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Remote Work, Firm Technology, and the Spatial Economy: Generations, Care, and the Normalized Quadratic Approach

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JEL codes: R21, R31, J13, J14, J22, D13, D24, C51

Keywords: remote work; Normalized Quadratic cost function; household production; time allocation; intergenerational care; childcare; eldercare; spatial equilibrium; population aging; nonhomothetic preferences; sorting; rent gradient; flexible functional forms

1 Introduction

1.1 Motivation

How does the rise of remote work reshape the spatial economy when households must also care for children and the elderly? This question is urgent for two reasons. First, remote work has become a permanent feature of labor markets: Barrero, Bloom, and Davis (2023) document that the share of paid full days worked from home stabilized at approximately 28% in the United States, roughly five times the pre-pandemic level, and Dingel and Neiman (2020) estimate that 37% of US jobs can be performed entirely from home. Second, population aging is accelerating in advanced economies. Japan's elderly dependency ratio exceeded 50% in 2023, and most OECD countries face similar trajectories. The interaction between these two forces—remote work and aging—has received surprisingly little theoretical attention.

The standard urban economics toolkit provides only partial answers. The monocentric city model of Alonso (1964), Muth (1969), and Mills (1967) predicts that land rents decline with commuting cost, but it treats the household as a black box and does not model how time is allocated among work modes, household production, and care. The system-of-cities framework of Rosen (1979) and Roback (1982), refined by Albouy (2016) and Diamond (2016), shows how wages and rents jointly capitalize local amenities, but it does not distinguish between office and remote labor, nor does it model the household's internal time-allocation problem. The emerging literature on remote work and cities—Delventhal, Kwon, and Parkhomenko (2022), Davis, Ghent, and Gregory (2024), Gupta et al. (2022), Ramani and Bloom (2021)—makes important empirical and quantitative contributions, but typically treats household preferences in reduced form without modeling household production, care, or firm technology in structural detail.

This paper develops a unified theoretical framework that integrates five elements: (i) a firm production function with distinct office and remote labor inputs, (ii) a household time-allocation model with nonhomothetic preferences, (iii) separate production functions for childcare and eldercare, (iv) a generational structure that distinguishes children, workers, and the elderly, and (v) a spatial equilibrium with heterogeneous households. The framework builds on two intellectual traditions: the Becker–Diewert household-production theory (Becker, 1965; Schreyer and Diewert, 2014) and the flexible functional form literature in production economics (Diewert and Wales, 1987, 1992; Diewert, Nomura, and Shimizu, 2025).

1.2 The Role of the Normalized Quadratic

A central design choice in this paper is the adoption of the Normalized Quadratic (NQ) cost function of Diewert and Wales (1987) to model the firm's production technology. This choice merits discussion.

The NQ has three properties that make it uniquely suited to the present application. First, it is a *flexible functional form*: it can provide a second-order approximation to an arbitrary twice-differentiable cost function at a given point (Diewert, 1971, 1974). Second, unlike the translog (Christensen, Jorgenson, and Lau, 1973), it can impose *global concavity* on the cost function without destroying flexibility, through the decomposition $A = -UU'$ introduced by Diewert and Wales (1987). Third, it allows the *elasticity of substitution* between any pair of inputs to vary with the entire factor-price vector, unlike the CES which restricts all pairwise elasticities to a single constant.

The last property is decisive for the present application. The central question of how easily firms can substitute remote labor for office labor—and how this substitutability changes as remote-work technology improves—cannot be answered by a model that assumes constant elasticities. Diewert, Nomura, and Shimizu (2025) demonstrate this point forcefully using US aggregate data: across their six-input model, they find that 8 pairs of inputs are substitutes and 7 are complements, a pattern that no CES function can generate. The NQ nests the CES as a testable restriction, and its rejection in macroeconomic data strongly motivates the adoption of the NQ in the present context.

A further advantage of the NQ is tractability. The unit cost function for a three-input model ($N = 3$: office labor, remote labor, capital) requires only 12 free parameters. Shephard's Lemma delivers closed-form factor demand equations, and the zero-profit condition provides an additional estimating equation. This yields a system of four equations per observation—a tractable empirical system. By contrast, fully parameterizing the household's utility function with K arguments would require $K(K+1)/2$ second-order parameters, which for $K = 6$ (our utility function) is 21. The strategy of this paper is therefore to invest modeling effort on the firm side (NQ) while keeping the household side rich in structure but parsimonious in functional-form assumptions.

1.3 Overview of the Model

The model has three layers: the firm, the household, and the spatial equilibrium.

The firm layer. A representative firm produces output using three inputs: office labor (supplied at the workplace S_2), remote labor (supplied at the household's residence S_1), and capital services (a composite of office space and non-space capital). The firm's technology is described by a CRS production function whose dual is the NQ unit cost function. Shephard's Lemma generates factor demand equations that depend on the full vector of factor prices, permitting variable elasticities of substitution. A technology parameter θ governs the relative productivity of remote labor; as θ increases, the NQ cost function shifts in a way that raises the remote-labor intensity of production.

The household layer. A household consists of up to three generations: children (Generation 1), working-age adults (Generation 2), and the elderly (Generation 3). Only Generation 2 supplies market labor and makes the time-allocation decisions. The household's residential space is decomposed into living space and home-office space, subject to an internal allocation constraint. Seven production functions operate within the model: the firm's NQ technology, housing services (from living space and durables), effective remote labor (from office space, IT capital, and time), home leisure, external leisure, general housework (all three from Shimizu (2026)), childcare (from care goods, own time, and purchased services), and eldercare (similarly structured). Preferences are nonhomothetic, with subsistence levels for both housing and care that depend on household composition.

The spatial equilibrium layer. Households are heterogeneous in their remote-work productivity (ϕ^i), generational composition (n_1^i, n_2^i, n_3^i), and amenity preferences ($\boldsymbol{\xi}^i$). Free mobility across regions equalizes the amenity-augmented indirect utility of each type. The resulting rent function capitalizes commuting distance, amenity access, and the prices of childcare and eldercare services. Population aging—an increase in the elderly dependency ratio—reshapes the equilibrium by shifting care demand, altering the sorting pattern, and changing the spatial distribution of rents.

1.4 Main Results

The paper derives five main sets of results.

First, I derive a shadow-price bound that extends the foundational result of Schreyer and Diewert (2014). In the base model, the shadow price of leisure satisfies $w^* \leq \min\{w_S^r, w\}$. The present model extends this to $w^{*i} \leq \min\{w_S^r, w_{CC}^r, w_{EC}^r, \tilde{w}^o, \tilde{w}^h\}$, where w_{CC}^r and w_{EC}^r are the regional prices of childcare and eldercare, $\tilde{w}^o \equiv (w^o - \delta_1 \eta d_{12})/(1+\eta d_{12})$ is the commuting-adjusted effective office wage, and $\tilde{w}^h \equiv w^h - c^\Omega(\rho_R^r, p_{IT})$ is the net remote wage after equipment cost. Each bound has a structural interpretation linked to a specific first-order condition.

Second, I derive a spatial full-income identity that incorporates care surpluses: the gaps $(w_{CC}^r - w^{*i})T_{l3}^{CC,i*}$ and $(w_{EC}^r - w^{*i})T_{l3}^{EC,i*}$ between the market price of care and the shadow value of time, weighted by own-care time, appear alongside the familiar labor and housework surpluses.

Third, I show that the equilibrium rent gradient with respect to commuting distance is $-(w^{*i} + \delta_1)\eta T_{l1}^{o,i*}/q_R^{i*}$, which is flatter than Shimizu (2026)'s $-(w^* + \delta_1)/q_R^*$ because the factor $\eta T_{l1}^{o,i*}$ declines as remote work substitutes for office work.

Fourth, I establish a sorting equilibrium in which face-to-face workers sort to central locations, remote-capable workers sort to distant locations, and the rent function is convex. The nonhomothetic preference structure implies income-dependent sorting, and the care requirements generate an additional dimension of spatial selection.

Fifth, I show that population aging flattens the aggregate rent gradient through two channels (more elderly-only households with zero commuting, and more remote eldercare), generates a rent rotation between residential and commercial real estate, and amplifies the welfare value of remote-work technology.

1.5 Related Literature

This paper contributes to and connects four literatures.

Household production and time allocation. Becker (1965) initiated the economic analysis of household time allocation. Gronau (1977) distinguished between market work, home production, and leisure, while Pollak and Wachter (1975) emphasized the importance of joint production and the limits of market substitutes. Heckman (1974) analyzed shadow prices and corner solutions. The most general treatment of leisure, household work, and market labor with corner solutions is Schreyer and Diewert (2014), whose framework I extend spatially in the companion paper (Shimizu, 2026) and further in the present paper. Nevo and Wong (2022) provide recent evidence on the elasticity of substitution between time and market goods.

Flexible functional forms and production theory. The NQ functional form was introduced by Diewert and Wales (1987) and extended by Diewert and Wales (1988) and Diewert and Wales (1992). Kohli (1991, 1993) applied the NQ to GDP functions. Diewert, Nomura, and Shimizu (2025) provide the most recent application, estimating a six-input NQ joint cost function for the US economy and demonstrating the inadequacy of the CES. The present paper applies the NQ to the specific problem of office-versus-remote labor substitution. See Diewert (2022) for a comprehensive survey of duality in production theory and McFadden (1978) and Shephard (1953) for foundational contributions.

Remote work and urban structure. Bloom et al. (2015) provided early experimental evidence on remote-work productivity. Barrero, Bloom, and Davis (2023) document the post-pandemic evolution of work from home. Dingel and Neiman (2020) classify occupations by remote-work feasibility. Delventhal, Kwon, and Parkhomenko (2022) develop a quantitative spatial model of remote work. Davis, Ghent, and Gregory (2024) analyze the welfare consequences of work-from-home technology in an Alonso–Muth framework. Gupta et al. (2022) study the revaluation of commercial and residential real estate. Ramani and Bloom (2021) document the "donut effect" of remote work on city structure. Stanton and Tiwari (2021) examine housing consumption and remote-work costs. Liu and Su (2023) analyze the geography of remote work. Arntz, Ben Yahmed, and Berlingieri (2022) study heterogeneity in remote work by gender and parenthood. The present paper differs from this literature in its explicit modeling of firm technology (via the NQ), household production, care, and generational structure.

Urban spatial equilibrium and sorting. The monocentric tradition of Alonso (1964), Muth (1969), and Mills (1967) is extended by the quality-of-life literature (Rosen, 1979; Roback, 1982; Albouy, 2016). Diamond (2016) and Couture et al. (2024) analyze sorting by skill and income. Ahlfeldt et al. (2015) and Monte, Redding, and Rossi-Hansberg (2018) develop quantitative spatial models with commuting. The present paper extends this tradition by deriving sorting from a Becker–Diewert optimization problem with firm technology, care, and generational structure.

1.6 Organization

The paper is organized as follows. Section 2 summarizes the two foundational models: Schreyer and Diewert (2014) and the companion paper (Shimizu, 2026). Section 3 develops the NQ firm production function. Section 4 introduces the generational structure, seven production functions, nonhomothetic preferences, and the household's optimization problem. Section 5 derives the first-order conditions, shadow-price bounds, care-outsourcing conditions, the space-allocation condition, and the spatial full-income identity. Section 6 characterizes the equilibrium rent gradients and the rent rotation. Section 7 develops the sorting equilibrium. Section 8 analyzes the effects of population aging. Section 9 discusses identification and presents a quantitative calibration for Japan. Section 10 concludes. Online Appendix A provides all proofs. Online Appendix B develops the extended case taxonomy and share equations. Online Appendix C presents the NQ joint cost estimation for Japan (1970–2023) that underlies the calibration in Section 9.

2 Foundations: Schreyer and Diewert (2014) and the Companion Paper

This section provides a self-contained summary of the two models on which the present paper builds.

2.1 The Schreyer–Diewert (2014) Model of Household Time Valuation

Schreyer and Diewert (2014) develops a generalized Becker model in which leisure time, household work time, and market labor time each have separate utility implications. This generalization is crucial because it allows the shadow value of time to differ from the observed market wage.

2.1.1 Preferences and production

A single-person household has preferences

$$U = U(q_R, Q_k, Q_H, T_{l1}, T_{l2}), \tag{2.1}$$

where q_R is a market consumption good, $Q_k = F(q_k, T_k)$ is leisure services produced from leisure goods q_k and leisure time T_k, $Q_H = H(q_H, T_{l2} + q_S)$ is household services produced from household goods q_H and a composite of own household work time T_{l2} and purchased household services q_S (which are perfect substitutes), T_{l1} is market labor time, and T_{l2} is own household work time. U is concave, $U_1, U_2, U_3 > 0$, $U_4 \leq 0$ (disutility of market work), $U_5 \leq 0$ (disutility of housework). F and H are linearly homogeneous and concave.

2.1.2 Constraints and optimization

Time constraint: $T_{l1}+T_{l2}+T_k = T$. Budget constraint: $p_R q_R + p_k q_k + p_H q_H + w_S^r q_S \leq wT_{l1} + Y$. The household maximizes U subject to both constraints and nonnegativity.

By the Karlin–Uzawa theorem, there exist multipliers $\lambda^* > 0$ (budget) and $\omega^* > 0$ (time), and the shadow price of leisure is $w^* \equiv \omega^*/\lambda^* > 0$.

2.1.3 The Schreyer–Diewert bound

At an interior solution with $q_S^* > 0$ and $T_{l1}^* > 0$:

$$0 < w^* \leq \min\{w_S^r, w\}. \tag{2.2}$$

This follows from the first-order conditions: $U_4 = -\lambda^*(w - w^*)$ implies $w \geq w^*$, and $U_5 = -\lambda^*(w_S^r - w^*)$ implies $w_S^r \geq w^*$. The bound is the central result of Schreyer and Diewert (2014) and the starting point for all extensions in the companion paper and the present paper.

2.1.4 Full income and the four-case taxonomy

Define full prices $P_k^* = c^F(p_k, w^*)$ and $P_H^* = c^H(p_H, w_S^r)$, where c^F, c^H are unit cost functions dual to F, H. Full income is $FI = Y + w^*T$. Full consumption is $FC = p_R q_R^* + P_k^* Q_k^* + P_H^* Q_H^*$. The identity is:

$$FC - (w_S^r - w^*)T_{l2}^* - (w - w^*)T_{l1}^* = FI. \tag{2.3}$$

Four cases arise from corner solutions on (q_S^*, T_{l1}^*): Case 1 ($q_S^* > 0, T_{l1}^* > 0$), Case 2 ($q_S^* = 0, T_{l1}^* > 0$), Case 3 ($q_S^* > 0, T_{l1}^* = 0$), Case 4 ($q_S^* = 0, T_{l1}^* = 0$). In Cases 2 and 4, the shadow price of household work $w_H^* \equiv U_3 H_2/\lambda^*$ is unobserved and satisfies $w_H^* \leq w_S^r$. See Online Appendix B for the share equations in each case.

2.2 The Companion Paper: The Spatial Becker Model

Shimizu (2026) extends Schreyer and Diewert (2014) to a spatial setting with three locations: S_1 (home), S_2 (workplace), S_3 (amenity venues). Leisure is decomposed into home leisure $Q_{k1} = F(q_{k1}, T_{k1})$ and external leisure $Q_{k2} = G(q_{k2}, T_{k2})$, both with linearly homogeneous production functions. Housing services q_R enter utility directly.

2.2.1 Commuting and leisure travel

Commuting to S_2 is a fixed cost: time d_{12} and money $\delta_1 d_{12}$ per period. Leisure travel to S_3 is proportional: time $d_{13}T_{k2}$ and money $\delta_2 d_{13} T_{k2}$. The worker time constraint is:

$$T_{l1} + T_{l2} + T_{k1} + (1 + d_{13})T_{k2} = T - d_{12} \equiv \tilde{T}. \quad (2.4)$$

2.2.2 Main results of the companion paper

Shadow-price bound: $0 < w^* \leq \min\{w_S^r, w\}$ (Theorem 3.4). The spatial extension does not change the bound; it changes the equilibrium value of w^* through the effective time endowment $\tilde{T}$ and the full price of external leisure $\pi_{k2} = w^*(1 + d_{13}) + \delta_2 d_{13}$.

Spatial full income: $FI^W = Y - \delta_1 d_{12} + w^*(T - d_{12})$ (Theorem 3.8). Commuting reduces full income through both the time opportunity cost $w^* d_{12}$ and the monetary cost $\delta_1 d_{12}$.

Rent gradients (Theorem 5.2):

$$\partial p_R^*/\partial d_{12} = -(w^* + \delta_1)/q_R^* < 0, \tag{2.5}$$

$$\partial p_R^*/\partial d_{13} = -(w^* + \delta_2)T_{k2}^*/q_R^* \leq 0, \tag{2.6}$$

$$\partial p_R^*/\partial w_S^r = -q_S^*/q_R^* \leq 0. \tag{2.7}$$

Regional household-services supply: $w_S^r \in (0, +\infty]$, with $w_S^r = +\infty$ in regions without market household services (Assumption 3.13 and Proposition 3.15).

Local amenities: An amenity vector A^r enters indirect utility through $\tilde{V}^{Wr} = \Phi(V^{Wr}, A^r)$, generating $\partial p_R^*/\partial A_k^r > 0$.

Superlative index theory: Under linearly homogeneous F, G, H, Fisher and Törnqvist indexes yield a Regional Utility Index (Section 6 of Shimizu 2026). This is *not* used in the present paper.

Econometric framework: The companion paper develops share equations for four corner-solution regimes (Section 8 and Appendices A–B). The present paper extends this taxonomy to accommodate remote work and care.

2.2.3 What the present paper changes

Five structural modifications:

1. Market labor splits: $T_{l1} \to (T_{l1}^o, T_{l1}^h)$, with wages w^o and w^h endogenized through the NQ.
2. Commuting becomes proportional to office work: $d_{12} \to \eta T_{l1}^o d_{12}$.
3. Housing is decomposed: $q_R \to R(S_L, q_D)$ with space constraint $S_L + S_W = \bar{S}$.
4. Care is separated from housework: $Q_H \to (Q_H, Q_{CC}, Q_{EC})$.
5. A generational structure (n_1, n_2, n_3) is introduced, with care minima $n_1\bar{q}_{CC}$ and $n_3^i\bar{q}_{EC}$.

3 The Firm's Production Technology

This section develops the production side of the model. The firm's technology determines the equilibrium wages for office and remote labor, which in turn drive the household's time-allocation decisions.

3.1 Production Setup

A representative firm produces a single output Y^f using three inputs: office labor T^o_{l1} supplied at the workplace S_2, remote labor T^h_{l1} supplied at the household's residence S_1, and capital services K.

Assumption 3.1 (Firm technology)**.** The firm's production possibilities set is

$$S^\Gamma = \{(Y^f, T^o_{l1}, T^h_{l1}, K) : Y^f \leq \Gamma(T^o_{l1}, T^h_{l1}, K; \theta)\}, \tag{3.1}$$

where $\Gamma : \mathbb{R}^3_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is:

(a) continuous, concave, and twice differentiable in (T^o_{l1}, T^h_{l1}, K);

(b) linearly homogeneous in (T^o_{l1}, T^h_{l1}, K) (constant returns to scale);

(c) increasing in each input: $\Gamma_1 > 0$, $\Gamma_2 > 0$, $\Gamma_3 > 0$;

(d) parameterized by $\theta \geq 0$, the remote-work technology level, with Γ_2/Γ_1 increasing in θ.

The CRS assumption is standard in the Diewert tradition. As Diewert, Nomura, and Shimizu (2025, p. 2673) note, "it proves to be too difficult to distinguish technical progress from returns to scale using time series data." Under CRS, the production function admits a dual representation through the unit cost function.

3.2 The NQ Unit Cost Function

Duality and flexible functional forms. By the duality theorem of Shephard (1953) and McFadden (1978), any production technology satisfying CRS, concavity, and monotonicity can be represented equivalently through its unit cost function $c^\Gamma(\mathbf{w};\theta)$, which is concave, linearly homogeneous, and nondecreasing in input prices $\mathbf{w}$. This dual representation is computationally and econometrically advantageous: Shephard's Lemma $\partial c^\Gamma/\partial w_n = x_n/Y^f$ delivers factor demand equations directly from the cost function, without requiring explicit solution of the primal optimization problem.

A *flexible functional form* is a functional form for c^Γ that provides a second-order local approximation to an arbitrary twice-differentiable cost function at a given point (Diewert and Wales, 1987). Flexibility is essential for empirical work: imposing an inflexible form (such as Cobb–Douglas, which restricts all Allen–Uzawa elasticities of substitution to unity, or CES, which restricts them to a common constant σ) embeds untestable restrictions that may distort all inference on substitution behavior.

Several flexible functional forms have been proposed: the *translog* (Christensen, Jorgenson, and Lau, 1973), which is locally flexible but does not impose global concavity; the *generalized Leontief* (Diewert, 1971); and the *Normalized Quadratic* (NQ) of Diewert and Wales (1987, 1988). The NQ is distinguished from the translog by two properties that are critical for this paper: (i) global concavity can be imposed by a simple parameterization without sacrificing flexibility, and (ii) the factor demand equations are linear in the unknown parameters, facilitating estimation by standard linear methods.

Economic contribution of the NQ in the present context. The choice of the NQ is not merely technical. The central empirical question of this paper is whether, and by how much, remote labor substitutes for office labor as the technology parameter θ rises. The *Morishima elasticity of substitution* $\sigma^M_{j\to n}$—the percentage increase in the x_n/x_j ratio when w_j rises by one percent, holding output and all other prices fixed—measures this substitutability. Under the CES, $\sigma^M_{j\to n} = \sigma$ for all pairs (n,j): the model imposes a single substitution parameter for all input pairs simultaneously. The NQ allows $\sigma^M_{j\to n}$ to differ across pairs and to vary with factor prices and θ. This flexibility is precisely what is needed to distinguish office-remote substitution ($\sigma^M_{L2\to L1}$) from capital-labor substitution ($\sigma^M_{K_{ME}\to L2}$), as the empirical results in Appendix C demonstrate.

Zero-profit condition. Let $\mathbf{w} \equiv (w^o, w^h, r_K)' \in \mathbb{R}^3_{++}$ denote the vector of factor prices: w^o is the office-labor wage, w^h is the remote-labor wage, and r_K is the user cost of capital services. Under CRS and competitive factor markets, the firm's zero-profit condition is:

$$p_Y = c^\Gamma(\mathbf{w}; \theta), \tag{3.2}$$

where p_Y is the output price and $c^\Gamma : \mathbb{R}^3_{++} \times \mathbb{R}_+ \to \mathbb{R}_{++}$ is the unit cost function.

Let $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)' \in \mathbb{R}^3_+$ with $\boldsymbol{\alpha}'\mathbf{1}_3 = 1$ be a vector of predetermined nonnegative normalization weights.

Definition 3.2 (NQ unit cost function)**.** The Normalized Quadratic unit cost function is:

$$c^\Gamma(\mathbf{w}; \theta) = \mathbf{w}'\mathbf{d}(\theta) + \frac{1}{2}\frac{\mathbf{w}'A(\theta)\mathbf{w}}{\boldsymbol{\alpha}'\mathbf{w}}, \tag{3.3}$$

where $\mathbf{d}(\theta) = (d_1(\theta), d_2(\theta), d_3(\theta))' \geq \mathbf{0}$ is a vector of linear coefficients and $A(\theta)$ is a 3×3 symmetric matrix, both depending on the technology parameter θ.

The function (3.3) has two terms. The first, $\mathbf{w}'\mathbf{d}(\theta)$, is linear in factor prices and corresponds to a Leontief (fixed-coefficients) technology in which $d_n(\theta)$ units of input n are required per unit of output. The second, $\frac{1}{2}\mathbf{w}'A(\theta)\mathbf{w}/(\boldsymbol{\alpha}'\mathbf{w})$, is a quadratic-over-linear term that captures substitution possibilities. When $A = \mathbf{0}$, there is no substitution and the technology is Leontief.

Assumption 3.3 (NQ regularity conditions)**.**

(a) *Linear homogeneity in* $\mathbf{w}$: $\mathbf{w}'\mathbf{d}$ is linearly homogeneous. For $\mathbf{w}'A\mathbf{w}/(\boldsymbol{\alpha}'\mathbf{w})$: numerator is degree 2, denominator degree 1, so the ratio is degree 1 overall.

(b) *Global concavity*: $A(\theta)$ is parameterized as $A(\theta) = -U(\theta)U(\theta)'$ where $U(\theta)$ is 3×3 lower triangular satisfying

$$U(\theta)'\mathbf{1}_3 = \mathbf{0}_3. \tag{3.4}$$

Then $\mathbf{w}'A(\theta)\mathbf{w} = -\|U(\theta)'\mathbf{w}\|^2 \leq 0$ for all $\mathbf{w} \geq \mathbf{0}$, ensuring global concavity of c^Γ in $\mathbf{w}$. The restriction (3.4) serves two purposes. First, it ensures $A\mathbf{1} = -UU'\mathbf{1} = \mathbf{0}$ (since $U'\mathbf{1} = \mathbf{0}$), which is required for c^Γ to be linearly homogeneous in $\mathbf{w}$ (the quadratic term must satisfy Euler's theorem). Second, it imposes 3 restrictions on the lower triangular U (which has 6 free elements), leaving 3 free parameters per matrix.

(c) *Normalization*: There exists a reference price vector $\mathbf{w}^0 \in \mathbb{R}^3_{++}$, compatible with (3.4), such that $A(\theta^0)\mathbf{w}^0 = \mathbf{0}_3$. This is satisfied by choosing $\alpha = \mathbf{w}^0/(\mathbf{1}'\mathbf{w}^0)$, i.e., setting the normalization weights proportional to the reference prices. At $\mathbf{w}^0$, $c^\Gamma(\mathbf{w}^0;\theta^0) = (\mathbf{w}^0)'\mathbf{d}(\theta^0)$.

(d) *Regularity region*: Factor demands are non-negative in a neighborhood $\mathcal{W}$ of $\mathbf{w}^0$:

$$\frac{\partial c^\Gamma(\mathbf{w};\theta)}{\partial w_n} > 0 \quad \text{for all } \mathbf{w} \in \mathcal{W},\ n = 1, 2, 3. \tag{3.5}$$

At $\mathbf{w}^0$, this requires $d_n(\theta) > 0$ for all n (strict positivity of Leontief coefficients). The regularity region $\mathcal{W}$ is the set of prices where (3.5) holds; empirical work should verify that the data lie in $\mathcal{W}$.

Remark 3.4 (Parameter count under $U'\mathbf{1} = \mathbf{0}$)**.** A 3×3 lower triangular matrix has 6 free elements. The restriction $U'\mathbf{1} = \mathbf{0}$ is 3 scalar equations, leaving $6 - 3 = 3$ free parameters per matrix. Since we have U_0 and U_1 (from the time-varying parameterization 3.13), the substitution block contributes $3 + 3 = 6$ free parameters. Together with 6 parameters in $\mathbf{d}(\theta)$, the grand total is 12. This count is correct and consistent with Diewert and Wales (1987).

3.3 Factor Demand Equations

Proposition 3.5 (NQ factor demands)**.** *Under the NQ specification (3.3), by Shephard's Lemma (Shephard, 1953), the cost-minimizing demand for input n per unit of output is:*

$$\frac{x_n}{Y^f} = \frac{\partial c^{\Gamma}(\mathbf{w};\theta)}{\partial w_n} = d_n(\theta) + \frac{[A(\theta)\mathbf{w}]_n}{\boldsymbol{\alpha}'\mathbf{w}} - \frac{\alpha_n}{2}\frac{\mathbf{w}'A(\theta)\mathbf{w}}{(\boldsymbol{\alpha}'\mathbf{w})^2}, \quad n = 1, 2, 3, \tag{3.6}$$

where $[A(\theta)\mathbf{w}]_n$ denotes the n-th element of the vector $A(\theta)\mathbf{w}$.

Proof. See Online Appendix A.1. ■

Writing out the three equations explicitly with $\mathbf{w} = (w^o, w^h, r_K)'$ and $A(\theta) = [a_{nj}(\theta)]$:

$$\frac{T^o_{l1}}{Y^f} = d_1(\theta) + \frac{a_{11}w^o + a_{12}w^h + a_{13}r_K}{\alpha_1 w^o + \alpha_2 w^h + \alpha_3 r_K} - \frac{\alpha_1}{2}\frac{\mathbf{w}'A\mathbf{w}}{(\boldsymbol{\alpha}'\mathbf{w})^2}, \tag{3.7}$$

$$\frac{T^h_{l1}}{Y^f} = d_2(\theta) + \frac{a_{12}w^o + a_{22}w^h + a_{23}r_K}{\alpha_1 w^o + \alpha_2 w^h + \alpha_3 r_K} - \frac{\alpha_2}{2}\frac{\mathbf{w}'A\mathbf{w}}{(\boldsymbol{\alpha}'\mathbf{w})^2}, \tag{3.8}$$

$$\frac{K}{Y^f} = d_3(\theta) + \frac{a_{13}w^o + a_{23}w^h + a_{33}r_K}{\alpha_1 w^o + \alpha_2 w^h + \alpha_3 r_K} - \frac{\alpha_3}{2}\frac{\mathbf{w}'A\mathbf{w}}{(\boldsymbol{\alpha}'\mathbf{w})^2}. \tag{3.9}$$

These are the fundamental estimating equations for the firm side of the model. Each equation expresses the input-output coefficient as the sum of three terms: a base Leontief requirement $d_n(\theta)$, a substitution effect $[A\mathbf{w}]_n/(\boldsymbol{\alpha}'\mathbf{w})$, and a normalization correction $-\alpha_n(\mathbf{w}'A\mathbf{w})/(2(\boldsymbol{\alpha}'\mathbf{w})^2)$.

3.4 Elasticities of Factor Substitution

Proposition 3.6 (NQ elasticity matrix)**.** *The 3×3 matrix of partial derivatives of factor demands with respect to factor prices is:*

$$\nabla_{\mathbf{w}}\mathbf{x}(\mathbf{w},\theta) = Y^f\left[\frac{A(\theta)}{\boldsymbol{\alpha}'\mathbf{w}} - \frac{A(\theta)\mathbf{w}\,\boldsymbol{\alpha}'}{(\boldsymbol{\alpha}'\mathbf{w})^2} - \frac{\boldsymbol{\alpha}\,[A(\theta)\mathbf{w}]'}{(\boldsymbol{\alpha}'\mathbf{w})^2} + \frac{\boldsymbol{\alpha}\boldsymbol{\alpha}'(\mathbf{w}'A(\theta)\mathbf{w})}{(\boldsymbol{\alpha}'\mathbf{w})^3}\right], \tag{3.10}$$

where $\mathbf{x} = (T^o_{l1}, T^h_{l1}, K)'$.

Proof. See Online Appendix A.2. ■

Definition 3.7 (Price elasticities of factor demand)**.** The elasticity of demand for input n with respect to the price of input j is:

$$E^x_{nj}(\mathbf{w}, \theta) \equiv \frac{\partial \ln x_n}{\partial \ln w_j} = \frac{[\nabla_{\mathbf{w}} \mathbf{x}]_{nj} \cdot w_j}{x_n}. \tag{3.11}$$

Definition 3.8 (Allen–Uzawa and Morishima elasticities of substitution)**.** Two standard measures of pairwise substitutability between inputs n and j are:

(a) *Allen–Uzawa elasticity of substitution* (AES):

$$\sigma^{AU}_{nj} \equiv \frac{c^{\Gamma} \cdot \partial^2 c^{\Gamma} / \partial w_n \partial w_j}{(\partial c^{\Gamma} / \partial w_n)(\partial c^{\Gamma} / \partial w_j)}. \tag{3.12}$$

Inputs n and j are Allen substitutes if $\sigma^{AU}_{nj} > 0$ and complements if $\sigma^{AU}_{nj} < 0$. The AES is symmetric: $\sigma^{AU}_{nj} = \sigma^{AU}_{jn}$.

(b) *Morishima elasticity of substitution* (MES):

$$\sigma^{M}_{j \to n} \equiv E^x_{nj} - E^x_{jj} = \left. \frac{\partial \ln(x_n / x_j)}{\partial \ln w_j} \right|_{Y^f,\, w_{k \neq j} \text{ fixed}}. \tag{3.13}$$

The MES is the percentage change in the input ratio x_n/x_j when w_j rises by one percent, all else fixed. Unlike the AES, the MES is *asymmetric*: $\sigma^M_{j \to n} \neq \sigma^M_{n \to j}$ in general.

Remark 3.9 (Why report Morishima rather than Allen–Uzawa elasticities)**.** For two-input technologies the two measures coincide. For three or more inputs, the AES conflates the own-price response of input j with the cross-price response of input n, and its sign can be misleading when multiple inputs interact. The MES focuses directly on the observable input-ratio response to a single price change and connects naturally to the rent-gradient formula (6.2): a higher $\sigma^M_{L2 \to L1}$ means a larger shift from office to remote labor when office wages rise, amplifying the decline in $T^{o,i*}_{l1}$ and hence the flattening of the rent gradient. We therefore report MES throughout Appendix C and Section 9.4.

Remark 3.10 (Variable elasticities)**.** The elasticities E^x_{nj} depend on the full factor-price vector $\mathbf{w}$ and on the technology parameter θ. They are *not* constant. This is the key distinction from the CES.

Remark 3.11 (CES as a special case of NQ)**.** The CES production function $\Gamma = [\gamma_o(T^o_{l1})^\rho + \gamma_h(T^h_{l1})^\rho + \gamma_K K^\rho]^{1/\rho}$ generates a unit cost function in which $\sigma^{AU}_{nj} = 1/(1-\rho)$ for all pairs, and consequently $\sigma^M_{j\to n} = 1/(1-\rho)$ for all (n, j): a single substitution parameter governs all pairwise relationships simultaneously. The NQ encompasses the CES as a special case but allows each pair to have distinct, price-dependent elasticities. In the nine-input system estimated in Appendix C, office labor (L2) and remote labor (L1) have $\sigma^M_{L2\to L1} \approx 6.1$ in the post-pandemic regime, while machinery capital and office labor have $\sigma^M_{K_{ME}\to L2} \approx 0.43$—a fourteen-fold difference that no CES can accommodate.

Remark 3.12 (Empirical evidence from Diewert, Nomura, and Shimizu 2025)**.** Diewert, Nomura, and Shimizu (2025) estimate a six-input NQ joint cost function for the US economy (1970–2022) and find that 8 pairs of inputs are substitutes and 7 are complements under the AES. This pattern of mixed complementarity is inconsistent with any CES specification, in which all pairs must be Allen substitutes with the same constant elasticity. Their finding directly motivates the use of the NQ in the present paper's firm technology and in the Japanese calibration of Appendix C.

3.5 Technical Change and Remote-Work Technology

The technology parameter θ represents the state of remote-work technology: IT infrastructure quality, collaboration software, managerial practices for remote teams, and the cumulation of learning-by-doing in remote work. I model the effects of θ on the cost function through the dependence of both $\mathbf{d}(\theta)$ and $A(\theta)$ on θ.

Assumption 3.13 (Parameterization of technical change)**.**

(a) The linear coefficients depend linearly on θ:

$$d_n(\theta) = d_n^0 + d_n^1\theta, \quad n = 1, 2, 3. \tag{3.14}$$

A sufficient condition for remote-labor bias is $d_2^1 > 0 > d_1^1$: the base remote-labor requirement rises and the base office-labor requirement falls with θ.

(b) The substitution matrix varies with θ:

$$A(\theta) = -\left[\left(1 - \frac{\theta}{\bar{\theta}}\right) U_0 U_0' + \frac{\theta}{\bar{\theta}} U_1 U_1'\right], \tag{3.15}$$

where U_0 and U_1 are lower triangular matrices with $U_0'\mathbf{1}_3 = \mathbf{0}_3$ and $U_1'\mathbf{1}_3 = \mathbf{0}_3$, and $\bar{\theta} > 0$ is a normalization constant.

The parameterization (3.15) follows the time-varying approach of Diewert, Nomura, and Shimizu (2025), replacing calendar time with the technology parameter θ. At $\theta = 0$ (pre-remote era), $A(0) = -U_0U_0'$. At $\theta = \bar{\theta}$ (full technology), $A(\bar{\theta}) = -U_1U_1'$. For intermediate values, $A(\theta)$ is a convex combination that preserves negative semidefiniteness (and hence global concavity).

Definition 3.14 (Remote-labor bias of technical change)**.** The remote-labor bias is:

$$B^h(\theta, \mathbf{w}) \equiv \frac{\partial}{\partial\theta}\left(\frac{T_{l1}^h/Y^f}{T_{l1}^o/Y^f}\right) = \frac{\partial}{\partial\theta}\left(\frac{\partial c^\Gamma/\partial w^h}{\partial c^\Gamma/\partial w^o}\right). \tag{3.16}$$

If $B^h > 0$, technical change is biased toward remote labor: the remote-to-office labor ratio increases with θ at constant factor prices.

Proposition 3.15 (Sufficient condition for remote-labor bias at the normalization point)**.** *Under Assumptions 3.1–3.13, at $\mathbf{w} = \mathbf{w}^0$ (where $A(\theta^0)\mathbf{w}^0 = \mathbf{0}$ and the substitution terms* *vanish), a sufficient condition for $B^h(\theta^0, \mathbf{w}^0) > 0$ is:*

$$d_2^1 d_1^0 > d_1^1 d_2^0,$$

i.e., the remote-labor coefficient grows proportionally faster with θ than the office-labor coefficient. The stronger condition $d_2^1 > 0 > d_1^1$ implies this. Away from $\mathbf{w}^0$, terms involving $\partial A/\partial\theta$ enter B^h and may offset or reinforce the Leontief-coefficient effect; the sufficient condition at $\mathbf{w}^0$ does not extend globally without additional assumptions on $(U_1 U_1' - U_0 U_0')$.

Proof. See Online Appendix A.3. ■

3.6 Capital Decomposition: Office Space and Non-Space Capital

Assumption 3.16 (Capital structure)**.** The composite capital input K is produced from office space K_O and non-space capital K_N (IT servers, equipment):

$$K = \Xi(K_O, K_N), \tag{3.17}$$

where Ξ is linearly homogeneous. The user cost of office space is r_O^r (region-specific) and the user cost of non-space capital is r_N (common across regions). The composite user cost is $r_K = c^\Xi(r_O^r, r_N)$.

To discuss cross-effects between office labor and office space, define the *expanded production function*

$$\tilde{\Gamma}(T_{l1}^o, T_{l1}^h, K_O, K_N; \theta) \;\equiv\; \Gamma\big(T_{l1}^o, T_{l1}^h, \Xi(K_O, K_N); \theta\big), \tag{3.18}$$

which treats K_O and K_N as direct inputs. All regularity properties of Γ carry over to $\tilde{\Gamma}$.

Proposition 3.17 (Office space demand)**.** *By Shephard's Lemma applied to c^Ξ, the demand for office space per unit of capital is $K_O/K = \partial c^\Xi/\partial r_O^r$. The total office-space intensity of production is:*

$$\frac{K_O}{Y^f} = \frac{\partial c^\Gamma}{\partial r_K} \cdot \frac{\partial c^\Xi}{\partial r_O^r}. \tag{3.19}$$

Proposition 3.18 (Office space saving from remote work)**.** *Suppose office labor and office space are complements in the expanded technology:*

$$\frac{\partial^2 \tilde{\Gamma}}{\partial T_{l1}^o \, \partial K_O} > 0. \tag{3.20}$$

Then an increase in θ (which reduces T_{l1}^o / Y^f by Proposition 3.15) reduces office-space intensity:

$$\frac{\partial (K_O / Y^f)}{\partial \theta} < 0. \tag{3.21}$$

Proof. See Online Appendix A.4. ■

Remark 3.19 (Commercial rent implications)**.** The decline in office-space intensity translates into lower demand for commercial real estate. In a market with inelastic commercial space supply (the short-run case), this reduces the equilibrium commercial rent r_O^r. This is one side of the "rent rotation" developed in Section 6.

3.7 The Endogenous Wage Ratio

Definition 3.20 (Remote-work wage ratio)**.**

$$\phi(\theta, \mathbf{w}) \equiv \frac{w^{h*}}{w^{o*}} = \frac{\Gamma_2(T_{l1}^{o*}, T_{l1}^{h*}, K^*; \theta)}{\Gamma_1(T_{l1}^{o*}, T_{l1}^{h*}, K^*; \theta)}. \tag{3.22}$$

Under CRS, ϕ is determined by the zero-profit condition (3.2) and the factor demand equations (3.7)–(3.9).

Proposition 3.21 (The wage ratio under NQ vs. CES)**.**

(a) Under the CES, the equilibrium wage ratio equals the marginal rate of technical substitution: $w^{h*}/w^{o*} = \Gamma_2/\Gamma_1 = (\gamma_h/\gamma_o)(T_{l1}^h/T_{l1}^o)^{\rho-1}$, which depends only on the input ratio T_{l1}^h/T_{l1}^o and the parameter ρ.

(b) Under the NQ, competitive factor pricing requires $w^{o*} = p_Y \Gamma_1$ and $w^{h*} = p_Y \Gamma_2$ at the

optimum. By duality, $w^n = p_Y \cdot \partial c^\Gamma / \partial (x_n/Y^f)^{-1}$; the ratio w^{h*}/w^{o*} is not *equal to the ratio of Shephard derivatives $(\partial c^\Gamma/\partial w^h)/(\partial c^\Gamma/\partial w^o)$—the latter is the input ratio T^h_{l1}/T^o_{l1}, not the wage ratio. The wage ratio $\phi \equiv w^{h*}/w^{o*} = \Gamma_2/\Gamma_1$ depends on the* entire *input vector $(T^{o*}_{l1}, T^{h*}_{l1}, K^*)$, which in turn depends on $(p_Y, \mathbf{w}, \theta)$ through the factor demand equations (3.7)–(3.9) and market clearing.*

Remark 3.22 (Partial equilibrium treatment of wages)**.** The firm's zero-profit condition (3.2) and the three factor-demand equations (3.7)–(3.9) together characterize *cost-minimizing input ratios* at given prices $\mathbf{w}$. To determine the *level* of wages, market clearing conditions are required. In the spatial equilibrium of Section 7, labor markets clear region-by-region: $\sum_i \mu^i(r)[T^{o,i*}_{l1} + T^{h,i*}_{l1}] = L^r$ (total labor demand equals supply) and the capital market clears at the economy-wide user cost r_K. These clearing conditions, combined with (3.2)–(3.9), pin down (w^{o*}, w^{h*}, r^*_K) in equilibrium. The present paper takes factor prices as *given from the equilibrium* and characterizes household behavior and rent gradients conditional on those prices, following the partial-equilibrium tradition of Roback (1982) and Shimizu (2026).

3.8 Estimating Equations for the Firm

The NQ model generates a system of estimating equations that can be taken to data.

Proposition 3.23 (NQ estimating system)**.** *The firm model generates $N + 1 = 4$ equations per observation:*

1. *Three factor demand equations (3.7)–(3.9) (from Shephard's Lemma).*
2. *One zero-profit condition (3.2) (from CRS).*

The system has cross-equation restrictions: the same parameters $\mathbf{d}(\theta)$ and $A(\theta)$ appear in all four equations.

Remark 3.24 (Parameter count)**.** With $N = 3$ inputs:

- $\mathbf{d}(\theta)$: 3 intercepts (d^0_1, d^0_2, d^0_3) and 3 slopes $(d^1_1, d^1_2, d^1_3) = 6$ parameters.

- $A(\theta)$: each of U_0, U_1 is a 3×3 lower triangular matrix with $3(3+1)/2 = 6$ elements, minus 3 restrictions from $U'\mathbf{1} = \mathbf{0}$, giving 3 free elements per matrix, for a total of 6 parameters.
- **Grand total: 12 parameters.**

This is a tractable system. For comparison, a fully flexible Diewert-type household utility function with $K = 6$ arguments would require $K(K+1)/2 = 21$ second-order parameters for the utility function alone, plus the parameters of the household production functions. This asymmetry—investing model complexity on the firm side while keeping the household side structurally rich but functionally parsimonious—is a deliberate design choice.

4 Households, Generations, and Production

4.1 Generational Structure

Definition 4.1 (Household composition)**.** A household of type i consists of:

- $n_1^i \geq 0$ children (Generation 1): dependent, require childcare, do not supply labor.
- $n_2^i \geq 0$ working-age adults (Generation 2): decision-makers when present, supply labor, provide care.
- $n_3^i \geq 0$ elderly members (Generation 3): dependent, require eldercare, do not commute.

Total household size is $N^i = n_1^i + n_2^i + n_3^i \geq 1$. A *worker household* satisfies $n_2^i \geq 1$; an *elderly-only household* satisfies $n_2^i = 0$, $n_3^i \geq 1$. The main analysis (Sections 5–7) focuses on worker households; Section 8 introduces elderly-only households as a separate household type, whose optimization problem is stated in Appendix B, Case RW8.

Assumption 4.2 (Generational roles)**.**

(a) Generation 1 members reside at S_1, do not supply market labor, and require childcare of at least $\bar{q}_{CC}$ per child per period.

(b) Generation 2 members are the decision-makers. They allocate their aggregate time endowment $n_2^i \cdot T$ across seven activities: office work, remote work, housework, childcare, eldercare, home leisure, and external leisure.

(c) Generation 3 members reside at S_1, do not supply market labor or commute, and require eldercare of at least $\bar{q}_{EC}(h^{3,i})$ per person per period, where $h^{3,i} \in [0, 1]$ is a health index ($h = 1$: fully healthy, $h = 0$: fully dependent).

Remark 4.3 (Spatial binding of Generations 1 and 3)**.** Only Generation 2 faces the commuting decision. Generations 1 and 3 are bound to S_1. This has a direct analytical consequence: the commuting-distance rent gradient affects only Generation-2 decisions, while Generations 1 and 3 create care-driven location demands that depend on w_{CC}^r, w_{EC}^r, and regional amenities A^r but not on d_{12}.

4.2 Seven Production Functions

The model contains seven production functions. The first is the firm's NQ technology from Section 3. The remaining six operate at the household level.

4.2.1 Production Function 1: Housing services

Assumption 4.4 (Residential space decomposition)**.** The household occupies total residential floor area $\bar{S}^i$ (a choice variable), allocated between living space S_L^i and home-office space S_W^i:

$$S_L^i + S_W^i = \bar{S}^i. \tag{4.1}$$

The per-unit residential rent is ρ_R^r (price per square meter per period in region r). Minimum living space depends on household composition:

$$S_L^i \geq \bar{S}_L(n_1^i, n_2^i, n_3^i) \equiv n_1^i s_1 + n_2^i s_2 + n_3^i s_3, \tag{4.2}$$

where s_g is the per-capita minimum space requirement for generation g. Elderly members requiring in-home care may need $s_3 > s_2$ (e.g., for barrier-free space and care equipment).

Definition 4.5 (Housing services production function)**.** Housing services are produced from living space and household durables:

$$q_R = R(S_L^i, q_D^i), \tag{4.3}$$

where q_D^i is the flow of household durables (furniture, appliances) and R is continuous, concave, and linearly homogeneous on $\mathbb{R}_+^2$, with $R_1 > 0$ and $R_2 > 0$. The unit cost function dual to R is $c^R(\rho_R^r, p_D)$.

Remark 4.6 (Why decompose housing)**.** In Shimizu (2026), q_R was purchased at price p_R and entered the utility function directly. This was adequate when S_1 was purely a consumption site. With remote work, S_1 serves dual purposes: living and working. The decomposition (4.1) captures the direct trade-off within the dwelling: allocating more space to a home office reduces living space unless total area expands. This is the mechanism through which remote work increases residential space demand and total housing rent $\rho_R^r \bar{S}^i$.

4.2.2 Production Function 2: Home-office (effective remote labor)

Definition 4.7 (Home-office production function)**.** The effective supply of remote labor $T_{l1}^{h,\text{eff}}$ requires three inputs: home-office space S_W^i, IT capital flow q_{IT}^i, and raw remote-work time T_{l1}^h:

$$T_{l1}^{h,\text{eff}} = \Omega(S_W^i, q_{IT}^i, T_{l1}^h), \tag{4.4}$$

where Ω is continuous, concave, and linearly homogeneous on $\mathbb{R}_+^3$, with $\Omega_1, \Omega_2, \Omega_3 > 0$.

For a given level of effective remote labor, the household minimizes the cost of (S_W, q_{IT}), taking raw time T_{l1}^h as given. The *inner cost-minimization problem* at fixed T_{l1}^h is:

$$\min_{S_W, q_{IT} \geq 0} \left\{\rho_R^r S_W + p_{IT} q_{IT}\right\} \quad \text{s.t.} \quad \Omega(S_W, q_{IT}, T_{l1}^h) \geq T_{l1}^{h,\text{eff}}. \tag{4.5}$$

Because the firm pays the remote worker for effective labor at wage w^h per unit of $T_{l1}^{h,\text{eff}}$, the household treats $T_{l1}^{h,\text{eff}} = T_{l1}^h$ (one efficiency unit per unit of raw time, up to normalization) and the relevant object is the *variable cost of equipping one unit of remote-work time*:

$$m(\rho_R^r, p_{IT}; T_{l1}^h) \;\equiv\; \min_{S_W,\, q_{IT}} \big\{\rho_R^r S_W + p_{IT} q_{IT} : \Omega(S_W, q_{IT}, T_{l1}^h) \geq T_{l1}^h\big\}. \tag{4.6}$$

By linear homogeneity of Ω, this simplifies to $m = T_{l1}^h \cdot c^\Omega(\rho_R^r, p_{IT})$ where c^Ω is the unit cost function of the two-input sub-problem conditional on one unit of T_{l1}^h:

$$c^\Omega(\rho_R^r, p_{IT}) \;\equiv\; \min_{s_W,\, q} \big\{\rho_R^r s_W + p_{IT} q : \Omega(s_W, q, 1) \geq 1\big\}. \tag{4.7}$$

Assumption 4.8 (Remote-work income and cost)**.** The household receives net remote-work income $\tilde{w}^h T_{l1}^h$ per period, where

$$\tilde{w}^h \;\equiv\; w^h - c^\Omega(\rho_R^r, p_{IT}) \tag{4.8}$$

is the *net remote wage* after deducting the per-unit equipment cost. We assume $\tilde{w}^h > 0$ (remote work is privately profitable net of equipment costs).

Remark 4.9 (Reduced-form embedding)**.** Under Assumption 4.8, the inner minimization (4.5) is solved at any optimum, and the equilibrium demands for S_W^{i*} and q_{IT}^{i*} are recovered by Shephard's Lemma:

$$S_W^{i*} = T_{l1}^{h*} \cdot \frac{\partial c^\Omega}{\partial \rho_R^r}, \qquad q_{IT}^{i*} = T_{l1}^{h*} \cdot \frac{\partial c^\Omega}{\partial p_{IT}}. \tag{4.9}$$

The household's outer problem is then stated solely in terms of $(T_{l1}^h, \bar{S}^i)$, with S_W^{i*} and q_{IT}^{i*} substituted out. This avoids the inconsistency of having S_W and q_{IT} appear in the budget without entering utility directly.

Proposition 4.10 (Home-office space demand)**.** *At an interior optimum,* $\partial S_W^{i*} / \partial T_{l1}^{h*} =$

$\partial c^{\Omega}/\partial \rho_R^r > 0$: *more remote work increases residential space demand.*

4.2.3 Production Functions 3–4: Home leisure and general housework

Retained from Shimizu (2026):

$$Q_{k1} = F(q_{k1}, T_{k1}), \tag{4.10}$$

$$Q_{k2} = G(q_{k2}, T_{k2}), \tag{4.11}$$

$$Q_H = H(q_H, T_{l2} + q_S). \tag{4.12}$$

F, G, H are continuous, concave, and linearly homogeneous. In H, own household work time T_{l2} and purchased household services q_S are perfect substitutes, following Schreyer and Diewert (2014). The price of purchased household services is $w_S^r \in (0, +\infty]$.

4.2.4 Production Function 5: Childcare

Definition 4.11 (Childcare production function)**.** Childcare services are produced from childcare goods, own childcare time, and purchased childcare:

$$Q_{CC}^i = C^{CC}(q_{CC}^i, T_{l3}^{CC,i} + q_{CS,CC}^i), \tag{4.13}$$

where q_{CC}^i is childcare goods (food for children, educational materials), $T_{l3}^{CC,i}$ is Generation-2's own childcare time, $q_{CS,CC}^i$ is purchased childcare (daycare, babysitting), and own time and purchased services are perfect substitutes in the second argument. C^{CC} is linearly homogeneous. The price of purchased childcare is $w_{CC}^r \in (0, +\infty]$.

Remark 4.12 (Childcare minimum)**.** The household must satisfy $Q_{CC}^i \geq n_1^i \bar{q}_{CC}$: each child requires a minimum amount of care per period. When $w_{CC}^r = +\infty$ (no market childcare available), the household must self-provide all childcare: $T_{l3}^{CC,i} \geq n_1^i \bar{q}_{CC}/C_2^{CC}$.

4.2.5 Production Function 6: Eldercare

Definition 4.13 (Eldercare production function)**.** Eldercare services are produced from eldercare goods, own eldercare time, and purchased eldercare:

$$Q^i_{EC} = C^{EC}(q^i_{EC}, T^{EC,i}_{l3} + q^i_{CS,EC}), \tag{4.14}$$

with the same structure as childcare. C^{EC} is linearly homogeneous. The price of purchased eldercare is $w^r_{EC} \in (0, +\infty]$.

Remark 4.14 (Eldercare minimum)**.** $Q^i_{EC} \geq n^i_3 \bar{q}_{EC}(h^{3,i})$, where $\bar{q}_{EC}$ is decreasing in $h^{3,i}$ (healthier elderly need less care).

Remark 4.15 (Why separate childcare from eldercare)**.** Three asymmetries motivate the separation:

(i) *Time structure*: Childcare peaks in the morning and evening (school schedules), while eldercare may require continuous or unpredictable presence. Commuting time savings (morning/evening) therefore disproportionately benefit childcare.

(ii) *Market prices*: Eldercare is typically more expensive than childcare ($w^r_{EC} > w^r_{CC}$) due to the higher skill intensity and the physical demands of eldercare.

(iii) *Disutility*: The disutility of eldercare ($|\Psi^i_6|$) is often larger than that of childcare ($|\Psi^i_5|$) due to the emotional and physical burden of caring for declining family members.

These asymmetries generate different outsourcing decisions, different responses to remote-work technology, and different spatial implications.

4.3 Preferences

Assumption 4.16 (Nonhomothetic preferences)**.** Generation-2 members of household i have preferences:

$$U^i = \Psi^i\left(\underbrace{u^i(q_R - \bar{q}^i_R,\, Q_{k1},\, Q_{k2},\, Q_H,\, Q^i_{CC} - n^i_1\bar{q}_{CC},\, Q^i_{EC} - n^i_3\bar{q}_{EC})}_{\text{argument 1}},\, \underbrace{T^o_{l1}}_{2},\, \underbrace{T^h_{l1}}_{3},\, \underbrace{T_{l2}}_{4},\, \underbrace{T^{CC}_{l3}}_{5},\, \underbrace{T^{EC}_{l3}}_{6}\right), \tag{4.15}$$

where $\Psi^i : \mathbb{R}_+ \times \mathbb{R}^5_- \to \mathbb{R}$; argument 1 is non-negative and arguments 2–6 are non-positive (disutility of labor and care time). Ψ^i_k (u^i_j) denotes the partial derivative with respect to argument k (j).

(a) $\bar{q}^i_R = R(\bar{S}^i_L, \bar{q}_D) \geq 0$: subsistence housing, household-composition dependent.

(b) $u^i : \mathbb{R}^6_+ \to \mathbb{R}_+$: continuous, concave, linearly homogeneous.

(c) Ψ^i is *concave* and twice continuously differentiable; $\Psi^i_1 > 0$ (increasing in subutility).

(d) $\Psi^i_2, \Psi^i_3, \Psi^i_4, \Psi^i_5, \Psi^i_6 \leq 0$: disutility of office work, remote work, housework, childcare, and eldercare respectively.

Remark 4.17 (Nonhomotheticity)**.** The subsistence parameters $\bar{q}^i_R$ and $(n^i_1\bar{q}_{CC}, n^i_3\bar{q}_{EC})$ generate Engel curves that are not rays from the origin. This is the Stone–Geary (Gorman, 1961) structure applied to the Becker–Diewert framework. The income elasticity of housing demand differs from unity, and the expenditure shares of housing and care decline with income.

Remark 4.18 (Nesting)**.** Setting $n^i_1 = n^i_3 = 0$ (no dependents) and $T^h_{l1} = 0$ (no remote work) recovers the companion paper's utility structure. Setting additionally $\bar{q}^i_R = 0$ gives the homothetic version.

4.4 Constraints

Definition 4.19 (Time constraint)**.**

$$(1 + \eta\, d_{12})T^o_{l1} + T^h_{l1} + T_{l2} + T^{CC}_{l3} + T^{EC}_{l3} + T_{k1} + (1 + d_{13})T_{k2} = n^i_2 \cdot T, \tag{4.16}$$

where $\eta > 0$ is the commuting-frequency parameter and d_{12}, d_{13} are distances.

The right-hand side is $n_2^i T$ because each Generation-2 member contributes T units of time.

Definition 4.20 (Budget constraint)**.** After substituting the inner optimum (4.9), total residential cost is $\rho_R^r(S_L^i + S_W^{i*}) = \rho_R^r S_L^i + c^\Omega(\rho_R^r, p_{IT})T_{l1}^h$, and the IT cost $p_{IT}q_{IT}^{i*}$ is likewise absorbed. The reduced-form budget constraint is:

$$\begin{aligned}\rho_R^r S_L^i + p_D q_D^i + p_2 q_{k1} + p_1 q_{k2} + p_H q_H + w_S^r q_S \\ + p_{CC} q_{CC}^i + w_{CC}^r q_{CS,CC}^i + p_{EC} q_{EC}^i + w_{EC}^r q_{CS,EC}^i + \delta_1 \eta T_{l1}^o d_{12} + \delta_2 d_{13} T_{k2} \\ \leq w^o T_{l1}^o + \tilde{w}^h T_{l1}^h + n_3^i P_{\text{pension}} + Y^i,\end{aligned} \quad (4.17)$$

where $\tilde{w}^h \equiv w^h - c^\Omega(\rho_R^r, p_{IT})$ is the net remote wage (Assumption 4.8), and $p_R \equiv c^R(\rho_R^r, p_D)$ is the unit cost of housing services from living space and durables. The housing expenditure $\rho_R^r S_L^i + p_D q_D^i = p_R q_R^{i*}$ at the inner optimum for housing services.

Definition 4.21 (Space constraint)**.** $S_L^i + S_W^i = \bar{S}^i$, with $S_L^i \geq \bar{S}_L^i$ and $S_W^i \geq 0$.

Definition 4.22 (Care minima)**.** $Q_{CC}^i \geq n_1^i \bar{q}_{CC}$ and $Q_{EC}^i \geq n_3^i \bar{q}_{EC}(h^{3,i})$.

4.5 The Optimization Problem

Generation-2 of household i solves:

$$\max\ U^i \qquad \text{s.t.} \quad (4.1),\ (4.16),\ (4.17),\ Q_{CC}^i \geq n_1^i \bar{q}_{CC},\ Q_{EC}^i \geq n_3^i \bar{q}_{EC},\ \text{and nonnegativity.} \quad (4.18)$$

Proposition 4.23 (Existence of saddle point)**.** *Under Assumption 4.16 and the maintained assumptions on $R, F, G, H, C^{CC}, C^{EC}, \Omega$ (all continuous, concave, linearly homogeneous), the problem (4.18) is a concave programming problem with a convex feasible set. Slater's* *constraint qualification holds: there exists a strictly feasible point (e.g., set all goods and time variables to small positive values, set* $S_L^i > \bar{S}_L^i$ *to satisfy the space minimum, and set* $T_{l3}^{CC,i} = n_1^i \bar{q}_{CC}/C_2^{CC} + \epsilon$ *and* $T_{l3}^{EC,i} = n_3^i \bar{q}_{EC}/C_2^{EC} + \epsilon$ *for small* $\epsilon > 0$ *to satisfy the care minima strictly). By the Karlin–Uzawa Saddle Point Theorem (Karlin, 1959; Uzawa, 1958), there exist unique multipliers* $\lambda^{i*} > 0$ *(budget),* $\omega^{i*} > 0$ *(time), and* $\mu_S^{i*} \geq 0$ *(space) such that the KKT conditions are necessary and sufficient for a global optimum.*

Proof. See Online Appendix A.5. ■

Definition 4.24 (Shadow price of leisure)**.** $w^{*i} \equiv \omega^{i*}/\lambda^{i*} > 0$.

5 Equilibrium Characterization

5.1 First-Order Conditions

At an interior solution with $T_{l1}^{o*} > 0$, $T_{l1}^{h*} > 0$, $q_S^* > 0$, $q_{CS,CC}^* > 0$, $q_{CS,EC}^* > 0$ (Case RW1, the "full interior" case), the first-order conditions for the Lagrangian of problem (4.18) are derived in Online Appendix A.6. Throughout, $\lambda^{i*} > 0$ denotes the (raw, unnormalized) budget multiplier, $\omega^{i*} > 0$ the time multiplier, and $w^{*i} \equiv \omega^{i*}/\lambda^{i*}$ the shadow price of time.

Goods:

$$\text{Housing:} \qquad \Psi_1^i u_1^i = \lambda^{i*} p_R, \quad \text{where } p_R \equiv c^R(\rho_R^r, p_D), \tag{5.1}$$

$$\text{Home-leisure goods:} \qquad \Psi_1^i u_2^i F_1 = \lambda^{i*} p_2, \tag{5.2}$$

$$\text{External-leisure goods:} \qquad \Psi_1^i u_3^i G_1 = \lambda^{i*} p_1, \tag{5.3}$$

$$\text{Housework goods:} \qquad \Psi_1^i u_4^i H_1 = \lambda^{i*} p_H, \tag{5.4}$$

$$\text{Purchased housework:} \qquad \Psi_1^i u_4^i H_2 = \lambda^{i*} w_S^r, \tag{5.5}$$

$$\text{Childcare goods:} \qquad \Psi_1^i u_5^i C_1^{CC} = \lambda^{i*} p_{CC}, \tag{5.6}$$

$$\text{Purchased childcare:} \qquad \Psi_1^i u_5^i C_2^{CC} = \lambda^{i*} w_{CC}^r, \tag{5.7}$$

$$\text{Eldercare goods:} \qquad \Psi_1^i u_6^i C_1^{EC} = \lambda^{i*} p_{EC}, \tag{5.8}$$

$$\text{Purchased eldercare:} \qquad \Psi_1^i u_6^i C_2^{EC} = \lambda^{i*} w_{EC}^r. \tag{5.9}$$

Time:

$$\text{Home-leisure time:} \qquad \Psi_1^i u_2^i F_2 = \lambda^{i*} w^{*i}, \tag{5.10}$$

$$\text{External-leisure time:} \qquad \Psi_1^i u_3^i G_2 = \lambda^{i*} \pi_{k2}^i, \tag{5.11}$$

$$\text{Housework time:} \qquad \Psi_1^i u_4^i H_2 + \Psi_4^i = \lambda^{i*} w^{*i}, \tag{5.12}$$

$$\text{Childcare time:} \qquad \Psi_1^i u_5^i C_2^{CC} + \Psi_5^i = \lambda^{i*} w^{*i}, \tag{5.13}$$

$$\text{Eldercare time:} \qquad \Psi_1^i u_6^i C_2^{EC} + \Psi_6^i = \lambda^{i*} w^{*i}, \tag{5.14}$$

$$\text{Office work:} \qquad \Psi_2^i = -\lambda^{i*}(1 + \eta d_{12})(\tilde{w}^o - w^{*i}), \tag{5.15}$$

$$\text{Remote work:} \qquad \Psi_3^i = -\lambda^{i*}(\tilde{w}^h - w^{*i}), \tag{5.16}$$

where

$$\tilde{w}^o \equiv \frac{w^o - \delta_1 \eta d_{12}}{1 + \eta d_{12}} \tag{5.17}$$

is the commuting-adjusted effective office wage (see eq. 5.15),

$$\tilde{w}^h \equiv w^h - c^{\Omega}(\rho_R^r, p_{IT}) \tag{5.18}$$

is the net remote wage after equipment cost (Assumption 4.8), and

$$\pi_{k2}^i \equiv w^{*i}(1 + d_{13}) + \delta_2 d_{13} \tag{5.19}$$

is the full price of external leisure time (as in Shimizu 2026).

Remark 5.1 (FOC for office work: derivation of $\tilde{w}^o$)**.** The FOC for T_{l1}^o from the raw Lagrangian is $\Psi_2^i + \lambda^{i*}(w^o - \delta_1 \eta d_{12}) - \omega^{i*}(1 + \eta d_{12}) = 0$, which rearranges to (5.15). There is no multiplier renormalization; the factor $(1 + \eta d_{12})$ is carried explicitly. For the bound derivation, $\Psi_2^i \leq 0$ and $\lambda^{i*}(1 + \eta d_{12}) > 0$ imply $\tilde{w}^o \geq w^{*i}$.

Space: The space constraint $S_L^i + S_W^{i*}(T_{l1}^h) = \bar{S}^i$ (where S_W^{i*} is given by (4.9)) yields a single binding constraint on $\bar{S}^i$. Let $\mu_S^{i*} \geq 0$ be the Lagrange multiplier on $S_L^i + S_W^{i*} \leq \bar{S}^i$ and define $\hat{\mu}_S^{i*} \equiv \mu_S^{i*}/\lambda^{i*}$. The space FOC for living space S_L^i is:

$$\frac{\Psi_1^i u_1^i R_1(S_L^{i*}, q_D^{i*})}{\lambda^{i*}} = \hat{\mu}_S^{i*} = \rho_R^r. \tag{5.20}$$

The optimal $\bar{S}^i$ is determined by balancing the marginal utility of space (via R_1) against the per-unit rental cost ρ_R^r. The allocation of $\bar{S}^i$ between S_L^i and S_W^{i*} is then recovered from (4.9).

5.2 Shadow-Price Bounds

Theorem 5.2 (Shadow-price bound under remote work and care)**.** *At an interior solution (Case RW1):*

$$0 < w^{*i} \leq \min\{w_S^r,\ w_{CC}^r,\ w_{EC}^r,\ \tilde{w}^o,\ \tilde{w}^h\}. \tag{5.21}$$

Proof. From (5.15): $\Psi_2^i \leq 0$, $\lambda^{i*}(1+\eta d_{12}) > 0$ imply $\tilde{w}^o \geq w^{*i}$. From (5.16): $\Psi_3^i \leq 0$, $\lambda^{i*} > 0$ imply $\tilde{w}^h \geq w^{*i}$. From (5.5) and (5.12): $\Psi_1^i u_4^i H_2 = \lambda^{i*} w_S^r$ substituted into (5.12) gives $\Psi_4^i = -\lambda^{i*}(w_S^r - w^{*i})$; since $\Psi_4^i \leq 0$, $w_S^r \geq w^{*i}$. From (5.7) and (5.13): $\Psi_5^i = -\lambda^{i*}(w_{CC}^r - w^{*i})$; since $\Psi_5^i \leq 0$, $w_{CC}^r \geq w^{*i}$. From (5.9) and (5.14): $\Psi_6^i = -\lambda^{i*}(w_{EC}^r - w^{*i})$; since $\Psi_6^i \leq 0$, $w_{EC}^r \geq w^{*i}$. Positivity follows from $\omega^{i*}, \lambda^{i*} > 0$. See Online Appendix A.7 for full details. ■

Remark 5.3 (Genealogy of the bound)**.** Schreyer and Diewert (2014, p. 97): $w^* \leq \min\{w_S^r, w\}$. Shimizu (2026, Theorem 3.4): $w^* \leq \min\{w_S^r, w\}$. Present paper: $w^{*i} \leq \min\{w_S^r, w_{CC}^r, w_{EC}^r, \tilde{w}^o, \tilde{w}^h\}$, adding care prices, decomposing the wage into commuting-adjusted office wage and net remote wage.

5.3 Care Outsourcing Conditions

Proposition 5.4 (Housework outsourcing)**.** $q_S^{i*} > 0$ *if and only if* $w^{*i} + |\Psi_4^i|/\lambda^{i*} \geq w_S^r$.

Proposition 5.5 (Childcare outsourcing)**.** $q_{CS,CC}^{i*} > 0$ *if and only if*

$$w^{*i} + \frac{|\Psi_5^i|}{\lambda^{i*}} \geq w_{CC}^r. \tag{5.22}$$

Proposition 5.6 (Eldercare outsourcing)**.** $q_{CS,EC}^{i*} > 0$ *if and only if*

$$w^{*i} + \frac{|\Psi_6^i|}{\lambda^{i*}} \geq w_{EC}^r. \tag{5.23}$$

Each condition balances two benefits of own-time provision (shadow value of released time w^{*i} plus avoided disutility $|\Psi_k^i|/\lambda^{i*}$) against the market price. When $|\Psi_5^i|$ is small (parents enjoy childcare), the outsourcing threshold is low. When $|\Psi_6^i|$ is large (eldercare is burdensome), the threshold is high, but the higher market price w_{EC}^r may still preclude outsourcing.

Proof. See Online Appendix A.8. ■

5.4 Internal Shadow Price of Home-Office Space

Proposition 5.7 (Space allocation equilibrium)**.** *At an interior optimum with* $S_L^{i*} > \bar{S}_L^i$ *and* $T_{l1}^{h*} > 0$*, the housing FOC (5.20) and the envelope of the inner minimization jointly determine the space allocation:*

$$\underbrace{\frac{\Psi_1^i u_1^i R_1(S_L^{i*}, q_D^{i*})}{\lambda^{i*}}}_{\text{shadow value of living space (in monetary units)}} = \rho_R^r = w^h \cdot \left.\frac{\partial c^\Omega / \partial \rho_R^r}{\partial c^\Omega / \partial w^h}\right|_{\rho_R^r, p_{IT}}. \tag{5.24}$$

The three-way equality states: (i) the monetized marginal utility of living space equals the rental price ρ_R^r*; (ii) the rental price equals the shadow cost of office space per unit of remote time, derived from the inner minimization. The home-office space* $S_W^{i*} = T_{l1}^{h*} \cdot \partial c^\Omega / \partial \rho_R^r$ *is then recovered from Shephard's Lemma on* c^Ω*.*

Proof. See Online Appendix A.9. ■

Corollary 5.8 (Remote work increases total space demand)**.** *An increase in* T_{l1}^{h*} *raises* $S_W^{i*} = T_{l1}^{h*} \cdot \partial c^\Omega / \partial \rho_R^r$ *proportionally. Total housing expenditure* $\rho_R^r(S_L^{i*} + S_W^{i*})$ *rises with remote-work intensity.*

5.5 Spatial Full Income

Theorem 5.9 (Spatial full-income identity)**.** *For a Generation-2 household at an interior solution, using the reduced-form budget constraint (4.17) with net remote wage* $\tilde{w}^h$*:*

$$\begin{aligned} FC^{i,RW} - (w_S^r - w^{*i})T_{l2}^{i*} - (w_{CC}^r - w^{*i})T_{l3}^{CC,i*} - (w_{EC}^r - w^{*i})T_{l3}^{EC,i*} \\ - (\tilde{w}^o - w^{*i})(1 + \eta d_{12})T_{l1}^{o,i*} - (\tilde{w}^h - w^{*i})T_{l1}^{h,i*} = n_3^i P_{\text{pension}} + Y^i + w^{*i} n_2^i T \;\equiv\; FI^{i,RW}, \end{aligned} \tag{5.25}$$

where full consumption is

$$FC^{i,RW} \equiv p_R q_R^{i*} + P_{k1}^{i*} Q_{k1}^{i*} + \tilde{P}_{k2}^{i*} Q_{k2}^{i*} + P_H^{i*} Q_H^{i*} + P_{CC}^{i*} Q_{CC}^{i*} + P_{EC}^{i*} Q_{EC}^{i*} \tag{5.26}$$

with full prices $P^*_{k1} = c^F(p_2, w^{*i})$, $\tilde{P}^*_{k2} = c^G(p_1, \pi^i_{k2})$, $P^*_H = c^H(p_H, w^r_S)$, $P^*_{CC} = c^{CC}(p_{CC}, w^r_{CC})$, $P^*_{EC} = c^{EC}(p_{EC}, w^r_{EC})$, *and* $p_R = c^R(\rho^r_R, p_D)$.

Proof. See Online Appendix A.10. The home-office equipment cost $c^\Omega(\rho^r_R, p_{IT})T^{h*}_{l1}$ appears on the expenditure side of (4.17) and is exactly cancelled by the difference $w^h - \tilde{w}^h$, so it drops from the identity. ■

Remark 5.10 (Office-work surplus term)**.** The term $(\tilde{w}^o - w^{*i})(1 + \eta d_{12})T^{o,i*}_{l1}$ carries the factor $(1+\eta d_{12})$ because the raw FOC for T^o_{l1} does. This correctly accounts for the time cost of commuting in the full-income accounting.

Remark 5.11 (Care surpluses)**.** The terms $(w^r_{CC} - w^{*i})T^{CC,i*}_{l3}$ and $(w^r_{EC} - w^{*i})T^{EC,i*}_{l3}$ are "care surpluses": the implicit income gain from self-providing care rather than purchasing it at market prices.

5.6 Nonhomothetic Expenditure Shares

Proposition 5.12 (Income-dependent housing share)**.** *Under Assumption 4.16, optimal housing demand is:*

$$q^{i*}_R = \bar{q}^i_R + (FI^{i,RW} - c^R\bar{q}^i_R) \cdot s^i_R(\mathbf{P}^i), \tag{5.27}$$

where $s^i_R(\mathbf{P}^i)$ *is the housing expenditure share of the homothetic subutility* u^i *evaluated at the full-price vector* $\mathbf{P}^i$*. The expenditure share of housing in full income declines with* $FI^{i,RW}$*.*

Proof. See Online Appendix A.11. ■

Remark 5.13 (Welfare comparison without superlative indexes)**.** Shimizu (2026) uses superlative index theory to construct a Regional Utility Index. The present paper does not pursue that route due to parameter proliferation under nonhomothetic preferences. Instead, welfare comparisons can use the *equivalent variation* derived from $V^{i,r}$: for a change from θ_0 to θ_1, the EV is the income transfer ΔY satisfying $V^{i,r}(\ldots, \theta_0, Y^i + \Delta Y) = V^{i,r}(\ldots, \theta_1, Y^i)$. The full-income identity (5.25) provides the structural link.

6 Equilibrium Rent Gradients

6.1 Indirect Utility and Free Mobility

Let $V^{i,r}$ denote the indirect utility of type i in region r. The amenity-augmented indirect utility is:

$$\tilde{V}^{i,r} = \Phi^i(V^{i,r}, A^r; \boldsymbol{\xi}^i), \tag{6.1}$$

where A^r is the regional amenity vector and $\boldsymbol{\xi}^i$ governs type-i's sensitivity. Free mobility requires $\tilde{V}^{i,r} = \bar{V}^i$ for all r with positive type-i population.

6.2 The Rent Gradient

Theorem 6.1 (Rent gradient under remote work)**.** *At an interior equilibrium:*

$$\frac{\partial p_R^{*r}}{\partial d_{12}^r} = -\frac{(w^{*i} + \delta_1)\eta\, T_{l1}^{o,i*}}{q_R^{i*}}. \tag{6.2}$$

Proof. See Online Appendix A.12. ∎

Remark 6.2 (Flattening relative to the companion paper)**.** Shimizu (2026)'s gradient (2.5) was $-(w^* + \delta_1)/q_R^*$. In the present model, the factor $\eta T_{l1}^{o,i*}$ replaces the implicit factor of 1. Since $T_{l1}^{o,i*} < n_2^i T$ (office work is only part of total time) and $\eta < 1$ in typical calibrations, the gradient is flatter. As θ increases and $T_{l1}^{o,i*}$ falls, the gradient flattens further.

6.3 Additional Rent Gradients

Proposition 6.3 (Care capitalization into rents)**.**

$$\partial p_R^{*r}/\partial d_{13}^r = -(w^{*i} + \delta_2)T_{k2}^{i*}/q_R^{i*} \leq 0, \tag{6.3}$$

$$\partial p_R^{*r}/\partial w_S^r = -q_S^{i*}/q_R^{i*} \leq 0, \tag{6.4}$$

$$\partial p_R^{*r}/\partial w_{CC}^r = -q_{CS,CC}^{i*}/q_R^{i*} \leq 0, \tag{6.5}$$

$$\partial p_R^{*r}/\partial w_{EC}^r = -q_{CS,EC}^{i*}/q_R^{i*} \leq 0, \tag{6.6}$$

$$\partial p_R^{*r}/\partial A_k^r > 0 \textit{ for valued amenities.} \tag{6.7}$$

Proof. See Online Appendix A.13. ■

Remark 6.4 (Micro-foundation for care rent premia)**.** Equations (6.5)–(6.6) show that access to affordable childcare and eldercare is capitalized into rents. This provides a structural foundation for the empirical observation that urban areas with abundant care infrastructure command rent premia.

6.4 Rent Rotation

Proposition 6.5 (Rent rotation)**.** *An increase in θ simultaneously:*

(a) raises residential rents (through Corollary 5.8: more residential space demand);

(b) lowers commercial rents (through Proposition 3.18: less office space demand).

The residential increase is concentrated in suburban regions (where remote-capable households sort), and the commercial decrease is concentrated in CBDs.

Proof. See Online Appendix A.14. ■

7 Sorting Equilibrium

7.1 Household Types

Definition 7.1 (Type)**.** A household type is $i = (\phi^i, n_1^i, n_2^i, n_3^i, h^{3,i}, \boldsymbol{\xi}^i)$, where $\phi^i \equiv w^{h,i}/w^{o,i} \in [0,1]$ is the remote-work wage ratio determined by the NQ firm technology and the household's occupation.

7.2 Spatial Equilibrium with Heterogeneous Types

Definition 7.2 (Spatial equilibrium)**.** A spatial equilibrium consists of a rent vector $\mathbf{p}_R^* = (p_R^{*1}, \ldots, p_R^{*M})'$ and an allocation $\mu^i(r)$ (type-i household measure in region r) satisfying:

(i) $\tilde{V}^{i,r}(p_R^{*r}, \ldots) = \bar{V}^i$ for all r with $\mu^i(r) > 0$;

(ii) $\tilde{V}^{i,r}(p_R^{*r}, \ldots) \leq \bar{V}^i$ for all r with $\mu^i(r) = 0$;

(iii) $\sum_i \mu^i(r) q_R^{i*}(r) = H^r$ for all r (housing market clearing).

Remark 7.3 (Existence of equilibrium)**.** Existence follows when the number of types is finite and housing supply H^r is continuous and strictly increasing in p_R^r. The indirect utility $\tilde{V}^{i,r}$ is continuous and strictly decreasing in p_R^r (envelope theorem, $q_R^{i*} > 0$), so each type's bid-rent function $\psi^i(d_{12})$ is well-defined. The equilibrium rent is the upper envelope; market clearing pins down reservation utilities. Uniqueness requires stronger conditions (e.g., gross substitutability), not imposed here. The sorting results hold for any equilibrium satisfying Definition 7.2.

7.3 Sorting by Remote-Work Capacity

Theorem 7.4 (Sorting by ϕ)**.** *Suppose regions are ordered by $d_{12}^1 < \cdots < d_{12}^M$ and households are of two types: face-to-face ($\phi^i = 0$, type F) and remote-capable ($\phi^i > 0$, type R). Assume that at each d_{12}, w^{i*}, q_R^{i*} and $T_{l1}^{o,i*}$ are continuous in d_{12}, and that $T_{l1}^{o,R*} < T_{l1}^{o,F*}$ at every distance (type R works less in the office than type F). Define the bid-rent function $\psi^i(d_{12})$*

as the rent that equalizes $\tilde{V}^{i,r} = \bar{V}^i$. Then:

(a) Type F has bid-rent slope $\partial\psi^F/\partial d_{12} = -(w^{*F} + \delta_1)\eta T_{l1}^{o,F*}/q_R^{F*} < 0$.

(b) Type R has $|\partial\psi^R/\partial d_{12}| < |\partial\psi^F/\partial d_{12}|$ because $T_{l1}^{o,R*} < T_{l1}^{o,F*}$ and other determinants are comparable.

(c) If additionally $\psi^F(0) \geq \psi^R(0)$ (type F outbids type R at the center), then at least one crossing $\hat{d}_{12}$ exists; type F occupies central regions and type R occupies distant regions in any equilibrium consistent with the bid-rent ordering.

Remark 7.5 (Limitations of the sorting result)**.** Parts (a)–(b) follow from the FOCs and the rent-gradient theorem. Part (c) requires the auxiliary assumption $\psi^F(0) \geq \psi^R(0)$, which holds if the productivity advantage of face-to-face work at the CBD is large enough. Uniqueness of the crossing and convexity of the upper envelope $p_R^* = \max\{\psi^F, \psi^R\}$ require additional parametric conditions (e.g., affine bid-rents) that are not imposed in the general model.

Proof. See Online Appendix A.15. ■

7.4 Sorting by Care Needs

Proposition 7.6 (Care-driven sorting under tight constraints)**.**

(a) Under tight time or budget constraints (formally, when the care minima $n_1^i\bar{q}_{CC}$ and $n_3^i\bar{q}_{EC}$ bind and own-care time is insufficient to satisfy the minima alone), households with $n_1^i > 0$ sort toward regions with $w_{CC}^r < \infty$, and households with $n_3^i > 0$ sort toward regions with $w_{EC}^r < \infty$.

(b) Remote-capable households with care needs face a relaxed constraint: the commuting-time saving $(1 + \eta d_{12})\Delta T_{l1}^o$ can be redirected to own care, making self-care feasible at greater distances from the CBD. Thus θ expansion weakly expands the feasible location set for care-constrained households.

Proof. See Online Appendix A.16. ■

7.5 Elderly-Only Households

Proposition 7.7 (Elderly-only households)**.** *Households with* $(n_1^i, n_2^i, n_3^i) = (0, 0, n_3)$ *have zero commuting, so* $\partial p_R^* / \partial d_{12} = 0$ *for these households. They respond only to amenity access and eldercare availability. As their population share rises, the aggregate rent gradient flattens.*

Proof. See Online Appendix A.17. ■

7.6 Nonhomothetic Effects on Sorting

Proposition 7.8 (Income-dependent sorting)**.** *Under nonhomothetic preferences, among type-R households, those with higher* $FI^{i,RW}$ *have lower housing expenditure shares. These high-income remote workers sort to suburban locations that combine lower density with good amenities, rather than simply maximizing distance from* S_2*. The subsistence parameter* $\bar{q}_R^i$ *creates a minimum housing demand that constrains sorting for large households.*

Proof. Follows from Proposition 5.12 and the bid-rent argument in Online Appendix A.15. ■

8 Population Aging and Spatial Equilibrium

8.1 Parameterizing Demographic Change

Define the dependency ratios:

$$\delta_C \equiv \frac{\sum_i n_1^i \mu^i}{\sum_i n_2^i \mu^i}, \qquad \delta_E \equiv \frac{\sum_i n_3^i \mu^i}{\sum_i n_2^i \mu^i}. \tag{8.1}$$

Population aging: $\delta_C \downarrow$ and $\delta_E \uparrow$.

8.2 Effects on Care Markets

Proposition 8.1 (Care market restructuring)**.**

(a) $\delta_C \downarrow$: aggregate childcare demand falls. In thick markets, w^r_{CC} declines. In thin markets, childcare providers exit and $w^r_{CC} \to +\infty$.

(b) $\delta_E \uparrow$: aggregate eldercare demand rises. w^r_{EC} increases (care worker shortage). Excess demand appears in supply-constrained regions.

(c) Combined: spatial pattern of "eldercare-rich, childcare-poor" regions emerges.

8.3 Effects on the Spatial Equilibrium

Theorem 8.2 (Aging and spatial equilibrium)**.** *Suppose the elderly dependency ratio $\delta_E = n^i_3/n^i_2$ rises exogenously. Assume (i) housing supply is upward-sloping in each region; (ii) care-market wages w^r_{EC} are region-specific and do not adjust instantaneously; (iii) the preference for eldercare proximity is monotone in n^i_3. Then:*

(a) The aggregate rent gradient flattens through two channels: (i) more elderly-only households (with $\partial p^*_R/\partial d_{12} = 0$); (ii) more Generation-2 workers who shift time to eldercare (reducing $T^{o,i*}_{l1}$, flattening their individual bid-rent slope).

(b) Under assumption (ii), equilibrium rents in regions with low w^r_{EC} (eldercare-rich) rise relative to eldercare-poor regions, as more households bid for access to care services.

(c) Under mild regularity conditions (see Appendix A.18), the marginal welfare gain $\partial V^{i,r}/\partial\theta$ is larger when $n^i_3 > 0$ than when $n^i_3 = 0$: remote work allows commuting time to be redirected to eldercare, a use with higher marginal value in aging households.

Proof. See Online Appendix A.18. ■

Remark 8.3 (Remote work as a care-enabling technology)**.** In a young society with small δ_E, remote work is primarily a commuting-saving and residential-sorting device. In an aging society with large δ_E, remote work becomes a *care-enabling technology*: it allows Generation-2 workers to remain employed while providing eldercare at S_1. Without remote work, these workers face the binary choice between employment (requiring S_2 commuting) and eldercare (requiring S_1 presence)—the "care-or-work" dilemma that leads to labor-force withdrawal ("care retirement"). Remote work breaks this binary by allowing T_{l1}^{o*} to decline smoothly rather than jump to zero.

8.4 Interaction with the NQ Firm Technology

Proposition 8.4 (Aging and the NQ)**.** *Aging reduces the labor force ($\sum_i n_2^i T$ declines), raising (w^o, w^h). Through the NQ factor demand equations (3.7)–(3.9), this induces substitution toward capital. The direction of substitution between office and remote labor depends on $a_{12}(\theta)$: if $a_{12} > 0$ (office and remote labor are gross substitutes), aging raises the remote-labor share. The NQ allows this cross-elasticity to be estimated rather than assumed.*

Proof. See Online Appendix A.19. ■

9 Identification, Calibration, and Empirical Relevance

This section serves two purposes. First, it outlines the identification strategy for taking the model to data. Second, it presents a quantitative calibration for the Japanese economy—using the NQ joint cost estimates reported in Online Appendix C and publicly available microdata—to demonstrate that the model's key predictions are empirically grounded.

9.1 Mobile-Phone Carrier Data

Following Shimizu (2026) (Section 7), Kreindler and Miyauchi (2023), and Miyauchi, Nakajima, and Redding (2025), carrier data classify stays into home, workplace, and other locations. The observables are:

1. Weekday S_2 time $\rightarrow T_{l1}^{o,i}$ (directly identified).
2. Weekday S_1 time during work hours $\rightarrow T_{l1}^{h,i} + T_{k1}^{i} + T_{l2}^{i} + T_{l3}^{CC,i} + T_{l3}^{EC,i}$ (five-activity mixture).

3. S_1–S_2 travel $\to \eta T_{l1}^{o,i} d_{12}^i$ (directly identified).
4. S_3 time $\to T_{k2}^i$ (directly identified).

The companion paper faced a two-activity conflation ($T_{k1}+T_{l2}$). The present model has a *five*-activity conflation, making identification more demanding. Partial resolution strategies include: (i) weekday-weekend differencing (T_{l1}^h is weekday-concentrated); (ii) time-of-day analysis (T_{l3}^{CC} peaks in morning/evening); (iii) industry-based ϕ^i assignment from census data.

9.2 Economic Census with Spatial Coordinates

Geocoded establishment data provide: (i) S_2 coordinates for precise d_{12} computation; (ii) industry codes for ϕ^i assignment; (iii) wage and employment data for the NQ factor demand equations.

9.3 Testable Predictions

The model generates five testable predictions:

1. The rent gradient with respect to d_{12} is flatter for workers in remote-capable industries than for workers in face-to-face industries (Theorem 7.4).
2. Residential space demand increases with T_{l1}^h, controlling for income (Corollary 5.8).
3. Eldercare-rich regions command rent premia for elderly-care households (Eq. 6.6).
4. The aggregate rent gradient flattens more in aging prefectures (Theorem 8.2).
5. The CES restriction on the NQ substitution matrix is rejected in macroeconomic data (Remark 3.11).

9.4 Calibration: Japan 2019–2023

We calibrate the model to the Greater Tokyo metropolitan area using two data sources: (i) the NQ joint cost estimates reported in Online Appendix C, which apply the curvatureregular NQ system of Section 3 to Japanese macro data (1970–2023) with age-disaggregated labor; and (ii) publicly available statistics on wages, commuting, land prices, and remote-work penetration. The purpose is not structural estimation but rather a *proof-of-concept* calibration that demonstrates the quantitative magnitudes predicted by Theorem 6.1.

Firm-side parameters. Table 1 reports the NQ Morishima elasticities estimated in Online Appendix C, Specification 3 (nine inputs including age-disaggregated labor and land services; preferred rank pair $(r_A, r_B) = (4, 3)$). We interpret the 30–39 age cohort (L2) as *office labor* (T^o_{l1}) and the under-29 cohort (L1) as the marginal cohort that disproportionately shifts to *remote labor* (T^h_{l1}), consistent with the fact that younger workers exhibit the largest increase in remote-work capacity in Japanese census data.

Table 1: NQ Morishima elasticities: Japan macro technology (authors' estimates; see Online Appendix C)

Elasticity	Symbol	1970–2023 avg.	2008–2019	2020–2023
Office → Remote substitution	$\sigma^M_{L2\to L1}$	3.071	4.671	6.104
Machinery → Office labor	$\sigma^M_{K_{ME}\to L2}$	0.718	0.439	0.434
Office labor → Machinery	$\sigma^M_{L2\to K_{ME}}$	1.944	2.580	3.490

Notes: Morishima elasticity $\sigma^M_{j\to i} \equiv E^x_{ij} - E^x_{jj}$; positive values indicate that input i substitutes for input j when the wage of j rises. L1: age ≤ 29; L2: age 30–39; K_{ME}: machinery and equipment capital services. Regime averages correspond to the spline knots at 1973, 1990, 2008, and 2020 (see Online Appendix C, Table C.5). Authors' estimates from the NQ joint cost system with rank-expansion curvature correction; full parameter estimates and standard errors are in Online Appendix C.

Three results from Table 1 are directly relevant to the present model. First, $\sigma^M_{L2\to L1}$ rises sharply from 4.671 in 2008–2019 to 6.104 in 2020–2023, indicating that office–remote labor substitutability accelerated after the pandemic—precisely the increase in θ that Assumption 3.13 models. Second, the capital-deepening response to prime-age wage pressure ($\sigma^M_{L2\to K_{ME}}$) strengthens across regimes, confirming that the NQ substitution matrix is not well approximated by a constant-elasticity CES specification. Third, the machinery-to-labor response ($\sigma^M_{K_{ME}\to L2}$) weakens over time, consistent with diminishing returns to substitution as the remote-work margin matures.

Rent-gradient calibration. We now calibrate Theorem 6.1,

$$\frac{\partial p_R^{*r}}{\partial d_{12}^r} = -\frac{(w^{o*} + \delta_1)\,\eta\,T_{l1}^{o,i*}}{q_R^{i*}}, \tag{9.1}$$

for the Greater Tokyo area in 2019 (pre-pandemic) and 2023 (post-pandemic stabilization). The parameters and their sources are listed in Table 2.

Table 2: Rent-gradient calibration: Greater Tokyo area, 2019 vs. 2023

Parameter	Symbol	2019	2023	Source
Office wage (JPY/hour)	w^{o*}	1,938	2,044	MHLW Wage Structure Survey (age 30–39)
Commute speed (km/h)	—	40	40	MLIT Urban Transport Census
Time cost per km, round-trip (h/km)	η	0.050	0.050	Derived (= 2/40)
Transport cost per km, round-trip (JPY/km)	δ_1	20	20	Commuter rail fare (self-share)
Office attendance (fraction of full week)	$T_{l1}^{o,i*}$	1.00	0.60	MIC Labour Force Survey; Persol Research (2023)
Residential floor space (m^2)	q_R^{i*}	60	60	Standard household assumption
Rent gradient ($JPY/m^2/km$)	$\partial p_R^*/\partial d_{12}$	**−1.95**	**−1.22**	Eq. (9.1), this paper

Notes: JPY = Japanese yen. MHLW = Ministry of Health, Labour and Welfare. MLIT = Ministry of Land, Infrastructure, Transport and Tourism. MIC = Ministry of Internal Affairs and Communications. w^{o*}: scheduled cash earnings for full-time workers aged 30–39, divided by scheduled hours (160 h/month). δ_1: self-paid commuter rail fare per km (round-trip). $T_{l1}^{o,i*}$: fraction of the standard five-day week in the office; the 2023 value (0.60) = three office days per week, consistent with Barrero, Bloom, and Davis (2023) and Persol Research Institute (2023). Gradient formula: eq. (9.1) in the main text; $(w^{o*}\eta+\delta_1)$ is total commuting cost per km (JPY/km).

The calibration yields a pre-pandemic gradient of -1.95 JPY/m^2/km and a post-pandemic gradient of -1.22 JPY/m^2/km—a **37 % flattening**. Two forces drive this result. First, the shift from five to three office days per week reduces $T_{l1}^{o,i*}$ from 1.00 to 0.60 (a 40 % decline), directly compressing the effective commuting cost that capitalizes into rents. Second, the modest wage increase (from JPY 1,938 to JPY 2,044 per hour) is dominated by the attendance effect, so the net gradient narrows substantially.

The 37 % reduction in gradient steepness is consistent with the empirical finding of Barrero, Bloom, and Davis (2023) that the share of paid days worked from home stabilized at roughly five times its pre-pandemic level, and with the estimates of Delventhal, Kwon, and Parkhomenko (2022) for US metropolitan areas, who document significant flattening of rent gradients relative to pre-pandemic trends. For Japan specifically, the acceleration of $\sigma^M_{L2\to L1}$ from 4.671 (2008–2019) to 6.104 (2020–2023) in Table 1—and in the underlying NQ estimates of Online Appendix C—implies that the firm-side substitutability between office and remote labor increased by 31 % between regimes, reinforcing the household-side effect captured by the fall in $T^{o,i*}_{l1}$.

Remark 9.1 (Calibration as a lower bound on flattening)**.** The 37 % estimate should be interpreted as a conservative lower bound. The calibration holds δ_1 and q^{i*}_R fixed across years and abstracts from sorting-induced compositional changes in the residential population (Section 7). Equation (6.2) shows that sorting amplifies the gradient change: as remote-capable households (higher ϕ^i) sort to peripheral locations, the average $T^{o,i*}_{l1}$ of the marginal mover declines further, steepening the flattening relative to the average computed here. Endogenizing sorting and care demand (Sections 7–8) provides additional amplification channels that a full structural estimation would capture.

10 Conclusion

This paper has developed a spatial model of remote work, firm technology, and intergenerational care within the Becker–Diewert household-production tradition.

The NQ firm technology is the distinctive methodological contribution. It allows the substitutability between office and remote labor to vary with the factor-price vector, yielding variable elasticities that the CES cannot produce. The NQ generates a tractable system of 12 parameters—far fewer than a fully flexible household utility specification would require. This asymmetry reflects a deliberate design choice that exploits the comparative advantage of the Diewert–Nomura–Shimizu research program.

The generational structure introduces care as a spatial force. The model predicts that population aging flattens the rent gradient through two channels (more elderly-only households and more remote eldercare) and generates a rent rotation between residential and commercial real estate. Remote work emerges not merely as a commuting-saving device but as a care-enabling technology: in an aging society, the marginal value of remote work is amplified because commuting savings are redirected to eldercare. This prediction—that the welfare value of remote-work technology is *increasing* in the elderly dependency ratio—is perhaps the most novel contribution of the paper.

The model is rich enough to generate nontrivial predictions about sorting, rent gradients, care outsourcing, and the interaction between firm technology and household time allocation, yet tractable enough that all results are derived in closed form from a single optimization problem. The seven production functions (firm NQ, housing, home office, home leisure, housework, childcare, eldercare) and the generational structure provide a comprehensive framework for analyzing the spatial consequences of the two defining transformations of the 21st-century economy: the remote-work revolution and population aging.

Three extensions remain for future work. First, the entry of care-service providers could be endogenized, linking w^r_{CC} and w^r_{EC} to the spatial distribution of demand. Second, the model could be embedded in a quantitative spatial framework with continuous geography for full numerical evaluation; the calibration in Section 9.4 provides a starting point, showing that the 37 % post-pandemic flattening of the Tokyo rent gradient is quantitatively consistent with the model's predictions. Third, the NQ could be extended to multiple output sectors following Diewert, Nomura, and Shimizu (2025), capturing inter-industry heterogeneity in remote-work feasibility.

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Online Appendix

Chihiro Shimizu — Hitotsubashi University — March 2026

Appendix A: Proofs

A.1: Proof of Proposition 3.5 (NQ Factor Demands)

Differentiate (3.3) with respect to w_n. Write $f(\mathbf{w}) = \frac{1}{2}\mathbf{w}'A\mathbf{w}$ and $g(\mathbf{w}) = \boldsymbol{\alpha}'\mathbf{w}$. Then $\partial f/\partial w_n = [A\mathbf{w}]_n$ (using symmetry of A) and $\partial g/\partial w_n = \alpha_n$. By the quotient rule:

$$\frac{\partial}{\partial w_n}\left[\frac{f}{g}\right] = \frac{[A\mathbf{w}]_n(\boldsymbol{\alpha}'\mathbf{w}) - \frac{1}{2}(\mathbf{w}'A\mathbf{w})\alpha_n}{(\boldsymbol{\alpha}'\mathbf{w})^2} = \frac{[A\mathbf{w}]_n}{\boldsymbol{\alpha}'\mathbf{w}} - \frac{\alpha_n}{2}\frac{\mathbf{w}'A\mathbf{w}}{(\boldsymbol{\alpha}'\mathbf{w})^2}.$$

Adding $d_n(\theta)$ gives (3.6). ∎

A.2: Proof of Proposition 3.6 (NQ Elasticity Matrix)

Define $v_n \equiv [A\mathbf{w}]_n$ and $a \equiv \boldsymbol{\alpha}'\mathbf{w}$ and $Q \equiv \mathbf{w}'A\mathbf{w}$. From A.1, $x_n/Y^f = d_n + v_n/a - \alpha_n Q/(2a^2)$. Differentiating with respect to w_j:

Term from v_n/a:

$$\frac{\partial}{\partial w_j}\left[\frac{v_n}{a}\right] = \frac{a_{nj}}{a} - \frac{v_n\alpha_j}{a^2}.$$

Term from $-\alpha_n Q/(2a^2)$:

$$-\frac{\alpha_n}{2}\frac{\partial}{\partial w_j}\left[\frac{Q}{a^2}\right] = -\frac{\alpha_n}{2}\left[\frac{2v_j}{a^2} - \frac{2Q\alpha_j}{a^3}\right] = -\frac{\alpha_n v_j}{a^2} + \frac{\alpha_n\alpha_j Q}{a^3}.$$

Summing: $\partial(x_n/Y^f)/\partial w_j = a_{nj}/a - v_n\alpha_j/a^2 - \alpha_n v_j/a^2 + \alpha_n\alpha_j Q/a^3$.

In matrix form: $\nabla_{\mathbf{w}}(\mathbf{x}/Y^f) = A/a - A\mathbf{w}\,\boldsymbol{\alpha}'/a^2 - \boldsymbol{\alpha}(A\mathbf{w})'/a^2 + \boldsymbol{\alpha}\boldsymbol{\alpha}'Q/a^3$. Multiplying by Y^f gives (3.10). ∎

A.3: Proof of Proposition 3.15 (Remote-Labor Bias)

The remote-to-office ratio is $\rho \equiv (\partial c^\Gamma/\partial w^h)/(\partial c^\Gamma/\partial w^o)$. Differentiating with respect to θ at $\mathbf{w} = \mathbf{w}^0$ (where $A(\theta^0)\mathbf{w}^0 = \mathbf{0}$ by normalization):

$$\left.\frac{\partial \rho}{\partial \theta}\right|_{\mathbf{w}^0} = \frac{d_2^1 \cdot d_1(\theta^0) - d_1^1 \cdot d_2(\theta^0)}{(d_1(\theta^0))^2} + \text{terms involving } \partial A/\partial \theta.$$

At the normalization point, the substitution terms vanish because $A(\theta^0)\mathbf{w}^0 = \mathbf{0}$. The sign of $\partial\rho/\partial\theta$ is determined by $d_2^1 d_1^0 - d_1^1 d_2^0$. A sufficient condition for positivity is $d_2^1/d_2^0 > d_1^1/d_1^0$, i.e., the proportional effect of θ on the remote coefficient exceeds that on the office coefficient. The stronger sufficient condition $d_2^1 > 0 > d_1^1$ is immediate. ■

A.4: Proof of Proposition 3.18 (Office Space Saving)

From (3.19), $K_O/Y^f = (\partial c^\Gamma/\partial r_K) \cdot (\partial c^\Xi/\partial r_O)$. Differentiating with respect to θ:

$$\frac{\partial(K_O/Y^f)}{\partial\theta} = \frac{\partial^2 c^\Gamma}{\partial r_K \partial\theta} \cdot \frac{\partial c^\Xi}{\partial r_O}.$$

(The second factor $\partial c^\Xi/\partial r_O$ does not depend on θ.) Now $\partial^2 c^\Gamma/\partial r_K\partial\theta < 0$ if θ improvement reduces the marginal cost contribution of capital. Under complementarity between office labor and capital ($\partial^2 c^\Gamma/\partial w^o \partial r_K > 0$), a reduction in office labor demand (from higher θ) reduces the derived demand for capital, and specifically for the office-space component of capital. ■

A.5: Proof of Proposition 4.23 (Saddle Point)

The objective U^i is concave in the choice variables: u^i is concave as a linearly homogeneous concave function of concave functions $(R, F, G, H, C^{CC}, C^{EC})$; Ψ^i is concave and increasing in u^i; the composition is concave. The constraints are linear (or concave), so the feasible set is convex. The Slater CQ holds: set all quantities to small positive values with both constraints holding strictly. By Karlin (1959) and Uzawa (1958), saddle-point multipliers $\lambda^{i*} > 0$ and $\omega^{i*} > 0$ exist. Strict monotonicity of Ψ^i in u^i ensures $\lambda^{i*} > 0$ and $\omega^{i*} > 0$. ■

A.6: Derivation of First-Order Conditions (Section 5.1)

The household's problem is stated in reduced form after substituting the inner optima for S_W^{i*} and q_{IT}^{i*} (Remark 4.9). The Lagrangian for the outer problem is:

$$\begin{aligned}\mathcal{L}^i &= \Psi^i\big(u^i(\cdots), T_{l1}^o, T_{l1}^h, T_{l2}, T_{l3}^{CC}, T_{l3}^{EC}\big)\\ &\quad + \lambda\Big[w^o T_{l1}^o + \tilde{w}^h T_{l1}^h + n_3^i P_{\text{pension}} + Y^i - p_R q_R - p_2 q_{k1} - p_1 q_{k2} - p_H q_H - w_S^r q_S\\ &\qquad\quad - p_{CC} q_{CC} - w_{CC}^r q_{CS,CC} - p_{EC} q_{EC} - w_{EC}^r q_{CS,EC} - \delta_1 \eta T_{l1}^o d_{12} - \delta_2 d_{13} T_{k2}\Big]\\ &\quad + \omega\Big[n_2^i T - (1{+}\eta d_{12}) T_{l1}^o - T_{l1}^h - T_{l2} - T_{l3}^{CC} - T_{l3}^{EC} - T_{k1} - (1{+}d_{13}) T_{k2}\Big],\end{aligned}$$

where $\tilde{w}^h \equiv w^h - c^\Omega(\rho_R^r, p_{IT})$ is the net remote wage, and $p_R = c^R(\rho_R^r, p_D)$ the housing unit cost. There is no space multiplier in the outer problem: the total space $\bar{S}^i = S_L^{i*} + S_W^{i*}(T_{l1}^h)$ is determined by the housing FOC and the inner minimization. We define $w^{*i} \equiv \omega^*/\lambda^* > 0$ with *raw* (unnormalized) multipliers $\lambda^*, \omega^* > 0$.

FOC for goods (all standard, giving equations (5.1)–(5.9)):

$$q_R: \ \Psi_1^i u_1^i \frac{\partial u^i}{\partial q_R} = \lambda^* p_R, \quad q_{k1}: \ \Psi_1^i u_2^i F_1 = \lambda^* p_2, \quad \ldots$$

FOC for T_{l1}^o:

$$\begin{aligned}&\Psi_2^i + \lambda^*(w^o - \delta_1 \eta d_{12}) - \omega^*(1 + \eta d_{12}) = 0\\ \Longrightarrow\ &\Psi_2^i = -\lambda^*(1 + \eta d_{12})\left(\frac{w^o - \delta_1 \eta d_{12}}{1 + \eta d_{12}} - w^{*i}\right) = -\lambda^*(1 + \eta d_{12})(\tilde{w}^o - w^{*i}),\end{aligned}$$

giving (5.15). No renormalization of λ^* is used; the factor $(1 + \eta d_{12})$ is carried explicitly.

FOC for T_{l1}^h:

$$\Psi_3^i + \lambda^* \tilde{w}^h - \omega^* = 0 \implies \Psi_3^i = -\lambda^*(\tilde{w}^h - w^{*i}),$$

giving (5.16). The net remote wage $\tilde{w}^h$ appears directly because $p_{IT} q_{IT}$ and $\rho_R^r S_W$ have been absorbed via the inner minimization.

FOC for $T_{l2}, T_{l3}^{CC}, T_{l3}^{EC}$: standard, giving (5.12)–(5.14).

FOC for housing space S_L^i: The living space enters the outer problem through $q_R = R(S_L^i, q_D^i)$ (after housing cost minimization). At the optimum:

$$\lambda^* p_R \cdot \frac{\partial q_R}{\partial S_L} = \lambda^* p_R \cdot \frac{R_1}{c^R} \cdot \rho_R^r = \text{shadow value of extra } \bar{S}^i,$$

which simplifies (using $p_R = c^R(\rho_R^r, p_D)$ and Shephard's Lemma $\partial c^R / \partial \rho_R^r = S_L / q_R$) to the condition $\Psi_1^i u_1^i R_1 / \lambda^* = \rho_R^r$, i.e., the marginal utility of living space (monetized) equals the per-unit rent (5.20).

Recovery of S_W^{i*} and q_{IT}^{i*}: by Shephard's Lemma applied to c^Ω:

$$S_W^{i*} = T_{l1}^{h*} \cdot \frac{\partial c^\Omega(\rho_R^r, p_{IT})}{\partial \rho_R^r}, \qquad q_{IT}^{i*} = T_{l1}^{h*} \cdot \frac{\partial c^\Omega(\rho_R^r, p_{IT})}{\partial p_{IT}}.$$

These are not FOCs of the outer problem; they are the solutions to the inner minimization (4.5). ■

A.7: Proof of Theorem 5.2 (Shadow-Price Bound)

From (5.15): $\Psi_2^i = -\lambda^*(1 + \eta d_{12})(\tilde{w}^o - w^{*i})$. Since $\Psi_2^i \leq 0$ and $\lambda^*(1 + \eta d_{12}) > 0$, we get $\tilde{w}^o \geq w^{*i}$.

From (5.16): $\Psi_3^i = -\lambda^*(\tilde{w}^h - w^{*i})$. Since $\Psi_3^i \leq 0$ and $\lambda^* > 0$, we get $\tilde{w}^h \geq w^{*i}$.

From (5.5) and (5.12): $\Psi_1^i u_4^i H_2 = \lambda^* w_S^r$ and $\Psi_1^i u_4^i H_2 + \Psi_4^i = \lambda^* w^{*i}$, giving $\Psi_4^i = -\lambda^*(w_S^r - w^{*i})$. Since $\Psi_4^i \leq 0$: $w_S^r \geq w^{*i}$.

Similarly from (5.7) and (5.13): $w_{CC}^r \geq w^{*i}$; from (5.9) and (5.14): $w_{EC}^r \geq w^{*i}$.

Combining: $w^{*i} \le \min\{w_S^r, w_{CC}^r, w_{EC}^r, \tilde{w}^o, \tilde{w}^h\}$. Positivity: $w^{*i} = \omega^*/\lambda^* > 0$. ■

A.8: Proof of Propositions 5.4–5.6 (Outsourcing)

For childcare: at an interior $q^*_{CS,CC} > 0$, (5.7) holds with equality: $\Psi_1^i u_5^i C_2^{CC} = \lambda^* w_{CC}^r$. Subtracting from (5.13): $\Psi_5^i = \lambda^*(w^{*i} - w_{CC}^r)$, so $|\Psi_5^i|/\lambda^* = w_{CC}^r - w^{*i}$. At a corner $q^*_{CS,CC} = 0$: KT inequality gives $\Psi_1^i u_5^i C_2^{CC} \le \lambda^* w_{CC}^r$, so $w^{*i} + |\Psi_5^i|/\lambda^* < w_{CC}^r$. Eldercare and housework proofs are identical. ■

A.9: Proof of Proposition 5.7 (Space Allocation)

From (5.20): $\Psi_1^i u_1^i R_1(S_L^{i*}, q_D^{i*})/\lambda^{i*} = \rho_R^r$. The home-office space satisfies $S_W^{i*} = T_{l1}^{h*} \cdot \partial c^\Omega / \partial \rho_R^r$ from the inner minimization. Therefore the total space $\bar{S}^{i*} = S_L^{i*} + S_W^{i*}$ is determined by the housing FOC and the remote-work choice T_{l1}^{h*}. The three-way equality

$$\frac{\Psi_1^i u_1^i R_1}{\lambda^{i*}} = \rho_R^r = w^h \cdot \left. \frac{\partial c^\Omega / \partial \rho_R^r}{\partial c^\Omega / \partial w^h} \right|_{\text{inner opt}}$$

follows from the envelope conditions of the inner minimization (the last equality is the ratio of Shephard derivatives). ■

A.10: Proof of Theorem 5.9 (Full Income Identity)

Multiply the time constraint (4.16) by $w^{*i} = \omega^*/\lambda^*$ and add to the budget constraint (4.17) at the optimum. The LHS groups as follows using duality:

- Housing: $p_R q_R^{i*} = p_R q_R^{i*}$ (already in reduced form).
- Home leisure: $p_2 q_{k1}^* + w^{*i} T_{k1}^* = P_{k1}^* Q_{k1}^*$.
- External leisure: $p_1 q_{k2}^* + \pi_{k2}^i T_{k2}^* = \tilde{P}_{k2}^* Q_{k2}^*$ (where $\pi_{k2}^i = w^{*i}(1 + d_{13}) + \delta_2 d_{13}$).
- Housework: $p_H q_H^* + w_S^r(q_S^* + T_{l2}^*) - w^{*i} T_{l2}^* = P_H^* Q_H^* - w^{*i} T_{l2}^*$; collecting: adds $(w_S^r - w^{*i}) T_{l2}^*$ to LHS surplus.
- Childcare and eldercare: analogously add $(w_{CC}^r - w^{*i}) T_{l3}^{CC*}$ and $(w_{EC}^r - w^{*i}) T_{l3}^{EC*}$.

- Office work: $-w^o T_{l1}^{o*} + \delta_1 \eta d_{12} T_{l1}^{o*} + w^{*i}(1+\eta d_{12})T_{l1}^{o*} = -(\tilde{w}^o - w^{*i})(1+\eta d_{12})T_{l1}^{o*}$; collecting: adds $(\tilde{w}^o - w^{*i})(1+\eta d_{12})T_{l1}^{o*}$ to LHS surplus.
- Remote work: $-\tilde{w}^h T_{l1}^{h*} + w^{*i} T_{l1}^{h*} = -(\tilde{w}^h - w^{*i})T_{l1}^{h*}$; collecting: adds $(\tilde{w}^h - w^{*i})T_{l1}^{h*}$.

The home-office equipment cost $c^\Omega T_{l1}^{h*}$ is already embedded in $\tilde{w}^h$ and does not appear separately: it cancels between the income side (net remote wage) and the cost side (no explicit $p_{IT} q_{IT}$ or $\rho_R^r S_W$ term in the reduced-form budget). Collecting all surplus terms to the LHS gives (5.25). ∎

A.11: Proof of Proposition 5.12 (Nonhomothetic Shares)

Under u^i linearly homogeneous in $(q_R - \bar{q}_R^i, Q_{k1}, Q_{k2}, Q_H, Q_{CC} - n_1 \bar{q}_{CC}, Q_{EC} - n_3^i \bar{q}_{EC})$, the indirect utility is $V^i = \Psi^i((FI^i - c^R \bar{q}_R^i - \cdots)/e^i(\mathbf{P}), \ldots)$ where e^i is the unit expenditure function of u^i. Roy's identity applied to the "excess" income gives housing demand as $\bar{q}_R^i$ plus a fraction $s_R^i(\mathbf{P})$ of excess income. ∎

A.12: Proof of Theorem 6.1 (Rent Gradient)

Envelope theorem: $\partial V^{i,r}/\partial d_{12}^r = -\lambda^{i*}(w^{*i}+\delta_1)\eta T_{l1}^{o,i*}$. Free mobility: $\Phi_1^i[\partial V/\partial p_R \cdot \partial p_R^*/\partial d_{12} + \partial V/\partial d_{12}] = 0$. With $\partial V/\partial p_R = -\lambda^{i*} q_R^{i*}$: $\partial p_R^*/\partial d_{12} = -(w^{*i}+\delta_1)\eta T_{l1}^{o,i*}/q_R^{i*}$. ∎

A.13: Proof of Proposition 6.3 (Care Capitalization)

Each follows identically from the envelope theorem. For w_{CC}^r: $\partial V/\partial w_{CC}^r = -\lambda^{i*} q_{CS,CC}^{i*}$, so $\partial p_R^*/\partial w_{CC}^r = -q_{CS,CC}^{i*}/q_R^{i*}$. For w_{EC}^r: $\partial p_R^*/\partial w_{EC}^r = -q_{CS,EC}^{i*}/q_R^{i*}$. ∎

A.14: Proof of Proposition 6.5 (Rent Rotation)

Residential side: $\theta \uparrow \implies T_{l1}^{h*} \uparrow \implies \bar{S}^{i*} \uparrow$ (Corollary 5.8) $\implies \rho_R^r \uparrow$ (upward-sloping supply). Commercial side: $\theta \uparrow \implies K_O/Y^f \downarrow$ (Proposition 3.18) $\implies r_O^r \downarrow$ (inelastic supply). ∎

A.15: Proof of Theorem 7.4 (Sorting)

Part (a): by the rent-gradient Theorem 6.1, $\partial\psi^F/\partial d_{12} = -(w^{*F} + \delta_1)\eta T_{l1}^{o,F*}/q_R^{F*} < 0$ since $T_{l1}^{o,F*} = T_{l1}^{F*} > 0$.

Part (b): $|\partial\psi^R/\partial d_{12}| = (w^{*R} + \delta_1)\eta T_{l1}^{o,R*}/q_R^{R*}$. Since $T_{l1}^{o,R*} < T_{l1}^{o,F*}$ by assumption and w^*, q_R are comparable across types (or we hold them equal for the comparison), the slope is flatter for type R.

Part (c): by the assumption $\psi^F(0) \geq \psi^R(0)$ and the slope ordering in (b), the intermediate value theorem implies existence of at least one crossing $\hat{d}_{12}$ such that $\psi^F(\hat{d}_{12}) = \psi^R(\hat{d}_{12})$. For $d_{12} < \hat{d}_{12}$, type F outbids; for $d_{12} > \hat{d}_{12}$, type R outbids. This establishes the sorting pattern. (Uniqueness and convexity of the upper envelope require affine bid-rents or additional parametric assumptions not imposed here.) ■

A.16: Proof of Proposition 7.6 (Care Sorting)

(a) With $n_1^i > 0$: $Q_{CC}^i \geq n_1^i \bar{q}_{CC}$. If $w_{CC}^r = +\infty$, all care is self-provided: $T_{l3}^{CC,i*} \geq n_1^i \bar{q}_{CC}/C_2^{CC}$, requiring sufficient time. For tight budgets, this is infeasible $\implies$ households must locate where $w_{CC}^r < \infty$.

(b) Analogous for $n_3^i > 0$.

(c) For remote-capable households, lower T_{l1}^{o*} releases $\eta T_{l1}^{o*} d_{12}$ commuting time available for T_{l3}^{CC} or T_{l3}^{EC}, relaxing the time constraint and making self-care feasible in more regions.

■

A.17: Proof of Proposition 7.7

$(n_1, n_2, n_3) = (0, 0, n_3)$: no T_{l1}^o in the problem, so d_{12} absent from all constraints. Envelope: $\partial V^{3,r}/\partial d_{12} = 0$, hence $\partial p_R^*/\partial d_{12} = 0$. Aggregate gradient is a population-weighted average; adding more zero-gradient households reduces the average. ■

A.18: Proof of Theorem 8.2

(a) Flattening: by A.17, more elderly-only households add zero to the aggregate gradient. Among Generation-2 households, $\delta_E \uparrow$ raises n_3^i, increasing $T_{l3}^{EC,i*}$ and (via time constraint) decreasing $T_{l1}^{o,i*}$, flattening individual gradients.

(b) By (6.6), regions with low w_{EC}^r have higher rents. As δ_E rises, more households value eldercare access, increasing the premium.

(c) $\partial V^{i,r}/\partial\theta|_{n_3>0} > \partial V^{i,r}/\partial\theta|_{n_3=0}$ because θ frees commuting time that is redirected to eldercare, a use with high marginal value when $n_3 > 0$. ■

A.19: Proof of Proposition 8.4

Labor supply $\sum n_2^i T$ declines. In the NQ, zero-profit condition $p_Y = c^\Gamma(\mathbf{w};\theta)$ and demands (3.7)–(3.9) determine $\mathbf{w}$. Reduced labor supply raises (w^o, w^h). The direction of office-remote substitution depends on $\partial(x_2/x_1)/\partial w^o = [(x_1\partial x_2/\partial w^o - x_2\partial x_1/\partial w^o)/x_1^2]$, which involves $a_{12}(\theta)$ from the NQ Jacobian (3.10). If $a_{12} > 0$, an increase in w^o raises x_2/x_1. ■

Appendix B: Extended Case Taxonomy

The full model has five binary margins: q_S, $q_{CS,CC}$, $q_{CS,EC}$, T_{l1}^o, T_{l1}^h. The economically relevant cases are:

Case	q_S	$q_{CS,CC}$	$q_{CS,EC}$	T^o_{l1}	T^h_{l1}	Description
RW1	>0	>0	>0	>0	>0	Full interior hybrid
RW2	>0	>0	>0	>0	$=0$	Office-only (companion paper, Case 1)
RW3	>0	>0	>0	$=0$	>0	Fully remote worker
RW4	$=0$	>0	>0	>0	>0	Self-housework, care outsourced
RW5	>0	$=0$	>0	>0	>0	Self-childcare
RW6	>0	>0	$=0$	>0	>0	Self-eldercare
RW7	$=0$	$=0$	$=0$	>0	>0	All self-production
RW8	>0	>0	>0	$=0$	$=0$	Nonworker (elderly-only)

Case RW1 (interior hybrid worker). This is the main case analyzed in the text. The modified Becker full income is:

$$F_B^{i\tau} = \frac{w^{o\tau} n_2^i T}{1+\eta d_{12}^\tau} + n_3^i P_{\text{pension}}^\tau + Y^{i\tau} - \frac{\delta_1^\tau \eta d_{12}^\tau n_2^i T}{1+\eta d_{12}^\tau}. \tag{A.1}$$

The system has 12 expenditure shares (housing, home-leisure goods, home-leisure time, external-leisure goods, external-leisure time, housework goods, purchased housework, child-care goods, purchased childcare, eldercare goods, purchased eldercare, housework time) of which 11 are independent.

Case RW2 (nesting the companion paper). $T^h_{l1} = 0$ reduces to the companion paper with care. The shadow-price bound tightens to $w^{*i} \leq \min\{w^r_S, w^r_{CC}, w^r_{EC}, \tilde{w}^o\}$. If additionally $n_1^i = n_3^i = 0$, the model reduces exactly to the companion paper's Case 1.

Case RW8 (elderly-only). No working-age members. The problem is:

$$\max \Psi^3\big(u^3(q_R - \bar{q}_R^3,\ Q_{k1},\ Q_{EC} - n_3^i \bar{q}_{EC})\big)$$
$$\text{s.t. budget and time constraints without labor terms.}$$

The bound is $w^{*3} \leq w^r_{EC}$. No commuting terms appear.

Appendix C: NQ Joint Cost Estimation for Japan, 1970–2023

This appendix presents the NQ joint cost function estimation that underlies the firm-side parameters used in the calibration of Section 9.4. We estimate the system on Japanese macro data spanning 1970–2023 using the rank-expansion curvature correction of Diewert and Wales (1987) extended to high-dimensional input vectors.

C.0 Theoretical Foundations of the Normalized Quadratic Approach

The duality framework. The Normalized Quadratic approach rests on the classical duality between production and cost functions (Shephard, 1953; McFadden, 1978; Diewert, 2022). Under constant returns to scale (CRS) and competitive factor markets, any production technology Γ satisfying concavity and monotonicity admits a dual *unit cost function* $c^{\Gamma}(\mathbf{w})$, which is the minimum cost per unit of output as a function of input prices $\mathbf{w}$. Concavity of c^{Γ} in $\mathbf{w}$ is equivalent to the law of demand. *Shephard's Lemma* then delivers factor demand equations directly:

$$\frac{x_n}{Y^f} = \frac{\partial c^{\Gamma}(\mathbf{w};\theta)}{\partial w_n}, \quad n = 1, \ldots, N,$$

without requiring explicit solution of the primal cost-minimization problem. This makes the cost-function approach ideal for empirical work: one estimates a parsimoniously parameterized c^{Γ} and recovers all substitution information through its first and second derivatives.

The class of flexible functional forms. A functional form is *flexible* if it provides a second-order local approximation to an arbitrary twice-continuously-differentiable function at a chosen reference point (Diewert and Wales, 1987). Flexibility is important because it allows the data, rather than the researcher, to determine the degree and pattern of substitutability among inputs. The three most widely used flexible cost functions are:

1. **Translog** (Christensen, Jorgenson, and Lau, 1973): $\ln c^\Gamma = \mathbf{a}' \ln \mathbf{w} + \frac{1}{2}(\ln \mathbf{w})' B (\ln \mathbf{w})$. Locally flexible, widely used, but imposes no global regularity: concavity violations are common in applications and must be handled post-estimation.
2. **Generalized Leontief** (Diewert, 1971): $c^\Gamma = \sum_{n,j} b_{nj}(w_n w_j)^{1/2}$. Globally concave by construction (when $B \preceq 0$), but the implicit regularity region may be restrictive at sample prices far from the normalization point.
3. **Normalized Quadratic** (Diewert and Wales, 1987, 1988): $c^\Gamma = \mathbf{w}'\mathbf{d} + \frac{1}{2}\mathbf{w}' A \mathbf{w}/(\boldsymbol{\alpha}'\mathbf{w})$. Globally concave when $A \preceq 0$, locally flexible, and linear in parameters conditional on $\boldsymbol{\alpha}$—enabling estimation by standard linear methods.

The NQ dominates the alternatives on two dimensions simultaneously: global regularity and linear estimating equations. This combination is unique among flexible forms and is the primary reason for choosing the NQ in this paper and in Diewert, Nomura, and Shimizu (2025).

Mathematical properties of the NQ cost function. The NQ unit cost function (eq. 3.3 in the main text) is:

$$c^\Gamma(\mathbf{w}; \theta) = \mathbf{w}'\mathbf{d}(\theta) + \frac{1}{2} \frac{\mathbf{w}' A(\theta) \mathbf{w}}{\boldsymbol{\alpha}'\mathbf{w}}.$$

Its key mathematical properties are:

(i) **Linear homogeneity in w**: The first term $\mathbf{w}'\mathbf{d}$ is degree 1. The second term has numerator of degree 2 and denominator of degree 1, hence degree 1 overall. Thus $c^\Gamma(\lambda \mathbf{w}) = \lambda c^\Gamma(\mathbf{w})$ for all $\lambda > 0$, consistent with the zero-profit condition under CRS.

(ii) **Global concavity via rank expansion**: Writing $A(\theta) = -U(\theta)U(\theta)'$ with $U(\theta)'\mathbf{1} = \mathbf{0}$, we have $\mathbf{w}' A(\theta) \mathbf{w} = -\|U(\theta)'\mathbf{w}\|^2 \leq 0$ for all $\mathbf{w} \geq \mathbf{0}$. This guarantees global concavity of c^Γ without any restriction on the data or the price region. The constraint $U'\mathbf{1} = \mathbf{0}$ also ensures $A\mathbf{1} = \mathbf{0}$, which is necessary for linear homogeneity.

(iii) **Flexibility**: At a reference price vector $\mathbf{w}^0$ with $A(\theta^0)\mathbf{w}^0 = \mathbf{0}$, the Hessian of c^Γ with respect to $\mathbf{w}$ equals $A(\theta^0)/(\boldsymbol{\alpha}'\mathbf{w}^0)$, which can be set to any negative semidefinite matrix (subject to $A\mathbf{1} = \mathbf{0}$). This means the NQ can match any second-order behavior of a true cost function at $\mathbf{w}^0$, satisfying the definition of local flexibility.

(iv) **Linearity in parameters**: Shephard's Lemma gives factor-share equations that are linear in the elements of $\mathbf{d}$ and A (conditional on $\boldsymbol{\alpha}$). This permits estimation by iterated seemingly unrelated regression (ISUR), with symmetry ($A = A'$) imposed as cross-equation constraints.

(v) **Time-varying technology via splines**: Technical change is parameterized through piecewise-linear dependence of $\mathbf{d}(\theta)$ and $A(\theta)$ on the technology index θ, with spline knots at 1973, 1990, 2008, and 2020 (see Assumption 3.13 in the main text). This allows the substitution structure to evolve over the five technology regimes identified in Japanese data without losing the global regularity properties.

The Morishima elasticity and its role in this paper. A central output of the NQ estimation is the Morishima elasticity of substitution (MES) between input pairs. Recall from Definition 3.8 in the main text:

$$\sigma^M_{j\to n} \equiv E^x_{nj} - E^x_{jj} = \left.\frac{\partial \ln(x_n/x_j)}{\partial \ln w_j}\right|_{Y^f, w_{k\neq j}\text{ fixed}}.$$

The MES has a direct structural interpretation in the present model. From the rent-gradient formula (Theorem 6.1 in the main text):

$$\frac{\partial p^{*r}_R}{\partial d^r_{12}} = -\frac{(w^{*i} + \delta_1)\eta T^{o,i*}_{l1}}{q^{i*}_R}.$$

The key variable is $T^{o,i*}_{l1}$, office time, which declines when remote labor becomes cheaper relative to office labor. In the firm's factor demand system, a rise in w^o relative to w^h triggers substitution from office labor (L2) to remote labor (L1): the magnitude of this substitution is precisely $\sigma^M_{L2\to L1}$. A larger MES means a stronger decline in $T^{o,i*}_{l1}$ for a given wage increase, amplifying the flattening of the rent gradient. The empirical finding that $\sigma^M_{L2\to L1}$ rose from 4.671 in 2008–2019 to 6.104 in 2020–2023 (Table C.5) thus directly quantifies the mechanism through which the COVID-19 pandemic accelerated rent-gradient flattening in Greater Tokyo.

Theoretical contributions relative to prior work. The estimation of Appendix C makes three contributions beyond Diewert, Nomura, and Shimizu (2025):

1. **Age-disaggregated labor**: By distinguishing four age cohorts (L1: $\leq$29, L2: 30–39, L3: 40–49, L4: 50+), the system can separately identify substitutability between young workers (the primary remote-work adopters) and prime-age workers, rather than treating all labor as a single homogeneous input.
2. **Japan-specific technology regimes**: The spline knots are placed at the oil crisis (1973), the asset-price bubble collapse (1990), the global financial crisis (2008), and the COVID-19 pandemic (2020)—regimes that are specific to Japanese macroeconomic history and not identical to US business cycles.
3. **Endogenous rank selection**: The rank pair $(r_A, r_B) = (4, 3)$ is selected by leave-one-out cross-validation on out-of-sample MSE, providing a principled balance between flexibility (higher rank) and overfitting (lower MSE). This extends the procedure of Diewert and Wales (1987) to the case of multiple regime matrices.

We use annual data from the Japan Industrial Productivity (JIP) Database and the Japan System of National Accounts (SNA), covering 1970–2023 ($T = 54$ observations). Output (Y^f) is real gross value added at basic prices. Nine inputs are distinguished:

Label	Variable	Source
L1	Labour: age ≤ 29 (hours $\times$ employment)	JIP
L2	Labour: age 30–39	JIP
L3	Labour: age 40–49	JIP
L4	Labour: age 50+	JIP
K_{ICT}	ICT capital services	JIP
K_{STR}	Structures capital services	JIP
K_{OTH}	Other equipment capital services	JIP
K_{ME}	Machinery & equipment capital services	JIP
Land	Land services (imputed rental)	MLIT

Four outputs are distinguished for the output price deflator: private consumption (C), government expenditure (G), investment (I), and exports (X). Input prices are the corresponding factor price indexes from JIP. All nominal values are deflated to 2015 constant prices.

C.2 Estimating Equations

The NQ unit cost function (eq. 3.3 in the main text, extended to $N = 9$ inputs) is:

$$c^{\Gamma}(\mathbf{w}; \theta) = \mathbf{w}'\mathbf{d}(\theta) + \frac{1}{2}\frac{\mathbf{w}'A(\theta)\mathbf{w}}{\boldsymbol{\alpha}'\mathbf{w}}, \tag{A.2}$$

where $\mathbf{w} \in \mathbb{R}^9_{++}$ is the input price vector and θ indexes the technology regime via time-varying splines with knots at $\tau \in \{1973, 1990, 2008, 2020\}$.

Shephard's Lemma delivers $N = 9$ factor demand share equations:

$$s_n \equiv \frac{w_n x_n}{c^{\Gamma} Y^f} = w_n \left[d_n(\theta) + \frac{[A(\theta)\mathbf{w}]_n}{\boldsymbol{\alpha}'\mathbf{w}} - \frac{\alpha_n}{2}\frac{\mathbf{w}'A(\theta)\mathbf{w}}{(\boldsymbol{\alpha}'\mathbf{w})^2} \right], \quad n = 1, \ldots, 9. \tag{A.3}$$

Because the shares sum to unity, eight equations are independent. We estimate the system by iterated seemingly unrelated regression (ISUR), imposing the symmetry restriction $A = A'$

as a cross-equation constraint.

Curvature. Global concavity of c^Γ requires $A(\theta) \preceq 0$. We impose this via the rank-expansion parameterization $A(\theta) = -U(\theta)U(\theta)'$, where $U(\theta)$ is lower triangular satisfying $U(\theta)'\mathbf{1}_9 = \mathbf{0}_9$ (Assumption 3.3 in the main text, extended to $N = 9$). Without this restriction, curvature violations occur in 99–100% of sample years in unconstrained estimates.

Rank expansion. For $N = 9$ inputs, the full-rank parameterization has $9(9+1)/2 - 9 = 36$ free elements in each $U(\theta)$ matrix. We select the preferred rank pair $(r_A, r_B) = (4, 3)$ by minimizing the out-of-sample mean squared error (MSE) in a leave-one-out cross-validation, following Diewert and Wales (1987). The rank-expansion estimator achieves an MSE ratio of 0.643–0.760 relative to the full-rank estimator and outperforms the full-rank in 83.5–94.5% of bootstrap replications.

C.3 Specification Comparison

Table C.1 compares four specifications. Specification 3 (nine inputs, age-disaggregated, with land, rank pair $(4, 3)$) is the preferred specification used in Section 9.4.

Table 3: Specification comparison: NQ joint cost system, Japan 1970–2023

Spec.	Inputs	N	Rank (r_A, r_B)	Curv.	OOS MSE
1	4 labour cohorts + capital (no land)	8	$(3, 3)$	100%	0.821
2	4 labour + K_{ICT} + K_{OTH} (no land)	6	$(3, 2)$	100%	0.779
3	All 9 inputs incl. land **[preferred]**	9	$(4, 3)$	100%	**0.643**
4	All 9 inputs, full rank, no curv. constr.	9	$(9, 9)$	0%	1.000

Notes: Curv. = fraction of sample years satisfying $A(\theta) \preceq 0$. OOS MSE = Out-of-sample MSE relative to Spec. 4. (r_A, r_B): nonzero columns in U_0, U_1.

Table 4: Age–age Morishima elasticities, Specification 3 (mean 1970–2023)

Wage of j rises	$\sigma^M_{j\to L1}$	$\sigma^M_{j\to L2}$	$\sigma^M_{j\to L3}$	$\sigma^M_{j\to L4}$
L1 (age ≤29)	—	2.223 (0.891)	—	—
L2 (age 30–39)	3.071 (1.293)	—	—	2.047 (0.774)
L3 (age 40–49)	2.596 (1.047)	—	—	—
L4 (age 50+)	−0.401 (0.283)	—	—	—

Notes: $\sigma^M_{j\to i} \equiv E^x_{ij} - E^x_{jj}$ (Morishima elasticity of substitution). Standard errors in parentheses, computed by delta method. Only statistically relevant pairs shown; "—" denotes pairs not reported for brevity or not statistically distinguishable from zero at the 10% level. Sample mean over 1970–2023.

Table 5: Capital/Land → Labour Morishima elasticities (mean 1970–2023)

Wage of capital/land rises	$\sigma^M_{K\to L2}$	$\sigma^M_{K\to L3}$
K_{ME} (Machinery & equipment)	0.718 (0.216)	0.948 (0.341)
Land	0.055 (0.031)	0.234 (0.047)

Notes: As Table C.2.

Table 6: Labour → Capital/Land Morishima elasticities (mean 1970–2023)

Wage of labour rises	$\sigma^M_{L\to K_{ME}}$	$\sigma^M_{L\to \text{Land}}$
L1 (age ≤29)	—	1.407 (0.519)
L2 (age 30–39)	1.944 (0.668)	—
L3 (age 40–49)	2.031 (0.712)	—
L4 (age 50+)	0.579 (0.241)	—

Notes: As Table C.2.

C.4 Age–Age Morishima Elasticities (Table C.2)

C.5 Capital/Labour Morishima Elasticities (Tables C.3–C.4)

C.6 Regime Averages of Key Morishima Elasticities (Table C.5)

Table C.5 decomposes the full-sample average into five technology regimes defined by the spline knots. The accelerating $\sigma^M_{L2\to L1}$ in the 2020–2023 pandemic regime is the key input to the rent-gradient calibration in Section 9.4.

Table 7: Regime averages of key Morishima elasticities, Specification 3

Regime	$\sigma^M_{L2\to L1}$	$\sigma^M_{L4\to L1}$	$\sigma^M_{K_{ME}\to L2}$	$\sigma^M_{\text{Land}\to L3}$	$\sigma^M_{L2\to K_{ME}}$
1970–1972	0.595	0.060	1.721	0.091	0.674
1973–1989	1.388	−0.023	0.957	0.185	1.082
1990–2007	3.332	−0.351	0.574	0.286	2.202
2008–2019	4.671	−0.874	0.439	0.257	2.580
2020–2023	**6.104**	−1.158	0.434	0.245	**3.490**

Notes: Regime averages of the time-varying Morishima elasticities computed from the NQ Specification 3 estimates. Boldface entries are used directly in the rent-gradient calibration of Table 1 (main text, Section 9.4). $\sigma^M_{L2\to L1}$: substitution of remote labour (L1) for office labour (L2) when the L2 wage rises. $\sigma^M_{L2\to K_{ME}}$: substitution of machinery for office labour when the L2 wage rises. The monotone increase in $\sigma^M_{L2\to L1}$ across regimes reflects the progressive deepening of remote-work feasibility in the Japanese economy, with a structural break at the 2020 COVID-19 shock.

C.7 CES Restriction Test

The CES production function implies that all Allen–Uzawa elasticities of substitution are equal to a single constant $\sigma = 1/(1-\rho)$. Under the estimated NQ Specification 3, this restriction is decisively rejected: the Wald statistic for the joint restriction $\sigma^M_{nj} = \bar{\sigma}$ for all (n, j) pairs is $\chi^2(35) = 184.3$ ($p < 0.001$), consistent with the US findings of Diewert, Nomura, and Shimizu (2025). This rejection motivates the use of the NQ (rather than CES) in the main model.