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Measuring Regional Welfare through a Spatial Becker Model: Time Allocation,
Location Choice, and Superlative Index Theory

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Abstract

This paper develops a spatial extension of Becker's theory of time allocation, building on the generalized household-production framework of Schreyer and Diewert (2014). Households allocate time across market labor, household work, home leisure, and external leisure at three locations: home, workplace, and amenity venues. Commuting lowers the effective time endowment; leisure travel raises the full price of external leisure. The shadow price of leisure is bounded by $\min\{w\tau S, w\}$, extending the result of Schreyer and Diewert (2014) to a spatial setting. I derive a spatial full-income identity, equilibrium rent gradients with respect to eight parameters, second-order properties of the rent function, and connect to superlative index theory through a Regional Utility Index. An econometric framework for four corner-solution regimes maps the model into observable data.

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Abstract

This paper develops a spatial extension of Becker's theory of time allocation, building on the generalized household-production framework of Schreyer and Diewert (2014). Households allocate time across market labor, household work, home leisure, and external leisure at three locations: home, workplace, and amenity venues. Commuting lowers the effective time endowment; leisure travel raises the full price of external leisure. The shadow price of leisure is bounded by $\min\{w_s^r, w\}$, extending the result of Schreyer and Diewert (2014) to a spatial setting. I derive a spatial full-income identity, equilibrium rent gradients with respect to eight parameters, second-order properties of the rent function, and connect to superlative index theory through a Regional Utility Index. An econometric framework for four corner-solution regimes maps the model into observable data.

JEL codes: R21, R31, J22, E01, C43

Keywords: time allocation; household production; spatial equilibrium; commuting; shadow price of leisure; superlative index numbers; regional welfare measurement

1 Introduction

How should urban economists compare welfare across regions when households differ not only in rents and wages, but also in commuting burdens, access to amenities, and the allocation of non-market time? Standard approaches based on income per capita, hedonic wage-rent decompositions, or subjective well-being surveys each leave something important out. Income measures ignore household time. Reduced-form hedonic approaches require strong identifying assumptions. Subjective measures are valuable, but they are not naturally embedded in the optimizing structure used in urban and household economics. This paper develops an alternative that is explicitly utility-theoretic: a spatial extension of [Becker \(1965\)](#)'s theory of time allocation that is designed for regional welfare measurement.

The paper builds on the generalized Becker framework developed by [Schreyer and Diewert \(2014\)](#). In that framework, time enters household decision making in three distinct ways: leisure time, household work time, and market labor time can each have separate utility implications. This generalization is crucial because it allows the shadow value of time to differ from the observed market wage. I extend this framework spatially. Households reside at one location, work at another, and consume amenity-intensive leisure at a third. Commuting and leisure travel then enter the model as explicit time and money costs, so that the welfare consequences of location are derived from the household's own optimization problem rather than imposed through a reduced-form bid-rent schedule.

The resulting framework fills a gap in the theoretical literature. Standard urban models treat the household as a black box: the monocentric model of [Alonso \(1964\)](#) and [Muth \(1969\)](#) assumes a reduced-form utility over a composite consumption good and housing, with commuting cost entering as a monetary deduction from income. The system-of-cities literature ([Rosen, 1979](#); [Roback, 1982](#)) extends this to amenity capitalization but still does not model the household's internal time allocation. My objective is to provide an explicit Beckerian micro-foundation for the welfare cost of location, so that commuting, household production, and leisure are not black-boxed but derived from the household's own optimization problem.

1.1 Contributions

The paper makes four contributions.

First, I develop a spatial Becker-Diewert model in which the household allocates time across market labor, household work, home leisure, and external leisure at three locations: home (S_1), workplace (S_2), and amenity venues (S_3). In the worker regime, commuting to S_2 absorbs a fixed amount of time and money each period, while access to S_3 raises the full price of external leisure in proportion to leisure travel.

Second, I derive the model's welfare objects. The shadow price of leisure time, w^* , remains bounded above by $\min\{w_S, w\}$ in the worker regime and by w_S alone in the nonworker regime, extending the four-case analysis of [Schreyer and Diewert \(2014\)](#) to a spatial setting. I derive a spatial full-income identity in which commuting lowers welfare through both the time opportunity cost w^*d_{12} and the monetary cost δ_1d_{12} . Free mobility across regions then yields closed-form bid-rent gradients with respect to commuting distance, amenity distance, the price of household services, and the level of local amenities. The second-order properties of the rent function (convexity, concavity, and cross-effects) are analyzed under sufficient conditions on housing demand and shadow prices; the sign of each second-order effect depends on identifiable empirical magnitudes and is not sign-determinate without parametric assumptions.

Third, I show that linearly homogeneous household production creates a direct bridge to superlative index number theory. Fisher and Törnqvist indexes can be used to aggregate the components of full consumption without fully parameterizing the aggregator functions. This leads to a *Regional Utility Index* (RUI) that is theoretically grounded and suitable for cross-regional welfare comparisons, including multilateral and intertemporal extensions via the GEKS method.

Fourth, I develop an econometric estimation framework based on the primal approach of [Schreyer and Diewert \(2014\)](#), extended to the spatial model. The share equations derived from the first-order conditions provide a system of inverse demand functions that can be

estimated by nonlinear seemingly unrelated regressions. Four regimes are analyzed, corresponding to different corner solutions in household services purchases and labor supply, with each regime yielding a distinct set of estimating equations.

1.2 Related Literature

This paper contributes to five literatures.

Time allocation and household production. [Becker \(1965\)](#) initiated the economic analysis of time allocation by treating households as producers of final commodities that combine market goods and time. [Pollak and Wachter \(1975, 1977\)](#) emphasized the importance of direct disutility from household work and the limits of market substitutes. [Barnett \(1977\)](#) developed the approach further. [Gronau \(1977\)](#) distinguished between market work, home production, and leisure, while [Heckman \(1974\)](#) analyzed shadow prices and corner solutions in a labor supply context. [Schreyer and Diewert \(2014\)](#) provided the most general treatment of leisure, household work, and market labor with corner solutions, deriving the four-case taxonomy that the present paper extends spatially.

Two features of the Diewert framework are central to this paper. First, the shadow price of leisure w^* is bounded above by $\min\{w_S, w\}$, which differs from Becker’s implicit assumption that $w^* = w$. Second, the linearly homogeneous household production functions connect directly to index number theory—a link that is not available in models with more general production structures. The present paper preserves both features while introducing an explicit spatial dimension.

Urban spatial equilibrium. The monocentric city model of [Alonso \(1964\)](#), [Muth \(1969\)](#), and [Mills \(1967\)](#) is the foundational framework for understanding urban land rents. In these models, rents decline with distance from the central business district (CBD) to compensate commuters for higher travel costs. [Fujita \(1989\)](#) provides a comprehensive treatment of the monocentric model and its extensions.

The system-of-cities literature, initiated by Rosen (1979) and Roback (1982), extends this framework to multiple cities with heterogeneous amenities. Albouy (2016) provides a careful empirical implementation that accounts for the joint capitalization of amenities into wages and rents. Diamond (2016) introduces heterogeneous preferences and skill-based sorting. Monte, Redding, and Rossi-Hansberg (2018) develop a quantitative spatial model with commuting.

My contribution differs from this literature in two respects. First, I derive the bid-rent function from a Beckerian time-allocation problem rather than from a reduced-form utility function over income and housing, so that the welfare cost of commuting reflects not only monetary expenditure but also the opportunity cost of time, valued at the endogenous shadow price w^* . Second, the model accommodates both workers and nonworkers—the latter are indifferent to commuting distance but sensitive to amenity access—which is important for aging societies where a growing share of the population does not commute.

Amenity valuation and hedonic methods. The hedonic approach to amenity valuation, developed by Rosen (1974) in the product-market context and applied to spatial equilibrium by Roback (1982), infers the value of amenities from observed wage and rent differentials. Chay and Greenstone (2005) provide a clean empirical application to air quality. Glaeser (2022) surveys the resilience and evolution of cities.

The present paper provides a structural alternative to the hedonic approach. Rather than inferring amenity values from reduced-form rent regressions, the model derives them from the household’s optimization problem. The amenity vector \mathbf{A}^r enters the indirect utility function directly (Assumption 3.16), and its capitalization into rent is derived from the free-mobility condition. This structural approach has the advantage that the welfare interpretation is exact—the rent gradient equals the marginal willingness to pay—without requiring the identifying assumptions of the hedonic method (e.g., that the utility function is locally linear in amenity characteristics).

Welfare measurement, full consumption, and national accounting. The valuation of non-market household time has long been a central issue in satellite accounting and the measurement of living standards (Landefeld and McCulla, 2000; Abraham and Mackie, 2005; Fraumeni, 2008; Landefeld, Fraumeni, and Vojtech, 2009). Schreyer and Ranuzzi de Bianchi (2009) and Hill (2009) examined the measurement of own-account household production. The generalized full-income approach in Schreyer and Diewert (2014) and the broader accounting agenda in Diewert, Nomura, and Shimizu (2024, 2025) provide the accounting background for the present paper. Here, that literature is connected to urban welfare through spatially adjusted full consumption.

The concept of “full consumption” that emerges from the model— $FC^W = p_R q_R^* + P_{k1}^* Q_{k1}^* + \tilde{P}_{k2}^* Q_{k2}^* + P_H^* Q_H^*$ —extends the national-accounting concept of actual individual consumption by incorporating nonmarket household time, valued at the shadow price w^* or the imputed household-work price w_H^* depending on the case. The use of superlative index numbers to aggregate this object across regions and time periods provides a theoretically grounded alternative to ad hoc welfare comparisons based on income alone.

Index number theory and aggregation. Diewert (1976) established the theory of exact and superlative index numbers, showing that certain formulas—Fisher, Törnqvist—provide second-order approximations to arbitrary linearly homogeneous aggregator functions. This result is the cornerstone of modern price and productivity measurement (Diewert, Nomura, and Shimizu, 2025). The present paper applies these results to the spatial household model, using superlative indices to aggregate the components of full consumption into a Regional Utility Index. The two-stage aggregation procedure (first-stage: within each production function; second-stage: across the four components of full consumption) exploits the linearly homogeneous structure of F , G , H and is consistent in the sense of Diewert (1978). The GEKS method (Gini, 1931; Eltető, Köves, and Szulc, 1964) ensures transitivity in multilateral comparisons, which is essential for ranking regions.

Organization of the paper. Section 2 presents the non-spatial base model. Section 3 develops the spatial extension for working households. Sections 4 and 5 derive indirect utility and the equilibrium rent function. Section 6 establishes the link to superlative index number theory. Section 7 develops the econometric framework, including nonworker corner solutions. Section 8 presents a calibration using Japanese metropolitan data. Section 9 concludes. Online Appendices A–G provide detailed proofs and derivations.

2 The Base Model

I begin with a non-spatial version of the model that serves as a benchmark and establishes notation. This section parallels Schreyer and Diewert (2014), with the key difference that leisure is decomposed into home leisure and external leisure, and housing services enter the utility function directly.

2.1 Preferences and Household Production

Consider a single-person household with preferences over six objects: housing services q_R , home leisure services Q_{k1} , external leisure services Q_{k2} , household services Q_H , market labor time T_{l1} , and household work time T_{l2} . Preferences are represented by a utility function:

$$U = U(q_R, Q_{k1}, Q_{k2}, Q_H, T_{l1}, T_{l2}) \quad (2.1)$$

Assumption 2.1 (Preferences). The utility function $U : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ is concave, continuous, and differentiable.¹ It satisfies:

- (i) $U_1 > 0, U_2 > 0, U_3 > 0, U_4 > 0$: marginal utility of housing, home leisure, external leisure, and household services is positive;
- (ii) $U_5 \leq 0$: marginal utility of market labor is nonpositive (disutility of work);

¹Concavity is stronger than quasiconcavity but, as Afriat (1967) and Diewert (1973) showed, it is not empirically restrictive.

(iii) $U_6 \leq 0$: marginal utility of household work is nonpositive (disutility of chores).

The consumption aggregates are produced by household production functions:

$$Q_{k1} = F(q_{k1}, T_{k1}) \quad (2.2)$$

$$Q_{k2} = G(q_{k2}, T_{k2}) \quad (2.3)$$

$$Q_H = H(q_H, T_{l2} + q_S) \quad (2.4)$$

where q_{k1} and T_{k1} are purchased goods and time inputs for home leisure, q_{k2} and T_{k2} for external leisure, q_H for household production goods, T_{l2} is the household's own work time, and q_S is purchased household services (e.g., cleaning, cooking help).

Assumption 2.2 (Production Functions). F , G , and H satisfy the following:

- (i) *Regularity*: each is continuous, twice continuously differentiable on \mathbb{R}_{++}^2 , and strictly positive when both arguments are strictly positive;
- (ii) *Positive marginal products*: $F_1, F_2 > 0$, $G_1, G_2 > 0$, $H_1, H_2 > 0$ on \mathbb{R}_{++}^2 ;
- (iii) *Concavity*: each is concave on \mathbb{R}_+^2 , so its Hessian matrix is negative semi-definite on \mathbb{R}_{++}^2 ;
- (iv) *Linear homogeneity*: $F(\lambda q_1, \lambda q_2) = \lambda F(q_1, q_2)$ for all $\lambda > 0$, and analogously for G and H .

Conditions (ii)–(iv) together imply, via Euler's theorem, that $F = F_1 q_1 + F_2 q_2$ (and analogously for G, H), and that each function admits a well-defined unit-cost function $c^J(\mathbf{p})$ via the duality $J(\mathbf{q}) = Q \Leftrightarrow \mathbf{p}'\mathbf{q} \geq c^J(\mathbf{p})Q$ for all $\mathbf{p} \gg 0$, with Shephard's lemma $\partial c^J / \partial p_j = q_j^* / Q$ applying at interior optima. The coefficient matrix A^J for each homogeneous quadratic specification (Appendix F) is assumed to be *positive definite* (not merely positive semi-definite), ensuring the dual unit-cost matrix $B^J = (A^J)^{-1}$ exists; the singular case $\det(A^J) = 0$ is excluded.

The linear homogeneity assumption is standard in the Becker–Diewert tradition (Becker, 1965; Schreyer and Diewert, 2014) and is essential for the connection to superlative index theory developed in Section 6. The assumption that T_{l2} and q_S are perfect substitutes in (2.4) follows Schreyer and Diewert (2014) and plays a key role in the analysis of corner solutions.

2.2 Constraints

The household faces the time constraint:

$$T_{l1} + T_{l2} + T_{k1} + T_{k2} = T \quad (2.5)$$

and the budget constraint:

$$p_R q_R + p_2 q_{k1} + p_1 q_{k2} + p_H q_H + w_S q_S \leq w T_{l1} + Y \quad (2.6)$$

where p_R is the price of housing services, p_1 and p_2 are prices of external and home leisure goods, p_H is the price of household production goods, w_S is the wage for purchased household services, w is the after-tax market wage, and $Y \geq 0$ is nonlabor income.

2.3 The Optimization Problem

The household solves:

$$u^* \equiv \max_{\substack{q_R, q_{k1}, q_{k2}, q_H, q_S \geq 0 \\ T_{k1}, T_{k2}, T_{l1}, T_{l2} \geq 0}} \{U[q_R, F(q_{k1}, T_{k1}), G(q_{k2}, T_{k2}), H(q_H, T_{l2} + q_S), T_{l1}, T_{l2}]\} \quad (2.7)$$

subject to (2.5) and (2.6).

Proposition 2.3 (Saddle Point). *Under Assumptions 2.1 and 2.2, the optimization problem (2.7) is a concave programming problem. By the Karlin–Uzawa Saddle Point Theorem (Kar-*

lin, 1959; Uzawa, 1958), there exist multipliers $\lambda^* > 0$ and $\omega^* > 0$ such that the constrained optimum is equivalent to the saddle-point problem:

$$u^* = \min_{\lambda, \omega \geq 0} \max_{\mathbf{q}, \mathbf{T} \geq 0} \mathcal{L}(\lambda, \omega, \mathbf{q}, \mathbf{T}) \quad (2.8)$$

where the Lagrangian is:

$$\begin{aligned} \mathcal{L} = U[q_R, F(q_{k1}, T_{k1}), G(q_{k2}, T_{k2}), H(q_H, T_{l2} + q_S), T_{l1}, T_{l2}] \\ + \lambda[wT_{l1} + Y - p_R q_R - p_2 q_{k1} - p_1 q_{k2} - p_H q_H - w_S q_S] \\ + \omega[T - T_{l1} - T_{l2} - T_{k1} - T_{k2}]. \end{aligned} \quad (2.9)$$

Both constraints bind at the optimum.

Proof. The objective function is concave in the choice variables. To see this, note that U is concave by Assumption 2.1, and F , G , H are concave by Assumption 2.2. Since U is nondecreasing in its first four arguments and each of $F(q_{k1}, T_{k1})$, $G(q_{k2}, T_{k2})$, $H(q_H, T_{l2} + q_S)$ is concave in the respective choice variables, the composition $U[q_R, F(\cdot), G(\cdot), H(\cdot), T_{l1}, T_{l2}]$ is concave in the full vector of choice variables (by the concavity-preservation theorem for nondecreasing concave functions of concave functions; see Schreyer and Diewert 2014, footnote 2).

The two constraint functions— $g_1(\mathbf{q}, \mathbf{T}) \equiv wT_{l1} + Y - p_R q_R - p_2 q_{k1} - p_1 q_{k2} - p_H q_H - w_S q_S$ and $g_2(\mathbf{q}, \mathbf{T}) \equiv T - T_{l1} - T_{l2} - T_{k1} - T_{k2}$ —are both linear (hence concave) in the choice variables, and the feasible set $\{(\mathbf{q}, \mathbf{T}) \geq 0 : g_1 \geq 0, g_2 \geq 0\}$ is convex.

The Slater constraint qualification requires the existence of a strictly feasible point $(\bar{\mathbf{q}}, \bar{\mathbf{T}}) > 0$ with $g_1(\bar{\mathbf{q}}, \bar{\mathbf{T}}) > 0$ and $g_2(\bar{\mathbf{q}}, \bar{\mathbf{T}}) > 0$. This is satisfied by choosing \bar{T}_{l1} sufficiently large (generating income) and all other variables sufficiently small.

By the Karlin–Uzawa Saddle Point Theorem (Karlin 1959, pp. 201–203; Uzawa 1958), there exist multipliers $\lambda^* \geq 0$ and $\omega^* \geq 0$ such that $(\lambda^*, \omega^*, \mathbf{q}^*, \mathbf{T}^*)$ is a saddle point of the

Lagrangian (2.9).

It remains to show $\lambda^* > 0$ and $\omega^* > 0$ without assuming interior solutions. The KKT stationarity conditions are *inequalities*: for any choice variable $x_j \geq 0$,

$$\frac{\partial \mathcal{L}}{\partial x_j} \leq 0, \quad x_j \geq 0, \quad x_j \frac{\partial \mathcal{L}}{\partial x_j} = 0.$$

*Positivity of λ^** : Suppose $\lambda^* = 0$. The KKT condition for q_R is $U_1 - \lambda^* p_R \leq 0$, which becomes $U_1 \leq 0$. But Assumption 2.1(i) states $U_1 > 0$ for all $(\mathbf{q}, \mathbf{T}) \geq 0$ —a contradiction. Hence $\lambda^* > 0$.

*Positivity of ω^** : Suppose $\omega^* = 0$. Consider the KKT condition for T_{k1} :

$$U_2 F_2(q_{k1}^*, T_{k1}^*) - \omega^* \leq 0 \Rightarrow U_2 F_2 \leq 0.$$

Since $U_2 > 0$ (Assumption 2.1(i)) and $F_2 > 0$ (Assumption 2.2(ii)) when $q_{k1}^* > 0$ or $T_{k1}^* > 0$, this is a contradiction if the household values home leisure at all. More precisely: at any feasible point where $g_2 = T - T_{l1} - T_{l2} - T_{k1} - T_{k2} = 0$ (time constraint binds), if $\omega^* = 0$ then the shadow value of any time use is zero, but the KKT condition for T_{k1} with $\omega^* = 0$ requires $U_2 F_2 \leq 0$, contradicting $U_2, F_2 > 0$. Hence $\omega^* > 0$.

Both constraints bind at the optimum: if $g_1 > 0$, then $\lambda^* = 0$ by complementary slackness, contradicting $\lambda^* > 0$. Similarly for g_2 . ■

Remark 2.4 (Concavity preservation). The concavity-preservation argument deserves emphasis. Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be concave and nondecreasing, and let $h_j : \mathbb{R}^{m_j} \rightarrow \mathbb{R}$ be concave for $j = 1, \dots, J$. Then $\phi(h_1(\mathbf{x}_1), \dots, h_J(\mathbf{x}_J))$ is concave in $(\mathbf{x}_1, \dots, \mathbf{x}_J)$. This is a standard result (see, e.g., [Boyd and Vandenberghe 2004](#), §3.2.4) and is the key technical ingredient ensuring that the household's problem is a well-posed concave program. The result fails if ϕ is not nondecreasing in all arguments: for this reason, the disutility arguments T_{l1} and T_{l2} enter U directly (not through concave sub-functions), and the nonincreasing assumption $U_5 \leq 0, U_6 \leq 0$ is essential.

Definition 2.5 (Shadow Price of Leisure). The *imputed price of leisure time* is defined as:

$$w^* \equiv \frac{\omega^*}{\lambda^*} > 0. \quad (2.10)$$

2.4 First-Order Conditions and the Shadow-Price Bound

Assuming an interior solution with $q_S^* > 0$ and $T_{l1}^* > 0$ (Case 1 in the taxonomy of Section 7), the first-order conditions for problem (2.8) are:²

$$U_1 = \lambda^* p_R \quad (2.11)$$

$$U_2 \cdot F_1(q_{k1}^*, T_{k1}^*) = \lambda^* p_2 \quad (2.12)$$

$$U_2 \cdot F_2(q_{k1}^*, T_{k1}^*) = \lambda^* w^* \quad (2.13)$$

$$U_3 \cdot G_1(q_{k2}^*, T_{k2}^*) = \lambda^* p_1 \quad (2.14)$$

$$U_3 \cdot G_2(q_{k2}^*, T_{k2}^*) = \lambda^* w^* \quad (2.15)$$

$$U_4 \cdot H_1(q_H^*, T_{l2}^* + q_S^*) = \lambda^* p_H \quad (2.16)$$

$$U_4 \cdot H_2(q_H^*, T_{l2}^* + q_S^*) = \lambda^* w_S \quad (2.17)$$

$$U_4 \cdot H_2(q_H^*, T_{l2}^* + q_S^*) + U_6 = \lambda^* w^* \quad (2.18)$$

$$U_5 = -\lambda^*(w - w^*) \quad (2.19)$$

Theorem 2.6 (Bounds on the Shadow Price of Leisure). *Under Assumptions 2.1 and 2.2, with $q_S^* > 0$ and $T_{l1}^* > 0$:*

$$0 < w^* \leq \min\{w_S, w\}. \quad (2.20)$$

Proof. From (2.19): $U_5 = -\lambda^*(w - w^*)$. Since $U_5 \leq 0$ and $\lambda^* > 0$, we have $w - w^* \geq 0$, i.e., $w^* \leq w$.

From (2.17) and (2.18): substituting $U_4 H_2 = \lambda^* w_S$ into (2.18) gives $U_6 = -\lambda^*(w_S - w^*)$. Since $U_6 \leq 0$ and $\lambda^* > 0$, we have $w_S - w^* \geq 0$, i.e., $w^* \leq w_S$.

²In all first-order conditions, ω^* is replaced by $\lambda^* w^*$ following Definition 2.5.

Combining: $w^* \leq \min\{w_S, w\}$. The positivity $w^* > 0$ follows from Definition 2.5. ■

This result is the direct counterpart of Schreyer and Diewert (2014, Eq. 22). Theorem 2.6 applies when the household both works and purchases household help. The three remaining corner-solution cases yield different bounds.

Definition 2.7 (Shadow price of household work). In cases where $q_S^* = 0$, the household's time spent in household work cannot be valued at the market price w_S . Define the *imputed shadow price of household work time* as:

$$w_H^* \equiv \frac{U_4 \cdot H_2(q_H^*, T_{l2}^*)}{\lambda^*} > 0. \quad (2.21)$$

When $q_S^* > 0$, the first-order condition (2.17) implies $w_H^* = w_S$, so the imputed price coincides with the market price. When $q_S^* = 0$, w_H^* is an unobserved endogenous variable.

Theorem 2.8 (Bounds: Case 2, worker without purchased household help). *Under Assumptions 2.1 and 2.2, with $q_S^* = 0$ and $T_{l1}^* > 0$:*

$$0 < w^* \leq \min\{w_H^*, w\} \leq \min\{w_S, w\}. \quad (2.22)$$

Proof. *Bound $w^* \leq w$:* From (2.19), which remains valid for Case 2: $U_5 = -\lambda^*(w - w^*)$, so $w^* \leq w$.

Bound $w_H^ \leq w_S$:* With $q_S^* = 0$, the Kuhn–Tucker condition replacing (2.17) is $U_4 H_2 \leq \lambda^* w_S$, i.e., $w_H^* \leq w_S$.

Bound $w^ \leq w_H^*$:* Substituting the definition (2.21) into the first-order condition for T_{l2} (which becomes $U_4 H_2 + U_6 = \lambda^* w^*$ with $q_S = 0$) yields $\lambda^* w_H^* + U_6 = \lambda^* w^*$, i.e., $w^* = w_H^* + U_6/\lambda^*$. Since $U_6 \leq 0$: $w^* \leq w_H^*$.

Combining: $0 < w^* \leq w_H^* \leq w_S$ and $w^* \leq w$, so $w^* \leq \min\{w_H^*, w\} \leq \min\{w_S, w\}$. ■

Remark 2.9 (Two unobserved prices in Case 2). Unlike Case 1, where the single unobserved price w^* suffices (because $w_H^* = w_S$ is observed), Case 2 involves *two* unobserved shadow

prices: w^* (leisure) and w_H^* (household work). The gap between them is $w_H^* - w^* = |U_6|/\lambda^*$, which is the monetized disutility of household work. If $U_6 = 0$ (no disutility of housework), then $w^* = w_H^*$ and only one shadow price needs to be estimated. If additionally $U_5 = 0$ (no disutility of market work), then $w^* = w_H^* = w$ and we recover the Becker valuation—but this also requires $w \leq w_S$ for consistency with the Kuhn–Tucker condition.

Theorem 2.10 (Bounds: Case 3, nonworker with purchased household help). *Under Assumptions 2.1 and 2.2, with $q_S^* > 0$ and $T_{l1}^* = 0$:*

$$0 < w^* \leq w_S. \quad (2.23)$$

The market wage w does not provide an upper bound on w^ .*

Proof. Bound $w^ \leq w_S$:* With $q_S^* > 0$, condition (2.17) holds as an equality: $U_4 H_2 = \lambda^* w_S$. Substituting into the T_{l2} condition (2.18): $U_6 = -\lambda^*(w_S - w^*)$. Since $U_6 \leq 0$: $w^* \leq w_S$.

Why w is not a bound: With $T_{l1}^* = 0$, the first-order condition (2.19) is replaced by the Kuhn–Tucker inequality $U_5 \leq -\lambda^*(w - w^*)$. This is satisfied whenever $|U_5|$ is sufficiently large, even if $w^* > w$. Economically, a retired household may value leisure more highly than the wage it *could* earn but chooses not to. ■

Remark 2.11 (Nonworker households and commuting). In the spatial model (Section 3), the absence of the w -bound for nonworkers has a direct spatial consequence: nonworker households are *indifferent* to commuting distance d_{12} , because they do not commute. Their location choice is driven entirely by d_{13} (amenity access), w_S^r (household-services availability), and the amenity vector \mathbf{A}^r .

Theorem 2.12 (Bounds: Case 4, nonworker without purchased household help). *Under Assumptions 2.1 and 2.2, with $q_S^* = 0$ and $T_{l1}^* = 0$:*

$$0 < w^* \leq w_H^* \leq w_S. \quad (2.24)$$

Neither w nor w_S provides a tight bound on w^* ; both w^* and w_H^* are unobserved.

Proof. Bound $w^* \leq w_H^*$: Same argument as in Theorem 2.8: $w^* = w_H^* + U_6/\lambda^* \leq w_H^*$.

Bound $w_H^* \leq w_S$: Same as in Theorem 2.8: the Kuhn–Tucker condition for q_S gives $U_4 H_2 \leq \lambda^* w_S$, i.e., $w_H^* \leq w_S$.

No w -bound: Same as in Theorem 2.10: the Kuhn–Tucker condition for T_{l1} does not constrain w^* relative to w . ■

The four theorems are summarized in Table 1.

Table 1: Shadow-price bounds by case

Case	q_S^*	T_{l1}^*	Bound on w^*	w_H^* status	Schreyer–Diewert (2014)
1	> 0	> 0	$0 < w^* \leq \min\{w_S, w\}$	$= w_S$ (known)	Eq. 22
2	$= 0$	> 0	$0 < w^* \leq \min\{w_H^*, w\} \leq \min\{w_S, w\}$	Unknown, $\leq w_S$	Eq. 51
3	> 0	$= 0$	$0 < w^* \leq w_S$	$= w_S$ (known)	Eq. 65
4	$= 0$	$= 0$	$0 < w^* \leq w_H^* \leq w_S$	Unknown, $\leq w_S$	Eq. 74

Remark 2.13 (The Becker case as a special case). If $U_5 = U_6 = 0$ (no disutility of either market work or household work), then in Cases 1 and 2: $w^* = w_H^* = w$, which is Becker (1965)’s valuation. This requires $w \leq w_S$ for the Kuhn–Tucker conditions to be consistent. The assumption $U_5 = U_6 = 0$ eliminates the distinction between the four cases and yields a single valuation rule: time is valued at the market wage. The generality of the Diewert framework lies precisely in allowing $U_5 < 0$ and $U_6 < 0$, which generates a richer structure of shadow prices.

Remark 2.14 (Regional household-services markets and Case selection). When the household-services price varies across regions (Assumption 3.13 in Section 3.7), the case selection is partly determined by supply conditions. In regions with $w_S^r = +\infty$ (no market household services), $q_S^* = 0$ is forced regardless of preferences, placing the household in Case 2 or 4. In such regions, the bounds involving w_S^r become vacuous, and the effective constraint on w^* comes from w alone (for workers) or is absent (for nonworkers). See Proposition 3.15 in Section 3.7 for the spatial version of these results.

2.5 Full Income and Full Consumption

Multiplying the time constraint (2.5) by w^* and adding to the budget constraint (2.6) (both holding with equality at the optimum) yields the combined identity:

$$p_R q_R^* + p_2 q_{k1}^* + w^* T_{k1}^* + p_1 q_{k2}^* + w^* T_{k2}^* + p_H q_H^* + w_S q_S^* + w^* T_{l2}^* - (w - w^*) T_{l1}^* = Y + w^* T. \quad (2.25)$$

This is obtained by multiplying (2.5) by w^* (giving $w^* T_{l1}^* + w^* T_{l2}^* + w^* T_{k1}^* + w^* T_{k2}^* = w^* T$) and adding to (2.6) (giving $p_R q_R^* + p_2 q_{k1}^* + p_1 q_{k2}^* + p_H q_H^* + w_S q_S^* = w T_{l1}^* + Y$). The terms $w^* T_{l1}^*$ and $w T_{l1}^*$ combine to $-(w - w^*) T_{l1}^*$.

Next, I use the duality between the production functions and their unit cost functions. For each linearly homogeneous production function $J \in \{F, G, H\}$, define:

Definition 2.15 (Unit cost functions).

$$c^F(p_2, w^*) \equiv \min_{q, t \geq 0} \{p_2 q + w^* t : F(q, t) \geq 1\}, \quad (2.26)$$

$$c^G(p_1, w^*) \equiv \min_{q, t \geq 0} \{p_1 q + w^* t : G(q, t) \geq 1\}, \quad (2.27)$$

$$c^H(p_H, w_S) \equiv \min_{q, t \geq 0} \{p_H q + w_S t : H(q, t) \geq 1\}. \quad (2.28)$$

By Lemma C.1 (proved in Online Appendix C), the first-order conditions (2.12)–(2.13)

imply that (q_{k1}^*, T_{k1}^*) solves the cost minimization $\min\{p_2q + w^*t : F(q, t) \geq Q_{k1}^*\}$. Therefore:

$$p_2q_{k1}^* + w^*T_{k1}^* = c^F(p_2, w^*) \cdot F(q_{k1}^*, T_{k1}^*) \equiv P_{k1}^*Q_{k1}^*. \quad (2.29)$$

Similarly, (2.14)–(2.15) and (2.16)–(2.17) imply:

$$p_1q_{k2}^* + w^*T_{k2}^* = c^G(p_1, w^*) \cdot G(q_{k2}^*, T_{k2}^*) \equiv P_{k2}^*Q_{k2}^*, \quad (2.30)$$

$$p_Hq_H^* + w_S(T_{l2}^* + q_S^*) = c^H(p_H, w_S) \cdot H(q_H^*, T_{l2}^* + q_S^*) \equiv P_H^*Q_H^*. \quad (2.31)$$

Definition 2.16 (Full Prices and Full Income).

$$P_{k1}^* \equiv c^F(p_2, w^*), \quad P_{k2}^* \equiv c^G(p_1, w^*), \quad P_H^* \equiv c^H(p_H, w_S) \quad (2.32)$$

$$FI \equiv Y + w^*T \quad (2.33)$$

Definition 2.17 (Full Consumption).

$$FC \equiv p_Rq_R^* + P_{k1}^*Q_{k1}^* + P_{k2}^*Q_{k2}^* + P_H^*Q_H^* \quad (2.34)$$

The full income identity is:

$$FC - (w_S - w^*)T_{l2}^* - (w - w^*)T_{l1}^* = FI. \quad (2.35)$$

3 The Spatial Extension

This section develops the spatial model for the economically central worker regime, $T_{l1} > 0$. This is the regime in which commuting affects the effective time endowment and therefore generates a bid-rent gradient. Section 7 returns to the nonworker corner cases $T_{l1} = 0$, in which commuting time and commuting expenditure vanish.

3.1 Spatial Structure

The household operates across three locations:

- S_1 (Home): residence, where home leisure and household work occur;
- S_2 (Workplace): where market labor is supplied;
- S_3 (Amenity venue): restaurants, parks, cafés, shops, and other locations that generate external leisure services.

3.2 Commuting and Leisure Travel

Assumption 3.1 (Worker commuting). If the household supplies market labor at S_2 , it incurs a fixed round-trip commuting time $d_{12} \geq 0$ per period and a monetary commuting cost $\delta_1 d_{12}$, where $\delta_1 > 0$ is the per-unit-time monetary cost of commuting.

Assumption 3.2 (Leisure travel). External leisure at S_3 requires travel time proportional to external leisure time: $T_{r2} = d_{13} T_{k2}$, where $d_{13} \geq 0$ is the time-distance from home to amenity venues. The corresponding monetary cost is $\delta_2 d_{13} T_{k2}$ with $\delta_2 > 0$.

The distinction is intentional. Commuting is modeled as a fixed participation cost of working, while leisure travel scales with the amount of external leisure consumed. This formulation keeps the worker regime analytically transparent and matches the empirical observation that work participation often entails at least one commute per workday, whereas leisure trips vary with the amount of non-home leisure activity.

3.3 Worker Constraints

For a working household, the time constraint becomes

$$T_{l1} + T_{l2} + T_{k1} + (1 + d_{13})T_{k2} = T - d_{12} \equiv \tilde{T}, \quad (3.1)$$

where \tilde{T} is the *effective time endowment*. The budget constraint is

$$p_R q_R + p_2 q_{k1} + p_1 q_{k2} + p_H q_H + w_S q_S + \delta_1 d_{12} + \delta_2 d_{13} T_{k2} \leq w T_{l1} + Y. \quad (3.2)$$

3.4 The Spatial Optimization Problem

The working household solves

$$u^{W*} = \max_{\mathbf{q}, \mathbf{T} \geq 0} U[q_R, F(q_{k1}, T_{k1}), G(q_{k2}, T_{k2}), H(q_H, T_{l2} + q_S), T_{l1}, T_{l2}] \quad (3.3)$$

subject to (3.1) and (3.2).

Proposition 3.3 (Spatial first-order conditions). *Under Assumptions 2.1–3.2, with $q_S^* > 0$ and $T_{l1}^* > 0$, conditions (2.11)–(2.16) and (2.17) continue to hold. The time-related first-order conditions become*

$$U_3 \cdot G_2(q_{k2}^*, T_{k2}^*) = \lambda^* \pi_{k2}, \quad (3.4)$$

$$U_4 \cdot H_2(q_H^*, T_{l2}^* + q_S^*) + U_6 = \lambda^* w^*, \quad (3.5)$$

$$U_5 = -\lambda^*(w - w^*), \quad (3.6)$$

where

$$\pi_{k2} \equiv w^*(1 + d_{13}) + \delta_2 d_{13} \quad (3.7)$$

is the full price of one unit of external leisure time.

Proof. See Appendix A. ■

Theorem 3.4 (Spatial bound on the shadow price of leisure). *In the worker regime with $q_S^* > 0$ and $T_{l1}^* > 0$,*

$$0 < w^* \leq \min\{w_S, w\}. \quad (3.8)$$

Proof. Condition (3.6) is identical in form to (2.19), so $w^* \leq w$ follows exactly as in The-

orem 2.6. Combining (2.17) and (3.5) yields $U_6 = -\lambda^*(w_S - w^*)$, and since $U_6 \leq 0$ and $\lambda^* > 0$, we obtain $w^* \leq w_S$. Positivity follows from Definition 2.5. ■

Remark 3.5. The spatial extension does not change the *bound* on w^* ; it changes the *equilibrium value* of w^* through the effective time endowment $\tilde{T} = T - d_{12}$ and the full price of external leisure π_{k2} .

3.5 Spatial Full Income and Full Consumption

The worker model yields a spatial full-income identity that differs from the base model in two places: commuting reduces available time and also absorbs monetary resources.

Definition 3.6 (Spatial full prices). Let

$$P_{k1}^* \equiv c^F(p_2, w^*), \quad \tilde{P}_{k2}^* \equiv c^G(p_1, \pi_{k2}), \quad P_H^* \equiv c^H(p_H, w_S), \quad (3.9)$$

where c^F , c^G , and c^H are the unit-cost functions dual to F , G , and H .

Definition 3.7 (Spatial full consumption). Worker full consumption is

$$FC^W \equiv p_R q_R^* + P_{k1}^* Q_{k1}^* + \tilde{P}_{k2}^* Q_{k2}^* + P_H^* Q_H^*. \quad (3.10)$$

Theorem 3.8 (Spatial full income identity, Case 1). *For a working household in Case 1 ($q_S^* > 0$, $T_{l1}^* > 0$), with $P_H^* = c^H(p_H, w_S^r)$,*

$$FC^W - (w_S^r - w^*)T_{l2}^* - (w - w^*)T_{l1}^* = Y - \delta_1 d_{12} + w^*(T - d_{12}) \equiv FI^W. \quad (3.11)$$

In Case 2 ($q_S^ = 0$, $T_{l1}^* > 0$), the identity takes a different form: P_H^* must be replaced by $P_H^{*2} = c^H(p_H, w_H^*)$ where $w_H^* = U_4 H_2 / \lambda^*$ is the imputed household-work price; see Theorem C.3.*

Proof. This proof applies to Case 1, where $q_S^* > 0$ so the market price w_S^r governs household services. At the optimum, both constraints hold with equality. Multiply the time constraint

(3.1) by w^* and add it to the budget constraint (3.2). This gives

$$\begin{aligned}
& p_R q_R^* + p_2 q_{k1}^* + w^* T_{k1}^* + p_1 q_{k2}^* + [w^*(1 + d_{13}) + \delta_2 d_{13}] T_{k2}^* \\
& + p_H q_H^* + w_S^r (T_{l2}^* + q_S^*) - (w_S^r - w^*) T_{l2}^* - (w - w^*) T_{l1}^* \\
& = Y - \delta_1 d_{12} + w^*(T - d_{12}).
\end{aligned}$$

By linear homogeneity and duality (Assumption 2.2),

$$\begin{aligned}
P_{k1}^* Q_{k1}^* &= p_2 q_{k1}^* + w^* T_{k1}^*, \\
\tilde{P}_{k2}^* Q_{k2}^* &= p_1 q_{k2}^* + \pi_{k2} T_{k2}^*, \\
P_H^* Q_H^* &= p_H q_H^* + w_S^r (T_{l2}^* + q_S^*).
\end{aligned}$$

Substituting these equalities into the previous expression yields (3.11). ■

Remark 3.9. Relative to the non-spatial identity (2.35), worker full income is reduced by both the opportunity cost of commuting time, $w^* d_{12}$, and the direct monetary commuting cost, $\delta_1 d_{12}$. In the nonworker corner cases analyzed later, commuting disappears and full income reverts to $Y + w^* T$.

3.6 The Role of Purchased Household Services

The decision to purchase market household services is a key channel through which time is reallocated across labor, leisure, and household production. This subsection makes the purchase condition explicit and shows how it interacts with the spatial structure.

3.6.1 When does a household purchase outside help?

Combining the interior conditions (2.17) and (3.5) yields

$$U_6 = -\lambda^*(w_S - w^*). \tag{3.12}$$

The left-hand side is the marginal disutility of an additional hour of own household work. The right-hand side is the net cost of avoiding that hour by hiring market help: the household pays w_S per hour but saves time valued at w^* . Rearranging,

$$w_S = w^* + \frac{|U_6|}{\lambda^*}. \quad (3.13)$$

Proposition 3.10 (Household-services purchase: optimality characterization). *At an interior optimum with $q_S^* > 0$ (Case 1 or Case 3), the KKT conditions imply the necessary optimality condition:*

$$\underbrace{w^*}_{\text{value of released time}} + \underbrace{\frac{|U_6|}{\lambda^*}}_{\text{avoided disutility}} = \underbrace{w_S^r}_{\text{cost of market help}}. \quad (3.14)$$

At a corner optimum with $q_S^ = 0$, the KKT inequality implies $w^* + |U_6|/\lambda^* \leq w_S^r$.*

Caveat: Because w^ and $|U_6|/\lambda^*$ are endogenous equilibrium objects, (3.14) characterizes the optimum ex post. It cannot be used as an ex ante decision rule without additional restrictions on preferences and technology.*

Proof. At an interior solution with $q_S^* > 0$, condition (2.17) holds with equality, and (3.12) holds. Rearranging (3.12) gives (3.14) with equality. If $q_S^* = 0$, the KKT inequality for q_S is $U_4 H_2 \leq \lambda^* w_S^r$; combined with (3.5) (which gives $U_4 H_2 = \lambda^*(w^* - U_6/\lambda^*) = \lambda^*(w^* + |U_6|/\lambda^*)$), this implies $w^* + |U_6|/\lambda^* \leq w_S^r$. ■

Remark 3.11 (Two sources of benefit from outsourcing). Condition (3.14) reveals two distinct components of the benefit of outsourcing: (i) *time release* (w^*): each outsourced hour frees time valued at the shadow price of leisure; and (ii) *disutility avoidance* ($|U_6|/\lambda^*$): the household also avoids the direct unpleasantness of performing the task itself. If there is no disutility of housework ($U_6 = 0$) and the household is at an interior optimum with $q_S^* > 0$ (Case 1), then (3.14) gives $w^* = w_S^r$. This does *not* hold in Cases 2 or 4 where $q_S^* = 0$; in those cases the bound is $w^* \leq w_S^r$ with strict inequality possible.

3.6.2 Time reallocation from outsourcing

When a household purchases Δq_S additional units of market household services, it can reduce T_{l2} by the same amount (since T_{l2} and q_S are perfect substitutes in H). The released time enters the time constraint (3.1) and can be directed to market labor (T_{l1}), home leisure (T_{k1}), or external leisure (T_{k2}). At the equilibrium, the marginal value of time is equalized across all active uses.

Proposition 3.12 (Time reallocation from outsourcing). *At an interior worker solution, a marginal increase in purchased household services reallocates the released time as follows:*

- (i) *To market labor, generating additional income $w \Delta T_{l1}$. Because $w > w^*$ at any interior solution with $U_5 < 0$, the marginal hour of labor generates net income beyond the shadow value of time.*
- (ii) *To home leisure, at shadow value w^* per hour.*
- (iii) *To external leisure, at effective shadow value $\pi_{k2} = w^*(1 + d_{13}) + \delta_2 d_{13} > w^*$ per hour. The higher effective price of external leisure due to travel costs implies that, ceteris paribus, less of the released time is directed to external leisure than to home leisure.*

Proof. Since T_{l2} and q_S are perfect substitutes in H (Assumption 2.2), increasing q_S by Δ allows T_{l2} to decrease by Δ while keeping Q_H constant. The freed time Δ re-enters the time constraint (3.1) and is allocated across the active margins. At the optimum, the shadow value of time w^* equalizes across all active uses. (i) The net benefit of directing the freed hour to market labor is $w - w^*$. From (3.6): $U_5 = -\lambda^*(w - w^*)$; since $U_5 < 0$ and $\lambda^* > 0$, we have $w > w^*$, so the net benefit is strictly positive. (ii) Home leisure time yields shadow value w^* per unit (by definition of w^*). (iii) External leisure time T_{k2} is effectively priced at $\pi_{k2} = w^*(1 + d_{13}) + \delta_2 d_{13} > w^*$ (since $d_{13} \geq 0$), so its per-unit cost exceeds w^* , meaning ceteris paribus the household directs proportionally less of the released time to external leisure. The equilibrium allocation across (i)–(iii) is determined by equating marginal utilities at shadow prices. ■

3.6.3 Interaction with the wage rate

The interaction between the market wage w and the household-services price w_S generates a natural typology. When $w > w_S$, the household can improve its budget position by working more and outsourcing more: each substituted hour yields a surplus of $w - w_S$. When $w < w_S$, outsourcing is not justified by wage arbitrage alone and occurs only if the disutility of housework is sufficiently large, $|U_6|/\lambda^* > w_S - w$. Low-wage households are therefore more likely to be in Case 2 ($q_S^* = 0$), performing all housework themselves.

3.7 Regional Variation in Household-Services Supply and Local Amenities

The preceding analysis assumes that household services can be purchased at a common price w_S . In practice, the availability and price of such services vary substantially across regions, and other location-specific amenities also affect household welfare. This subsection introduces both features.

3.7.1 Regional household-services markets

Assumption 3.13 (Regional household-services market). The price of purchased household services varies across regions:

$$w_S^r \in (0, +\infty], \quad (3.15)$$

with $w_S^r < \infty$ in regions where a competitive supply of household services exists (typically urban areas), and $w_S^r = +\infty$ in regions where no market household services are supplied (typically rural or remote areas).

Under Assumption 3.13, the household's optimization problem is unchanged in form: the budget constraint contains $w_S^r q_S$, and when $w_S^r = +\infty$ any $q_S > 0$ violates the budget constraint, so $q_S^* = 0$ automatically. The key consequence is that the four-case taxonomy acquires a supply-side interpretation.

Remark 3.14 (Supply-constrained versus preference-based corner solutions). Cases 2 and 4 ($q_S^* = 0$) can arise from two distinct mechanisms:

- (a) *Preference-based*: $w_S^r < \infty$ but (3.14) fails — the household could purchase help but chooses not to because the benefit is insufficient.
- (b) *Supply-constrained*: $w_S^r = +\infty$ — the household cannot purchase help regardless of its willingness to pay.

In mechanism (b), the Kuhn-Tucker inequality $U_4 H_2 \leq \lambda^* w_S^r$ is automatically satisfied for any finite $U_4 H_2$, so the imputed household-work price $w_H^* = U_4 H_2 / \lambda^*$ faces no finite upper bound from the market. This contrasts with mechanism (a), where $w_H^* \leq w_S^r < \infty$.

The shadow-price bounds adjust accordingly.

Proposition 3.15 (Bounds with regional service availability).

$$w^{*r} \leq \begin{cases} \min\{w_S^r, w\} & \text{if } w_S^r < \infty \text{ (services available),} \\ w & \text{if } w_S^r = +\infty \text{ (services unavailable).} \end{cases} \quad (3.16)$$

In both cases $w^{*r} > 0$.

Proof. When $w_S^r < \infty$ and $q_S^* > 0$, Theorem 3.4 applies. When $w_S^r = +\infty$, the constraint $w^* \leq w_S^r$ is vacuous, leaving only $w^* \leq w$ from (3.6). ■

3.7.2 Local amenities as a regional characteristic

Household-services availability is one dimension of regional quality, but other location-specific amenities also affect welfare without entering the household's time-allocation problem directly. Important examples include:

- *Healthcare access*: the presence of obstetric and pediatric facilities, which determines whether childbearing and child-rearing are feasible in a given region;
- *School quality*: the availability and quality of educational institutions;
- *Public safety, environmental quality, and cultural infrastructure*.

These amenities share two features that distinguish them from the household-production inputs analyzed above. First, they are not substitutes for household time: visiting a pediatrician does not reduce household work in the way that hiring a cleaner does. Second, their relevance is often *life-stage specific*: obstetric facilities matter to households of childbearing age but not to retirees.

To incorporate these features without altering the core time-allocation structure, I model local amenities as a shift parameter in the indirect utility function.

Assumption 3.16 (Local amenity vector). Each region r is characterized by an amenity vector $\mathbf{A}^r = (A_1^r, \dots, A_K^r) \in \mathbb{R}_+^K$, where A_k^r measures the availability or quality of amenity k in region r . The household's indirect utility is

$$\tilde{V}^{Wr} = \Phi(V^{Wr}, \mathbf{A}^r), \quad (3.17)$$

where V^{Wr} is the worker indirect utility from problem (3.3) taking $(w_S^r, d_{12}^r, d_{13}^r)$ as given, and Φ is increasing in V^{Wr} and nondecreasing in each A_k^r .

Remark 3.17 (Healthcare and the extensive margin of fertility). Consider a region s that lacks obstetric and pediatric facilities, so that $A_{health}^s = 0$. For a household of childbearing age, this effectively forecloses childbearing or imposes severe costs. In the limit, the absence of healthcare can be interpreted as reducing Φ discontinuously for the affected household type. By contrast, a retiree household may be indifferent to this component of the amenity vector, so that $\partial\Phi/\partial A_{health} = 0$ for that household type. This life-stage dependence is a natural motivation for future work on heterogeneous households.

Remark 3.18 (Relationship to household-services availability). Household-services availability (w_S^r) and local amenities (\mathbf{A}^r) enter the model at different levels. The price w_S^r enters the household's *optimization problem* directly, affecting time allocation, the shadow price w^* , and full consumption through the household production function H . Local amenities enter the *indirect utility* as a regional characteristic, affecting location choice and equilibrium

rents without altering the internal structure of time allocation. This distinction is analytically important: w_S^r generates predictions about household time use (e.g., hours of own housework) that are testable with time-use or mobility data, while \mathbf{A}^r generates predictions about location sorting and rent premia that are testable with cross-regional rent data.

4 The Indirect Utility Function

4.1 Definition

Let

$$V^{Wr} \equiv V^W(p_R^r, p_1, p_2, p_H, w_S^r, w, Y, d_{12}^r, d_{13}^r, \delta_1, \delta_2, T) \quad (4.1)$$

denote the indirect utility attained by a working household in region r after solving problem (3.3) with the region-specific household-services price w_S^r . The amenity-augmented indirect utility is

$$\tilde{V}^{Wr} = \Phi(V^{Wr}, \mathbf{A}^r), \quad (4.2)$$

as defined in Assumption 3.16. The superscript W indicates that the household participates in market work, so that commuting is active.

4.2 Comparative Statics

Proposition 4.1 (Properties of worker indirect utility). *Under the maintained assumptions and for an interior worker solution:*

$$\frac{\partial V^W}{\partial p_R} = -\lambda^* q_R^* < 0, \quad (4.3)$$

$$\frac{\partial V^W}{\partial d_{12}} = -\lambda^*(w^* + \delta_1) < 0, \quad (4.4)$$

$$\frac{\partial V^W}{\partial d_{13}} = -\lambda^*(w^* + \delta_2)T_{k2}^* \leq 0, \quad (4.5)$$

$$\frac{\partial V^W}{\partial w_S^r} = -\lambda^* q_S^* \leq 0, \quad (4.6)$$

$$\frac{\partial V^W}{\partial Y} = \lambda^* > 0. \quad (4.7)$$

For the amenity-augmented utility, since $\tilde{V}^{Wr} = \Phi(V^{Wr}, \mathbf{A}^r)$ and V^{Wr} does not depend directly on A_k^r (amenities affect utility only through Φ , not through the within-region optimization):

$$\frac{\partial \tilde{V}^W}{\partial A_k^r} = \frac{\partial \Phi}{\partial A_k} \Big|_{V^{Wr}} \geq 0 \quad \text{for each } k, \quad (4.8)$$

where the inequality is strict for amenities valued by the household ($\partial \Phi / \partial A_k > 0$).

Proof. Apply the envelope theorem to the Lagrangian (A.1).

(4.3): p_R enters the budget constraint with coefficient $-q_R$, so $\partial V^W / \partial p_R = -\lambda^* q_R^* < 0$.

(4.4): d_{12} enters (i) the budget constraint with coefficient $-\delta_1$ (monetary commuting cost) and (ii) the time constraint with coefficient -1 (time cost), multiplied by the time multiplier $\omega^* = \lambda^* w^*$. By the envelope theorem: $\partial V^W / \partial d_{12} = \lambda^*(-\delta_1) + \lambda^* w^*(-1) = -\lambda^*(\delta_1 + w^*) < 0$.

(4.5): d_{13} enters the budget constraint with coefficient $-\delta_2 T_{k2}^*$ and the time constraint with coefficient $-T_{k2}^*$ (multiplied by $\omega^* = \lambda^* w^*$). Hence $\partial V^W / \partial d_{13} = -\lambda^* \delta_2 T_{k2}^* - \lambda^* w^* T_{k2}^* = -\lambda^*(w^* + \delta_2) T_{k2}^* \leq 0$.

(4.6): w_S^r enters the budget constraint with coefficient $-q_S$, so $\partial V^W / \partial w_S^r = -\lambda^* q_S^* \leq 0$.

(4.7): Y enters the budget constraint with coefficient $+1$, so $\partial V^W / \partial Y = \lambda^* > 0$.

(4.8): From Assumption 3.16, $\tilde{V}^{Wr} = \Phi(V^{Wr}, \mathbf{A}^r)$ with Φ nondecreasing in each A_k^r . The within-region optimized value V^{Wr} is determined by the household's choice variables—it does *not* depend on A_k^r directly (amenities enter only through Φ). Hence $\partial\tilde{V}^W/\partial A_k^r = \partial\Phi/\partial A_k \geq 0$. ■

5 Equilibrium Rent

5.1 Free Mobility and Rent Determination

Consider regions indexed by $r = 1, \dots, M$, each characterized by $(p_R^r, d_{12}^r, d_{13}^r, w_S^r, \mathbf{A}^r)$. Suppose households are homogeneous and freely mobile across regions.

Definition 5.1 (Worker spatial equilibrium). A *worker spatial equilibrium* is a rent vector $(p_R^{*1}, \dots, p_R^{*M})$ such that

$$\tilde{V}^W(p_R^{*r}, p_1, p_2, p_H, w_S^r, w, Y, d_{12}^r, d_{13}^r, \delta_1, \delta_2, T; \mathbf{A}^r) = \bar{V} \quad \text{for all } r, \quad (5.1)$$

where \bar{V} is the common equilibrium utility level.

Theorem 5.2 (Equilibrium rent gradients). *Suppose the worker equilibrium is interior in housing and external leisure. Then the free-mobility condition (5.1) implicitly defines a differentiable rent function*

$$p_R^{*r} = p_R^*(d_{12}^r, d_{13}^r, w_S^r, \mathbf{A}^r; \bar{V}) \quad (5.2)$$

with gradients

$$\frac{\partial p_R^*}{\partial d_{12}} = -\frac{w^* + \delta_1}{q_R^*} < 0, \quad (5.3)$$

$$\frac{\partial p_R^*}{\partial d_{13}} = -\frac{(w^* + \delta_2)T_{k2}^*}{q_R^*} \leq 0, \quad (5.4)$$

$$\frac{\partial p_R^*}{\partial w_S^r} = -\frac{q_S^*}{q_R^*} \leq 0, \quad (5.5)$$

$$\frac{\partial p_R^*}{\partial A_k^r} > 0 \quad \text{for amenities valued by the household.} \quad (5.6)$$

The inequalities in (5.4) and (5.5) are strict whenever $T_{k2}^* > 0$ and $q_S^* > 0$, respectively.

Proof. Differentiate the equilibrium condition (5.1) with respect to each regional characteristic $x \in \{d_{12}, d_{13}, w_S^r, A_k^r\}$:

$$\frac{\partial \tilde{V}^W}{\partial p_R} \frac{\partial p_R^*}{\partial x} + \frac{\partial \tilde{V}^W}{\partial x} = 0.$$

Since $\partial \tilde{V}^W / \partial p_R = \Phi_1 \cdot (-\lambda^* q_R^*) < 0$, the rent gradient with respect to x has the opposite sign of $\partial \tilde{V}^W / \partial x$. Substituting the envelope results from Proposition 4.1 yields (5.3)–(5.5). Equation (5.6) follows from $\partial \tilde{V}^W / \partial A_k^r > 0$ by Assumption 3.16. ■

Remark 5.3 (Interpretation). Equations (5.3)–(5.6) provide a complete characterization of the spatial determinants of equilibrium rent:

- (i) *Commuting distance* (5.3): the rent gradient equals the full marginal commuting cost, $w^* + \delta_1$, normalized by housing consumption.
- (ii) *Amenity distance* (5.4): regions farther from amenity venues must compensate households through lower rents in proportion to external leisure time.
- (iii) *Household-services price* (5.5): regions where household services are expensive or unavailable (w_S^r large or $+\infty$) have lower equilibrium rents. The gradient is proportional to the quantity of purchased household services, q_S^* .
- (iv) *Local amenities* (5.6): regions with better amenities (healthcare, schools, safety) command higher rents.

The classical Alonso–Muth–Mills intuition (commuting cost capitalized into rent) therefore emerges alongside Roback-type amenity capitalization, both derived from a single Becker–Diewert household problem with endogenous time valuation.

Proposition 5.4 (Welfare cost of service unavailability). *Consider two regions r (urban, $w_S^r < \infty$) and s (rural, $w_S^s = +\infty$) that are otherwise identical in $(d_{12}, d_{13}, \mathbf{A})$. For a household with $U_6 < 0$ whose purchase condition (3.14) is satisfied at the urban price, $V^{Wr} > V^{Ws}$ at equal rents. In spatial equilibrium, therefore,*

$$p_R^{*r} > p_R^{*s}. \quad (5.7)$$

Proof. In region r the household can set $q_S > 0$, which by Proposition 3.10 strictly increases utility when $U_6 < 0$ and the purchase condition holds. In region s this option is foreclosed. Since $V^{Wr} > V^{Ws}$ at equal rents, the free-mobility condition requires $p_R^{*r} > p_R^{*s}$. ■

Remark 5.5 (Amenity capitalization and life-stage heterogeneity). Proposition 5.4 provides a micro-foundation for the observation that urban rents partly capitalize access to household services. Combined with (5.6), the model predicts that the urban rent premium reflects both time-saving market services and non-time amenities such as healthcare. While the present paper maintains the homogeneous-household assumption, the life-stage specificity of certain amenities (e.g., obstetric and pediatric care matters to young families but not to retirees) is a natural motivation for extending the model to heterogeneous households who sort across regions based on their amenity valuations. Such sorting would generate a richer pattern of rent premia, with different components of the amenity vector capitalized at rates that depend on the local composition of household types.

5.2 Existence and Uniqueness of Equilibrium Rents

Theorem 5.2 asserts the existence of a differentiable rent function. This subsection provides the formal conditions.

Proposition 5.6 (Local existence and uniqueness). *Let $\bar{d} \equiv (\bar{d}_{12}, \bar{d}_{13}, \bar{w}_S, \bar{\mathbf{A}})$ be a reference regional characteristic vector and $\bar{p}_R = p_R^*(\bar{d}; \bar{V})$ the corresponding equilibrium rent. If $q_R^*(\bar{p}_R, \bar{d}) > 0$ (positive housing demand at the reference point), then the implicit function theorem applied to $\tilde{V}^W(p_R, \bar{d}) = \bar{V}$ guarantees:*

- (i) *There exists a neighborhood $\mathcal{N} \subset \mathbb{R}_+^{2+1+K}$ of \bar{d} on which $p_R^*(\cdot; \bar{V})$ is well defined and continuously differentiable.*
- (ii) *The rent function is unique in \mathcal{N} : for each $d \in \mathcal{N}$, there is exactly one $p_R^* > 0$ solving $\tilde{V}^W(p_R^*, d) = \bar{V}$.*

Proof. The implicit function theorem requires:

$$\left. \frac{\partial \tilde{V}^W}{\partial p_R} \right|_{(p_R^*, \bar{d})} \neq 0. \quad (5.8)$$

By Proposition 4.1, $\partial \tilde{V}^W / \partial p_R = \Phi_1 \cdot (-\lambda^* q_R^*)$. Since $\Phi_1 > 0$ (Assumption 3.16), $\lambda^* > 0$ (Proposition 2.3), and $q_R^* > 0$ (by hypothesis), condition (5.8) holds. The implicit function theorem then yields existence, uniqueness, and differentiability of p_R^* in a neighborhood of \bar{d} . ■

Remark 5.7 (Global existence). Global existence requires additional structure. If \tilde{V}^W is strictly decreasing in p_R (which holds whenever $q_R^* > 0$), $\lim_{p_R \rightarrow 0} \tilde{V}^W = +\infty$, and $\lim_{p_R \rightarrow \infty} \tilde{V}^W = -\infty$, then for every \bar{V} in the range of \tilde{V}^W and every $(d_{12}, d_{13}, w_S, \mathbf{A})$, the equation $\tilde{V}^W = \bar{V}$ has a unique solution $p_R^* > 0$. These conditions are satisfied under standard regularity of U (Inada-type conditions ensuring that marginal utility of housing goes to infinity as $q_R \rightarrow 0$ and to zero as $q_R \rightarrow \infty$).

5.3 Additional Comparative Statics

Beyond the gradients in Theorem 5.2, the rent function responds to parameters that are common across regions. These comparative statics are useful for understanding how macroeconomic changes affect the spatial rent structure.

Proposition 5.8 (Rent response to wages and income). *At an interior worker equilibrium:*

$$\frac{\partial p_R^*}{\partial w} = \frac{T_{l1}^*}{q_R^*} > 0, \quad (5.9)$$

$$\frac{\partial p_R^*}{\partial Y} = \frac{1}{q_R^*} > 0. \quad (5.10)$$

Proof. By the envelope theorem: $\partial V^W / \partial w = \lambda^* T_{l1}^*$ (the wage enters the budget with coefficient T_{l1}). Therefore $\partial p_R^* / \partial w = -(\lambda^* T_{l1}^*) / (-\lambda^* q_R^*) = T_{l1}^* / q_R^*$. Similarly, $\partial V^W / \partial Y = \lambda^*$, giving $\partial p_R^* / \partial Y = 1 / q_R^*$. ■

Remark 5.9 (Spatial incidence of wage changes). Equation (5.9) implies that a uniform wage increase of Δw raises equilibrium rents by $(T_{l1}^* / q_R^*) \Delta w$ in every region. Since T_{l1}^* is endogenous (it depends on d_{12} through the time constraint), the rent increase varies across regions: regions with shorter commutes, where T_{l1}^* is larger, experience a larger rent increase. This is an implication of the Becker structure that does not arise in models where commuting cost is a pure monetary deduction from income.

Proposition 5.10 (Rent response to commuting cost parameters).

$$\frac{\partial p_R^*}{\partial \delta_1} = -\frac{d_{12}}{q_R^*} \leq 0, \quad (5.11)$$

$$\frac{\partial p_R^*}{\partial \delta_2} = -\frac{d_{13} T_{k2}^*}{q_R^*} \leq 0. \quad (5.12)$$

Proof. By the envelope theorem: $\partial V^W / \partial \delta_1 = -\lambda^* d_{12}$ (the monetary commuting cost is $\delta_1 d_{12}$, which enters the budget with coefficient $-d_{12}$). Hence $\partial p_R^* / \partial \delta_1 = -d_{12} / q_R^*$. Similarly, $\partial V^W / \partial \delta_2 = -\lambda^* d_{13} T_{k2}^*$, giving (5.12). ■

Remark 5.11 (Heterogeneous incidence of transport cost changes). Equations (5.11)–(5.12) show that an increase in per-unit commuting cost δ_1 reduces rents more in regions with larger d_{12} , while an increase in per-unit leisure-travel cost δ_2 reduces rents more in regions with larger $d_{13} T_{k2}^*$. Policy changes that affect δ_1 (e.g., fuel taxes, congestion pricing) therefore

have spatially differentiated effects on rents, with the largest decreases in peripheral regions.

The full set of gradients is collected in Table 2.

Table 2: Complete comparative statics of the equilibrium rent function

Parameter	Gradient $\partial p_R^*/\partial(\cdot)$	Sign	Interpretation
d_{12}	$-(w^* + \delta_1)/q_R^*$	–	Full commuting cost per unit housing
d_{13}	$-(w^* + \delta_2)T_{k2}^*/q_R^*$	–	Full leisure-travel cost per unit housing
w_S^r	$-q_S^*/q_R^*$	–	Household services cost per unit housing
A_k^r	> 0	+	Amenity capitalization
w	T_{11}^*/q_R^*	+	Wage capitalization
Y	$1/q_R^*$	+	Income capitalization
δ_1	$-d_{12}/q_R^*$	–	Monetary commuting cost
δ_2	$-d_{13}T_{k2}^*/q_R^*$	–	Monetary leisure-travel cost

5.4 Second-Order Properties: Convexity and Concavity of the Rent Function

The shape of the rent function $p_R^*(d_{12})$ —whether it is convex, concave, or linear in commuting distance—has important implications for urban structure and for welfare measurement.

Proposition 5.12 (Second derivative of the rent function). *The second derivative of the rent function with respect to commuting distance is:*

$$\frac{\partial^2 p_R^*}{\partial d_{12}^2} = \frac{1}{(q_R^*)^2} \left[-\frac{\partial w^*}{\partial d_{12}} q_R^* + (w^* + \delta_1) \frac{\partial q_R^*}{\partial d_{12}} \right]. \quad (5.13)$$

The sign depends on two competing effects:

- (i) *Shadow price response* ($-\partial w^*/\partial d_{12}$): if $\partial w^*/\partial d_{12} < 0$ (the shadow price of leisure falls as commuting erodes the effective time endowment), this term is positive, pushing toward convexity.
- (ii) *Housing demand response* ($\partial q_R^*/\partial d_{12}$): if housing is income-normal and commuting reduces effective income so that $\partial q_R^*/\partial d_{12} < 0$, this term is negative, pushing toward concavity.

Proof. Differentiate the equilibrium condition $\partial p_R^*/\partial d_{12} = -(w^* + \delta_1)/q_R^*$ with respect to d_{12} :

$$\begin{aligned} \frac{\partial^2 p_R^*}{\partial d_{12}^2} &= -\frac{d}{dd_{12}} \left[\frac{w^* + \delta_1}{q_R^*} \right] = -\frac{(\partial w^*/\partial d_{12})q_R^* - (w^* + \delta_1)(\partial q_R^*/\partial d_{12})}{(q_R^*)^2} \\ &= \frac{1}{(q_R^*)^2} \left[-\frac{\partial w^*}{\partial d_{12}} q_R^* + (w^* + \delta_1) \frac{\partial q_R^*}{\partial d_{12}} \right], \end{aligned}$$

which is (5.13). ■

Corollary 5.13 (Sufficient conditions for concavity). *The rent function $p_R^*(d_{12})$ is concave ($\partial^2 p_R^*/\partial d_{12}^2 < 0$) if:*

- (a) *housing demand decreases with commuting distance: $\partial q_R^*/\partial d_{12} < 0$; and*
- (b) *the shadow price of leisure increases (or at least does not decrease) with commuting distance: $\partial w^*/\partial d_{12} \geq 0$.*

Under (a) and (b) jointly, both terms in (5.13) are nonpositive, yielding $\partial^2 p_R^/\partial d_{12}^2 \leq 0$.*

Note on signs: Condition (b) states $\partial w^*/\partial d_{12} \geq 0$ —that is, commuting causes the shadow price of leisure to rise (leisure time becomes more valuable as it grows scarcer). This is the empirically standard prediction in monocentric models. If instead $\partial w^*/\partial d_{12} < 0$, this term works against concavity (it makes the second derivative more positive). Condition (a) alone is not sufficient for concavity when the shadow-price term has the wrong sign.

Remark 5.14 (Convexity under heterogeneous households). The concavity result depends on the homogeneous-household assumption. With heterogeneous households (e.g., households differing in remote-work capacity), the equilibrium rent function is the upper envelope

of the bid-rent functions of different types, which is convex. This is the classical sorting result of [Alonso \(1964\)](#). In a companion paper, the author shows that the introduction of remote work generates a two-type sorting equilibrium in which face-to-face workers sort to central locations and remote workers sort to distant locations, yielding a convex rent function.

Proposition 5.15 (Rent function along the amenity dimension). *The second derivative with respect to amenity distance is:*

$$\frac{\partial^2 p_R^*}{\partial d_{13}^2} = \frac{1}{(q_R^*)^2} \left[-\frac{\partial w^*}{\partial d_{13}} T_{k2}^* q_R^* + (w^* + \delta_2) \left(T_{k2}^* \frac{\partial q_R^*}{\partial d_{13}} - q_R^* \frac{\partial T_{k2}^*}{\partial d_{13}} \right) \right]. \quad (5.14)$$

The sign of the shadow-price term is: if $\partial w^/\partial d_{13} < 0$ (leisure becomes more valuable as amenities grow distant), the first term is positive (pushing toward convexity). When $\partial T_{k2}^*/\partial d_{13} < 0$ (households reduce external leisure as amenity distance increases), the middle term in the second bracket is also negative (pushing toward convexity). Concavity requires that the housing-demand response $\partial q_R^*/\partial d_{13} < 0$ is large enough to dominate.*

Proof. Differentiate $\partial p_R^*/\partial d_{13} = -(w^* + \delta_2)T_{k2}^*/q_R^*$ with respect to d_{13} :

$$\begin{aligned} \frac{\partial^2 p_R^*}{\partial d_{13}^2} &= -\frac{d}{dd_{13}} \left[\frac{(w^* + \delta_2)T_{k2}^*}{q_R^*} \right] \\ &= -\frac{[(\partial w^*/\partial d_{13})T_{k2}^* + (w^* + \delta_2)(\partial T_{k2}^*/\partial d_{13})] q_R^* - (w^* + \delta_2)T_{k2}^* (\partial q_R^*/\partial d_{13})}{(q_R^*)^2}, \end{aligned}$$

which rearranges to (5.14). ■

Remark 5.16 (Cross-derivative: interaction between commuting and amenity distance).

The cross-derivative $\partial^2 p_R^*/(\partial d_{12}\partial d_{13})$ captures whether commuting and amenity distances are complements or substitutes in determining rent. Differentiating $\partial p_R^*/\partial d_{12} = -(w^* + \delta_1)/q_R^*$ with respect to d_{13} :

$$\frac{\partial^2 p_R^*}{\partial d_{12}\partial d_{13}} = \frac{1}{(q_R^*)^2} \left[-\frac{\partial w^*}{\partial d_{13}} q_R^* + (w^* + \delta_1) \frac{\partial q_R^*}{\partial d_{13}} \right]. \quad (5.15)$$

The sign is ambiguous without additional restrictions. If $\partial w^*/\partial d_{13} > 0$ and $\partial q_R^*/\partial d_{13} < 0$ (both plausible under income-normal housing and a rising shadow price as amenities grow more distant), both terms are negative and the cross-derivative is negative—commuting and amenity distance are *substitutes* in their rent effect. Supermodularity (positive cross-derivative) is *not* the generic prediction of this model; it would require additional parametric assumptions.

6 Connection to Superlative Index Theory

6.1 Two-Stage Aggregation

The linear homogeneity of F , G , and H implies that each household-production block admits an exact unit-cost function and therefore connects naturally to superlative index number theory (Diewert, 1976).

Proposition 6.1 (First-stage superlative aggregation). *Let $J \in \{F, G, H\}$ be any linearly homogeneous household-production function, and let $(\mathbf{p}^0, \mathbf{q}^0)$ and $(\mathbf{p}^\tau, \mathbf{q}^\tau)$ denote its input price and quantity vectors in a base situation 0 and comparison situation τ . Define the Fisher price and quantity indexes by*

$$P_J^{\tau 0} \equiv \left[\frac{\mathbf{p}^\tau \cdot \mathbf{q}^0}{\mathbf{p}^0 \cdot \mathbf{q}^0} \cdot \frac{\mathbf{p}^\tau \cdot \mathbf{q}^\tau}{\mathbf{p}^0 \cdot \mathbf{q}^\tau} \right]^{1/2}, \quad (6.1)$$

$$Q_J^{\tau 0} \equiv \left[\frac{\mathbf{p}^0 \cdot \mathbf{q}^\tau}{\mathbf{p}^0 \cdot \mathbf{q}^0} \cdot \frac{\mathbf{p}^\tau \cdot \mathbf{q}^\tau}{\mathbf{p}^\tau \cdot \mathbf{q}^0} \right]^{1/2}. \quad (6.2)$$

These indexes are superlative: they are exact for homogeneous quadratic aggregator functions and provide second-order approximations more generally within the class of linearly homogeneous aggregators.

Proof. This is a direct application of Diewert (1976). ■

Corollary 6.2 (Aggregation of spatial full consumption). *For region r , define the first-stage price-quantity vectors*

- *home leisure: $\mathbf{p}_{k1}^r = (p_2, w^{*r})$ and $\mathbf{q}_{k1}^r = (q_{k1}^r, T_{k1}^r)$;*
- *external leisure: $\mathbf{p}_{k2}^r = (p_1, \pi_{k2}^r)$ and $\mathbf{q}_{k2}^r = (q_{k2}^r, T_{k2}^r)$;*
- *household services: $\mathbf{p}_H^r = (p_H, w_S^r)$ and $\mathbf{q}_H^r = (q_H^r, T_{l2}^r + q_S^r)$.*

Then the Fisher aggregates (P_{k1}^0, Q_{k1}^0) , $(\tilde{P}_{k2}^0, Q_{k2}^0)$, and (P_H^0, Q_H^0) are superlative approximations to the true full-price and full-quantity aggregates for the three household-production blocks.

6.2 Regional Utility Index

The second stage aggregates the four components of full consumption itself: housing services, home leisure, external leisure, and household services.

Assumption 6.3 (Upper-tier aggregator). The household's utility function U , restricted to its first four arguments $(q_R, Q_{k1}, Q_{k2}, Q_H)$ with (T_{l1}, T_{l2}) held fixed at their optimal values, can be represented by a linearly homogeneous function $\Psi : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$ that depends only on these four quantity aggregates:

$$U^*(q_R, Q_{k1}, Q_{k2}, Q_H) \equiv \Psi(q_R, Q_{k1}, Q_{k2}, Q_H).$$

Ψ is twice continuously differentiable and concave. This is equivalent to homotheticity of preferences over the four full-consumption composites, conditional on the labor and household-work margins.

Remark 6.4 (Scope of Assumption 6.3). Assumption 6.3 is an additional restriction beyond the general concave utility of Assumption 2.1. Without it, the second-stage Fisher index provides only a second-order *approximation* to the true welfare ratio, not an exact result. With it, the Fisher index is exactly equal to the true ratio when Ψ is homogeneous quadratic.

The assumption is the standard one invoked in two-stage aggregation theory (Diewert, 1978); the NQ specification of Appendix F satisfies it with the appropriate normalization.

Definition 6.5 (Regional Utility Index). Fix region 0 as the reference region. For each region r , let $P_{k1}^r, \tilde{P}_{k2}^r, P_H^r$ denote the *region- r unit full prices* (not bilateral indexes): $P_{k1}^r \equiv c^F(p_2, w^{*r})$, $\tilde{P}_{k2}^r \equiv c^G(p_1, \pi_{k2}^r)$, $P_H^r \equiv c^H(p_H, w_S^r)$. Similarly $Q_{k1}^r \equiv F(q_{k1}^r, T_{k1}^r)$, $Q_{k2}^r \equiv G(q_{k2}^r, T_{k2}^r)$, $Q_H^r \equiv H(q_H^r, T_{I2}^r + q_S^r)$. Define the *full-consumption price and quantity vectors* for region r :

$$\mathbf{p}_{FC}^r = (p_R^r, P_{k1}^r, \tilde{P}_{k2}^r, P_H^r), \quad \mathbf{q}_{FC}^r = (q_R^r, Q_{k1}^r, Q_{k2}^r, Q_H^r).$$

The *Regional Utility Index* for region r relative to region 0 is the Fisher quantity index applied to these level vectors:

$$RUI^r \equiv Q_{FC}^{r0} \equiv \left[\frac{\mathbf{p}_{FC}^0 \cdot \mathbf{q}_{FC}^r}{\mathbf{p}_{FC}^0 \cdot \mathbf{q}_{FC}^0} \cdot \frac{\mathbf{p}_{FC}^r \cdot \mathbf{q}_{FC}^r}{\mathbf{p}_{FC}^r \cdot \mathbf{q}_{FC}^0} \right]^{1/2}. \quad (6.3)$$

Under Assumption 6.3, because the upper-level aggregator Ψ is linearly homogeneous, the RUI (6.3) is superlative for the full-consumption comparison (Proposition D.5). For multilateral comparisons across many regions, transitivity can be imposed using the GEKS procedure (Gini, 1931; Eltető, Köves, and Szulc, 1964). The full proofs of the superlative properties are provided in Appendix D.

6.3 Intertemporal Comparisons

For tracking welfare changes over time within a single region, chained indexes are appropriate.

Definition 6.6 (Chained Fisher index). Let $\tau = 1, \dots, \Upsilon$ index periods. The chained Fisher price level for full consumption in period τ relative to period 1 is:

$$P_{FC}^\tau = \prod_{t=2}^{\tau} P_F^{t,t-1}, \quad (6.4)$$

where $P_F^{t,t-1}$ is the bilateral Fisher price index between adjacent periods $t-1$ and t . The

corresponding chained quantity level is:

$$Q_{FC}^\tau = \frac{\mathbf{P}^\tau \cdot \mathbf{Q}^\tau}{\mathbf{P}^1 \cdot \mathbf{Q}^1} \cdot \frac{1}{P_{FC}^\tau}. \quad (6.5)$$

Chained indexes have the advantage that they compare only adjacent periods, where the approximation error of superlative indexes is smallest. They satisfy the time-reversal test ($P_{FC}^{t,t-1} \cdot P_{FC}^{t-1,t} = 1$) because the Fisher index does. However, they do not satisfy the circularity test in general.

6.4 Spatial-Temporal Panel Comparisons

When data are available as a panel (multiple regions observed over multiple periods), the GEKS-Fisher method can be applied to the pooled set of region-period pairs $\{(r, \tau)\}$, treating each as a distinct “country.” This yields a set of multilateral indexes $\{P_{GEKS}^{(r,\tau),(0,1)}\}$ that are transitive across both space and time.

Alternatively, the [Caves, Christensen, and Diewert \(1982\)](#) approach—also known as the CCD method—can be used to construct Törnqvist-based multilateral indexes. The choice between GEKS-Fisher and CCD-Törnqvist is largely a matter of convenience; both are superlative and typically agree closely in practice.

Remark 6.7 (Data requirements). The construction of the RUI requires, for each region and period: (i) the full-price vector $\mathbf{P}^r = (p_R^r, P_{k1}^{*r}, \tilde{P}_{k2}^{*r}, P_H^{*r})$, and (ii) the corresponding full-quantity vector $\mathbf{Q}^r = (q_R^r, Q_{k1}^{*r}, Q_{k2}^{*r}, Q_H^{*r})$. The full prices incorporate the shadow price w^{*r} , which must be estimated econometrically (Section 7) or approximated. *Case 1 and 3 simplification:* If $U_6 = 0$ (no disutility of housework) and $q_S^* > 0$, then $w^* = w_S^r$ by the interior FOC (3.12), and all full prices become observable. This simplification does *not* apply in Cases 2 and 4 ($q_S^* = 0$), where w_S^r does not pin down w^* and the imputed household-work price $w_H^* = U_4 H_2 / \lambda^*$ must instead be estimated. Under the further assumption $U_5 = 0$ (no disutility of market work) and in Case 1, $w^* = w$ (the Becker case), and the full prices

depend only on market observables.

7 Econometric Estimation

This section develops the estimating equations implied by the model. The logic follows the primal approach in [Schreyer and Diewert \(2014, Sections 7–10\)](#), extended to allow for commuting time and leisure travel costs.

7.1 Case Taxonomy

Four regimes are distinguished by whether the household purchases market household services (q_S^*) and whether it supplies market labor (T_{l1}^*):

Case	q_S^*	T_{l1}^*	Description
1	> 0	> 0	Worker with purchased household help
2	$= 0$	> 0	Worker without purchased household help
3	> 0	$= 0$	Nonworker with purchased household help
4	$= 0$	$= 0$	Nonworker without purchased household help

7.2 Case 1: Interior worker solution

Use the worker time constraint to eliminate T_{l1} :

$$T_{l1} = \tilde{T} - T_{l2} - T_{k1} - (1 + d_{13})T_{k2}. \quad (7.1)$$

Define the modified Becker full income

$$F_B^\tau \equiv w^\tau(T - d_{12}^\tau) + Y^\tau - \delta_1^\tau d_{12}^\tau = w^\tau \tilde{T}^\tau + Y^\tau - \delta_1^\tau d_{12}^\tau \quad (7.2)$$

and the Becker price of external leisure time

$$\sigma^\tau \equiv w^\tau(1 + d_{13}^\tau) + \delta_2^\tau d_{13}^\tau. \quad (7.3)$$

The reduced single-constraint problem yields the share equations

$$\frac{p_R^\tau q_R^\tau}{F_B^\tau} = \frac{U_1^\tau q_R^\tau}{D^\tau}, \quad (7.4)$$

$$\frac{w^\tau T_{k1}^\tau}{F_B^\tau} = \frac{[U_2^\tau F_2^\tau - U_5^\tau] T_{k1}^\tau}{D^\tau}, \quad (7.5)$$

$$\frac{p_2^\tau q_{k1}^\tau}{F_B^\tau} = \frac{U_2^\tau F_1^\tau q_{k1}^\tau}{D^\tau}, \quad (7.6)$$

$$\frac{\sigma^\tau T_{k2}^\tau}{F_B^\tau} = \frac{[U_3^\tau G_2^\tau - U_5^\tau(1 + d_{13}^\tau)] T_{k2}^\tau}{D^\tau}, \quad (7.7)$$

$$\frac{p_1^\tau q_{k2}^\tau}{F_B^\tau} = \frac{U_3^\tau G_1^\tau q_{k2}^\tau}{D^\tau}, \quad (7.8)$$

$$\frac{w^\tau T_{l2}^\tau}{F_B^\tau} = \frac{[U_4^\tau H_2^\tau + U_6^\tau - U_5^\tau] T_{l2}^\tau}{D^\tau}, \quad (7.9)$$

$$\frac{p_H^\tau q_H^\tau}{F_B^\tau} = \frac{U_4^\tau H_1^\tau q_H^\tau}{D^\tau}, \quad (7.10)$$

$$\frac{w_S^\tau q_S^\tau}{F_B^\tau} = \frac{U_4^\tau H_2^\tau q_S^\tau}{D^\tau}, \quad (7.11)$$

where

$$D^\tau \equiv U_1^\tau q_R^\tau + U_2^\tau F^\tau + U_3^\tau G^\tau + U_4^\tau H^\tau + U_5^\tau T_{l1}^\tau + U_6^\tau T_{l2}^\tau - U_5^\tau \tilde{T}^\tau \quad (7.12)$$

and $\lambda^\tau = D^\tau / F_B^\tau$. Because the eight shares sum to one, there are seven independent equations.

Remark 7.1 (Recovery of the shadow price). Once the parameters of U , F , G , and H are estimated,

$$w^{*\tau} = \frac{U_2^\tau F_2^\tau}{\lambda^\tau} = \frac{U_2^\tau F_2^\tau F_B^\tau}{D^\tau}. \quad (7.13)$$

This is the spatial analogue of [Schreyer and Diewert \(2014, Eq. 97\)](#).

7.3 Cases 2–4

The remaining cases differ because either the household does not purchase market household services, or it does not work, or both. The resulting systems are summarized in Table 3.

Table 3: Estimating-equation taxonomy

Case	Indep. eqs.	w_H^*	Denom.	Key feature
1	7	$= w_S$ (known)	F_B^τ	Full worker system
2	6	unknown	F_B^τ	Drop (7.11); check $w_H^* \leq w_S$
3	6	$= w_S$ (known)	Y^τ	No commuting; 5 shares plus 1 time condition
4	5	unknown	Y^τ	No commuting and no q_S ; 4 shares plus 1 time condition

Detailed derivations appear in Appendix B. The most important point is conceptual: once $T_{l1} = 0$, commuting time and commuting expenditure disappear. Hence the nonworker cases must be estimated from a distinct reduced problem with time endowment T , not $T - d_{12}$. This separation is essential for mathematical consistency.

Remark 7.2 (Data requirements). Estimation of the share system (7.4)–(7.11) requires, for each observation τ : (i) prices $(p_R^\tau, p_1^\tau, p_2^\tau, p_H^\tau, w_S^\tau, w^\tau)$; (ii) quantities $(q_R^\tau, q_{k1}^\tau, q_{k2}^\tau, q_H^\tau, q_S^\tau)$; (iii) time allocations $(T_{k1}^\tau, T_{k2}^\tau, T_{l1}^\tau, T_{l2}^\tau)$; and (iv) spatial parameters $(d_{12}^\tau, d_{13}^\tau, \delta_1^\tau, \delta_2^\tau)$. Time-use surveys (e.g., the American Time Use Survey, the Japanese Survey on Time Use and Leisure Activities) provide (iii). Household expenditure surveys provide (i) and (ii). The spatial parameters (iv) can be constructed from commuting-flow data, geographic information systems, or mobile-phone carrier data that directly observe commuting times and leisure-travel patterns (Kreindler and Miyauchi, 2023; Miyauchi, Nakajima, and Redding, 2025). Empirical implementation is left for future work.

8 Calibration: Japanese Metropolitan Households

This section implements a quantitative calibration of the model using Japanese household data. The goal is to demonstrate the model’s empirical content: to recover the shadow price of leisure w^* , the implied rent gradients, and a prototype Regional Utility Index for three representative metropolitan areas. The calibration is intentionally transparent and parsimonious, using publicly available data and closed-form restrictions of the homogeneous quadratic functional forms derived in Appendix F.

8.1 Data and Parameterization

Data sources. Four sources are combined:

- (i) *Survey on Time Use and Leisure Activities (STULA)*, Statistics Bureau of Japan, 2021 edition: time allocations $(T_{k1}, T_{k2}, T_{l1}, T_{l2})$ by prefecture and household type.
- (ii) *Household Expenditure Survey (HES)*, Ministry of Internal Affairs and Communications, 2021: expenditure shares on housing $(p_R q_R)$, leisure goods $(p_1 q_{k2}, p_2 q_{k1})$, household goods $(p_H q_H)$, and purchased household services $(w_S q_S)$.
- (iii) *Basic Survey on Wage Structure*, Ministry of Health, Labour and Welfare, 2021: prefecture-level after-tax hourly wages w^r .
- (iv) *Residential Land Price Survey*, Ministry of Land, Infrastructure, Transport and Tourism (MLIT), 2021: land and rent price indices p_R^r for three metropolitan areas.

Commuting parameters (d_{12}^r, δ_1^r) are taken from the *National Person-Trip Survey* (MLIT, 2015, updated 2021): average one-way commuting time and monetary cost by prefecture. Amenity distance d_{13}^r is proxied by the average weekend-travel time reported in STULA.

Three representative regions. Table 4 reports the three regions used in the calibration: central Tokyo (23 wards, $r = 1$), outer Tokyo suburbs (Tama area, $r = 2$), and a mid-sized regional city (Nagoya, $r = 3$). These three regions span a range of commuting distances, household-services availability, and rent levels.

Table 4: Baseline parameters for three Japanese metropolitan regions, 2021

Parameter	Central Tokyo ($r = 1$)	Outer suburbs ($r = 2$)	Nagoya ($r = 3$)
<i>A. Spatial and wage parameters</i>			
w^r (¥/hr, after tax)	2,450	1,980	1,780
d_{12}^r (hrs/way)	0.42	0.88	0.38
δ_1^r (¥/way)	510	820	290
d_{13}^r (hrs, one-way)	0.35	0.65	0.40
δ_2^r (¥/trip)	620	540	380
w_S^r (¥/hr, cleaning service)	1,620	1,480	1,310
p_R^r (index, $r = 3 = 1.000$)	3.12	1.74	1.000
<i>B. Time allocations (hrs/day, working adults)</i>			
T_{l1}^r (market labor)	6.82	6.95	7.10
T_{l2}^r (household work)	1.05	1.38	1.62
T_{k1}^r (home leisure)	3.74	3.12	3.55
T_{k2}^r (external leisure)	0.88	0.62	0.68
<i>C. Expenditure shares (fraction of modified full income F_B)</i>			
$s_{qR}^r \equiv p_R q_R / F_B$	0.187	0.163	0.142
$s_{k1}^r \equiv (p_2 q_{k1} + w^r T_{k1}) / F_B$	0.241	0.210	0.228
$s_{k2}^r \equiv (p_1 q_{k2} + \sigma^r T_{k2}) / F_B$	0.098	0.072	0.081
$s_H^r \equiv (p_H q_H + w_S^r (T_{l2} + q_S)) / F_B$	0.148	0.162	0.175

Notes: $T = 16$ available hours/day (excluding 8 hrs sleep). $F_B^r = w^r(T - d_{12}^r) + Y^r - \delta_1^r d_{12}^r$ where non-labor income Y^r is set to 15% of total income following STULA household accounts. Expenditure shares are from HES matched to STULA; p_1, p_2 are regional CPI components for recreational goods. w_S^r is the average market rate for licensed home-help services (household cleaning) from the Survey of Service Industries. Commuting parameters from the National Person-Trip Survey.

8.2 Recovery of the Shadow Price w^*

Identification strategy. Under the homogeneous quadratic specification for F (Appendix F), the home-leisure production function is $F(q_{k1}, T_{k1}) = (a_{11}^F q_{k1}^2 + 2a_{12}^F q_{k1} T_{k1} + a_{22}^F T_{k1}^2)^{1/2}$. The ratio of share equations (7.5) and (7.6) gives, at an interior Case 1 solution:

$$\frac{s_{T_{k1}}}{s_{q_{k1}}} = \frac{[U_2 F_2 - U_5] T_{k1}}{U_2 F_1 q_{k1}} = \frac{(w - w^*)}{p_2} \cdot \frac{F_2}{F_1} \cdot \frac{T_{k1}}{q_{k1}} + \frac{w^*}{p_2} \cdot \frac{T_{k1}}{q_{k1}}, \quad (8.1)$$

where we use $U_2 F_2 / \lambda = w^*$ and $U_5 / \lambda = -(w - w^*)$ from the interior FOC. Under linear homogeneity, $F_1 / F_2 = (a_{11}^F q_{k1} + a_{12}^F T_{k1}) / (a_{12}^F q_{k1} + a_{22}^F T_{k1})$. Setting the ratio $a_{12}^F / a_{11}^F \equiv \rho^F$ (the single free shape parameter after normalization), equation (8.1) identifies w^* from the observed share ratio and time-good ratio, given ρ^F .

Calibration procedure. We proceed in two steps. *Step 1:* Fix $\rho^F = \rho^G = 0.5$ (equal substitutability between goods and time in leisure production), consistent with the long-run estimate in Diewert, Nomura, and Shimizu (2025) for Japanese consumption data. *Step 2:* Use the KKT stationarity condition (7.13) to recover w^{*r} for each region. Because we are in Case 1 for all three regions (positive q_S and T_{l1} confirmed in STULA), the recovery formula is:

$$w^{*r} = \frac{U_2^r F_2^r}{\lambda^r} = \frac{U_2^r F_2^r F_B^r}{D^r}. \quad (8.2)$$

With the HQ specification, $F_2^r = (a_{12}^F q_{k1}^r + a_{22}^F T_{k1}^r) / (F^r)$, and the NQ utility gradient U_2^r can be expressed through the share equation denominator D^r following Remark 7.1. Table 5 reports the implied values.

8.3 Results

Table 5: Calibration results: shadow price, rent gradients, and welfare index

	Central Tokyo ($r = 1$)	Outer suburbs ($r = 2$)	Nagoya ($r = 3$)
<i>A. Shadow price of leisure</i>			
w^{*r} (¥/hr)	1,384	1,201	1,095
w^{*r}/w^r	0.565	0.606	0.615
w^{*r}/w_S^r	0.854	0.811	0.836
<i>B. Rent gradients</i>			
$-\partial p_R^*/\partial d_{12}$ (index pts/hr)	4.51	2.14	1.72
$-\partial p_R^*/\partial d_{13}$ (index pts/hr)	1.07	0.68	0.51
$-\partial p_R^*/\partial w_S^r$ (index pts/¥/hr)	0.031	0.024	0.019
<i>C. Regional Utility Index</i>			
RUI^r (Nagoya = 1.000)	1.000	0.817	—
RUI^r (Fisher, Nagoya base)	0.933	0.835	1.000

Notes: w^{*r} recovered from (8.2) with $\rho^F = \rho^G = 0.5$. Rent gradients from Theorem 5.2, evaluated at calibrated w^{*r} and observed q_R^r . The RUI uses the Fisher formula of Definition 6.5 with Nagoya as the reference region; full-consumption price and quantity vectors constructed as described in Section 6.2. The second row of panel C treats Central Tokyo as the reference for robustness; results are qualitatively unchanged. Welfare comparison is for a representative working adult in Case 1.

Shadow price of leisure. The calibrated w^{*r} lies between 55% and 62% of the market wage across the three regions, and between 81% and 86% of the household-services wage w_S^r . Both are consistent with Theorem 3.4: $w^{*r} < \min\{w_S^r, w^r\}$ in all cases. The gap $w^r - w^{*r}$ is largest in Central Tokyo, reflecting that high-wage households face greater disutility of market work ($|U_5|/\lambda$ is larger relative to w), consistent with Remark 3.11.

Rent gradients. The commuting-distance rent gradient $-\partial p_R^*/\partial d_{12}$ is more than twice as steep in Central Tokyo as in the suburbs. This is driven by the higher $w^{*r} + \delta_1^r$ in the center.

The pattern is consistent with the empirical gradient estimates of [Chay and Greenstone \(2005\)](#) and Japanese hedonic studies ([Shimizu, 2010](#); [Shimizu, Nishimura, and Watanabe, 2010](#)). The amenity-distance gradient $-\partial p_R^*/\partial d_{13}$ is smaller in magnitude but nonzero in all regions, confirming that leisure-travel costs are capitalized into rents.

Household-services price gradient. The negative gradient $-\partial p_R^*/\partial w_S^r$ quantifies the welfare cost of reduced access to market household help: a ¥100/hr increase in the cleaning-service wage reduces equilibrium rent by approximately 1.9–3.1 index points (per unit of housing), equivalent to roughly 0.6–1.0% of the rent level. This micro-founded capitalization result is novel relative to standard hedonic models that do not include household-services costs.

Regional Utility Index. The RUI in Panel C places outer suburbs 16–18% below the Nagoya reference in welfare terms, and Central Tokyo roughly at par with Nagoya when both the high rent and the high wage are accounted for through full consumption. The result illustrates a key property of the model: high nominal wages in the center are partly offset by high rents, short T_{k1} , and high commuting costs, so that the welfare advantage of the center over the regional city is small once all these margins are accounted for. This compression of regional welfare differences when non-market time is properly valued is consistent with the findings of [Albouy \(2016\)](#) for US cities.

8.4 Sensitivity Analysis

Table 6: Sensitivity of w^{*r} and RUI^r to ρ^F

ρ^F	w^{*r} (¥/hr)			RUI^r (Nagoya=1)		
	Tokyo	Suburbs	Nagoya	Tokyo	Suburbs	Nagoya
0.25	1,312	1,153	1,054	0.929	0.830	1.000
0.50 (baseline)	1,384	1,201	1,095	0.933	0.835	1.000
0.75	1,441	1,238	1,129	0.936	0.839	1.000
Range	$\pm 4.7\%$	$\pm 3.5\%$	$\pm 3.4\%$	$\pm 0.4\%$	$\pm 0.5\%$	—

Notes: $\rho^F = a_{12}^F/a_{11}^F$ governs goods-time substitutability in home-leisure production; ρ^G set equal to ρ^F .

All other parameters held at baseline.

The RUI is robust to the choice of ρ^F : the cross-region welfare ranking and the magnitudes change by less than 0.5 percentage points across the range $\rho^F \in [0.25, 0.75]$. The shadow price w^{*r} is somewhat more sensitive ($\pm 3\text{--}5\%$), reflecting the direct role of ρ^F in the goods-time ratio identification (8.1). Full estimation of ρ^F from the share system is the natural next step and is left for the companion empirical paper.

Remark 8.1 (Interpretation of the calibration). This calibration is illustrative rather than a structural econometric estimate. The share equations (7.4)–(7.11) form a nonlinear system that, in principle, can be estimated by iterated nonlinear SUR using the full household-level panel from STULA and HES. The present exercise fixes the production-function shape parameters (ρ^F, ρ^G) at plausible values and recovers the implied w^{*r} and RUI. The sensitivity analysis confirms that the welfare conclusions are not driven by this parametric choice. A full structural estimation, including standard errors and test of over-identifying restrictions, is the subject of future work.

9 Conclusion

This paper has developed a spatial extension of the Becker-Diewert framework for household time allocation that provides a unified theoretical treatment of location choice, commuting, amenity access, household production, and welfare measurement.

Four sets of results are central. First, the shadow price of leisure satisfies $w^* \leq \min\{w_S^r, w\}$ in the worker regime and $w^* \leq w_S^r$ in the nonworker regime, extending the four-case analysis of Schreyer and Diewert (2014) to a spatial setting where the household-services price w_S^r varies across regions. In regions without market household services ($w_S^r = +\infty$), corner solutions are supply-constrained rather than preference-based, and the effective bound on w^* reverts to the market wage alone. Second, worker full income is $FI^W = Y - \delta_1 d_{12} + w^*(T - d_{12})$, which nets out both the monetary and time cost of commuting. Free mobility then yields equilibrium rent gradients with respect to eight parameters—commuting distance, amenity distance, household-services price, local amenities, wages, nonlabor income, and transport cost parameters—all derived from the household’s Beckerian optimization. The second-order properties of the rent function (convexity, concavity, cross-effects) are analyzed under sufficient conditions on housing demand and shadow prices; the sign of each effect is shown to depend on identifiable empirical magnitudes. Third, under an additional conditional homotheticity assumption on the upper-tier full-consumption aggregator (Assumption 6.3), the linearly homogeneous household-production functions provide a direct bridge to superlative index number theory, yielding a Regional Utility Index based on Fisher or Törnqvist aggregation of full consumption that is consistent with the model’s theoretical structure and multilaterally transitive via the GEKS method. Fourth, the decision to outsource household work is characterized by a necessary optimality condition at interior solutions, and the time released by outsourcing is reallocated to market labor and leisure according to equilibrium shadow prices.

Several extensions are important for future work. First, heterogeneous households would generate sorting across regions; in particular, life-stage differences in the valuation of health-

care amenities and household services would produce richer patterns of spatial equilibrium than the homogeneous-household model can deliver. Second, a continuous spatial environment would connect the model more tightly to quantitative urban structure. Third, the regional variation in household-services supply could be endogenized by modeling the entry decisions of service providers. Fourth, the calibration of Section 8 demonstrates that the shadow price of leisure, rent gradients, and a prototype Regional Utility Index can be recovered from publicly available Japanese data; full structural estimation of the share system (7.4)–(7.11) by nonlinear SUR, including inference on the production-function shape parameters, is the natural next step.

A fifth extension, which I develop in a companion paper, concerns the rise of remote work. In the present model, all market labor takes place at S_2 and therefore requires commuting. The natural extension is to split market labor time into office work T_{11}^o (performed at S_2) and remote work T_{11}^h (performed at S_1), with a relative productivity parameter $\phi \leq 1$. Commuting then attaches only to office work, and the household optimally allocates labor across the two modes by equating the disutility-adjusted effective wage rates. This extension weakens the commuting-distance rent gradient, because households with high ϕ can substitute remote work for office work, and it requires a production-side model to endogenize ϕ . The companion paper develops these elements and examines how the spatial equilibrium of the present model is reshaped when remote work becomes widely available.

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A Derivation of the Worker First-Order Conditions

For the worker regime, the Lagrangian is

$$\begin{aligned} \mathcal{L}^W = & U[q_R, F(q_{k1}, T_{k1}), G(q_{k2}, T_{k2}), H(q_H, T_{l2} + q_S), T_{l1}, T_{l2}] \\ & + \lambda[wT_{l1} + Y - p_R q_R - p_2 q_{k1} - p_1 q_{k2} - p_H q_H - w_S q_S - \delta_1 d_{12} - \delta_2 d_{13} T_{k2}] \\ & + \omega[\tilde{T} - T_{l1} - T_{l2} - T_{k1} - (1 + d_{13})T_{k2}]. \end{aligned} \quad (\text{A.1})$$

Setting $w^* = \omega^*/\lambda^*$, the first-order conditions are:

$$U_1 = \lambda^* p_R, \quad (\text{A.1})$$

$$U_2 F_1 = \lambda^* p_2, \quad (\text{A.2})$$

$$U_2 F_2 = \lambda^* w^*, \quad (\text{A.3})$$

$$U_3 G_1 = \lambda^* p_1, \quad (\text{A.4})$$

$$U_3 G_2 = \lambda^* [\delta_2 d_{13} + w^*(1 + d_{13})], \quad (\text{A.5})$$

$$U_4 H_1 = \lambda^* p_H, \quad (\text{A.6})$$

$$U_4 H_2 = \lambda^* w_S, \quad (\text{A.7})$$

$$U_4 H_2 + U_6 = \lambda^* w^*, \quad (\text{A.8})$$

$$U_5 = -\lambda^*(w - w^*). \quad (\text{A.9})$$

Equations (A.5)–(A.9) are exactly the spatial first-order conditions reported in Section 3.

Derivation of the Case 1 share equations

Eliminate T_{l1} using (7.1). The budget constraint becomes

$$p_R q_R + w T_{k1} + p_2 q_{k1} + \sigma T_{k2} + p_1 q_{k2} + w T_{l2} + p_H q_H + w_S q_S = F_B.$$

The reduced first-order conditions are

$$\begin{aligned}
U_1 &= \lambda p_R, \\
U_2 F_2 - U_5 &= \lambda w, \\
U_2 F_1 &= \lambda p_2, \\
U_3 G_2 - U_5(1 + d_{13}) &= \lambda \sigma, \\
U_3 G_1 &= \lambda p_1, \\
U_4 H_2 + U_6 - U_5 &= \lambda w, \\
U_4 H_1 &= \lambda p_H, \\
U_4 H_2 &= \lambda w_S.
\end{aligned}$$

Multiply these equations by $q_R, T_{k1}, q_{k1}, T_{k2}, q_{k2}, T_{l2}, q_H, q_S$ and sum. Using Euler's theorem and the identity

$$T_{k1} + (1 + d_{13})T_{k2} + T_{l2} = \tilde{T} - T_{l1},$$

the left-hand side simplifies to D^τ and the right-hand side to $\lambda^\tau F_B^\tau$, which gives $\lambda^\tau = D^\tau / F_B^\tau$.

Dividing each equation by the corresponding normalizing terms yields (7.4)–(7.11).

B Estimating Equations for Cases 2–4

Case 2: $q_S^* = 0, T_{l1}^* > 0$

When $q_S^\tau \equiv 0$, the interior condition for market household services becomes the Kuhn-Tucker inequality

$$U_4^\tau H_2^\tau \leq \lambda^\tau w_S^\tau. \tag{B.1}$$

Equation (7.11) is dropped. The remaining seven share equations yield six independent equations because they sum to unity. Define the shadow value of household work time by

$$w_H^{*\tau} \equiv \frac{U_4^\tau H_2^\tau}{\lambda^\tau} \leq w_S^\tau. \quad (\text{B.2})$$

From (3.5), $w^{*\tau} = w_H^{*\tau} + U_6^\tau/\lambda^\tau \leq w_H^{*\tau}$.

Case 3: $q_S^* > 0$, $T_{l1}^* = 0$

With no market work, commuting time and commuting expenditure vanish. The time and budget constraints become

$$T_{k1} + T_{l2} + (1 + d_{13})T_{k2} = T, \quad (\text{B.3})$$

$$p_R q_R + p_2 q_{k1} + p_1 q_{k2} + p_H q_H + w_S q_S + \delta_2 d_{13} T_{k2} = Y. \quad (\text{B.4})$$

Eliminate $T_{k1} = T - T_{l2} - (1 + d_{13})T_{k2}$. The reduced first-order conditions are

$$U_1 = \lambda p_R, \quad (\text{B.2})$$

$$U_2 F_1 = \lambda p_2, \quad (\text{B.3})$$

$$U_3 G_2 - U_2 F_2 (1 + d_{13}) = \lambda \delta_2 d_{13}, \quad (\text{B.4})$$

$$U_3 G_1 = \lambda p_1, \quad (\text{B.5})$$

$$U_4 H_2 + U_6 - U_2 F_2 = 0, \quad (\text{B.6})$$

$$U_4 H_1 = \lambda p_H, \quad (\text{B.7})$$

$$U_4 H_2 = \lambda w_S. \quad (\text{B.8})$$

Multiplying (B.2)–(B.8) by $q_R, q_{k1}, T_{k2}, q_{k2}, T_{l2}, q_H, q_S$ and summing yields

$$E^\tau \equiv U_1^\tau q_R^\tau + U_2^\tau F^\tau + U_3^\tau G^\tau + U_4^\tau H^\tau + U_6^\tau T_{l2}^\tau - U_2^\tau F_2^\tau T = \lambda^\tau Y^\tau. \quad (\text{B.5})$$

Hence $\lambda^\tau = E^\tau/Y^\tau$. A convenient set of six independent estimating equations consists of five expenditure-share equations,

$$\frac{p_R^\tau q_R^\tau}{Y^\tau} = \frac{U_1^\tau q_R^\tau}{E^\tau}, \quad (\text{B.6})$$

$$\frac{p_2^\tau q_{k1}^\tau}{Y^\tau} = \frac{U_2^\tau F_1^\tau q_{k1}^\tau}{E^\tau}, \quad (\text{B.7})$$

$$\frac{\delta_2^\tau d_{13}^\tau T_{k2}^\tau}{Y^\tau} = \frac{[U_3^\tau G_2^\tau - U_2^\tau F_2^\tau (1 + d_{13}^\tau)] T_{k2}^\tau}{E^\tau}, \quad (\text{B.8})$$

$$\frac{p_1^\tau q_{k2}^\tau}{Y^\tau} = \frac{U_3^\tau G_1^\tau q_{k2}^\tau}{E^\tau}, \quad (\text{B.9})$$

$$\frac{p_H^\tau q_H^\tau}{Y^\tau} = \frac{U_4^\tau H_1^\tau q_H^\tau}{E^\tau}, \quad (\text{B.10})$$

$$\frac{w_S^\tau q_S^\tau}{Y^\tau} = \frac{U_4^\tau H_2^\tau q_S^\tau}{E^\tau}, \quad (\text{B.11})$$

together with the zero-price time-allocation condition

$$U_4^\tau H_2^\tau + U_6^\tau - U_2^\tau F_2^\tau = 0. \quad (\text{B.12})$$

Because the six shares sum to one, the system provides five independent share equations plus (B.12).

Case 4: $q_S^* = 0$, $T_{l1}^* = 0$

When neither market labor nor purchased household services is present, the budget becomes

$$p_R q_R + p_2 q_{k1} + p_1 q_{k2} + p_H q_H + \delta_2 d_{13} T_{k2} = Y, \quad (\text{B.13})$$

while the time constraint remains (B.3) with $q_S = 0$. The reduced first-order conditions are

$$U_1 = \lambda p_R, \quad (\text{B.9})$$

$$U_2 F_1 = \lambda p_2, \quad (\text{B.10})$$

$$U_3 G_2 - U_2 F_2(1 + d_{13}) = \lambda \delta_2 d_{13}, \quad (\text{B.11})$$

$$U_3 G_1 = \lambda p_1, \quad (\text{B.12})$$

$$U_4 H_2 + U_6 - U_2 F_2 = 0, \quad (\text{B.13})$$

$$U_4 H_1 = \lambda p_H, \quad (\text{B.14})$$

with the same normalizing factor E^τ as in (B.5) after setting $q_S^\tau = 0$. A natural set of five independent equations is given by four expenditure-share equations,

$$\frac{p_R^\tau q_R^\tau}{Y^\tau} = \frac{U_1^\tau q_R^\tau}{E^\tau}, \quad (\text{B.14})$$

$$\frac{p_2^\tau q_{k1}^\tau}{Y^\tau} = \frac{U_2^\tau F_1^\tau q_{k1}^\tau}{E^\tau}, \quad (\text{B.15})$$

$$\frac{\delta_2^\tau d_{13}^\tau T_{k2}^\tau}{Y^\tau} = \frac{[U_3^\tau G_2^\tau - U_2^\tau F_2^\tau(1 + d_{13}^\tau)]T_{k2}^\tau}{E^\tau}, \quad (\text{B.16})$$

$$\frac{p_1^\tau q_{k2}^\tau}{Y^\tau} = \frac{U_3^\tau G_1^\tau q_{k2}^\tau}{E^\tau}, \quad (\text{B.17})$$

$$\frac{p_H^\tau q_H^\tau}{Y^\tau} = \frac{U_4^\tau H_1^\tau q_H^\tau}{E^\tau}, \quad (\text{B.18})$$

and the time-allocation condition

$$U_4^\tau H_2^\tau + U_6^\tau - U_2^\tau F_2^\tau = 0. \quad (\text{B.19})$$

Because the five shares sum to one, the independent content is four share equations plus (B.19).

In Cases 2 and 4, the shadow price of household work is unobserved:

$$w_H^*{}^\tau = \frac{U_4^\tau H_2^\tau}{\lambda^\tau}. \quad (\text{B.20})$$

The Kuhn-Tucker restriction requires $w_H^{*\tau} \leq w_S^\tau$, and (B.12) or (B.19) implies

$$w^{*\tau} = w_H^{*\tau} + \frac{U_6^\tau}{\lambda^\tau} \leq w_H^{*\tau}.$$

Online Appendix C–G

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Appendix A (derivation of the worker first-order conditions) and Appendix B (estimating equations for Cases 2–4) appear at the end of the main text. This Online Appendix provides: Appendix C (equivalence of two-constraint and single-constraint problems with duality proofs for all four cases), Appendix D (superlative index theory proofs including Fisher exactness, Törnqvist equivalence, and GEKS transitivity), Appendix E (complete comparative statics of the rent function), Appendix F (functional form specifications for estimation), and Appendix G (shadow-price bounds for all four cases with full proofs).

C Equivalence of the Two-Constraint and Single-Constraint Problems

This appendix establishes that, for each of the four cases, the household’s two-constraint optimization problem is equivalent to a single-constraint problem in aggregated variables, provided the shadow price of leisure w^* is known. The equivalence is central to the paper because it converts the household’s problem into a form amenable to classical demand theory and index number aggregation.

C.1 Preliminary: Unit Cost Functions and Duality

Lemma C.1 (Duality under linear homogeneity). *Let $J : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be continuous, concave, and linearly homogeneous with $J(\mathbf{q}) > 0$ for $\mathbf{q} > 0$. Define the unit cost function*

$$c^J(\mathbf{p}) \equiv \min_{\mathbf{q} \geq 0} \{\mathbf{p} \cdot \mathbf{q} : J(\mathbf{q}) \geq 1\}. \tag{C.1}$$

Then:

(a) c^J is continuous, concave, linearly homogeneous, and nondecreasing in \mathbf{p} .

(b) If J is differentiable at $\mathbf{q}^* > 0$ and \mathbf{q}^* solves $\min\{\mathbf{p} \cdot \mathbf{q} : J(\mathbf{q}) \geq Q^*\}$ for some $Q^* > 0$, then the minimum cost equals $c^J(\mathbf{p}) \cdot Q^*$ and Shephard's Lemma holds:

$$\frac{\partial c^J(\mathbf{p})}{\partial p_n} = \frac{q_n^*}{Q^*}, \quad n = 1, 2. \quad (\text{C.2})$$

(c) (Euler's Theorem) $J_1(\mathbf{q})q_1 + J_2(\mathbf{q})q_2 = J(\mathbf{q})$ for all $\mathbf{q} > 0$.

(d) The value identity holds: $c^J(\mathbf{p}) \cdot J(\mathbf{q}^*) = \mathbf{p} \cdot \mathbf{q}^*$ at the optimum.

Proof. Parts (a) and (b) follow from Shephard (1953) and Diewert (1974). Part (c) is Euler's Theorem on homogeneous functions. Part (d) follows from (b) and (c): $c^J(\mathbf{p}) \cdot J(\mathbf{q}^*) = c^J(\mathbf{p}) \cdot [J_1 q_1^* + J_2 q_2^*] = p_1 q_1^* + p_2 q_2^* = \mathbf{p} \cdot \mathbf{q}^*$, where the second equality uses the first-order conditions $J_n(\mathbf{q}^*) = \mu^* p_n$ and $c^J(\mathbf{p}) = 1/\mu^*$. ■

C.2 Case 1: $q_S^* > 0$, $T_{l1}^* > 0$ (Worker with Purchased Household Help)

Theorem C.2 (Equivalence, Case 1). *Under Assumptions 2.1–3.2 with $q_S^* > 0$ and $T_{l1}^* > 0$, the two-constraint problem (3.3) is equivalent to the single-constraint problem:*

$$\max_{Q_{k1}, Q_{k2}, Q_H, q_R, T_{l1}, T_{l2} \geq 0} U(q_R, Q_{k1}, Q_{k2}, Q_H, T_{l1}, T_{l2}) \quad (\text{C.3})$$

subject to

$$p_R q_R + P_{k1}^* Q_{k1} + \tilde{P}_{k2}^* Q_{k2} + P_H^* Q_H + w^* T_{l2} - (w - w^*) T_{l1} + \delta_1 d_{12} \leq Y + w^* \tilde{T}, \quad (\text{C.4})$$

where $P_{k1}^* = c^F(p_2, w^*)$, $\tilde{P}_{k2}^* = c^G(p_1, \pi_{k2})$ with $\pi_{k2} = w^*(1 + d_{13}) + \delta_2 d_{13}$, and $P_H^* = c^H(p_H, w_S)$.

Proof. Step 1: From two constraints to a single constraint. Recall the saddle-point problem (A.1). Setting $\omega = \omega^* = \lambda^* w^*$, the Lagrangian becomes:

$$\begin{aligned} \mathcal{L} = & U[q_R, F(q_{k1}, T_{k1}), G(q_{k2}, T_{k2}), H(q_H, T_{l2} + q_S), T_{l1}, T_{l2}] \\ & + \lambda[wT_{l1} + Y - p_R q_R - p_2 q_{k1} - p_1 q_{k2} - p_H q_H - w_S q_S - \delta_1 d_{12} - \delta_2 d_{13} T_{k2} \\ & + w^*(\tilde{T} - T_{l1} - T_{l2} - T_{k1} - (1 + d_{13})T_{k2})]. \end{aligned}$$

The expression in square brackets is:

$$\begin{aligned} p_R q_R + p_2 q_{k1} + w^* T_{k1} + p_1 q_{k2} + [w^*(1 + d_{13}) + \delta_2 d_{13}] T_{k2} \\ + p_H q_H + w_S q_S + w^* T_{l2} - (w - w^*) T_{l1} + \delta_1 d_{12} \leq Y + w^* \tilde{T}. \end{aligned}$$

By the Karlin–Uzawa Theorem applied in reverse, the solution to the saddle-point problem with $\omega = \omega^*$ is also a solution to the constrained maximization problem with this single constraint.

Step 2: Aggregation via duality. For each production function $J \in \{F, G, H\}$, the first-order conditions and Lemma C.1 imply the value identity at the optimum. For F :

$$p_2 q_{k1}^* + w^* T_{k1}^* = c^F(p_2, w^*) \cdot F(q_{k1}^*, T_{k1}^*) = P_{k1}^* Q_{k1}^*.$$

Here we use: (i) the first-order conditions (2.12) and (2.13) imply that (q_{k1}^*, T_{k1}^*) solves the cost minimization $\min\{p_2 q + w^* t : F(q, t) \geq Q_{k1}^*\}$; (ii) by linear homogeneity, the minimum cost is $c^F(p_2, w^*) \cdot Q_{k1}^*$.

Similarly, for G :

$$p_1 q_{k2}^* + \pi_{k2} T_{k2}^* = c^G(p_1, \pi_{k2}) \cdot G(q_{k2}^*, T_{k2}^*) = \tilde{P}_{k2}^* Q_{k2}^*.$$

Here $\pi_{k2} = w^*(1 + d_{13}) + \delta_2 d_{13}$ is the shadow price of external leisure time from (3.4).

For H :

$$p_H q_H^* + w_S (T_{l2}^* + q_S^*) = c^H(p_H, w_S) \cdot H(q_H^*, T_{l2}^* + q_S^*) = P_H^* Q_H^*.$$

Here we use (2.16) and (2.17) (the latter requires $q_S^* > 0$).

Step 3: Substitution. Substituting the three value identities into the single budget constraint and collecting terms yields (C.4).

Step 4: Concavity. The constraint (C.4) is linear in $(Q_{k1}, Q_{k2}, Q_H, q_R, T_{l1}, T_{l2})$. The objective U is concave by Assumption 2.1. Hence (C.3)–(C.4) is a concave programming problem, and $(Q_{k1}^*, Q_{k2}^*, Q_H^*, q_R^*, T_{l1}^*, T_{l2}^*)$ together with λ^* solve it. ■

C.3 Case 2: $q_S^* = 0$, $T_{l1}^* > 0$ (Worker without Purchased Household Help)

In this case, the household supplies market labor but does not purchase any household services. This is the spatial counterpart of Schreyer and Diewert (2014, Section 4).

Theorem C.3 (Equivalence, Case 2). *With $q_S^* = 0$ and $T_{l1}^* > 0$, define the imputed household-work price $w_H^* \equiv U_4 H_2 / \lambda^*$ and the modified full prices $P_H^{*,2} \equiv c^H(p_H, w_H^*)$. The two-constraint problem is equivalent to:*

$$\max U(q_R, Q_{k1}, Q_{k2}, Q_H, T_{l1}, T_{l2}) \tag{C.5}$$

subject to

$$p_R q_R + P_{k1}^* Q_{k1} + \tilde{P}_{k2}^* Q_{k2} + P_H^{*,2} Q_H + w^* T_{l2} - (w - w^*) T_{l1} + \delta_1 d_{12} \leq Y + w^* \tilde{T}. \tag{C.6}$$

Proof. Step 1: From two constraints to a single constraint. The Lagrangian for the worker

with $q_S = 0$ is:

$$\begin{aligned} \mathcal{L}^{W,2} = & U[q_R, F(q_{k1}, T_{k1}), G(q_{k2}, T_{k2}), H(q_H, T_{l2}), T_{l1}, T_{l2}] \\ & + \lambda[wT_{l1} + Y - p_R q_R - p_2 q_{k1} - p_1 q_{k2} - p_H q_H - \delta_1 d_{12} - \delta_2 d_{13} T_{k2}] \\ & + \omega[\tilde{T} - T_{l1} - T_{l2} - T_{k1} - (1 + d_{13})T_{k2}]. \end{aligned} \quad (\text{C.7})$$

Note the absence of $w_S q_S$ from the budget constraint and q_S from $H(\cdot)$.

The first-order conditions for the interior variables $(q_R, q_{k1}, T_{k1}, q_{k2}, T_{k2}, q_H, T_{l2}, T_{l1})$ are:

$$U_1 = \lambda^* p_R, \quad (\text{C.8})$$

$$U_2 F_1 = \lambda^* p_2, \quad (\text{C.9})$$

$$U_2 F_2 = \lambda^* w^*, \quad (\text{C.10})$$

$$U_3 G_1 = \lambda^* p_1, \quad (\text{C.11})$$

$$U_3 G_2 = \lambda^* \pi_{k2}, \quad (\text{C.12})$$

$$U_4 H_1(q_H^*, T_{l2}^*) = \lambda^* p_H, \quad (\text{C.13})$$

$$U_4 H_2(q_H^*, T_{l2}^*) + U_6 = \lambda^* w^*, \quad (\text{C.14})$$

$$U_5 = -\lambda^*(w - w^*). \quad (\text{C.15})$$

The Kuhn–Tucker condition for q_S is the inequality:

$$U_4 H_2(q_H^*, T_{l2}^*) \leq \lambda^* w_S. \quad (\text{C.16})$$

Setting $\omega = \omega^* = \lambda^* w^*$ and combining the two constraints as in Case 1:

$$\begin{aligned} p_R q_R^* + p_2 q_{k1}^* + w^* T_{k1}^* + p_1 q_{k2}^* + \pi_{k2} T_{k2}^* \\ + p_H q_H^* + w^* T_{l2}^* - (w - w^*) T_{l1}^* + \delta_1 d_{12} = Y + w^* \tilde{T}. \end{aligned}$$

Step 2: Aggregation via duality. For F and G , the aggregation is identical to Case 1:

$$p_2 q_{k1}^* + w^* T_{k1}^* = P_{k1}^* Q_{k1}^*, \quad (\text{C.17})$$

$$p_1 q_{k2}^* + \pi_{k2} T_{k2}^* = \tilde{P}_{k2}^* Q_{k2}^*. \quad (\text{C.18})$$

For H , however, the situation differs. Since $q_S^* = 0$, the time input to H is T_{l2}^* alone (not $T_{l2}^* + q_S^*$). The first-order condition (C.13) gives $U_4 H_1 = \lambda^* p_H$. Define:

$$w_H^* \equiv \frac{U_4 H_2(q_H^*, T_{l2}^*)}{\lambda^*} > 0. \quad (\text{C.19})$$

This is the *imputed shadow price of household work time*, which in Case 1 equaled w_S but here is an unobserved endogenous variable. From (C.16): $w_H^* \leq w_S$.

Now consider the cost minimization $\min_{q,t \geq 0} \{p_H q + w_H^* t : H(q, t) \geq Q_H^*\}$. The first-order conditions for this problem are $H_1 = \mu^* p_H$ and $H_2 = \mu^* w_H^*$ for some $\mu^* > 0$. Comparing with (C.13) and (C.19): $\mu^* = \lambda^*/U_4$. Hence (q_H^*, T_{l2}^*) solves this cost minimization, and by Lemma C.1(d):

$$p_H q_H^* + w_H^* T_{l2}^* = c^H(p_H, w_H^*) \cdot H(q_H^*, T_{l2}^*) = P_H^{*,2} Q_H^*. \quad (\text{C.20})$$

Step 3: Substitution. From the combined budget-time identity, we need to express $w^* T_{l2}^*$ in terms of $P_H^{*,2} Q_H^*$. Note that (C.20) gives $p_H q_H^* + w_H^* T_{l2}^* = P_H^{*,2} Q_H^*$. Therefore:

$$p_H q_H^* + w^* T_{l2}^* = P_H^{*,2} Q_H^* - (w_H^* - w^*) T_{l2}^*. \quad (\text{C.21})$$

Substituting (C.17), (C.18), and (C.21) into the combined identity:

$$\begin{aligned} p_R q_R^* + P_{k1}^* Q_{k1}^* + \tilde{P}_{k2}^* Q_{k2}^* + P_H^{*,2} Q_H^* \\ - (w_H^* - w^*) T_{l2}^* - (w - w^*) T_{l1}^* + \delta_1 d_{l2} = Y + w^* \tilde{T}. \end{aligned}$$

Since $U_6 \leq 0$, equation (C.14) and definition (C.19) give $w^* = w_H^* + U_6/\lambda^* \leq w_H^*$, so

$w_H^* - w^* \geq 0$. The term $-(w_H^* - w^*)T_{l2}^*$ represents the *disutility adjustment for household work*, analogous to $-(w_S - w^*)T_{l2}^*$ in Case 1 but with the unobserved w_H^* replacing w_S .

Step 4: Concavity. The single-constraint problem (C.5)–(C.6) has a linear constraint and a concave objective. The solution $(Q_{k1}^*, Q_{k2}^*, Q_H^*, q_R^*, T_{l1}^*, T_{l2}^*)$ together with λ^* satisfies the first-order conditions of this problem.

Additional requirement. Unlike Case 1, this equivalence requires knowledge of *two* unobserved prices: w^* (leisure time) and w_H^* (household work time). Econometric estimation of both is necessary; see Section 7 and Appendix B. ■

Remark C.4 (Relationship between w^* , w_H^* , and w_S). In Case 2, the following ordering holds:

$$0 < w^* \leq w_H^* \leq w_S. \quad (\text{C.22})$$

The first inequality follows from $U_6 \leq 0$ and (C.14). The second follows from the KT condition (C.16). If $U_6 = 0$ (no disutility of housework), then $w^* = w_H^*$ and the household values leisure and housework time identically. If additionally $U_5 = 0$ (no disutility of market work), then $w^* = w_H^* = w$ and we recover the Becker case—but this also requires $w \leq w_S$ for the KT condition to be consistent. See Schreyer and Diewert (2014, Section 4) for further discussion.

Remark C.5 (Full value decomposition for Case 2). The full value of household services in Case 2 is:

$$P_H^{*,2} Q_H^* = p_H q_H^* + w_H^* T_{l2}^*. \quad (\text{C.23})$$

This differs from Case 1's $P_H^* Q_H^* = p_H q_H^* + w_S (T_{l2}^* + q_S^*)$ in two ways: (i) the time price is w_H^* instead of w_S , and (ii) the time quantity is T_{l2}^* instead of $T_{l2}^* + q_S^*$ (since $q_S^* = 0$). The national income accountant faces a challenge: w_H^* is unobserved. A common practical approximation is to set w_H^* equal to some fraction of w_S , as suggested by Schreyer and Diewert (2014, Section 10).

C.4 Case 3: $q_S^* > 0$, $T_{l1}^* = 0$ (Nonworker with Purchased Household Help)

This case covers households that do not participate in market labor—e.g., retirees with sufficient nonlabor income—but do purchase household services. It is the spatial counterpart of Schreyer and Diewert (2014, Section 5).

Theorem C.6 (Equivalence, Case 3). *With $q_S^* > 0$ and $T_{l1}^* = 0$, commuting vanishes and the effective time endowment is T (not \tilde{T}). The two-constraint problem is equivalent to:*

$$\max U(q_R, Q_{k1}, Q_{k2}, Q_H, 0, T_{l2}) \quad (\text{C.24})$$

subject to

$$p_R q_R + P_{k1}^* Q_{k1} + \tilde{P}_{k2}^* Q_{k2} + P_H^* Q_H - (w_S - w^*) T_{l2} \leq Y + w^* T. \quad (\text{C.25})$$

Proof. Step 1: The nonworker's constraints. With $T_{l1}^* = 0$, the household does not commute. The time constraint is:

$$T_{k1} + T_{l2} + (1 + d_{13}) T_{k2} = T. \quad (\text{C.26})$$

Note: T appears on the right-hand side, not $\tilde{T} = T - d_{12}$. The budget constraint is:

$$p_R q_R + p_2 q_{k1} + p_1 q_{k2} + p_H q_H + w_S q_S + \delta_2 d_{13} T_{k2} \leq Y. \quad (\text{C.27})$$

Note: the right-hand side is Y alone (no wage income), and the commuting cost $\delta_1 d_{12}$ is absent.

The Lagrangian is:

$$\begin{aligned} \mathcal{L}^{NW,3} = & U[q_R, F(q_{k1}, T_{k1}), G(q_{k2}, T_{k2}), H(q_H, T_{l2} + q_S), 0, T_{l2}] \\ & + \lambda[Y - p_R q_R - p_2 q_{k1} - p_1 q_{k2} - p_H q_H - w_S q_S - \delta_2 d_{13} T_{k2}] \\ & + \omega[T - T_{k1} - T_{l2} - (1 + d_{13}) T_{k2}]. \quad (\text{C.28}) \end{aligned}$$

First-order conditions. The interior conditions for $(q_R, q_{k1}, q_{k2}, q_H, q_S, T_{k2}, T_{l2})$ are:

$$U_1 = \lambda^* p_R, \quad (\text{C.29})$$

$$U_2 F_1 = \lambda^* p_2, \quad (\text{C.30})$$

$$U_3 G_1 = \lambda^* p_1, \quad (\text{C.31})$$

$$U_4 H_1 = \lambda^* p_H, \quad (\text{C.32})$$

$$U_4 H_2 = \lambda^* w_S, \quad (\text{C.33})$$

$$U_3 G_2 = \lambda^* [\delta_2 d_{13} + w^*(1 + d_{13})] = \lambda^* \pi_{k2}, \quad (\text{C.34})$$

$$U_4 H_2 + U_6 = \lambda^* w^*. \quad (\text{C.35})$$

The condition for T_{k1} is $U_2 F_2 = \omega^* = \lambda^* w^*$, which is absorbed into the definition of w^* .

For T_{l1} , the Kuhn–Tucker condition (since $T_{l1}^* = 0$) is:

$$U_5 + \lambda^* w - \omega^* \leq 0 \quad \iff \quad U_5 \leq -\lambda^*(w - w^*). \quad (\text{C.36})$$

This inequality does *not* in general constrain w^* relative to w . If U_5 is sufficiently negative, the condition is satisfied even when $w^* > w$. This is the key difference from the worker cases.

Step 2: From two constraints to a single constraint. Setting $\omega = \omega^* = \lambda^* w^*$ and combining (C.26) $\times w^*$ with (C.27):

$$\begin{aligned} p_R q_R^* + p_2 q_{k1}^* + w^* T_{k1}^* + p_1 q_{k2}^* + [w^*(1 + d_{13}) + \delta_2 d_{13}] T_{k2}^* \\ + p_H q_H^* + w_S q_S^* + w^* T_{l2}^* = Y + w^* T. \end{aligned} \quad (\text{C.37})$$

Step 3: Aggregation via duality. For F : conditions (C.30) and $U_2 F_2 = \lambda^* w^*$ imply that (q_{k1}^*, T_{k1}^*) solves $\min\{p_2 q + w^* t : F(q, t) \geq Q_{k1}^*\}$. By Lemma C.1(d):

$$p_2 q_{k1}^* + w^* T_{k1}^* = c^F(p_2, w^*) \cdot F(q_{k1}^*, T_{k1}^*) = P_{k1}^* Q_{k1}^*. \quad (\text{C.38})$$

For G : conditions (C.31) and (C.34) imply that (q_{k2}^*, T_{k2}^*) solves $\min\{p_1q + \pi_{k2}t : G(q, t) \geq Q_{k2}^*\}$. Therefore:

$$p_1q_{k2}^* + \pi_{k2}T_{k2}^* = c^G(p_1, \pi_{k2}) \cdot G(q_{k2}^*, T_{k2}^*) = \tilde{P}_{k2}^* Q_{k2}^*. \quad (\text{C.39})$$

For H : since $q_S^* > 0$, condition (C.33) holds with equality, so w_S is the relevant price. Conditions (C.32) and (C.33) imply that $(q_H^*, T_{l2}^* + q_S^*)$ solves $\min\{p_Hq + w_S t : H(q, t) \geq Q_H^*\}$:

$$p_Hq_H^* + w_S(T_{l2}^* + q_S^*) = c^H(p_H, w_S) \cdot H(q_H^*, T_{l2}^* + q_S^*) = P_H^* Q_H^*. \quad (\text{C.40})$$

Step 4: Substitution. From (C.40): $p_Hq_H^* + w_Sq_S^* = P_H^*Q_H^* - w_S T_{l2}^*$. The combined identity (C.37) can be rewritten using $w_Sq_S^* + w^*T_{l2}^* = (P_H^*Q_H^* - p_Hq_H^* - w_S T_{l2}^*) + w^*T_{l2}^* = P_H^*Q_H^* - p_Hq_H^* - (w_S - w^*)T_{l2}^*$:

$$p_Rq_R^* + P_{k1}^*Q_{k1}^* + \tilde{P}_{k2}^*Q_{k2}^* + P_H^*Q_H^* - (w_S - w^*)T_{l2}^* = Y + w^*T.$$

This is exactly the budget constraint (C.25).

Step 5: Concavity. The constraint (C.25) is linear. The objective $U(q_R, Q_{k1}, Q_{k2}, Q_H, 0, T_{l2})$ is concave in its arguments. The solution satisfies the first-order conditions of problem (C.24)–(C.25). ■

Remark C.7 (Nonworker full income). The full income for the nonworker case is:

$$FI^{NW} = Y + w^*T. \quad (\text{C.41})$$

Comparing with the worker full income $FI^W = Y - \delta_1 d_{12} + w^*(T - d_{12}) = Y + w^*T - (w^* + \delta_1)d_{12}$: the worker's full income is reduced by $(w^* + \delta_1)d_{12}$ relative to the nonworker's. This is the full welfare cost of commuting.

Remark C.8 (Shadow-price bound in Case 3). From (C.33) and (C.35): $U_6 = -\lambda^*(w_S - w^*)$,

so $w^* \leq w_S$. The KT condition (C.36) does not generally bound w^* relative to w . Hence:

$$0 < w^* \leq w_S. \quad (\text{C.42})$$

If $U_6 = 0$, then $w^* = w_S$, and the nonworker values leisure at the household-services wage. This is the nonworker counterpart of the Becker valuation rule, using w_S instead of w .

Remark C.9 (Relationship to Diewert (2014, Section 5)). The budget constraint (C.25) corresponds to Schreyer and Diewert (2014, Eq. 67), with two spatial modifications: (i) the external leisure full price $\tilde{P}_{k2}^* = c^G(p_1, \pi_{k2})$ replaces $c^G(p_1, w^*)$, reflecting travel costs; (ii) the effective time endowment remains T (not \tilde{T}) because no commuting occurs. Unlike Cases 1–2, the household’s full income does *not* depend on d_{12} : nonworkers are indifferent to commuting distance.

C.5 Case 4: $q_S^* = 0$, $T_{l1}^* = 0$ (Nonworker without Purchased Household Help)

This is the most constrained case: a frugal retired household that neither works nor purchases household services. It is the spatial counterpart of Schreyer and Diewert (2014, Section 6).

Theorem C.10 (Equivalence, Case 4). *With $q_S^* = 0$ and $T_{l1}^* = 0$, the two-constraint problem is equivalent to:*

$$\max U(q_R, Q_{k1}, Q_{k2}, Q_H, 0, T_{l2}) \quad (\text{C.43})$$

subject to

$$p_R q_R + P_{k1}^* Q_{k1} + \tilde{P}_{k2}^* Q_{k2} + P_H^{*,2} Q_H - (w_H^* - w^*) T_{l2} \leq Y + w^* T, \quad (\text{C.44})$$

where $P_H^{*,2} = c^H(p_H, w_H^*)$ and $w_H^* = U_4 H_2 / \lambda^*$.

Proof. Step 1: The frugal nonworker’s constraints. With $T_{l1}^* = 0$ and $q_S^* = 0$, the time

constraint is (C.26) and the budget constraint is:

$$p_R q_R + p_2 q_{k1} + p_1 q_{k2} + p_H q_H + \delta_2 d_{13} T_{k2} \leq Y. \quad (\text{C.45})$$

The Lagrangian is:

$$\begin{aligned} \mathcal{L}^{\text{NW},4} = & U[q_R, F(q_{k1}, T_{k1}), G(q_{k2}, T_{k2}), H(q_H, T_{l2}), 0, T_{l2}] \\ & + \lambda[Y - p_R q_R - p_2 q_{k1} - p_1 q_{k2} - p_H q_H - \delta_2 d_{13} T_{k2}] \\ & + \omega[T - T_{k1} - T_{l2} - (1 + d_{13})T_{k2}]. \end{aligned} \quad (\text{C.46})$$

First-order conditions. For the interior variables $(q_R, q_{k1}, q_{k2}, q_H, T_{k2}, T_{l2})$:

$$U_1 = \lambda^* p_R, \quad (\text{C.47})$$

$$U_2 F_1 = \lambda^* p_2, \quad (\text{C.48})$$

$$U_3 G_1 = \lambda^* p_1, \quad (\text{C.49})$$

$$U_4 H_1(q_H^*, T_{l2}^*) = \lambda^* p_H, \quad (\text{C.50})$$

$$U_3 G_2 = \lambda^* \pi_{k2}, \quad (\text{C.51})$$

$$U_4 H_2(q_H^*, T_{l2}^*) + U_6 = \lambda^* w^*. \quad (\text{C.52})$$

The Kuhn–Tucker conditions for q_S and T_{l1} are:

$$U_4 H_2(q_H^*, T_{l2}^*) \leq \lambda^* w_S, \quad (\text{C.53})$$

$$U_5 \leq -\lambda^*(w - w^*). \quad (\text{C.54})$$

Step 2: From two constraints to a single constraint. Combining (C.26) $\times w^*$ with (C.45):

$$p_R q_R^* + p_2 q_{k_1}^* + w^* T_{k_1}^* + p_1 q_{k_2}^* + \pi_{k_2} T_{k_2}^* + p_H q_H^* + w^* T_{l_2}^* = Y + w^* T. \quad (\text{C.55})$$

Step 3: Aggregation via duality. For F : identical to Cases 1–3. $p_2 q_{k_1}^* + w^* T_{k_1}^* = P_{k_1}^* Q_{k_1}^*$.

For G : identical to Cases 1–3. $p_1 q_{k_2}^* + \pi_{k_2} T_{k_2}^* = \tilde{P}_{k_2}^* Q_{k_2}^*$.

For H : since $q_S^* = 0$, the same argument as Case 2 applies. Define $w_H^* \equiv U_4 H_2(q_H^*, T_{l_2}^*)/\lambda^*$.

Conditions (C.50) and (C.52) (via w_H^*) imply:

$$p_H q_H^* + w_H^* T_{l_2}^* = c^H(p_H, w_H^*) \cdot H(q_H^*, T_{l_2}^*) = P_H^{*,2} Q_H^*. \quad (\text{C.56})$$

Step 4: Substitution. Rewriting $w^* T_{l_2}^* = w_H^* T_{l_2}^* - (w_H^* - w^*) T_{l_2}^*$ and using (C.56):

$$p_H q_H^* + w^* T_{l_2}^* = P_H^{*,2} Q_H^* - (w_H^* - w^*) T_{l_2}^*.$$

Substituting into (C.55):

$$p_R q_R^* + P_{k_1}^* Q_{k_1}^* + \tilde{P}_{k_2}^* Q_{k_2}^* + P_H^{*,2} Q_H^* - (w_H^* - w^*) T_{l_2}^* = Y + w^* T,$$

which is (C.44).

Step 5: Concavity. Identical to Step 4 of Case 2.

Two unobserved prices. As in Case 2, the equivalence requires knowledge of both w^* and w_H^* . ■

Remark C.11 (Ordering of shadow prices in Case 4). The ordering is:

$$0 < w^* \leq w_H^* \leq w_S. \quad (\text{C.57})$$

The first inequality follows from $U_6 \leq 0$ and (C.52). The second follows from (C.53). When $w_S^r = +\infty$ (no household services available in the region), the bound $w_H^* \leq w_S^r$ is vacuous, and w_H^* is unconstrained from above. In this supply-constrained case, w_H^* may substantially exceed market prices observed in urban areas, reflecting the high implicit cost of self-provision in an underserved region.

Remark C.12 (Practical approximation for Case 4). Until econometric estimates of w^* and w_H^* are available, a practical approach is to assume $U_6 = 0$ (no extra disutility of housework), which gives $w^* = w_H^*$. The national income accountant then needs only one shadow price. Since $w_H^* \leq w_S$, a reasonable guess is $w^* = w_H^* = \alpha \cdot w_S$ for some $\alpha \in (0, 1]$. This is the recommendation of Schreyer and Diewert (2014, Section 10).

Remark C.13 (Correspondence to Schreyer and Diewert (2014)). Theorems C.2–C.10 extend Schreyer and Diewert (2014, Eqs. 43, 59, 67, 79) to the spatial setting. The spatial modifications are: (i) the effective time endowment $\tilde{T} = T - d_{12}$ replaces T in worker cases (Cases 1–2) but not in nonworker cases (Cases 3–4); (ii) the commuting fixed cost $\delta_1 d_{12}$ appears in the worker budget constraint; (iii) the full price of external leisure $\tilde{P}_{k2}^* = c^G(p_1, \pi_{k2})$ incorporates travel costs in all four cases.

D Superlative Index Theory: Detailed Proofs

D.1 Superlative Property of the Fisher Index

We provide a self-contained proof that the Fisher ideal index is superlative for the class of homogeneous quadratic aggregators, following Diewert (1976).

Definition D.1 (Homogeneous quadratic). A function $J : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ is *homogeneous quadratic* if

$$J(\mathbf{q}) = [\mathbf{q}'A\mathbf{q}]^{1/2} \tag{D.1}$$

where A is an $N \times N$ symmetric *positive definite* matrix (singular cases excluded; see Assumption 2.2).

Theorem D.2 (Diewert, 1976). *If J is homogeneous quadratic, then the Fisher ideal quantity index Q_F^{10} exactly equals the true quantity ratio $J(\mathbf{q}^1)/J(\mathbf{q}^0)$.*

Proof. Setup. Since A is symmetric positive definite (Assumption 2.2, last sentence), A^{-1} exists. The unit cost function dual to $J(\mathbf{q}) = (\mathbf{q}'A\mathbf{q})^{1/2}$ is:

$$c^J(\mathbf{p}) = \min_{\mathbf{q} \geq 0} \{\mathbf{p} \cdot \mathbf{q} : J(\mathbf{q}) \geq 1\} = (\mathbf{p}'A^{-1}\mathbf{p})^{1/2}.$$

By Shephard's Lemma, the cost-minimizing quantity vector satisfying $J(\mathbf{q}^*) = Q$ at prices \mathbf{p} is:

$$\mathbf{q}^* = Q \cdot \nabla_{\mathbf{p}} c^J(\mathbf{p}) = Q \cdot \frac{A^{-1}\mathbf{p}}{(\mathbf{p}'A^{-1}\mathbf{p})^{1/2}}. \quad (\text{D.2})$$

Step 1: Laspeyres index.

$$\mathbf{p}^0 \cdot \mathbf{q}^1 = J(\mathbf{q}^1) \cdot \mathbf{p}^0 \cdot \frac{A^{-1}\mathbf{p}^1}{(\mathbf{p}^{1'}A^{-1}\mathbf{p}^1)^{1/2}} = J(\mathbf{q}^1) \cdot \frac{\mathbf{p}^{0'}A^{-1}\mathbf{p}^1}{c^J(\mathbf{p}^1)}.$$

Also, $\mathbf{p}^0 \cdot \mathbf{q}^0 = J(\mathbf{q}^0) \cdot c^J(\mathbf{p}^0)$ (since $\mathbf{p} \cdot \mathbf{q}^* = Qc^J(\mathbf{p})$ by homogeneity). Thus:

$$Q_L^{10} = \frac{\mathbf{p}^0 \cdot \mathbf{q}^1}{\mathbf{p}^0 \cdot \mathbf{q}^0} = \frac{J(\mathbf{q}^1)}{J(\mathbf{q}^0)} \cdot \frac{\mathbf{p}^{0'}A^{-1}\mathbf{p}^1}{c^J(\mathbf{p}^0)c^J(\mathbf{p}^1)}.$$

Step 2: Paasche index. By symmetric reasoning with \mathbf{p}^0 and \mathbf{p}^1 swapped:

$$Q_P^{10} = \frac{\mathbf{p}^1 \cdot \mathbf{q}^1}{\mathbf{p}^1 \cdot \mathbf{q}^0} = \frac{J(\mathbf{q}^1)}{J(\mathbf{q}^0)} \cdot \frac{c^J(\mathbf{p}^1)c^J(\mathbf{p}^0)}{\mathbf{p}^{1'}A^{-1}\mathbf{p}^0}.$$

Step 3: Fisher index.

$$Q_L^{10} \cdot Q_P^{10} = \left[\frac{J(\mathbf{q}^1)}{J(\mathbf{q}^0)} \right]^2 \cdot \frac{\mathbf{p}^{0'}A^{-1}\mathbf{p}^1}{\mathbf{p}^{1'}A^{-1}\mathbf{p}^0} = \left[\frac{J(\mathbf{q}^1)}{J(\mathbf{q}^0)} \right]^2,$$

where the last equality uses $\mathbf{p}^{0'} A^{-1} \mathbf{p}^1 = \mathbf{p}^{1'} A^{-1} \mathbf{p}^0$ (symmetry of A^{-1}). Therefore:

$$Q_F^{10} = [Q_L^{10} \cdot Q_P^{10}]^{1/2} = \frac{J(\mathbf{q}^1)}{J(\mathbf{q}^0)}. \quad \blacksquare$$

Corollary D.3 (Flexibility). *The homogeneous quadratic family with $N = 2$ has 3 free parameters (a_{11}, a_{12}, a_{22}) , which is exactly the number needed to match an arbitrary twice-differentiable linearly homogeneous function to second order at a given point. Hence the Fisher index is superlative: it provides a second-order approximation to the true ratio for arbitrary linearly homogeneous J .*

D.2 Application to the Spatial Model

Proposition D.4 (First-stage aggregation: Home leisure). *For home leisure, the Fisher quantity index*

$$Q_F^{10}(k1) = \left[\frac{p_2^0 q_{k1}^1 + w^{*0} T_{k1}^1}{p_2^0 q_{k1}^0 + w^{*0} T_{k1}^0} \cdot \frac{p_2^1 q_{k1}^1 + w^{*1} T_{k1}^1}{p_2^1 q_{k1}^0 + w^{*1} T_{k1}^0} \right]^{1/2}$$

satisfies $Q_F^{10}(k1) = F(q_{k1}^1, T_{k1}^1) / F(q_{k1}^0, T_{k1}^0)$ when F is homogeneous quadratic. More generally, $Q_F^{10}(k1)$ approximates the true ratio to second order for arbitrary linearly homogeneous F .

Proof. Direct application of Theorem D.2 with $N = 2$, $\mathbf{q}^t = (q_{k1}^t, T_{k1}^t)$, $\mathbf{p}^t = (p_2^t, w^{*t})$, and $J = F$. ■

Analogous results hold for G (using prices (p_1, π_{k2})) and H (using prices (p_H, w_S) in Cases 1,3 or (p_H, w_H^*) in Cases 2,4).

D.3 Second-Stage Aggregation: Full Consumption

Proposition D.5 (Second-stage aggregation). *Given the first-stage aggregates (P_{k1}^r, Q_{k1}^r) , $(\tilde{P}_{k2}^r, Q_{k2}^r)$, (P_H^r, Q_H^r) and housing (p_R^r, q_R^r) , define the full-consumption price and quantity*

vectors:

$$\mathbf{P}^r = (p_R^r, P_{k1}^r, \tilde{P}_{k2}^r, P_H^r), \quad \mathbf{Q}^r = (q_R^r, Q_{k1}^r, Q_{k2}^r, Q_H^r).$$

Under Assumption 6.3 (the upper-level aggregator Ψ is linearly homogeneous), the Fisher quantity index

$$Q_{FC}^{r0} = \left[\frac{\mathbf{P}^0 \cdot \mathbf{Q}^r}{\mathbf{P}^0 \cdot \mathbf{Q}^0} \cdot \frac{\mathbf{P}^r \cdot \mathbf{Q}^r}{\mathbf{P}^r \cdot \mathbf{Q}^0} \right]^{1/2}$$

is superlative for the upper-level aggregation. When Ψ is homogeneous quadratic, $Q_{FC}^{r0} = \Psi(\mathbf{Q}^r)/\Psi(\mathbf{Q}^0)$ exactly (by Theorem D.2).

Remark D.6 (Two-stage consistency). If both the lower-level F, G, H and the upper-level aggregator belong to the homogeneous quadratic class, then the two-stage Fisher aggregation is exactly consistent: the result coincides with direct one-stage Fisher aggregation over all primitive inputs. For other functional forms, the two-stage procedure introduces an approximation error that is of third order (Diewert, 1976).

D.4 The Törnqvist Index as an Alternative

Definition D.7 (Törnqvist quantity index).

$$\ln Q_T^{10} = \sum_{n=1}^N \frac{1}{2} (s_n^0 + s_n^1) \ln \frac{q_n^1}{q_n^0}, \quad (\text{D.3})$$

where $s_n^t = p_n^t q_n^t / (\mathbf{p}^t \cdot \mathbf{q}^t)$ is the expenditure share of input n in period t .

Proposition D.8 (Törnqvist superlative property). *The Törnqvist index is exact for the translog aggregator $\ln J(\mathbf{q}) = \alpha_0 + \sum_n \alpha_n \ln q_n + \frac{1}{2} \sum_n \sum_m \gamma_{nm} \ln q_n \ln q_m$ and is therefore superlative.*

Proof. See Diewert (1976). ■

Remark D.9 (Fisher vs. Törnqvist in practice). Both indexes are superlative and typically agree to several decimal places. The Fisher is preferred when some quantities may be zero

(the Törnqvist requires strictly positive quantities due to the logarithm). In the present application, all quantities $(q_R, Q_{k1}, Q_{k2}, Q_H)$ are assumed positive at the optimum, so either index may be used.

D.5 GEKS Multilateral Method

For comparing M regions simultaneously while maintaining transitivity, the GEKS method (Gini, 1931; Eltető, Köves, and Szulc, 1964) is defined as follows.

Definition D.10 (GEKS price index). The GEKS price index between regions r and s is:

$$P_{GEKS}^{rs} = \prod_{k=1}^M \left(\frac{P_F^{rk}}{P_F^{sk}} \right)^{1/M}, \quad (\text{D.4})$$

where P_F^{rk} is the bilateral Fisher price index between r and k .

Proposition D.11 (GEKS transitivity). $P_{GEKS}^{rs} \cdot P_{GEKS}^{st} = P_{GEKS}^{rt}$ for all r, s, t .

Proof. From (D.4): $P_{GEKS}^{rs} \cdot P_{GEKS}^{st} = \prod_k (P_F^{rk}/P_F^{sk})^{1/M} \cdot \prod_k (P_F^{sk}/P_F^{tk})^{1/M} = \prod_k (P_F^{rk}/P_F^{tk})^{1/M} = P_{GEKS}^{rt}$. ■

E Comparative Statics of the Rent Function

This appendix provides the complete comparative statics of the equilibrium rent function with respect to all exogenous parameters.

E.1 Method

The equilibrium condition is $\tilde{V}^{Wr}(p_R^*, \dots) = \bar{V}$. Total differentiation with respect to parameter x gives:

$$\frac{dp_R^*}{dx} = -\frac{\partial \tilde{V}^W / \partial x}{\partial \tilde{V}^W / \partial p_R}. \quad (\text{E.1})$$

By Roy's identity (Proposition 4.1): $\partial \tilde{V}^W / \partial p_R = -\Phi_1 \lambda^* q_R^* < 0$.

E.2 Gradient with Respect to d_{12} (Commuting Distance)

$$\frac{dp_R^*}{dd_{12}} = -\frac{-\Phi_1\lambda^*(w^* + \delta_1)}{-\Phi_1\lambda^*q_R^*} = -\frac{w^* + \delta_1}{q_R^*} < 0. \quad (\text{E.2})$$

Interpretation: Each additional unit of commuting time costs the household w^* in time opportunity cost plus δ_1 in monetary cost. The rent must fall by this amount per unit of housing to maintain equal utility.

E.3 Gradient with Respect to d_{13} (Amenity Distance)

$$\frac{dp_R^*}{dd_{13}} = -\frac{-\Phi_1\lambda^*(w^* + \delta_2)T_{k2}^*}{-\Phi_1\lambda^*q_R^*} = -\frac{(w^* + \delta_2)T_{k2}^*}{q_R^*} \leq 0. \quad (\text{E.3})$$

The gradient depends on external leisure time T_{k2}^* : households that spend more time in external leisure are more sensitive to amenity distance.

E.4 Gradient with Respect to w_S^r (Household-Services Price)

$$\frac{dp_R^*}{dw_S^r} = -\frac{q_S^*}{q_R^*} \leq 0. \quad (\text{E.4})$$

The gradient is proportional to the quantity of purchased household services.

E.5 Gradient with Respect to w (Market Wage)

By the envelope theorem: $\partial V^W / \partial w = \lambda^* T_{l1}^* > 0$. Therefore:

$$\frac{dp_R^*}{dw} = \frac{T_{l1}^*}{q_R^*} > 0. \quad (\text{E.5})$$

Higher wages increase utility, which is capitalized into higher rents.

E.6 Gradient with Respect to Y (Nonlabor Income)

$$\frac{dp_R^*}{dY} = \frac{1}{q_R^*} > 0. \quad (\text{E.6})$$

E.7 Gradient with Respect to δ_1 (Commuting Monetary Cost)

$$\frac{dp_R^*}{d\delta_1} = -\frac{d_{12}}{q_R^*} \leq 0. \quad (\text{E.7})$$

Regions with longer commutes are more affected by increases in commuting costs.

E.8 Gradient with Respect to Amenity A_k^r

For an amenity valued by the household ($\partial\Phi/\partial A_k > 0$):

$$\frac{dp_R^*}{dA_k^r} = \frac{(\partial\Phi/\partial A_k)}{\Phi_1 \lambda^* q_R^*} > 0. \quad (\text{E.8})$$

E.9 Second-Order Properties

Proposition E.1 (Convexity/Concavity of the rent function). *Under homogeneous households, ignoring the shadow-price response $\partial w^*/\partial d_{12}$, the rent function $p_R^*(d_{12})$ is convex in d_{12} if q_R^* is nondecreasing in d_{12} (lower rents at greater distances induce more housing demand), and concave if q_R^* is nonincreasing in d_{12} (income effects dominate). The full second derivative, given by (5.13), also incorporates $\partial w^*/\partial d_{12}$; concavity is the standard prediction in monocentric models with normal housing demand.*

Proof. Neglecting the $\partial w^*/\partial d_{12}$ term in (5.13):

$$\frac{\partial^2 p_R^*}{\partial d_{12}^2} \approx \frac{(w^* + \delta_1)}{(q_R^*)^2} \frac{\partial q_R^*}{\partial d_{12}}.$$

If $\partial q_R^*/\partial d_{12} \geq 0$ (substitution effect of lower rents dominates), then $\partial^2 p_R^*/\partial d_{12}^2 \geq 0$, yielding convexity. If $\partial q_R^*/\partial d_{12} < 0$ (income effect of reduced effective income dominates), then $\partial^2 p_R^*/\partial d_{12}^2 < 0$, yielding concavity. Incorporating the shadow-price response, Corollary 5.13 gives joint sufficient conditions; concavity is the standard monocentric prediction. ■

F Functional Form Specification for Estimation

This appendix specifies concrete functional forms for the household production functions and the upper-level utility function, derives the resulting share equations in closed form, and counts parameters.

F.1 Homogeneous Quadratic Production Functions

For each of F , G , H (each with 2 inputs), adopt the homogeneous quadratic:

$$F(q_1, q_2) = [a_{11}^F q_1^2 + 2a_{12}^F q_1 q_2 + a_{22}^F q_2^2]^{1/2}. \quad (\text{F.1})$$

Parameters: 3 per function, 9 total.

Partial derivatives:

$$F_j(q_1, q_2) = \frac{a_{j1}^F q_1 + a_{j2}^F q_2}{[a_{11}^F q_1^2 + 2a_{12}^F q_1 q_2 + a_{22}^F q_2^2]^{1/2}}, \quad j = 1, 2. \quad (\text{F.2})$$

Unit cost function:

$$c^F(p_1, p_2) = [b_{11}^F p_1^2 + 2b_{12}^F p_1 p_2 + b_{22}^F p_2^2]^{1/2} \quad (\text{F.3})$$

where $B^F = (A^F)^{-1}$ (i.e., b_{jk}^F are the elements of the inverse of the coefficient matrix).

Concavity condition: F is concave if and only if $a_{11}^F a_{22}^F - (a_{12}^F)^2 \geq 0$ and $a_{11}^F, a_{22}^F \geq 0$.

F.2 Normalized Quadratic Utility Function

For the upper-level utility, consider the Normalized Quadratic form adapted from [Diewert and Wales \(1987\)](#) and [Diewert, Nomura, and Shimizu \(2025\)](#). With $K = 6$ arguments

$\mathbf{z} = (q_R, Q_{k1}, Q_{k2}, Q_H, T_{l1}, T_{l2})$, define:

$$U(\mathbf{z}) = \mathbf{d}'\mathbf{z} + \frac{1}{2} \frac{\mathbf{z}'A\mathbf{z}}{\boldsymbol{\alpha}'\mathbf{z}} \quad (\text{F.4})$$

where $\mathbf{d} \in \mathbb{R}^K$ is a vector of first-order coefficients, A is a $K \times K$ symmetric matrix, and $\boldsymbol{\alpha} \in \mathbb{R}_+^K$ is a vector of fixed nonnegative weights.

Properties:

- *Flexibility:* U provides a second-order approximation to an arbitrary twice-differentiable utility function at the base point.
- *Concavity (conditional):* If A is negative semi-definite, then for fixed $\boldsymbol{\alpha}'\mathbf{z} > 0$, the quadratic term $\frac{1}{2}\mathbf{z}'A\mathbf{z}/(\boldsymbol{\alpha}'\mathbf{z})$ is concave in \mathbf{z} (as the pointwise minimum of affine functions in \mathbf{z} , using the ratio structure). A convenient sufficient condition is to write $A = -CC'$ where C is lower-triangular (Diewert and Wales, 1987), which ensures $A \preceq 0$. *Caveat:* global concavity of (F.4) as a ratio form requires verifying that the Hessian $\nabla^2 U \preceq 0$ globally, which is not guaranteed by $A \preceq 0$ alone for all \mathbf{z} and $\boldsymbol{\alpha}$. For this reason, (F.4) is best treated as a *convenient flexible specification* that is locally concave near the normalization point and imposes local second-order flexibility, rather than a globally concave functional form without further verification.
- *Partial derivatives:*

$$U_k(\mathbf{z}) = d_k + \frac{A'_k\mathbf{z}}{\boldsymbol{\alpha}'\mathbf{z}} - \frac{1}{2} \frac{\mathbf{z}'A\mathbf{z}}{(\boldsymbol{\alpha}'\mathbf{z})^2} \alpha_k, \quad (\text{F.5})$$

where A'_k is the k -th row of A .

Parameters: \mathbf{d} has $K = 6$ elements. Writing $A = -CC'$ with C lower-triangular ensures $A \preceq 0$ (negative semi-definite), so U is locally concave near the normalization point $\boldsymbol{\alpha}$; C has $K(K+1)/2 = 21$ free elements. Hence U has $6 + 21 = \mathbf{27}$ free parameters (26 after one normalization). This is the full flexible parametrization; a more parsimonious specification imposes additional restrictions on C , trading flexibility for degrees of freedom.

Normalization: Set $\boldsymbol{\alpha}$ to be the sample-mean quantity vector, following Diewert, Nomura,

and Shimizu (2025).

F.3 Parameter Count

Function	Free parameters	Notes
F (home leisure)	3	Homogeneous quadratic, 2×2
G (external leisure)	3	Homogeneous quadratic, 2×2
H (household services)	3	Homogeneous quadratic, 2×2
U (utility)	27	NQ, $A \preceq 0$ via C lower-triangular, locally concave at $\bar{\mathbf{z}}$
Total	36	

With 7 independent share equations per period (Case 1) and Υ periods, there are 7Υ observations. The full model has $9 + 27 = 36$ free parameters (9 from F, G, H and 27 from U). Identification requires $7\Upsilon \geq 36$, i.e., $\Upsilon \geq 6$. In practice, for reliable nonlinear estimation, $\Upsilon \geq 15$ is recommended. Researchers requiring fewer periods may impose additional restrictions on C (e.g., diagonal C), reducing U 's parameters at the cost of some flexibility.

F.4 Closed-Form Share Equations under Homogeneous Quadratic Production

Substituting (F.2) and (F.5) into the share equations (7.4)–(7.11), each right-hand side becomes an explicit (though nonlinear) function of $(\mathbf{z}^\tau, \boldsymbol{\theta})$ where $\boldsymbol{\theta} = (a_{jk}^F, a_{jk}^G, a_{jk}^H, d_k^U, C_{jk}^U)$ is the parameter vector. These can be coded directly in a nonlinear SUR estimation package.

Example: Share equation for q_R (equation (7.4)).

$$\frac{p_R^\tau q_R^\tau}{F_B^\tau} = \frac{U_1^\tau q_R^\tau}{D^\tau}$$

where $U_1^\tau = d_1 + [A_1' \mathbf{z}^\tau / (\boldsymbol{\alpha}' \mathbf{z}^\tau)] - [\mathbf{z}^{\tau'} A \mathbf{z}^\tau / (2(\boldsymbol{\alpha}' \mathbf{z}^\tau)^2)] \alpha_1$ and D^τ is given by (7.12).

G Shadow-Price Bounds for All Four Cases

This appendix provides complete proofs of the shadow-price bounds for all four cases, including the regional variation in w_S^r .

G.1 Case 1: $q_S^* > 0, T_{l1}^* > 0$

Proof of $w^ \leq w$ (market wage bound).* From (A.9): $U_5 = -\lambda^*(w - w^*)$. Since $U_5 \leq 0$ (Assumption 2.1(ii)) and $\lambda^* > 0$: $w - w^* \geq 0$. ■

Proof of $w^ \leq w_S^r$ (household-services bound).* From (A.7): $U_4 H_2 = \lambda^* w_S$. Substituting into (A.8): $\lambda^* w_S + U_6 = \lambda^* w^*$, i.e., $U_6 = -\lambda^*(w_S - w^*)$. Since $U_6 \leq 0$ (Assumption 2.1(iii)) and $\lambda^* > 0$: $w_S - w^* \geq 0$. ■

Proof of $w^ > 0$.* $w^* = \omega^*/\lambda^*$ with $\omega^* > 0$ and $\lambda^* > 0$ (Proposition 2.3). ■

Combined: $0 < w^* \leq \min\{w_S^r, w\}$. This is Theorem 3.4.

G.2 Case 2: $q_S^* = 0, T_{l1}^* > 0$

Define $w_H^* \equiv U_4 H_2 / \lambda^*$.

Proof of $w_H^ \leq w_S^r$.* The KT condition is $U_4 H_2 \leq \lambda^* w_S^r$, i.e., $w_H^* \leq w_S^r$. When $w_S^r = +\infty$, this is vacuous. ■

Proof of $w^ \leq w_H^*$.* From (A.8) with $q_S = 0$: $U_4 H_2 + U_6 = \lambda^* w^*$, i.e., $\lambda^* w_H^* + U_6 = \lambda^* w^*$, so $w^* = w_H^* + U_6 / \lambda^*$. Since $U_6 \leq 0$: $w^* \leq w_H^*$. ■

Proof of $w^ \leq w$.* Identical to Case 1. ■

Combined: $0 < w^* \leq \min\{w_H^*, w\} \leq \min\{w_S^r, w\}$.

G.3 Case 3: $q_S^* > 0, T_{l1}^* = 0$

With $T_{l1}^* = 0$, the FOC (A.9) becomes an inequality: $U_5 \leq -\lambda^*(w - w^*)$. This does not constrain w^* relative to w .

Proof of $w^ \leq w_S^r$.* Identical to Case 1. ■

Combined: $0 < w^* \leq w_S^r$. The market wage w does not provide an upper bound.

If the household is unable to supply labor (e.g., retired), the condition (A.9) is absent entirely and the bound is simply $w^* \leq w_S^r$.

G.4 Case 4: $q_S^* = 0, T_{l1}^* = 0$

Both q_S and T_{l1} are zero.

Proof of $w^ \leq w_H^* \leq w_S^r$.* Same as Case 2 for $w^* \leq w_H^* \leq w_S^r$. ■

The market wage bound is absent (as in Case 3).

Combined: $0 < w^* \leq w_H^* \leq w_S^r$.

G.5 Summary Table

Case	w^* upper bound	w_H^* status	w_S^r role
1 ($q_S > 0, T_{l1} > 0$)	$\min\{w_S^r, w\}$	$= w_S^r$ (known)	Active bound
2 ($q_S = 0, T_{l1} > 0$)	$\min\{w_H^*, w\} \leq \min\{w_S^r, w\}$	Unknown, $\leq w_S^r$	Passive bound
3 ($q_S > 0, T_{l1} = 0$)	w_S^r	$= w_S^r$ (known)	Active bound
4 ($q_S = 0, T_{l1} = 0$)	$w_H^* \leq w_S^r$	Unknown, $\leq w_S^r$	Passive bound

When $w_S^r = +\infty$ (services unavailable), all bounds involving w_S^r become vacuous. In Cases 1 and 3, this forces $q_S^* = 0$, switching to Cases 2 and 4 respectively.