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GOLDEN RULE OPTIMALITY IN STOCHASTIC OLG ECONOMIES

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Abstract

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Golden Rule Optimality in Stochastic OLG Economies

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Abstract: This study examines conditional golden rule optimality (CGRO) in stochastic overlapping generations models to complement the existing results on conditional Pareto optimality (CPO). Although an example in which CPO implies CGRO is presented, it is shown that such a situation is avoidable under strictly convex preferences. Under such preferences, both CPO and CGRO are characterized by the conditions on the dominant root for the agents' common matrix of marginal rates of substitution. We demonstrate that CGRO requires the dominant root being exactly equal to one, whereas CPO allows it to be less than one. By adopting CGRO rather than CPO, we provide welfare theorems in the financial economy.

Keywords: Golden rule optimality; Pareto optimality; Dominant root; Welfare theorem; Stochastic overlapping generations model.

JEL Classification Numbers: D50; D62; D81; E40.

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1 Introduction

In the overlapping generations (OLG) economy, competitive equilibrium might not achieve an optimal allocation, even when markets operate perfectly, as in the Arrow-Debreu abstraction. It is understood that this sort of inefficiency is caused by the lack of a transversality condition at infinity. To design active policies (as social security) which remedy this type of inefficiency, it is important to identify optimality with easily verifiable conditions.

Many of studies have characterized optimality in stochastic OLG models: Peled (1984), Aiyagari and Peled (1991), Manuelli (1990), Chattopadhyay and Gottardi (1999), Chattopadhyay (2001, 2006), and Bloise and Calciano (2008) are such examples.[1] However, deterministic models have two primary optimality criteria: Pareto optimality and golden rule optimality. The above-mentioned studies focused on conditional Pareto optimality (CPO) and ignored conditional golden rule optimality (CGRO). Therefore, to complement the existing results on CPO, it seems worthwhile to study CGRO.

To do so, this study considers a simple, but rather, canonical stochastic OLG model such as that studied by Aiyagari and Peled (1991) and Chattopadhyay (2001). We then introduce two optimality criteria, CPO and CGRO for stationary feasible allocations.[2] According to these criteria, agents' welfare is evaluated by conditioning their utility on the state at the date of their birth. Agents are thus distinguished not only by their type and date of birth but also by the state at that date, and an agent's preference is defined over a set of contingent consumption streams available in the two periods of that agent's lifetime. The difference between these two criteria is clear: CPO copes with the welfare of the "initial old," while CGRO does not.

This study first discusses the relationship between CPO and CGRO. It is often believed that CGRO implies CPO and the set of CGRO allocations is strictly smaller than that of CPO allocations. However, this study presents two examples violating these intuitions. One of two examples is a situation wherein CGRO *does not* imply CPO. The other is a situation wherein the sets of CPO and CGRO allocations coincide with each other, and thus there is *no* CPO allocation which is not CGRO. We demonstrate that these two anomalous situations are avoidable by imposing, for example, strict quasi-concavity and boundary conditions on lifetime utility functions.

[1] These studies considered pure-endowment models, whereas Demange and Laroque (1999, 2000), Barbie et al. (2007), and Gottardi and Kubler (2011) studied models with production.

[2] CPO was first proposed by Muench (1977).

Under strict quasi-concavity of lifetime utility functions, this study characterizes both CPO and CGRO of "stationary feasible" allocations, whereas the existing literature focuses on the CPO of "(stationary) equilibrium" allocations. As an analogue of Pareto optimality in the standard Arrow-Debreu economy, the all agents' matrixes of marginal rates of substitution must coincide with one another at a CPO or CGRO allocation. Although CPO of a stationary feasible allocation is characterized by the dominant root of the matrix associated to the allocation being *less than or equal to unity*, we find that CGRO of a stationary feasible allocation is characterized by the dominant root of the matrix being exactly *equal to unity.* By these characterizations, we might say that CGRO is stronger than CPO as an optimality criterion.

It is known that a stationary equilibrium allocation with valued money, if any, is CPO.[3] By applying our results to the issue of equilibrium welfare, this study concludes that a stationary equilibrium allocation with valued money, if any, is not only CPO but also CGRO. This can be interpreted as the first welfare theorem of the stochastic OLG economy with money. By the first welfare theorem, we note that there might exist a CPO allocation that cannot be implemented by any stationary monetary equilibrium with transfers. That is, the second welfare theorem might not hold when we adopt CPO as an optimality criterion. By adopting CGRO, not CPO, this study also provides the second welfare theorem, i.e., any interior CGRO allocation can be achieved by a stationary monetary equilibrium under certain lump-sum transfers.

The organization of this paper is as follows: Section 2 presents details of the model. Section 3 defines CPO and CGRO and discusses the relationship between these two optimality criteria. Section 4 characterizes CPO and CGRO for stationary feasible allocations under strict quasi-concavity of utility functions. Section 5 introduces stationary equilibrium to the model and applies results given in the previous section to equilibrium allocations. Section 6 presents welfare theorems in the economy with financial assets. Section 7 concludes the paper. Proofs of results are provided in the Appendix.

2 The Economy

This study considers a stationary, one-good, finite-state, pure-endowment stochastic overlapping generations model with finitely lived agents, as studied by Aiyagari and Peled (1991) and Chattopadhyay (2001). Time is discrete and runs from $t = 0$ to infinity. Uncertainty is

[3] See also Sakai (1988).

modeled by a stationary Markov process with its finite state space S such that $0 \notin S$. For each $t \geq 0$, we denote by s_t the state realized in period t, called *period t state*, where the initial state $s_0 \in S$ is treated as given.[4]

After the realization of the state in each period $t \geq 1$, a new generation, the members of which are called *newly born agents* or simply *agents*, is born and lives for two periods. Let H be a nonempty finite set of members of each generation. We will assume that the economy is stationary, i.e., the endowments and preference structures of each agent depend only on the realizations of the Markov state during his/her lifetime, not on time or on past realizations. Thus, (i) the endowment stream of each agent $h \in H$ born at state $s \in S$ is denoted by $\omega_s^h = (\omega_s^{h1}, (\omega_{ss'}^{h2})_{s' \in S}) \in \Re_+ \times \Re_+^S$ and (ii) his/her lifetime utility function is denoted by $U^{hs} : \Re_+ \times \Re_+^S \to \Re$, where ω_s^{h1} and $(\omega_{ss'}^{h2})_{s' \in S}$ describe the endowments at birth and all states in the following period. It is assumed that $\omega_s^h \gg 0$ and U^{hs} is strictly monotone increasing, quasi-concave, and continuously differentiable on the interior of its domain.[5]

In addition, a one-period lived generation, the members of which are called *initial old agents* or simply *initial olds*, is born after the realization of period 1 state $s_1 \in S$. The set of the initial olds is given by H as defined above. Each initial old $h \in H$ born at period 1 state s_1 is assumed to be endowed with $\omega_{0s_1}^{2h} := \omega_{s_0 s_1}^{2h}$ units of the consumption good in his/her lifetime and his/her consumption streams $c_{0s_1}^{h2} \in \Re_+$ is ranked according to a utility function $u_0(c_{0s_1}^{h2}) := c_{0s_1}^{h2}$.

Let $S_0 := \{0\} \cup S$ and $\bar{\omega}_{ss'} := \sum_{h \in H} (\omega_{s'}^{1h} + \omega_{ss'}^{2h})$ for each $(s, s') \in S_0 \times S$, which is the total endowment when the current and preceding states are s' and s, respectively.[6] We concentrate our attention not on "all" feasible allocations but on "stationary" feasible allocations. A *stationary feasible allocation* of this economy is a family $c = \{c^{h1}, c^{h2}\}_{h \in H}$ of functions $c^{h1} : S \to \Re_+$ and $c^{h2} : S_0 \times S \to \Re_+$ such that

$$(\forall (s, s') \in S_0 \times S) \quad \sum_{h \in H} c_{s'}^{h1} + \sum_{h \in H} c_{ss'}^{h2} = \bar{\omega}_{ss'},$$

where $c_{0s_1}^{h2} \in \Re_+$ is the consumption of the initial old h born at period 1 state s_1, and $c_s^h = (c_s^{h1}, (c_{ss'}^{h2})_{s' \in S}) \in \Re_+ \times \Re_+^S$ is the consumption stream of the agent h born at the Markov state s. Let $\mathcal{A}$ be the set of all stationary feasible allocations with its generic element c. Note that $\mathcal{A}$ is

[4] This study implicitly considers a standard date-event tree as seen in, for example, Chattopadhyay (2001). Therefore, the initial state s_0 can be interpreted as the root of the date-event tree.

[5] In the rest of this study, we denote by $U_1^{hs}(c^1, c^2)$ and $U_{s'}^{hs}(c^1, c^2)$ the partial derivatives $\partial U^{hs}(c^1, c^2)/\partial c^1$ and $\partial U^{hs}(c^1, c^2)/\partial c_{s'}^2$ for all $h \in H$, all $s, s' \in S$, and all $(c^1, c^2) \in \Re_+ \times \Re_+^S$, respectively.

[6] We introduce S_0 to tell the consumption of the initial old from the agent's consumption in the second period of her life.

nonempty, bounded, closed, and convex. A stationary feasible allocation c is *interior* if $c_s^{h1} > 0$ and $c_{ss'}^{h2} > 0$ for all $s, s' \in S$ and all $h \in H$.

3 Optimality Criteria

This section introduces two criteria of optimality of stationary feasible allocations, CPO and CGRO, and we discuss the relationship between these two criteria.

For any two stationary feasible allocations $b, c \in \mathcal{A}$, we say that b *CPO-dominates* c if

$$(\forall (h,s) \in H \times S) \quad \begin{array}{l} b_{0s}^{h2} \geq c_{0s}^{h2}, \\ U^{hs}(b_s^h) \geq U^{hs}(c_s^h) \end{array}$$

with strict inequality somewhere. CPO is then defined as follows:

Definition 1 A stationary feasible allocation c is said to be *conditionally Pareto optimal* if there exists no other stationary feasible allocation b that CPO-dominates c.

CPO considers the welfare of the initial olds, whereas CGRO ignores it. For any two stationary feasible allocations $b, c \in \mathcal{A}$, we say that b *CGRO-dominates* c if

$$(\forall (h,s) \in H \times S) \quad U^{hs}(b_s^h) \geq U^{hs}(c_s^h)$$

with strict inequality somewhere. CGRO is then defined as follows:

Definition 2 A stationary feasible allocation c is said to be *conditionally golden rule optimal* if there exists no other stationary feasible allocation b that CGRO-dominates c.

These two criteria are analogues to those in a deterministic environment. In the rest of this study, we often call a stationary CPO (CGRO) allocation, a *CPO (CGRO) allocation.* Let CPO^* and $CGRO^*$ be the sets of CPO and CGRO allocations, respectively.

Next we discuss the relationship between CPO and CGRO. It is often considered that $CPO^* \supset CGRO^*$, i.e., a CGRO allocation is also CPO. However, the following example demonstrates that CGRO does not necessarily imply CPO.

Example 1 ($CPO^* \not\supset CGRO^*$) The first example illustrates an anomalous situation such that a CGRO allocation is not necessarily CPO. Consider the economy such that $S = \{\alpha, \beta\}$ and H is a singleton. In this example, we ignore superscripts h (of endowments, consumptions, and preferences), because H is a singleton. Suppose that $\omega_{ss'}^2$ is independent of the current state s for each $s, s' \in S$. Note that, in this economy, we can rewrite the total endowment as $\bar{\omega}_{ss'} \equiv \bar{\omega}_{s'}$ for all $s, s' \in S$. Also let $U^s(c_s) = 2c_s^1 + c_{s\alpha}^2 + c_{s\beta}^2$ for each $s \in S$. In the current setting, one can easily verify that, for every stationary feasible allocation c, it holds that $U^\alpha(c_s) = \bar{\omega}_\alpha + \bar{\omega}_\beta + c_\alpha^1 - c_\beta^1$ and $U^\beta(c_s) = \bar{\omega}_\alpha + \bar{\omega}_\beta - c_\alpha^1 + c_\beta^1$. Define the stationary feasible allocation $x = \{x^1, x^2\}$ by $x_{s'}^1 = \bar{\omega}_{s'}$ and $x_{ss'}^2 = 0$ for each $(s, s') \in S_0 \times S$. Obviously, allocation x is well-defined and CGRO, not CPO. In fact, for sufficiently small $\varepsilon > 0$, define the stationary feasible allocation $y = \{y^1, y^2\}$ by $y_{s'}^1 = \bar{\omega}_{s'} - \varepsilon$ and $y_{ss'}^2 = \varepsilon$ for each $(s, s') \in S_0 \times S$. This allocation is well-defined. Further, y CPO-dominates x, because it improves the initial old's welfare.

This example, surprisingly, also illustrates a situation such that $CPO^* \not\supset CGRO^*$ and $CPO^* \subset CGRO^*$, i.e., the set of CPO allocations is a proper subset of the set of CGRO allocations. In fact, all stationary feasible allocations are CGRO and the set of CPO allocations is given by $\{c \in \mathcal{A} : c_\alpha^1 = 0\} \cup \{c \in \mathcal{A} : c_\beta^1 = 0\}$ in this example. The following proposition provides a sufficient condition for avoiding situations illustrated in Example 1.

Proposition 1 *$CPO^* \supset CGRO^*$, i.e., every conditionally golden rule optimal allocation is conditionally Pareto optimal if either*

(a) *for any $x = (x^1, x^2), y = (y^1, y^2) \in \Re_+ \times \Re_+^S$ with $x^1 \neq y^1$, any $\alpha \in (0, 1)$, and any $(h, s) \in H \times S$, $U^{hs}(\alpha x + (1-\alpha)y) > \min\{U^{hs}(x), U^{hs}(y)\}$; or*

(b) *for any $x = (x^1, x^2), y = (y^1, y^2) \in \Re_+ \times \Re_+^S$ with $x^2 \neq y^2$, any $\alpha \in (0, 1)$, and any $(h, s) \in H \times S$, $U^{hs}(\alpha x + (1-\alpha)y) > \min\{U^{hs}(x), U^{hs}(y)\}$*

holds.

Proof of Proposition 1. See the Appendix. Q.E.D.

Note that both conditions (a) and (b) provided in this proposition hold, for example, under strict quasi-concavity of lifetime utility functions. Thus, as a corollary of Proposition 1, we can observe that $CPO^* \supset CGRO^*$, provided that lifetime utility functions are strictly quasi-concave. We can now say that CGRO is not a weaker criterion of optimality than CPO under strict quasi-concavity of lifetime utility functions.

We have provided sufficient conditions that ensure that CGRO implies CPO. The natural question following this result is whether CPO implies CGRO under such conditions. Although it may seem that there always exists a CPO allocation that is not CGRO, the following example illustrates an anomalous situation, wherein such a CPO allocation does not exist, because the sets of CPO and CGRO allocations coincide with each other.

Example 2 ($CPO^* = CGRO^*$) The second example illustrates an anomalous situation such that the sets of CPO and CGRO allocations coincide with each other. Consider the same economy as Example 1 except for preferences. Let $U^s(c_s) = c_s^1 + \bar{\omega}_\alpha \ln c_{s\alpha}^2 + \bar{\omega}_\beta \ln c_{s\beta}^2$ for each $s \in S$.[7] This lifetime utility function satisfies the condition (b) of Proposition 1. In the current setting, one can easily verify that, for every stationary feasible allocation c, it holds that $U^s(c_s) = c_s^1 + \bar{\omega}_\alpha \ln(\bar{\omega}_\alpha - c_\alpha^1) + \bar{\omega}_\beta \ln(\bar{\omega}_\beta - c_\beta^1) =: f^s(c_\alpha^1, c_\beta^1)$ for each $s \in S$. Note that $(\partial f^s/\partial c_s^1)(c_\alpha^1, c_\beta^1) = 1 - \bar{\omega}_s/(\bar{\omega}_s - c_s^1) < 0$ and $(\partial f^s/\partial c_{s'}^1)(c_\alpha^1, c_\beta^1) = -\bar{\omega}_{s'}/(\bar{\omega}_{s'} - c_{s'}^1) < 0$ for any stationary feasible allocation c with $0 < c_s^1 < \bar{\omega}_s$ and any $s \neq s' \in S$. Therefore, the stationary feasible allocation $x = \{x^1, x^2\}$ defined by $x_{s'}^1 = 0$ and $x_{ss'}^2 = \bar{\omega}_{s'}$ for each $(s, s') \in S_0 \times S$ is the unique CGRO allocation. Note that x is also the unique CPO allocation, and thus there is no CPO allocation that is not CGRO in this economy.

This example illustrates an anomalous situation such that the set of CPO allocations is too small to be distinguished from the set of CGRO allocations. In this example, such a situation is caused by quasi-linearity of lifetime utility functions. As shown in the following proposition, we can avoid situations such as those illustrated in Example 2 by imposing certain boundary conditions on preferences.

Proposition 2 *$CPO^* \supset CGRO^*$ but $CPO^* \not\subset CGRO^*$, i.e., there exists a conditionally Pareto optimal allocation that is not conditionally golden rule optimal if, in addition to the condition in the previous proposition, $\lim_{c_s^{h1} \downarrow 0} U_1^{hs}(c_s^{h1}, x_s^{h2}) = \infty$ for all $x_s^{h2} \in \Re_{++}^S$ and all $(h, s) \in H \times S$ holds.*

Proof of Proposition 2. See the Appendix. Q.E.D.

This section concludes with the following example of Proposition 2, demonstrating the existence of a CPO allocation that is not CGRO.

Example 3 ($CPO^* \supset CGRO^*$ **but** $CPO^* \not\subset CGRO^*$) The third example illustrates a situation such that the sets of CGRO allocations will be a proper subset of the set of CPO allocations.

[7] In this and the next examples, we consider that $\ln(0)$ to be well defined at $-\infty$.

Consider the same economy as Example 1 except for preferences. Let $U^s(c_s) = \bar{\omega}_s \ln c_s^1 + c_{s\alpha}^2 + c_{s\beta}^2$ for each $s \in S$. This lifetime utility function satisfies both the condition (a) of Proposition 1 and the boundary condition of Proposition 2. Note that the stationary feasible allocation $x = \{x^1, x^2\}$ defined by $x_{s'}^1 = 0$ and $x_{ss'}^2 = \bar{\omega}_{s'}$ for each $(s, s') \in S_0 \times S$ is a CPO allocation. In the current setting, one can easily verify that, for every stationary feasible allocation c, $U^s(c_s) = \bar{\omega}_s \ln c_s^1 - c_\alpha^1 - c_\beta^1 + (\bar{\omega}_\alpha + \bar{\omega}_\beta) =: g^s(c_\alpha^1, c_\beta^1)$ for each $s \in S$ holds. Note that $(\partial g^s/\partial c_s^1)(c_\alpha^1, c_\beta^1) = \bar{\omega}_s/c_s^1 - 1 > 0$ and $(\partial g^s/\partial c_{s'}^1)(c_\alpha^1, c_\beta^1) = -1 < 0$ for any stationary feasible allocation c with $0 < c_s^1 < \bar{\omega}_s$ and any $s \neq s' \in S$. Therefore, the allocation x is not CGRO. In fact, one can verify that, for sufficiently small $\varepsilon > 0$, the stationary feasible allocation $y = \{y^1, y^2\}$ defined by $y_{s'}^1 = \varepsilon$ and $y_{ss'}^2 = \bar{\omega}_{s'} - \varepsilon$ for each $(s, s') \in S_0 \times S$ is well-defined and CGRO-dominates x, because it improves the welfare of newly born agents.

4 Characterizations of Optimality Criteria

In the previous section, we examined the relationship between CPO and CGRO. It has been shown that CGRO implies CPO under strict convexity of preferences. Under such conditions, this section explores differences in the characterizations of these two optimality criteria. Thus, we strengthen the restrictions on the economy by assuming that U^{hs} is strictly quasi-concave for all $(h, s) \in H \times S$ in the rest of this paper.

Given an interior stationary feasible allocation c, let $m_{ss'}^h(c) = U_{s'}^{hs}(c_s^h)/U_1^{hs}(c_s^h)$ and let $M^h(c) = [m_{ss'}^h(c)]_{s,s' \in S}$, where $U_1^{hs}(c_s^h) = \partial U^{hs}(c_s^h)/\partial c_s^{h1}$ and $U_{s'}^{hs}(c_s^h) = \partial U^{hs}(c_s^h)/\partial c_{ss'}^{h2}$. The current restrictions on preferences imply that $M^h(c)$ is a positive square matrix. By the Perron-Frobenius theorem,[8] any positive square matrix M has a unique dominant root. This paper denotes by $\lambda^f(M)$ the dominant root of a positive square matrix M.

This study now characterizes CPO and CGRO. The following result extends Aiyagari and Peled (1991, Theorem 1) by characterizing the CPO of not an interior stationary "equilibrium" allocation but of an interior stationary "feasible" allocation.

Theorem 1 *An interior stationary feasible allocation c is conditionally Pareto optimal if and only if there exists a $S \times S$ matrix M with positive coefficients such that*

$$(\forall h \in H) \quad M = M^h(c)$$

[8]See, for example, Debreu and Herstein (1953) and Takayama (1974) for more details on the Perron-Frobenius theorem.

and its dominant root, $\lambda^f(M)$, is less than or equal to unity.

Proof of Theorem 1. See the Appendix. Q.E.D.

By Proposition 1 and Theorem 1, we now know that a CGRO allocation satisfies $\lambda^f(M) \leq 1$ for an appropriate positive square matrix M. However, can we characterize CGRO by a sharper condition than this? The next result gives us more information on the characterization of CGRO allocations, i.e., a CGRO allocation satisfies $\lambda^f(M) = 1$ for some appropriate positive square matrix M:

Theorem 2 *An interior stationary feasible allocation c is conditionally golden rule optimal if and only if there exists a $S \times S$ matrix M with positive coefficients such that*

$$(\forall h \in H) \quad M = M^h(c)$$

and its dominant root, $\lambda^f(M)$, is equal to unity.

Proof of Theorem 2. See the Appendix. Q.E.D.

This section characterized optimality criteria of stationary feasible allocations by the dominant root of a matrix related to them in an economy with strictly convex preferences. While the dominant root of a matrix related to a CGRO allocation must be equal to one, whereas that to a CPO allocation is allowed to be less than one.

Note that the dominant root criterion of CPO, not CGRO, provided in this section might not be applied to linear lifetime utility functions. We conclude this section by presenting such an example.

Example 4 (Linear Preferences) The fourth example illustrates an anomalous situation such that the dominant root criterion is inapplicable. Consider the same economy as Example 1 except for preferences. Let $U^s(c_s) = \delta c_s^1 + c_{s\alpha}^2 + c_{s\beta}^2$ for each $s \in S$, where $\delta > 0$. For every interior stationary feasible allocation c, one can easily calculate the dominant root λ^f of the matrix $M(c)$ as $\lambda^f = 2/\delta$. We can thus obtain that $\lambda^f < 1$ if $\delta > 2$, $\lambda^f = 1$ if $\delta = 2$, and otherwise $\lambda^f > 1$, where λ^f is the dominant root of the matrix $M(c)$. For $\delta \in (0,2) \cup (2,\infty)$, the dominant root criterion is applicable, i.e., all interior stationary feasible allocations are CPO if $\delta > 2$ but are not CPO if $\delta < 2$. However, for $\delta = 2$, all interior stationary feasible allocations are CGRO, but not CPO, as shown in Example 1, although $\lambda^f = 1$.

Strict quasi-concavity of lifetime utility functions, as assumed in this section, avoids the anomalous situation illustrated in this example.

5 Optimality of Stationary Equilibrium Allocations

The previous section characterized optimality criteria of stationary "feasible" allocations. These results also correspond to welfare analysis of stationary equilibrium. This section examines the relationship between optimality criteria and stationary "equilibrium" allocations.

This section defines a stationary equilibrium such that, in each single period, the one-period contingent claim market is complete:[9]

Definition 3 A pair (Π, c) of a positive price matrix $\Pi = [\pi_{ss'}]_{s,s' \in S}$ of contingent commodities and a stationary feasible allocation $c = (c_s^h)_{(h,s) \in H \times S}$ is called a *stationary equilibrium* if

- for all $(h,s) \in H \times S$, c_s^h belongs to the set

$$\operatorname*{arg\,max}_{(x_s^{h1}, x_s^{h2}) \in \Re_+ \times \Re_+^S} \left\{ U^{hs}(x_s^h) \,:\, x_s^{h1} + \sum_{s' \in S} x_{ss'}^{h2} \pi_{ss'} \leq \omega_s^{h1} + \sum_{s' \in S} \omega_{ss'}^{h2} \pi_{ss'} \right\},$$

- for all $s, s' \in S$, $\sum_{h \in H} (c_{s'}^{h1} + c_{ss'}^{h2}) = \bar{\omega}_{ss'}$.

In this definition, the former condition is the optimization problem of each agent $(h,s) \in H \times S$, and the latter is the market clearing conditions.

Let (Π, c) be a stationary equilibrium with $c_s^h \gg 0$ for all $(h,s) \in H \times S$, if any. Since, for all $(h,s) \in H \times S$, c_s^h must be a solution of the optimization problem of agent (h,s), it follows from the Kuhn-Tucker theorem that there exists some $\lambda^{hs} \geq 0$ such that

$$\begin{aligned} & U_1^{hs}(c_s^h) = \frac{\partial U^{hs}}{\partial c_s^{h1}}(c_s^h) = \lambda^{hs}, \\ (\forall s' \in S) \qquad & U_{s'}^{hs}(c_s^h) = \frac{\partial U^{hs}}{\partial c_{s'}^{h2}}(c_s^h) = \lambda^{hs} \pi_{ss'}, \end{aligned}$$

where λ^{hs} is the Lagrange multiplier of the lifetime budget constraint. Note that $\lambda^{hs} > 0$, because U^{hs} is strictly monotone increasing. Thus, we can observe that

$$(\forall h \in H) \quad \Pi = [\pi_{ss'}]_{s,s' \in S} = \left[\frac{U_{s'}^{hs}(c_s^h)}{U_1^{hs}(c_s^h)} \right]_{s,s' \in S} = M^h(c).$$

Therefore, the stationary equilibrium contingent claim price matrix Π can be always represented by the matrix of marginal rates of substitution at the stationary equilibrium allocation c. This

[9] That is, we consider economy with sequentially complete markets.

observation also states that all agents in the same generation must agree with the contingent claim price matrix at stationary equilibrium.

The next two propositions follow immediately from the previous observation and Theorems 1 and 2, respectively.

Proposition 3 (Aiyagari and Peled,1991) *For every stationary equilibrium* (Π, c) *with* $c_s^h \gg 0$ *for all* $(h, s) \in H \times S$, c *is conditionally Pareto optimal if and only if* $\lambda^f(\Pi) \leq 1$.

Proposition 4 *For every stationary equilibrium* (Π, c) *with* $c_s^h \gg 0$ *for all* $(h, s) \in H \times S$, c *is conditionally golden rule optimal if and only if* $\lambda^f(\Pi) = 1$.

These propositions characterize optimality of stationary equilibrium allocations. While the CPO of stationary equilibrium allocations is characterized by the dominant root of the contingent claim price matrix being less than or equal to one, their CGRO has the dominant root exactly equal to one. Note that Proposition 3 extends Theorem 1 of Aiyagari and Peled (1991) by allowing possibilities of heterogenous beliefs and nonseparability of preferences.

6 Welfare Theorems in Financial Economy

We described the optimality of stationary equilibrium allocations in the previous section. One of the important implications of the previous propositions is that stationary equilibrium allocations might be suboptimal even when markets are sequentially complete. It is known that an introduction of an infinitely lived outside asset such as *fiat money* can remedy such inefficiency. It is well known in the literature that an equilibrium with valued fiat money is CPO. In this section, we use the previous analysis to show that a monetary equilibrium is actually CGRO, and to show that the first and second theorems of welfare economics hold if the criterion of optimality is CGRO. In order to introduce the possibility transfers, needed for the second welfare theorem, we introduce in addition to fiat money a mondatory unfunded social security system.

Mandatory unfunded social security. We consider lump-sum transfers as a social security system. Each young $h \in H$ pays $\tau_{s'}^{h1}$ at the current state $s' \in S$. In contrast, each old $h \in H$ born at state $s \in S_0$ receives $\tau_{ss'}^{h2}$ at the current state $s' \in S$. It is assumed that the transfer $\tau = \{\tau^{h1}, \tau^{h2}\}_{h \in H}$ satisfies that $\tau_1^{h1} < \omega_s^{h1}$ and $\tau_{ss'}^{h2} > -\omega_{ss'}^{h2}$ for all $(s, s') \in S_0 \times S$.

It is also assumed that the authority's policy is balanced, i.e., $\sum_{h \in H} \tau^{h1}_{s'} = \sum_{h \in H} \tau^{h2}_{ss'}$ for all $(s, s') \in S_0 \times S$.

The financial asset. There exists an infinitely-lived outside asset, fiat money, which is available in positive supply, is normalized to unity, and yields no dividend. As in the previous section, we also consider sequentially complete contingent claim markets. We then introduce a stationary monetary equilibrium:

Definition 4 A triplet (q, Π, c) of a real price vector of money $q \in \Re^S_{++}$, a nonnegative matrix $\Pi = [\pi_{ss'}]_{s,s' \in S}$ of contingent claims, and a stationary feasible allocation $c = (c^h_s)_{(h,s) \in H \times S}$ is called a *stationary monetary equilibrium with transfers* τ if there exists money holdings $m \in \Re^{H \times S}_+$, and portfolios of contingent claims $[\theta^h_{ss'}]_{s,s' \in S, h \in H} \in (\Re^S)^{H \times S}$ such that

- for all $(h, s) \in H \times S$, $(c^h_s, m^h_s, \theta^h_s)$ belongs to the set

$$\underset{(x^h_s, z, \xi) \in (\Re_+ \times \Re^S_+) \times \Re \times \Re^S}{\arg\max} \left\{ U^{hs}(c) : \begin{array}{l} x^{h1}_s \leq \omega^{h1}_s - \tau^{h1}_s - q_s z - \sum_{s' \in S} \pi_{ss'} \xi_{s'} \\ (\forall s' \in S)\, x^{h2}_{ss'} \leq \omega^{h2}_{ss'} + \tau^{h2}_{ss'} + q_{s'} z + \xi_{s'} \end{array} \right\},$$

- for all $s, s' \in S$, $\sum_{h \in H}(c^{h1}_{s'} + c^{h2}_{ss'}) = \bar{\omega}_{ss'}$, $\sum_{h \in H} m^h_s = 1$, and $\sum_{h \in H} \theta^h_s = 0$.

In this definition, the former condition is the optimization problem of each agent $(h, s) \in H \times S$ with sequential budget constraints and the latter is the market clearing conditions. Remark that, per Gottardi (1996), a stationary monetary equilibrium with transfers exists generically and is locally isolated.[10]

We now demonstrate that the relationship between stationary monetary equilibrium and CGRO is analogous to the fundamental theorems of welfare economics (welfare theorems) in the standard Arrow-Debreu economy.

Theorem 3 *Every stationary monetary equilibrium with transfers, if it exists and its allocation is interior, achieves conditional golden rule optimality.*

This theorem is an analogue of the first welfare theorem. As mentioned above, it has been well-known that a stationary monetary equilibrium allocation (with sequentially complete market) is CPO. However, since the definition of a stationary equilibrium does not any special role

[10]Further, for a stochastic overlapping generations economy, in which preferences are additively separable and H is singleton, Ohtaki (2011) provides sufficient conditions for the existence and uniqueness of a stationary monetary equilibrium.

to an initial date, so that the equilibrium is implicitly considered on $(-\infty, +\infty)$, the equilibrium allocation is actually CGRO.

Note that there might exist a CPO allocation that cannot be implemented as a stationary monetary equilibrium, because stationary monetary equilibrium always achieves CGRO. Therefore, the second welfare theorem might not hold in stochastic OLG models if we adopt CPO as an optimality criterion. However, the following theorem shows that the second welfare theorem holds when we adopt CGRO, not CPO, as an optimality criterion.

Theorem 4 *Every interior conditionally golden rule optimal allocation, if any, can be achieved by a stationary monetary equilibrium with transfers.*

This theorem is an analogue of the second welfare theorem. As noted above, we might not be able to obtain this theorem when we adopt CPO instead of CGRO as an optimality criterion. Theorem 4 has an important implication, i.e., any interior CGRO allocation can be implemented as a stationary monetary equilibrium under an appropriate social security system.[11]

7 Conclusion

In a stochastic overlapping generations model, there exist two familiar criteria of optimality: conditional Pareto optimality (CPO) and conditional golden rule optimality (CGRO). This study has examined the relationship between these two criteria. Contrary to a familiar intuition, this study presents an example such that the set of CPO allocations become a proper subset of CGRO allocations. It has been shown that such an anomalous situation is avoidable by assuming strictly convex preferences.

The study has also examined how these two concepts are distinguished in their characterizations under such preferences. We have shown that both these criteria are characterized by conditions on the dominant root of the agents' common matrix of marginal rates of substitution. While CPO allows the dominant root of the matrix to be less than unity, CGRO requires that it is exactly equal to unity, because CPO copes with an initial condition, whereas CGRO does not. Thus, on the basis of their characteristics, we might say that CGRO is stronger than CPO as a criterion of optimality.

[11]Demange and Laroque (1999) also provided welfare theorems similar to ours. However, while their results were given in an economy with identical agents, our results are shown in an economy with heterogenous agents.

It has been known that a stationary monetary equilibrium achieves CPO. By applying our results to welfare on stationary monetary equilibrium, we can conclude that a stationary monetary equilibrium achieves not only CPO but also CGRO. This result can be interpreted as the first welfare theorem. Further, by adopting CGRO rather than CPO as an optimality criterion, this study has presented the second as well as the first welfare theorems in financial economy. These results complement existing results for CPO.

Appendix

Proof of Proposition 1. Suppose the contrary that there exists some CGRO allocation $c \in \mathcal{A}$ that is not CPO. Because c is not CPO, there exists some stationary feasible allocation b that CPO-dominates c, i.e., b satisfies that

$$(\forall (h,s) \in H \times S) \quad \begin{array}{l} b_{0s}^{h2} \geq c_{0s}^{h2}, \\ U^{hs}(b_s^h) \geq U^{hs}(c_s^h) \end{array}$$

with strict inequality somewhere. If b satisfies that

$$(\exists (h,s) \in H \times S) \quad U^{hs}(b_s^h) > U^{hs}(c_s^h),$$

then c is not CGRO and contradicts the hypothesis that c is also. Thus, we assume without loss of generality that

$$(\exists (k,\tau) \in H \times S) \quad b_{0\tau}^{k2} > c_{0\tau}^{k2}.$$

We first claim that $b_j^{i1} \neq c_j^{i1}$ for some $(i,j) \in H \times S$. Suppose the contrary that $b_s^{h1} = c_s^{h1}$ for all $(h,s) \in H \times S$. Because both c and b are stationary feasible allocations, it follows that

$$(\forall s' \in S) \quad \bar{\omega}_{0s'} - \sum_{h \in H} b_{0s'}^{h2} = \sum_{h \in H} b_{s'}^{h1} = \sum_{h \in H} c_{s'}^{h1} = \bar{\omega}_{0s'} - \sum_{h \in H} b_{0s'}^{h2},$$

which implies that $\sum_{h \in H} b_{0s'}^{h2} = \sum_{h \in H} b_{0s'}^{h2}$. However, this contradicts the hypothesis that $b_{0s}^{h2} \geq c_{0s}^{h2}$ for all (h,s) and $b_{0\tau}^{k2} > c_{0\tau}^{k2}$, which implies that $\sum_{h \in H} b_{0s'}^{h2} > \sum_{h \in H} b_{0s'}^{h2}$. Therefore, $b_j^{i1} \neq c_j^{i1}$ for some $(i,j) \in H \times S$.

We then claim that $b_{j'}^{i'2} \neq c_{j'}^{i'2}$ for some $(i',j') \in H \times S$. Suppose the contrary that $b_s^{h2} = c_s^{h2}$ for all $(h,s) \in H \times S$. Because both c and b are stationary feasible allocations, we obtain that, for all $s, s' \in S$,

$$\begin{aligned} & \sum_{h \in H} (c_{0s'}^{h2} - c_{ss'}^{h2}) \\ = \ & \bar{\omega}_{0s'} - \bar{\omega}_{ss'} \\ = \ & \sum_{h \in H} (b_{0s'}^{h2} - b_{ss'}^{h2}), \end{aligned}$$

which implies that

$$(\forall s' \in S) \quad \sum_{h\in H} c_{0s'}^{h2} = \sum_{h\in H} b_{0s'}^{h2},$$

because $b_s^{h2} = c_s^{h2}$ for all $(h,s) \in H \times S$. However, this contradicts the hypothesis that $b_{0s}^{h2} \geq c_{0s}^{h2}$ for all (h,s) and $b_{0\tau}^{k2} > c_{0\tau}^{k2}$. Therefore, $b_{j'}^{i'2} \neq c_{j'}^{i'2}$ for some $(i',j') \in H \times S$.

Now, let $d := \alpha c + (1-\alpha)b$ for some $\alpha \in (0,1)$. It is a stationary feasible allocation, because

$$\begin{aligned}
& \sum_{h\in H} d_{s'}^{h1} + \sum_{h\in H} d_{ss'}^{h2} \\
= \ & \alpha \sum_{h\in H} (c_{s'}^{h1} + c_{ss'}^{h2}) + (1-\alpha) \sum_{h\in H} (b_{s'}^{h1} + b_{ss'}^{h2}) \\
= \ & \bar{\omega}_{ss'}
\end{aligned}$$

for all $(s,s') \in S_0 \times S$. It also follows from quasi-concavity of utility functions that

$$(\forall (h,s) \in H \times S) \quad U^{hs}(d_s^h) \geq U^{hs}(c_s^h)$$

with strict inequality at either (i,j) or (i',j'), where the strict inequality follows from the restrictions (a) and (b) on lifetime utility functions, the facts that $b_j^{i1} \neq c_j^{i1}$ and $b_{j'}^{i'2} \neq c_{j'}^{i'2}$, and the hypothesis that $U^{hs}(b_s^h) \geq U^{hs}(c_s^h)$ for all $(h,s) \in H \times S$. This contradicts the hypothesis that c is a CGRO allocation. Q.E.D.

Proof of Proposition 2. It is obvious that the stationary feasible allocation $x = \{x^1, x^2\}$ defined by $x_{s'}^{h1} = 0$ and $x_{ss'}^{h2} = \bar{\omega}_{ss'}$ for every $(s,s') \in S_0 \times S$ is well-defined and CPO, because any other stationary feasible allocation revises the initial olds' welfare for the worse. It is sufficient to show that this x is not CGRO. However, the existence of a stationary feasible allocation that CGRO-dominates x follows immediately from the given boundary condition that $\lim_{c_s^{h1} \downarrow 0} U_1^{hs}(c_s^{h1}, c_s^{h2}) = \infty$ for all $c_s^{h2} \in \Re_{++}^S$ and all $(h,s) \in H \times S$. Q.E.D.

Proof of Theorem 1. Let c be an interior stationary feasible allocation. It is easy to verify that c is CPO if and only if there exist Pareto weights $\gamma : H \times S \to \Re_{++}$ and $\gamma_0 : H \times S \to \Re_+$ such that

$$c \in \arg\max_{b\in\mathcal{A}} \left(\sum_{(h,s)\in H\times S} \gamma^{hs} U^{hs}(b_s^h) + \sum_{(h,s)\in H\times S} \gamma_0^{hs} b_{0s}^{h2} \right).$$

Define the Lagrangian $\mathcal{L}$ by

$$\mathcal{L} \ = \sum_{(h,s)\in H\times S} \left(\gamma^{hs} U^{hs}(c_s^h) + \gamma_0^{hs} c_{0s}^{h2} \right) - \sum_{(s,s')\in S_0\times S} \lambda_{ss'} \left[\bar{\omega}_{ss'} - \sum_{s'\in S} (c_{s'}^{h1} + c_{ss'}^{h2}) \right],$$

where λ is the Lagrange multipliers for the resource constraint. Note that the objective function is strictly quasi-concave. Therefore, by Arrow and Enthoven (1961), the CPO of c can be completely characterized by the existence of Pareto weights $\gamma : H \times S \to \Re_{++}$ and $\gamma_0 : H \times S \to \Re_+$ and Lagrange multipliers $\lambda : S_0 \times S \to \Re_+$, which satisfy that

$$(\forall (h,s) \in H \times S) \quad \gamma^{hs} U_1^{hs}(c_s^h) = \sum_{s' \in S} \lambda_{s's} + \lambda_{0s}, \tag{1}$$

$$(\forall (h,s) \in H \times S)(\forall s' \in S) \quad \gamma^{hs} U_{s'}^{hs}(c_s^h) = \lambda_{ss'}, \tag{2}$$

$$(\forall (h,s) \in H \times S) \quad \gamma_0^{hs} - \lambda_{0s} \leq 0 \quad \text{with equality if} \quad c_{0s}^{h2} > 0. \tag{3}$$

Therefore, we should claim equivalence between the existence of γ, γ^0, and λ, satisfying Eqs.(1)–(3), and the existence of a positive square matrix M satisfying that

$$(\forall h \in H) \quad M = M^h(c)$$

and $\lambda^f(M) \leq 1$. This study, however, omits the proof of this claim because it is nearly identical to that of Theorem 1 of Aiyagari and Peled (1991). Q.E.D.

Proof of Theorem 2. We first claim that, for each interior stationary feasible allocation c and each $s \in S$, there exists some $h' \in H$ such that $c_{0s}^{h'2} > 0$. To verify this claim, let c be an interior stationary feasible allocation and $s \in S$. Because c is a stationary feasible allocation, we can obtain that, for all $s \in S$,

$$\begin{aligned} & \sum_{h \in H} c_s^{h1} + \sum_{h \in H} c_{\tilde{s}s}^{h2} \\ = \; & \omega_{\tilde{s}s} \\ = \; & \omega_{0s} \\ = \; & \sum_{h \in H} c_s^{h1} + \sum_{h \in H} c_{0s}^{h2}, \end{aligned}$$

which implies that

$$(\forall s \in S) \quad 0 < \sum_{h \in H} c_{\tilde{s}s}^{h2} = \sum_{h \in H} c_{0s}^{h2},$$

where the first inequality follows from the fact that c is interior. Therefore, for each $s \in S$, there exists at least one element $h' \in H$ such that $c_{0s}^{h'2} > 0$ because $c_{0s}^{h2} \geq 0$ for every $(h,s) \in H \times S$.

Let c be an interior stationary feasible allocation. It is easy to verify that c is a CGRO allocation if and only if there exist Pareto weights $\gamma : H \times S \to \Re_{++}$ such that

$$c \in \arg\max_{b \in \mathcal{A}} \sum_{(h,s) \in H \times S} \gamma^{hs} U^{hs}(b_s^h).$$

Define the Lagrangian $\mathcal{L}$ by

$$\mathcal{L} = \sum_{(h,s)\in H\times S} \gamma^{hs} U^{hs}(c_s^h) - \sum_{(s,s')\in S_0\times S} \lambda_{ss'} \left[\bar{\omega}_{ss'} - \sum_{s'\in S} (c_{s'}^{h1} + c_{ss'}^{h2}) \right],$$

where λ is the Lagrange multipliers for the resource constraint. Note that the objective function is strictly quasi-concave. Therefore, by Arrow and Enthoven (1961), the CGRO of c can be completely characterized by the existence of Pareto weights $\gamma : H \times S \to \Re_{++}$ and Lagrange multipliers $\lambda : S_0 \times S \to \Re_+$ which satisfy that

$$(\forall (h,s) \in H \times S) \quad \gamma^{hs} U_1^{hs}(c_s^h) = \sum_{s'\in S} \lambda_{s's} + \lambda_{0s}, \tag{4}$$

$$(\forall (h,s) \in H \times S)(\forall s' \in S) \quad \gamma^{hs} U_{s'}^{hs}(c_s^h) = \lambda_{ss'}, \tag{5}$$

$$(\forall (h,s) \in H \times S) \quad -\lambda_{0s} \le 0 \quad \text{with equality if} \quad c_{0s}^{h2} > 0. \tag{6}$$

Note that, by the previous claim, we can treat λ_{0s} as zero for each $s \in S$, because, for each $s \in S$, there exists $h' \in H$ such that $c_{0s}^{h'2} > 0$ and λ_{0s} is independent of index h. Therefore, we can ignore Eq.(6) and remove λ_{0s} from Eq.(4).

We should now claim the equivalence between the existence of γ and λ satisfying Eqs.(4) and (5) with $\lambda_{0s} = 0$ and the existence of positive square matrix M satisfying

$$(\forall h \in H) \quad M = M^h(c)$$

and $\lambda^f(M) = 1$. Assume the existence of γ and λ satisfying Eqs.(4) and (5) with $\lambda_{0s} = 0$ to show the existence of positive square matrix M satisfying $\lambda^f(M) = 1$. Note that, by strict monotonicity of U^{hs}, $m_{ss'}^h(c)$ is positive for all h and all $s, s' \in S$. We can then obtain from Eqs.(4) and (5) with $\lambda_{0s} = 0$ that

$$(\forall h \in H)(\forall s, s' \in S) \quad m_{ss'}^h(c) = \frac{\lambda_{ss'}}{\sum_{\tau\in S} \lambda_{\tau s}},$$

so that we can ignore the superscript h (and c) of $m_{ss'}^h(c)$, i.e., for all $s, s' \in S$, there exists some positive number $m_{ss'}$ such that $m_{ss'} = m_{ss'}^h(c)$ for all $h \in H$. Then, it follows that

$$(\forall s, s' \in S) \quad \lambda_{ss'} = \sum_{\tau\in S} \lambda_{\tau s} m_{ss'}.$$

Summing this equation over $s \in S$, we have

$$(\forall s, s' \in S) \quad \alpha = \alpha M,$$

where $\alpha_s := \sum_{\tau\in S} \lambda_{\tau s}$ and $M := [m_{ss'}]_{s,s'\in S} = M^h(c)$ for all $h \in H$. Note that M is an $S \times S$ matrix with positive coefficients. Therefore, it follows from the Perron-Frobenius theorem that $\lambda^f(M) = 1$.

Assume now that the existence of positive square matrix M satisfying

$$(\forall h \in H) \quad M = M^h(c)$$

and $\lambda^f(M) = 1$. Because M is an $S \times S$ matrix with positive coefficients, we can pick up the row eigenvector $\alpha \gg 0$ of M. Note that it satisfies that $\alpha \cdot (I - M) = 0$, where I is the $S \times S$ identity matrix. For all $h \in H$ and all $s, s' \in S$, define γ_{hs} and $\lambda_{ss'}$ by

$$\begin{aligned}
\gamma^{hs} &:= \frac{\alpha_s}{U_1^{hs}(c_s^h)}, \\
\lambda_{ss'} &:= \gamma^{hs} U_{s'}^{hs}(c_s^h).
\end{aligned}$$

By their definitions, we can obtain that

$$(\forall s, s' \in S) \quad \lambda_{ss'} = \alpha_s \frac{U_{s'}^{hs}(c_s^h)}{U_1^{hs}(c_s^h)} = \alpha_s m_{ss'},$$

so that $\alpha_{s'} = \sum_{s \in S} \lambda_{ss'}$ for all $s' \in S$. It is now easy to verify that γ and λ satisfies Eqs.(4) and (5) with $\lambda_{0s} = 0$. This completes the proof. Q.E.D.

Proof of Theorem 3. By the sequential budget constraints of an agent, we can obtain the agent's lifetime budget constraint such that: for all $(h, s) \in H \times S$,

$$d_s^{h1} + \sum_{s' \in S} d_{ss'}^{h2} \pi_{ss'} \leq \omega_s^{h1} - \tau_s^{h1} + \sum_{s' \in S} (\omega_{ss'}^{h2} + \tau_{ss'}^{hs}) \pi_{ss'} + \left(\sum_{s' \in S} \pi_{ss'} q_{s'} - q_s \right) z.$$

By this equation, we can obtain the no arbitrage condition when the money price is positive, i.e., $q = \Pi \cdot q$ for any stationary monetary equilibrium (q, Π, c) with transfers. To show this, we should verify that

$$(\forall s \in S) \quad q_s = \sum_{s' \in S} \pi_{ss'} q_{s'}.$$

Suppose the contrary that $q_s \neq \sum_{s' \in S} \pi_{ss'} q_{s'}$ for some $s \in S$. If $q_s < \sum_{s' \in S} \pi_{ss'} q_{s'}$, then, for all $h \in H$, agent (h, s) will choose ∞ as z and his/her optimization problem has no solution. However, if $q_s > \sum_{s' \in S} \pi_{ss'} q_{s'}$, then, for all $h \in H$, agent (h, s) will choose $-\infty$ as z and his/her optimization problem has no solution. In any cases, we obtain a contradiction, so that $q_s = \sum_{s' \in S} \pi_{ss'} q_{s'}$ for all $s \in S$.

Let (q, Π, c) be a stationary monetary equilibrium (q, Π, c) with $c_s^h \gg 0$ for all $(h, s) \in H \times S$. We have obtained that $\Pi \cdot q = q$. Because q_s is now positive for all $s \in S$, it follows from the Perron-Frobenius theorem that the $S \times S$ matrix Π with positive coefficients has the dominant root equal to unity. This completes the proof of Theorem 3. Q.E.D.

Proof of Theorem 4. Let $c = \{c^1, c^2\}$ be an interior CGRO allocation. Because it is CGRO, there exists some positive matrix $\Pi = [\pi_{ss'}]_{s,s' \in S}$ such that $\Pi = M^h(c)$ for all $h \in H$ and its dominant root is equal to one, i.e., $\lambda^f := \lambda^f(\Pi) = 1$. By the Perron-Frobenius theorem, there exists a unique $q \in \Re^S_{++}$ (up to normalization) such that $\Pi \cdot q = \lambda^f q$. Choose the Euclidean norm of q to be small enough and take any $m = \{m^h_s\}_{(h,s) \in H \times S} \in \Re^{H \times S}_{++}$, any $\tau = \{\tau^{h1}, \tau^{h2}\}_{h \in H}$, and any $\theta = [\theta^h_{s,s'}]_{s,s' \in S, h \in H}$ to satisfy (a) the budget constraint in the first period of each agent $(h,s) \in H \times S$:

$$c^{h1}_s = \omega^{h1}_s - \tau^{h1}_s - q_s m^h_s - \sum_{s' \in S} \pi_{ss'} \theta^h_{ss'}; \tag{7}$$

(b) $\omega^{h1}_s > \tau^{h1}_s$ and $\tau^{h2}_{ss'} > \omega^{h2}_{ss'}$ for $h \in H$ and $s, s' \in S$; (c) $\sum_{h \in H} m^h_s = 1$ and $\sum_{h \in H} \theta^h_s = 0$ for $s \in S$; and (d) $\sum_{h \in H} \tau^{h1}_s = \sum_{h \in H} \tau^{h2}_{s's}$ for all $(s', s) \in S_0 \times S$.[12] By their constructions, the first order conditions of all agents' optimization problems at the stationary monetary equilibrium are satisfied. Q.E.D.

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[12] Let $m^h_s := 1/|H|$, $\tau^{h1}_s := \omega^{h1}_s - c^{h1}_s - q_s/|H|$, and $\theta^h_{ss'} := 0$ for all $h \in H$ and all $s, s' \in S$. By choosing the Euclidean norm of q to be sufficiently small, m, τ^1, and θ can satisfy Eq.(7) and conditions (a)–(d) with some $\{\tau^{h2}\}_{h \in H}$ such that $\tau^{h2}_{ss'} > -\omega^{h2}_{ss'}$ and $\sum_{h \in H} \tau^{h1}_{s'} = \sum_{h \in H} \tau^{h2}_{ss'}$ for $(s, s') \in S_0 \times S$.

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