d.view for T_{D_i} among the i.d.views for T_{D_i} satisfying (6.1).

Consider the full domain case of Mike's Bike with recall-1. Then, (6.1) is violated since he receives the same piece at several lattice points. Suppose, however, that we give Mike a distance meter, and we skew the town so that the distance from SW to each lattice point differs. Let $d_{\langle \xi, M \rangle}$ be the distance through the path $\langle \xi, M \rangle$. Then the new information piece Mike receives at each lattice point is described as:

$$M \wedge d_{\langle \xi, M \rangle}. \tag{6.2}$$

With the distance meter and skewed town, Mike receives a different information piece at each lattice point, and so (6.1) is satisfied.

When Mike has only recall-1 ability with no distance meter, he finds various possible manners to connect his memory threads. However, with a distance meter, he can distinguish between each lattice point, and find a unique smallest way to connect his

[7] We find some analogy between this idea and the Eve-hypothesis in the recent biological antholopology. It is based on the assumption that some different antholopological histories inherited through women can be distinguished by some differences in their current mitochondoria. See Mithen [14].

memory threads. In fact, he succeeds in constructing the true map as a smallest one! This argument does not hold for the recall-0 case.

Consider another example of an information piece: Let the information piece at each lattice point describe the complete history to it; for example, if Mike reaches the northeast M through the path $\langle \xi, M \rangle = \langle (SW, e), (S, n), (M, e), (M, n), M \rangle$, then his new piece is

$$(SW, e) \wedge (S, n) \wedge (M, e) \wedge (M, n) \wedge M. \tag{6.3}$$

This piece contains all information about his previous moves and so it may be interpreted as expressing "perfect recall".

Although both (6.2) and (6.3) are entirely different, both these examples satisfy (6.1). Condition (6.1) treats them in the same manner. The common property in these examples captured by (6.1) is that these pieces are distinguished. Hence, we call it "distinguishability", rather than "perfect recall".

We will use the following lemmas in the proof of Theorem 6.1. Now, we fix the objective situation $(\Pi^o, \mathfrak{m}^o)$ and a closed domain D_i.

Lemma 6.2. If Π^o satisfies (6.1) for player i, then so does the PR-view Π^{PR}.

Proof. Let $\langle \xi, w \rangle, \langle \eta, v \rangle$ be decision-positions in Π^{PR} with $\langle \xi, w \rangle \neq \langle \eta, v \rangle$. Since Π^{PR} is the PR-view, there are two positions $\langle \xi', w \rangle, \langle \eta', v \rangle \in \Xi^o$ such that $\langle \xi', w \rangle_i = \langle \xi, w \rangle$ and $\langle \eta', v \rangle_i = \langle \eta, v \rangle$. Since $\langle \xi, w \rangle \neq \langle \eta, v \rangle$, we have $\langle \xi', w \rangle \neq \langle \eta', v \rangle$. By (6.1) for Π^o, we have $w \neq v$. ■

Condition (6.1) guarantees that the information pieces represent the positions.

Lemma 6.3. Let $(\Pi^i, \mathfrak{m}^i)$ be a personal view of player i satisfying (6.1). Then, the function φ defined by $\varphi\langle \xi, w \rangle = w$ for all $\langle \xi, w \rangle \in \Xi^i$ is a bijection from Ξ^i to W^i.

Proof. By IP3 for Π^i, φ is a surjection. Let $\langle \xi, w \rangle, \langle \eta, v \rangle \in \Xi^i$ with $\langle \xi, w \rangle \neq \langle \eta, v \rangle$. By (6.1), we have $w \neq v$. ■

Proof of Theorem 6.1. Since D_i is a closed domain and $\mathfrak{m}_i^o = \mathfrak{m}_i^{Rk}$ $(k \geq 1)$, $(\Pi^{PR}, \mathfrak{m}^{PI})$ is an i.d.view for T_{D_i} by Corollary 5.5 and it satisfies (6.1) by Lemma 6.2. Now, let $(\Pi^i, \mathfrak{m}^i)$ be any i.d.view $(\Pi^i, \mathfrak{m}^i)$ for T_{D_i} satisfying (6.1). Since Π^i and Π^{PR} are i.d.views for T_{D_i}, we have $W^i = W^{PR}$ by ID1. By Lemma 6.3, for each $w \in W^i = W^{PR}$, there is a unique position to w in Π^{PR}, and correspondingly, a unique position to w in Π^i. We prove $\prec^{PR} = \Delta\Xi^{PR} \subseteq \Delta\Xi^i = \prec^i$.

We show by induction on the length of positions that for each $w \in W^{PR}$, the position $\langle \xi, w \rangle$ to w in Ξ^{PR} is a subsequence of the position $\langle \eta, w \rangle$ to w in Ξ^i. This implies $\Delta\Xi^{PR} \subseteq \Delta\Xi^i$.

For the base case, let $\langle \xi, w \rangle$ be a position of length 1 to w in Ξ^{PR}, i.e., $\langle \eta, w \rangle = \langle w \rangle$. The unique position $\langle \eta, w \rangle$ to w in Ξ^i is a supersequence of $\langle w \rangle$.

Next, let $\langle \xi, w \rangle = \langle (w_1, a_1), ..., (w_{m-1}, a_{m-1}), w_m \rangle$ be a position of length $m > 1$ in Ξ^{PR}. The inductive hypothesis is that the position $\langle \xi', w_{m-1} \rangle = \langle (w_1, a_1), ..., (w_{m-2}, a_{m-2}), w_{m-1} \rangle$ in Ξ^{PR} is a subsequence of the position $\langle \eta', w_{m-1} \rangle$ in Ξ^i. Hence:

$$\langle \xi, w_m \rangle = \langle \xi', (w_{m-1}, a_{m-1}), w_m \rangle \text{ is a subsequence of } \langle \eta', (w_{m-1}, a_{m-1}), w_m \rangle. \quad (6.4)$$

Since player i has the recall-k ($k \geq 1$) memory function on a closed domain D_i, the sequence $\langle (w_{m-1}, a_{m-1}), w_m \rangle \in \Delta T_{D_i}$. By IP3, $\langle (w_{m-1}, a_{m-1}), w_m \rangle$ is a subsequence of the position $\langle \eta, w_m \rangle$ in Ξ^i. This together with the uniqueness of a position for each piece by Lemma 6.3 implies that the position $\langle \eta', w_{m-1} \rangle$ is an initial segment of $\langle \eta, w_m \rangle$. Hence, there is a $w' \in W^i$ such that $\langle \eta', (w_{m-1}, a_{m-1}), w' \rangle$ is an initial segment of $\langle \eta, w_m \rangle$. Hence, $\langle \eta', (w_{m-1}, a_{m-1}), w_m \rangle$ is a subsequence of $\langle \eta, w_m \rangle$. By (6.4), the position $\langle \xi, w_m \rangle$ in Ξ^{PR} is a subsequence of $\langle \eta, w_m \rangle$ in Ξ^i. ■

$$\begin{tikzcd} z_1 & z_2 & z_3 & z_4 \\ & w_1 \arrow[ul, "a"] \arrow[u, "b"] & w_2 \arrow[u, "a"] \arrow[ur, "b"] & \end{tikzcd}$$

Fig.6.1

$$\begin{tikzcd} z_1 & z_2 & z_3 & z_4 \\ & w_1 \arrow[ul, "a"] \arrow[u, "b"] \arrow[r, "b"] & w_2 \arrow[u, "a"] \arrow[ur, "b"] & \end{tikzcd}$$

Fig.6.2

By Lemma 6.3, the cardinalities of Ξ^i and Ξ^{PR} are the same as that of $W^i = W^{PR}$. This appears to imply $\prec^i = \prec^{PR}$, but we have a counterexample for this equivalence. Consider Example 3.1 with $Y_2 = \Xi_2^o$ and $\mathfrak{m}_2^{SPR}$. Then, the PR-view for player 2 is given as Fig.6.1. However, the protocol of Fig.6.2 is an i.d.view for T_{D_2} satisfying (6.1), and is strictly larger than Fig.6.1, though those views have the same number of positions. The PR-view violates Axiom N1, while the other does N2.

The PR-view (Π^{PR}, m^{PI}) is smallest among those with (6.1), but perhaps not among all i.d.views.

7. Mike's Bike Commuting (2): Evolution of a View

A player's i.d.view evolves together with his memory kit over time as he accumulates more experiences. This evolution process is related to his memory ability and his behavioral tendencies. Here we explore this process using Mike's bike commuting.

From the Cane Domain to Skinny Domains: Suppose he has the memory function of recall-1. In the beginning, his experienced domain D_1^{cane} is simply the regular route. One i.d.view $(\Pi^1, \mathfrak{m}^1)$ is this regular route together with the perfect-information memory function $\mathfrak{m}^1 = \mathfrak{m}^{PI}$, which is depicted in Fig.7.1.A.

After some time, the domain of accumulation has grown by one additional route with the bold dotted arrows through the southwest M. The new domain D_1' is given as

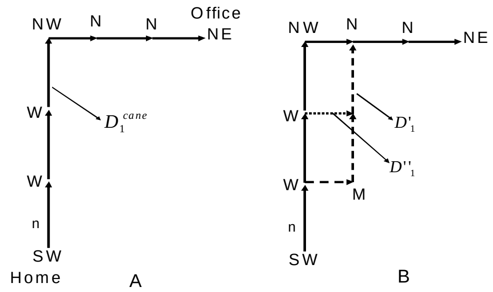

Figure 7.1: The Cane and Skinny Views

$D_1^{cane}\cup$the set of initial segments of $\langle(SW,n),(W,e),(M,n),(M,n),(N,e),(N,e),NE\rangle$, and is depicted in Fig.2.1.B. Here, the memory kit $T_{D_1'}$ is given as

$$T_{D_1'} = T_{D_1^{cane}} \cup \{\langle(W,e),M\rangle, \langle(M,n),M\rangle, \langle(M,n),N\rangle\}$$

This kit leads him to develop the expanded i.d.view of Fig.2.1.B[8].

Stagnation: If he tries another deviation from the north W, his experienced domain gains yet more positions and is given as $D_1'' = D_1'\cup$the set of initial segments of $\langle(SW,n),(W,n),(W,e),(M,n),(N,e),(N,e),NE\rangle$. See Fig.7.1.B. Since, however, the additional $\langle(W,e),M\rangle$ from the north W is already in memory kit $T_{D_1'}$, this does not change his memory kit, i.e., $T_{D_1'} = T_{D_1''}$. Hence, his i.d.view may be stagnant.

If his memory function is recall-2, then the newest memory kit $T_{D_1''}$ is strictly larger than the previous $T_{D_1'}$ and the original $T_{D_1^{cane}}$.

From Skinny Domain to the Full Domain: After many commutes, he has effectively experienced all places in the town. Consider the possible i.d.views when his memory is recall-k for small k.

Suppose that Mike has recall-1. First, the true map (Fig.2.1.A) and the larger one (Fig.7.2.A) are possible i.d.views. However, there are several minimal views, which are obtained by the procedure given in the proof of Theorem 4.1. Even if we restrict our attention to minimal views with the non-basic axioms N1-N3, we would find that Fig.2.1.A is not yet a minimal one. In this case, however, recall-2 is enough to guarantee that Fig.2.1.A is the smallest view. Thus additional requirements (or information) may help the player obtain a better view.

[8] Here, we represent this set of positions by the map of the form Fig.2.1.B. The positions themselves need not imply this representation. If Mike does this practice, he uses some additional assumptions on the town. Here, we simply use the map representation for simplicity.

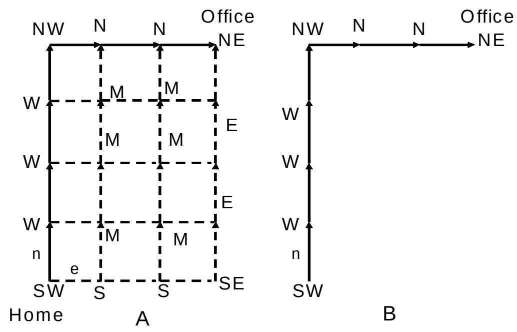

Figure 7.2: True and Imaginary

If we allow him to have a stronger memory, say recall-k but $k \leq 4$, then there is still a minimal i.d.view smaller than Fig.2.1.A. If he has memory function of recall-5 or higher (perfect-recall), then his smallest view is the true map.

True or Imaginary Structure: Let us return to the skinny domain case. Even though he trusts his own memory kit $T_{D_1^{cane}}$, there is another i.d.view having more repetitions of W and N. A possible i.d.view is depicted in Fig.7.2.B.

We can see this fact in the other way around: Suppose that the true town has the 5×4 street structure depicted as Fig.7.2.A. When Mike has the memory function of recall-1, the memory kit is the same as the previous memory kit $T_{D_1^{cane}}$ depicted in Fig.7.1.A. Hence, his i.d.view corresponding to Fig.7.1.A is an i.d.view in this case.

This interchangeability of the "true structure" and "an i.d.view" holds even when we go to the full domain. This fact means that with partiality in the memory ability, the truth is difficult to find.

8. Two Types of Behavioral Uses of I.D.Views

Now, a player brings and uses his view in the objective situation. We consider two such uses here. In Section 8.1, we study the problem of him checking his i.d.view with new experiences in the objective situation. We show that only the PR-view survives this checking when he checks his view in a sufficiently broad manner. In Section 8.2, we study how he may use his view to construct an optimal strategy for the objective situation. While his view may violate Axioms N1-N3, for optimal decision making, only the violation of N2 causes a serious problem.

8.1. Behavioral Checking of I.D.Views

When the memory function is partial, there may be multiple i.d.views for a player even if he focuses on minimal i.d.views. Multiplicity of i.d.views could be a serious problem if they suggest different behaviors. In this case, he may start looking for more clues to discriminate between those views. Here, we consider how he might use his new experiences to reject or accept some views.

Suppose that player i has an i.d.view Π^i, while keeping his regular behavior and making new trials within the domain D_i of accumulation. His memory is now aided by his view Π^i: At a position $\langle \xi, w \rangle$ in $(\Pi^o, \mathfrak{m}^o)$, he experiences his local memory $\mathfrak{m}_i^o\langle \xi, w \rangle$ and considers its relation to his view Π^i. He tries to identify each of his experiences with a position in his subjective view Π^i. Also, he checks successive positions in Π^i with successive experiences. In this process, he may find some incoherence between his view Π^i and experiences. If no such incoherence exists between them, he keeps $(\Pi^i, \mathfrak{m}^i)$.

Checking requires disciplined efforts for player i. It incurs large mental costs, which is contradictory to our basic presumption that the player has limited ability, time, and energy. Nevertheless, this is a matter of degree. Here, we explore the case where he is disciplined and has enough time and energy for sufficient checking. In this sense, the consideration here should be regarded as a limiting case.

To describe the above idea of successive checking, we define immediate successorship relations in Π^o (actually in D_i) and Π^i. We define the relation $\langle \xi, w \rangle <_a^{oI} \langle \eta, v \rangle$ in D_i iff $\langle \eta, v \rangle$ is an immediate successor of $\langle \xi, w \rangle$ in D_i with the choice of action a at w. Likewise, $\langle \xi', w' \rangle <_a^{iI} \langle \eta', v' \rangle$ is defined in Π^i, in which case, $\langle \xi', w' \rangle$ is, directly, the immediate predecessor position of $\langle \eta', v' \rangle$ with the choice a at $\langle \xi', w' \rangle$.

We say that player i cannot *falsify* (Π^i, m^i) *with his experiences* iff there is a function ψ from D_i to the set of positions Ξ^i in Π^i such that

F0: ψ is a surjection;

F1: for any $\langle \xi, w \rangle$ in D_i, if $\psi\langle \xi, w \rangle = \langle \eta, u \rangle$, then $w = u$;

F2: for any $\langle \xi, w \rangle, \langle \zeta, v \rangle$ in D_i, $\langle \xi, w \rangle <_a^{oI} \langle \zeta, v \rangle$ if and only if $\psi\langle \xi, w \rangle <_a^{iI} \psi\langle \zeta, v \rangle$.

The existence of ψ is required from the objective point of view, since player i does not know the structure of D_i. Nevertheless, F0, F1 and F2 describe the stability of an i.d.view against player i having the ability of effectively falsifying Π^i by his experiences. If F0 is violated, then he realizes after some time that some position in Π^i never occurs. Condition F1 means that he identifies his currently received piece u with some position ending with u in Π^i. Condition F2 is the requirement of player i's successive checking of his current and next positions in the objective Π^o and in his view Π^i.

The process of successive checking goes as follows. When he receives the first piece w in Π^o, he finds the minimal position $\langle w \rangle$ in Π^i. When he receives the next piece v after action a at w, he finds the immediate successor $\langle (w, a), v \rangle$ of $\langle w \rangle$ in Π^i. He continues this process, and when F0-F2 are satisfied, he finds no difficulties, and otherwise, he would find something wrong with his present view.

We say that the memory function $\mathfrak{m}_i^o$ is Y_i*-correct* iff $\mathfrak{m}_i^o\langle\xi, w\rangle$ is a subsequence of $\langle\xi, w\rangle_i$ for all $\langle\xi, w\rangle \in D_i$. The next theorem states that under the assumption of Y_i-correctness on $\mathfrak{m}_i^o$, the PR-view is the only i.d.view that cannot be falsified, which will be proved in the end of this subsection.

Theorem 8.1 (Falsification and the PR-View). Let D_i be a closed domain and $\mathfrak{m}_i^o$ a Y_i-correct memory function. Let $(\Pi^i, \mathfrak{m}^i)$ be an i.d.view from a memory kit T_{D_i}. Then $(\Pi^i, \mathfrak{m}^i)$ cannot be falsified with experiences if and only if $(\Pi^i, \mathfrak{m}^i)$ is the PR-view.

We now consider one important implication of Theorem 8.1. Suppose that player i considers his possible i.d.views from his memory kit and proceeds in the following way:

P1: his i.d.views $(\Pi^{i1}, \mathfrak{m}^{i1}), (\Pi^{i2}, \mathfrak{m}^{i2}), ...$ are enumerated;[9]

P2: If he brings the i.d.view $(\Pi^{ik}, \mathfrak{m}^{ik})$ with him to the objective situation and finds some incoherence with experiences, then he replaces it with the next view $(\Pi^{i(k+1)}, \mathfrak{m}^{i(k+1)})$.

If F0-F2 can be applied without errors, a consequence of Theorem 8.1 is that the above process terminates with the PR-view.

Nevertheless, the process of falsification may fail with some difficulties. As far as $(\Pi^i, \mathfrak{m}^i)$ is an i.d.view from the memory kit T_{D_i}, we find a function ψ satisfying requirement F1. Hence, we restrict our attention to a function ψ satisfying F1: Falsification itself is characterized by the negation of F0 or F2. The falsification of F2 is clear-cut: While he has received two successive memory threads $\mathfrak{m}_i^o\langle\xi, w\rangle$ and $\mathfrak{m}_i^o\langle\zeta, v\rangle$ with action a at w, $\psi\langle\xi, w\rangle$ and $\psi\langle\zeta, v\rangle$ do not successively occur in Π^i. On the other hand, falsification of F0 is more problematic: Trial-error has stochastic components, as described in Akiyama *et al* [1]. Even though some position in Π^{ik} has not occurred after many repetitions, player i may remain uncertain about whether it will ever occur. Here, he needs to make a doxastic decision (cf. Plato [16]) or a statistical decision to reject the present view $(\Pi^{ik}, \mathfrak{m}^{ik})$. There may be two types of errors as in statistical inference (cf., Rohatig [17], p.708). A Type I error occurs when player i waits for every position in Π^{ik} to occur and incorrectly does not reject the present (incorrect) view, and a Type II error occurs if he does not wait long enough for some position in $(\Pi^{ik}, \mathfrak{m}^{ik})$ and incorrectly rejects the (correct) PR-view. But once player i makes a doxastic decision that his PR-view is not falsified, it would be stable.

In Example 4.1 (Absent-minded Driver Game), player 1 has various minimal i.d.view such as Fig.4.2. Now, he brings this view in his mind when he drives. Then, unless he continues choosing c, he would not find anything wrong. But once he deviates to take

[9] Note that he does not need to enumerate all of these views before this process. Instead, he needs only some algorithm to have a "next" candidate from the present one.

action e, he would find his view could be incorrect. He may use a different one (or he may revise it in some way).

Proof of Theorem 8.1.(*If*): Let $(\Pi^i, \mathfrak{m}^i)$ be the PR-view. Then $\prec^i$ is given as $\Delta\{\langle \xi, w\rangle_i : \langle \xi, w\rangle \in D_i\}$, equivalently, the set of positions in Π^i is $\Xi^i = \{\langle \xi, w\rangle_i : \langle \xi, w\rangle \in D_i\}$. We define ψ by $\psi\langle \xi, w\rangle = \langle \xi, w\rangle_i$ for all $\langle \xi, w\rangle \in D_i$.Then, ψ satisfies F0 and F1. Consider F2. Suppose that $\langle \xi, w\rangle, \langle \eta, v\rangle$ in D_i and $\langle \xi, w\rangle <^{oI}_a \langle \eta, v\rangle$. Then, $\langle \xi, w\rangle_i, \langle \eta, v\rangle_i$ are positions in Π^i and $\psi\langle \xi, w\rangle = \langle \xi, w\rangle_i <^{iI}_a \langle \eta, v\rangle_i = \psi\langle \eta, v\rangle$. The converse can be seen by tracing back this argument.

(*Only-If*): Suppose that $(\Pi^i, \mathfrak{m}^i)$ cannot be falsified. Then there is a function ψ from D_i to Ξ^i satisfying F0, F1, and F2. We show by induction that $\psi\langle \xi, w\rangle = \langle \xi, w\rangle_i$ for all $\langle \xi, w\rangle \in D_i$.

Let $\langle \xi, w\rangle$ be a minimal position in D_i, i.e., no proper initial segment of $\langle \xi, w\rangle$ is in D_i. Then, $\langle \xi, w\rangle_i = \langle w\rangle$ since D_i is closed, i.e., it is a union of cane domains. Thus, $\mathfrak{m}^o_i\langle \xi, w\rangle = \langle w\rangle$ by Y_i-correctness. By F1, $\psi\langle \xi, w\rangle = \langle (w_1, a_1), ..., (w_m, a_m), u\rangle$ satisfies $u = w$. Now, suppose $\psi\langle \xi, w\rangle \neq \langle w\rangle$, i.e., $m \geq 1$. Since ψ is a surjection to Ξ^i by F0, there is a $\langle \eta, v\rangle$ in D_i such that $\psi\langle \eta, v\rangle = \langle (w_1, a_1), ..., (w_{m-1}, a_{m-1}), w_m\rangle$. Then, $\psi\langle \eta, v\rangle <^{iI}_{a_m} \psi\langle \xi, w\rangle$. Hence, by F2, we have $\langle \eta, v\rangle <^{oI}_{a_m} \langle \xi, w\rangle$, which contradicts the assumption that $\langle \xi, w\rangle$ is a minimal position in D_i. Hence, $\psi\langle \xi, w\rangle = \langle w\rangle = \langle \xi, w\rangle_i = \psi\langle \xi, w\rangle$.

Now, we suppose the inductive hypothesis that $\psi\langle \xi, w\rangle = \langle \xi, w\rangle_i$. Let $\langle \eta, v\rangle$ be the next position in D_i reached after taking a at w. Thus, $\langle \xi, w\rangle <^{oI}_a \langle \eta, v\rangle$. This implies $\langle \eta, v\rangle_i = \langle \langle \xi, w\rangle_i, a, v\rangle$. By F2, we have $\psi\langle \xi, w\rangle <^{iI}_a \psi\langle \eta, v\rangle$. It follows from this and the induction hypothesis that $\psi\langle \eta, v\rangle = \langle \psi\langle \xi, w\rangle, a, u\rangle$ for some u. Since $u = v$ by F1, we have $\psi\langle \eta, v\rangle = \langle \psi\langle \xi, w\rangle, a, v\rangle = \langle \langle \xi, w\rangle_i, a, v\rangle = \langle \eta, v\rangle_i$. ■

8.2. Violations of N1-N3 and their Effects on Decision Making with a View

The i.d.view he settles on may not be a full information protocol, as seen in Section 4.2. In this section, we discuss problems related to this. Suppose that player i finds an i.d.view (Π^i, m^i) by some method and decides to use it for his decision making. Then, this subjective view Π^i may violate Axioms N1-N3 even if it is the PR-view. We consider the problems arising from each violation:

Violation of N1(Root): The view has several trees;

Violation of N2(Determination): An exhaustive history does not determine a unique present information piece;

Violation of N3(History-Independent Extension): Some available actions at a position are not available at a position ending with the same information piece.

Since those violations are caused for different reasons, we should connect difficulties in decision making with the original objective situations causing the violations.

The violations of N1 and N2 may be caused by partial memory and the ignorance of another player, which are seen in Fig.4.6 and Fig.4.5. The main cause for the violation of N3 is the partiality of the domain D_i. It is easy to find an example of D_i and a memory function so that an i.d.view violates N3.

Now, we consider potential difficulties in decision making. If N3 is violated, the player should simply ignore the unused actions and he will face no serious problem in decision making. The violation of N2 is more serious as seen in Fig.4.5, where player 1 may not be able to decide between a and b. The violation of N1 may appear also to create difficulties with decision making, but the analysis below shows that this is not the case.

Let $(\Pi^i, \mathfrak{m}^i)$ be a personal view of player i, and let N^i be the player set of Π^i. In $(\Pi^i, \mathfrak{m}^i)$, the definition of a strategy needs a slight change: a *strategy* s_j for player $j \in N^i$ is defined by (3.11) and (8.1): for any position $\langle \xi, v\rangle \in \Xi_j^{iD}$,

$$s_j\langle \xi, v\rangle \in \{a : \langle \xi, (v, a), u\rangle \text{ is a position for some } u \text{ in } \Pi^i\}. \tag{8.1}$$

Since $(\Pi^i, \mathfrak{m}^i)$ is a subjective view, we use a different letter to denote a strategy. Now, we denote a profile of strategies for N^i by $s = (s_j)_{j\in N^i}$. Then, we say that a position $\langle \xi, (v_k, a_k), ..., (v_m, a_m), v_{m+1}\rangle$ is *s-compatible with* a position $\langle \xi, v_k\rangle$ iff $s_{\pi^i(v_k)}\langle \xi, v_k\rangle = a_k$, and $s_{\pi^i(v_t)}\langle \xi, (v_k, a_k), ..., (v_{t-1}, a_{t-1}), v_t\rangle = a_t$ for $t = k + 1, ..., m$.

Since Axiom N2 guarantees that for any $t = k+1, ..., m$, $\langle \xi, (v_{k-1}, a_{k-1}), ..., (v_{t-1}, a_{t-1}), v_t\rangle$ and a_t determine the unique piece v_{t+1}, we have the following.

Lemma 8.2 (Strategy-Determinancy). Let $(\Pi^i, \mathfrak{m}^i)$ be an i.d.view satisfying Axiom N2, and $s = (s_j)_{j\in N^i}$ a strategy profile Then, any position $\langle \xi, v_k\rangle$ uniquely determines an endposition which is s-compatible with $\langle \xi, v_k\rangle$.

For a position $\langle \xi, v\rangle$ and strategy profile $s = (s_j)_{j\in N^i}$, we define the *conditional payoff* $H_{i,\langle\xi,v\rangle}(s)$ to be the set of payoffs for player i given at the endpositions that are s-compatible with $\langle \xi, v\rangle$. In the example of Fig.4.4, $s_1(w_0) = a$ gives $H_{1,\langle w_0\rangle}(s) = \{3, 0\}$, and $s'_1(w_0) = b$ gives $H_{1,\langle w_0\rangle}(s') = \{1, 5\}$.

Suppose that s_{-i} is fixed. We say that a strategy s_i is *unambiguously optimal* at a position $\langle \xi, v\rangle$ iff for any strategy s'_i for player i,

$$\alpha \in H_{i,\langle\xi,v\rangle}(s_i, s_{-i}) \text{ and } \alpha' \in H_{i,\langle\xi,v\rangle}(s'_i, s_{-i}) \text{ imply } \alpha \geq \alpha'. \tag{8.2}$$

We say that s_i is *unambiguously optimal* iff it is unambiguously optimal at *all* decision positions $\langle \xi, v\rangle$ for player i in Π^i. These are relative concepts to the given s_{-i}. In other words, at any decision position of player i, the worst payoff from his given strategy is at least as good as the best from any alternative. In the example of Fig.4.4, no strategy is unambiguously optimal. Nevertheless, we have a guarantee that such a strategy exists for any i.d.view satisfying N2.[10].

Theorem 8.3 (Unambiguous Optimality with Axiom N2). Let $(\Pi^i, \mathfrak{m}^i)$ be an i.d.view that satisfies Axiom N2, and let s_{-i} be a profile of other players' strategies. Then, there is an unambiguously optimal strategy s_i for player i.

We remark that the theorem uses the fact that the subjective memory function $\mathfrak{m}^i$ is the perfect-information memory function $\mathfrak{m}^{PI}$. As mentioned earlier, since player i has this view in his mind, the perfect-information memory function makes sense. The violation of Axiom N2 still presents potential problems in this case.

If player i has a difficulty in decision making because his view violates N2, he may try to overcome it in various ways. He may modify his view to meet Axiom N2 such as in Theorem 4.4. Alternatively, he may use a weaker optimality criterion such as maximin optimality, i.e., he compares the worst payoffs compatible with each strategy. Another possibility is to look beyond his memory kit for some source of this indeterminacy, e.g., the move of an unobserved player.

9. Conclusions

First, we give an overall summary by highlighting the main findings along the steps given in Section 1.3.

Highlight 1: In Kaneko-Kline [8] and [10], an inductively derived view is effectively the same as the memory kit. This paper generalized the definition of an inductively derived view to allow a larger set of feasible sequences than the accumulated memory kit. This facilitates explorations of partiality in the objective memory function $\mathfrak{m}_i^o$.

Highlight 2: This generalized definition of an i.d.view allows general existence of an i.d.view, but there are multiple i.d.views. On the one hand, multiplicity may be regarded as a cost in that the analysis becomes more complicated. On the other hand, it leads us to a new frontier of inductive game theory that may help us to understand a variety of views observed in society.

Highlight 3: We considered minimal/smallest i.d.views. Minimality avoids large redundant views, but there may still be multiple minimal views. When $\mathfrak{m}_i^o$ has partiality, minimal views may not capture essential structures in that they are too small.

Highlight 4: Under Kuhn's distinguishability condition, a player may reach the PR-view as the smallest view. However, it is a demanding requirement for an information piece, and also the player is required to be able to analyze the hints hidden in each piece. In this sense, the result is not necessarily regarded as a resolution of multiplicity.

Highlight 5: Using Mike's bike commuting, we have shown that as the experienced domain is increased with time, a personal view is evolving, i.e., for some time, it is getting

[10] A proof is found in http://www.sk.tsukuba.ac.jp/SSM/libraries/pdf1201/1207.pdf

larger. However, he may get stuck even if he has more experiences.

Highlight 6: The next step is to check an i.d.view with new experiences in the objective situation. If he is fortunate, he reaches the PR-view and it becomes stable in the sense that he notices no incoherence between his view and his experiences. However, it could take a long time to reach the PR-view or he might even reject it or fail to reach it.

Highlight 7: Even if he takes a view as stable, e.g., the PR-view, he might meet some difficulties in his decision making. This is caused by the violations of Axioms N1-N3 for his view. The violation of Axiom N2 is more serious than the others: As long as Axiom N2 is satisfied, he can use his view for his payoff maximization.

We have many results on each step of the discourse, but there still remain many open problems. For example, what happens with the later part of this paper when the objective memory function has more incorrect components? For this problem, computer simulation may help. Another important problem is how each player gets the other player's understanding of the situation. We discuss this problem in Kaneko-Kline [11]. Nevertheless, treatments of individual experiences as well as individual views are basic for the further development of the new theory of other players' thoughts. We need to consider also interactions between various players' views and behavior. We anticipate that these explorations will lead to many new insights on human behavior and thought in society.

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