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TRACTABLE GRAPHICAL DEVICE FOR ANALYZING STATIONARY SOLG ECONOMIES

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Abstract

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Tractable Graphical Device for Analyzing Stationary SOLG Economies

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Keywords: Box diagram; Stochastic overlapping generations model.

JEL Classification Numbers: D50; D62; D81; E40.

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1 Introduction

This paper considers the stochastic overlapping generations (SOLG) economy. To our best knowledge, it has not been given any graphical devices yet, whereas it is one of the most important models in economics.[1] Therefore, this paper develops a new, tractable graphic device for analyzing the SOLG economy. This helps our intuitive understanding on the SOLG economy.

Our graphical device is similar to the Edgeworth box diagram in the sense that the set of all stationary feasible allocations is depicted in an appropriate box. Adding to indifference curves and the budget lines to the box diagram, we can draw the set of conditionally Pareto optimal allocations and can find stationary equilibrium with a valued outside asset.[2]

Our box diagram has great possibility of applications and extensions. It will be able to give us new insights to lots of economic phenomena. As an example of applications, this paper reconsiders the stochastic bubbles studied per Weil [20]. In his setting, the box diagram is drawn as a square. Stochastic bubbles are defined by a special case of stationary sunspot equilibrium. By using the box diagram, we can find *visually* a necessary and sufficient condition for the existence of stochastic bubbles.

The construction of this paper is as follows. Section 2 develops the graphical device: In Subsection 2.1, we present a detail of the economy considered in this paper. Subsection 2.2 introduces the box that depicts the set of allocations, and adds indifference curves in the box diagram. Subsection 2.3 draws CPO and CGRO allocations in the box diagram. Subsection 2.4 finds a stationary monetary equilibrium in the box. Here, we also demonstrate welfare theorems by using the box diagram. Subsection 2.5 presents a numerical example. Section 3 provides an application of the box diagram to the issue on the stochastic bubbles.

2 The Box Diagram

2.1 The Economy

This paper considers a stationary, two-state, one-good pure-endowment stochastic overlapping generations model with two-period-lived identical agents. Time is discrete and runs from $t = 0$ to infinity. Uncertainty is modeled by a stationary, two-state Markov process with its

[1] In fact, the SOLG economy is applied to a wide range of economics: social security designs [6, 7], financial mechanism designs [11], monetary theory [21], business cycle theory [17], and banking theory [8] are such examples.

[2] Lots of theoretical studies on SOLG models exist: [1, 5, 6, 7, 14, 16, 18, 19] for optimality of allocations; and [6, 9, 13, 14, 15] for existence of stationary monetary equilibrium, for example.

state space $S := \{\alpha, \beta\}$. The initial state $s_0 \in S$ is treated as given. For any $t \geq 1$, we denote by s_t the state realized in period t, which we simply call *period t state.*

The total endowment in period $t \geq 1$ is denoted by $\bar{\omega}_t \in \Re_{++}$. For any period $t \geq 1$, it is assumed that $\bar{\omega}_t$ depends only on the state s_t realized in that period, i.e., the total endowment in period t can be rewritten as $\bar{\omega}_t = \bar{\omega}_{s_t}$. After the realization of the state in each period, one new agent is born and lives for two periods. A contingent consumption stream of the agent born at period $t \geq 1$ is denoted by $c_t = (c_t^1, \{c_t^2(s_{t+1})\}_{s_{t+1} \in S}) \in \Re_+ \times \Re_+^S$. Agent $t \geq 1$ evaluates the contingent consumption streams c_t by a lifetime utility function $U^t : \Re_+ \times \Re_+^S \to \Re$. We assume that agent t's preference U^t depends on s_t rather than t itself, so that we rewrite agent t's preference as U^{s_t}. It is assumed that $U^s : \Re_+ \times \Re_+^S \to \Re$ is increasing, quasi-concave, and continuously differentiable for each $s \in S$. In addition, a one-period lived agent, called the *initial old*, is born after the realization of period 1 state $s_1 \in S$ and is assumed to rank her consumption streams $c_0^2(s_1) \in \Re_+$ according to a utility function $u_0(c_0^2(s_1)) := c_0^2(s_1)$.

Let $S_0 := \{0\} \cup S$. To provide a tractable graphical device, we concentrate our attention on not "all" feasible allocations but "stationary" feasible allocations. A *stationary feasible allocation* is a family $c = (c^1, c^2)$ of functions $c^1 : S \to \Re_+$ and $c^2 : S_0 \times S \to \Re_+$ such that

$$(\forall (s, s') \in S_0 \times S) \qquad c_{s'}^1 + c_{ss'}^2 = \bar{\omega}_{s'}, \tag{1}$$

where $c_{0s_1}^2$ is the consumption of the initial old at period 1 state s_1 and $(c_{s_t}^1, (c_{s_t s'}^2)_{s' \in S})$ is the contingent consumption stream of the agent born at s_t in period t. A stationary feasible allocation (c^1, c^2) is *interior* if $c_{s'}^1 > 0$ and $c_{ss'}^2 > 0$ for all $s, s' \in S$.

2.2 The Box and Indifference Curves

By Eq.(1), one can easily observe that every stationary feasible allocation (c^1, c^2) satisfies that $c_{0s'}^2 = \bar{\omega}_{s'} - c_{s'}^1 = c_{ss'}^2$ for each $s, s' \in S$. This observation says that an agent's consumption in the second period of her lifetime depends only on the state realized in that period, not on the state at which she is born. Henceforth, we often identify a stationary feasible allocation (c^1, c^2) with $(x^1, x^2) \in \Re_+^S \times \Re_+^S$ satisfying that $(x_{s'}^1, x_{s'}^2) = (c_{s'}^1, c_{0s'}^2)$ for all $s' \in S$. Thus, the condition (1) can degenerate into the system of at most two equations:

$$x_\alpha^1 + x_\alpha^2 = \bar{\omega}_\alpha \quad \text{and} \quad x_\beta^1 + x_\beta^2 = \bar{\omega}_\beta.$$

Note that these equations imply that, for any stationary feasible allocation, there is one-to-one relation between the first- and the second-period consumption vectors. Therefore, we can depict

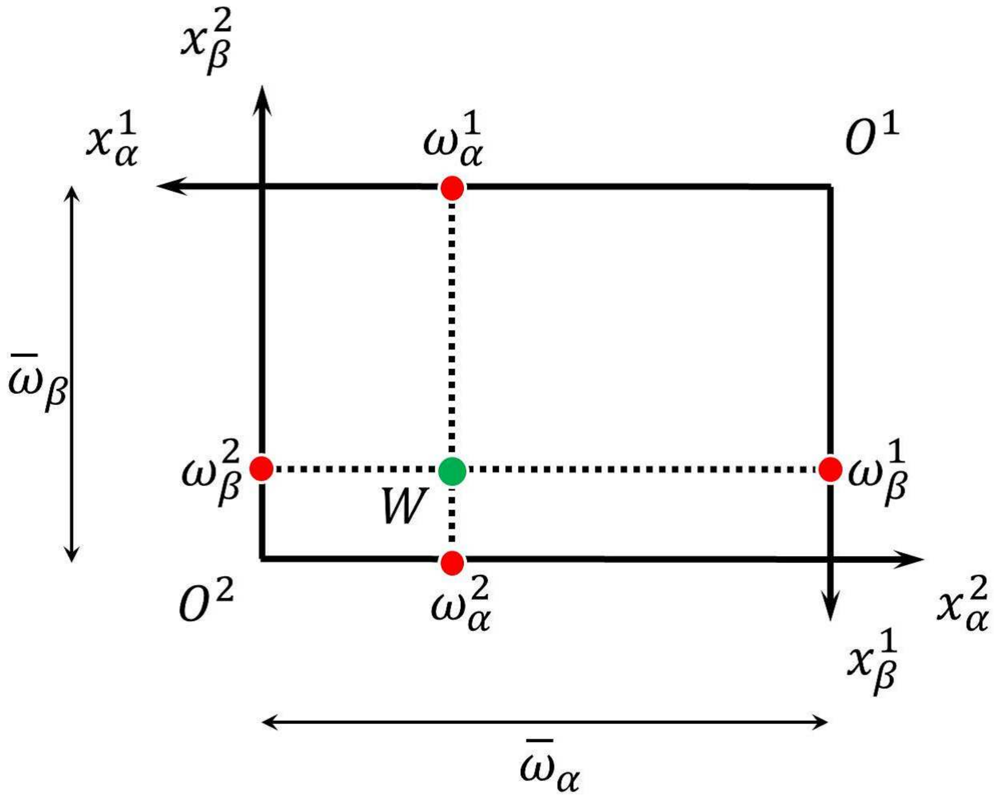


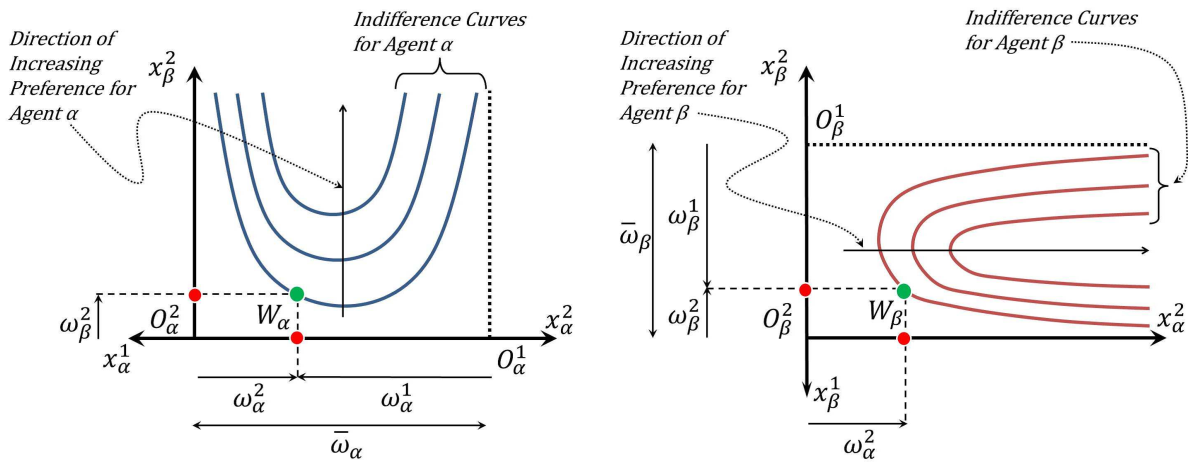
Figure 1: Range of Stationary Feasible Allocations

the range of stationary feasible allocations in the at most two-dimensional Euclidean space as shown in Figure 1. In the box of this figure, the vertical axis measures the total amount of the commodity β and the horizontal axis does that of the commodity α. The width and the height of the box are $\bar{\omega}_\alpha$, the total endowment of the commodity α, and $\bar{\omega}_\beta$, the total endowment of the commodity β. In the box, the southwest corner is the origin to measure the second-period contingent consumption plan, O^2, and the northeast corner is the origin to measure the first-period consumption contingent upon the realization of states, O^1. At the point W in the box, for example, agent born at state $s_t = s \in S$ consumes ω_s^1 and $(\omega_\alpha^2, \omega_\beta^2)$ in the first- and the second-period, respectively. Similarly, the initial old born at $s_1 = s$ consumed ω_s^2 at W.

We now add indifference curves to the box. Let $X_s := \{(x_\alpha^2, x_\beta^2) \in \Re_+^2 : x_s^2 \leq \bar{\omega}_s\}$ for $s \in S$. Since our attention is concentrated on the space of stationary feasible allocations, which is depicted by Figure 1, we restrict the preference of agent $s_t \in S$ as follows:

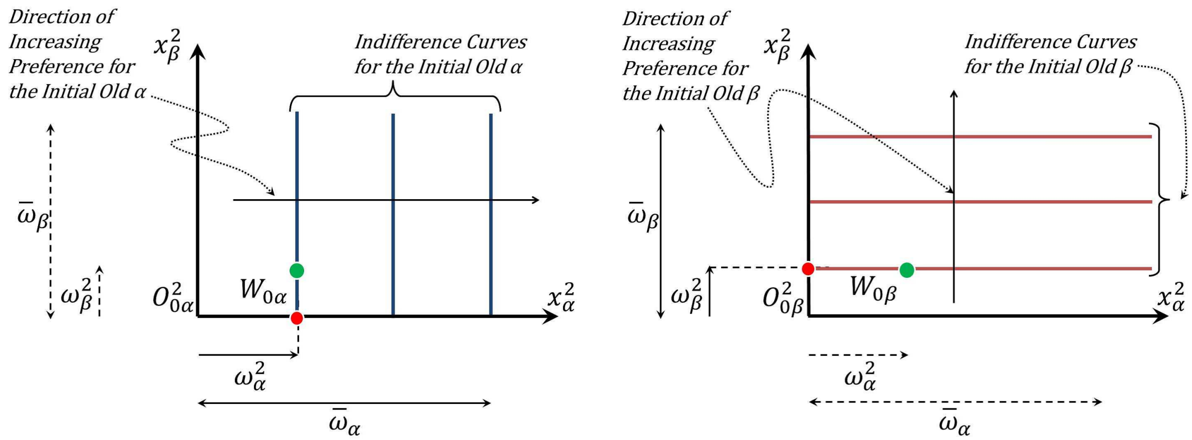
$$(\forall (x_\alpha^2, x_\beta^2) \in X_s) \quad \tilde{U}^s(x_\alpha^2, x_\beta^2) = U^s(\bar{\omega}_s - x_s^2, x_\alpha^2, x_\beta^2).$$

Since lifetime utility functions are quasi-concave, the upper contour set of $\tilde{U}^s$ with the utility level a, $\{x \in X_s : \tilde{U}^s(x) \geq a\}$, is a convex set for each $s \in S$. By totally differentiating the



(a) (b)

Figure 2: Indifference Curves: (a) For newly born agent α, (b) For β



(a) (b)

Figure 3: Indifference Curves: (a) For the initial old α, (b) For β

restricted lifetime utility function, we can obtain the slope of an indifference curve at $x^2 \in X_s$:

$$\frac{\Delta x_\beta^2}{\Delta x_\alpha^2} = \frac{U_1^\alpha(\bar{\omega}_\alpha - x_\alpha^2, x_\alpha^2, x_\beta^2) - U_\alpha^\alpha(\bar{\omega}_\alpha - x_\alpha^2, x_\alpha^2, x_\beta^2)}{U_\beta^\alpha(\bar{\omega}_\alpha - x_\alpha^2, x_\alpha^2, x_\beta^2)}$$

for the agent born at α and

$$\frac{\Delta x_\beta^2}{\Delta x_\alpha^2} = \frac{U_\alpha^\beta(\bar{\omega}_\beta - x_\beta^2, x_\alpha^2, x_\beta^2)}{U_1^\beta(\bar{\omega}_\beta - x_\beta^2, x_\alpha^2, x_\beta^2) - U_\beta^\beta(\bar{\omega}_\beta - x_\beta^2, x_\alpha^2, x_\beta^2)}$$

for the agent born at β, where $U_1^s = \partial U^s / \partial c_s^1$ and $U_{s'}^s = \partial U^s / \partial c_{ss'}^2$ for $s, s' \in S$. Therefore, by imposing the boundary conditions such that $\lim_{c_s^1 \downarrow 0} U_1^s(c_s^1, c_{s\alpha}^2, c_{s\beta}^2) = \infty$ and $\lim_{c_{ss}^2 \downarrow 0} U_s^s(x, c_{s\alpha}^2, c_{s\beta}^2) = \infty$ on lifetime utility functions in addition to (strict) quasi-concavity, we can observe that each of indifference curves of agents born in and after period 1 is depicted by the U-shaped curve

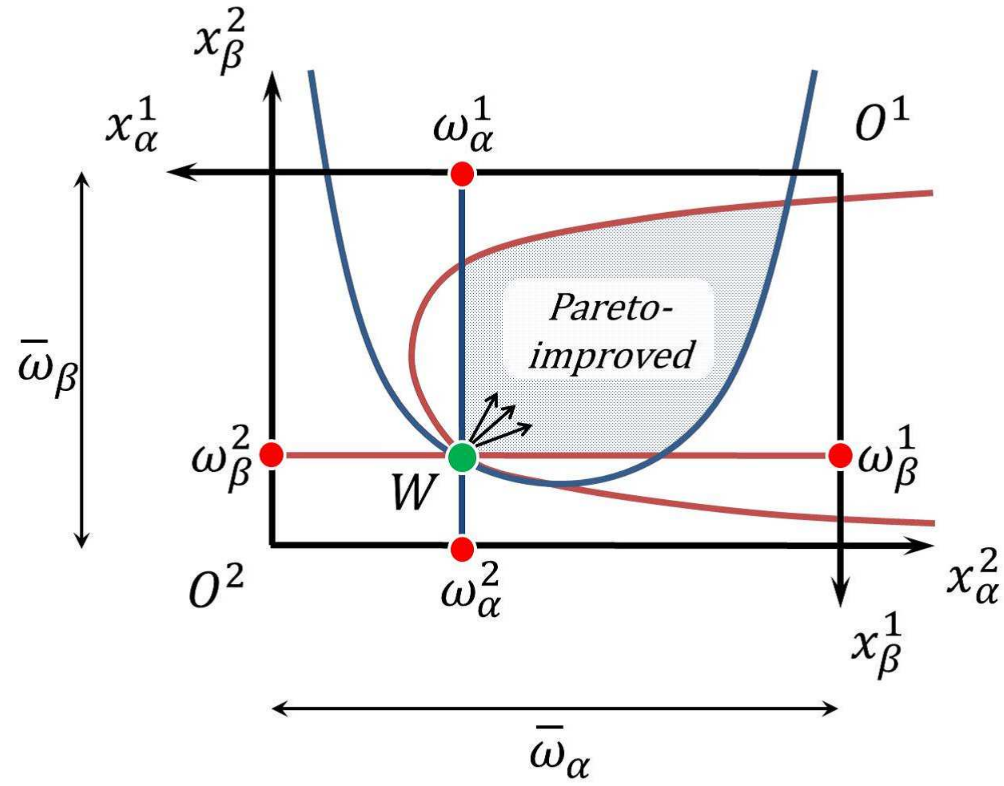


Figure 4: The Box Diagram

as in Figure 2.[3] As shown in this figure, one can easily verify that the lifetime utility of agent $s \in S$ increases when the consumption of commodity $s' \neq s$ increases. On the other hand, we can draw the initial olds' indifference curves as the straight lines as shown in Figure 3. This is because they are interested in only their consumption at the Markov state which they observed.

Figure 4 adds indifference curves to the box introduced above, i.e., the figure summarizes Figures 1, 2, and 3. This box diagram is our graphical device for analyzing SOLG economy.[4]

2.3 Conditional Optimality

This paper introduces two optimality criteria: conditional Pareto optimality (CPO) and conditional golden rule optimality (CGRO). This subsection depicts the sets of CPO and CGRO allocations in our box diagram. A stationary feasible allocation (c^1, c^2) is *conditionally Pareto optimal* if there is no other stationary feasible allocation (b^1, b^2) such that

$$(\forall s' \in S) \qquad b^2_{0s'} \geq c^2_{0s'},$$

$$(\forall s, s' \in S) \qquad U^s(b^1_s, b^2_{s\alpha}, b^2_{s\beta}) \geq U^s(c^1_s, c^2_{s\alpha}, c^2_{s\beta})$$

[3]If the boundary conditions are not satisfied, indifference sets will not necessarily be U-shaped curves. Quasi-linear utility functions are such examples.

[4]The author welcomes future generations who call our box diagram the *Ohtaki box*.

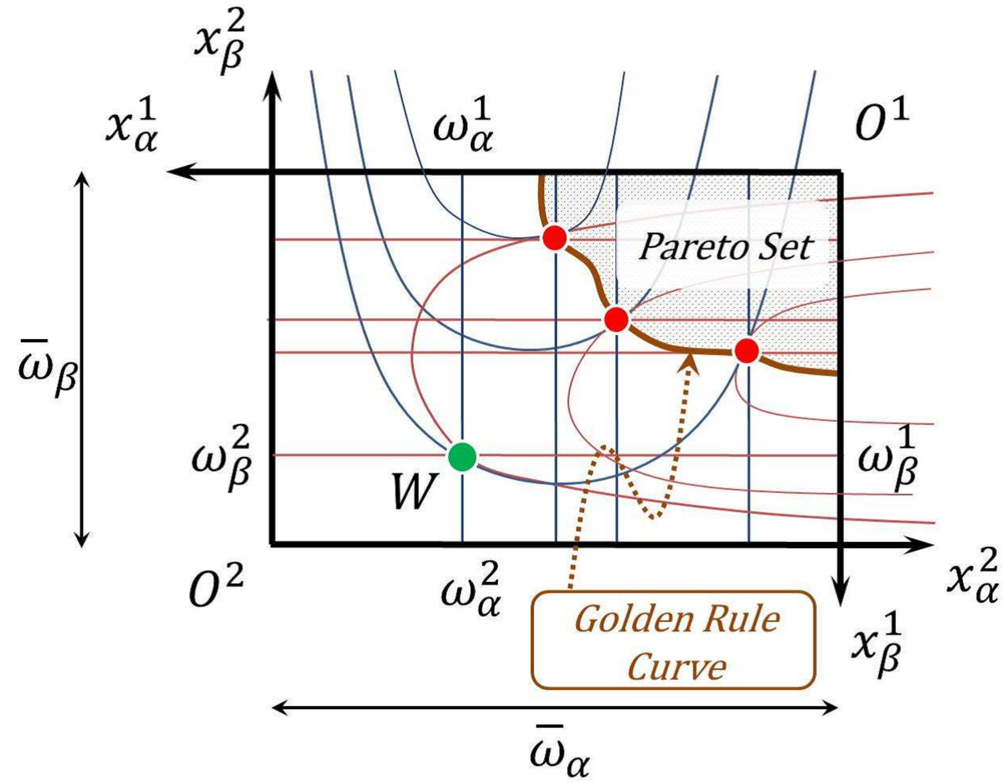


Figure 5: Conditional Pareto Optimality

with strict inequality somewhere. On the other hand, a stationary feasible allocation (c^1, c^2) is *conditionally golden rule optimal* if there is no other stationary feasible allocation (b^1, b^2) such that

$$(\forall s, s' \in S) \quad U^s(b^1_s, b^2_{s\alpha}, b^2_{s\beta}) \geq U^s(c^1_s, c^2_{s\alpha}, c^2_{s\beta})$$

with strict inequality somewhere.[5] In Figure 4, one can easily find that the given stationary feasible allocation W is neither CPO nor CGRO. In fact, the shaded area in Figure 4 depicts the set of stationary feasible allocations, which improve the stationary feasible allocation W in the sense of CPO.

We now turn to depict CPO and CGRO allocations in the box diagram. The set of CPO and CGRO allocations can be drawn as the *Pareto set* and the *golden rule curve* in Figure 5, respectively. One should notice that, by the definition of CGRO, each CGRO allocation can be drawn as the point, at which the indifference curves of both agents α and β are tangent to each other. This is an analogue of how to draw Pareto optimal allocations in the Edgeworth box. Also notice that CPO of shaded area in Figure 5 follows from the fact that, in the area, one

[5]CPO copes with welfare on the initial olds, while CGRO does not. The relationship between these two criteria was studied in detail by [16].

must make at least one agent or at least one initial old worth off to improve welfare of either of them.[6]

2.4 Stationary Monetary Equilibrium and Welfare Theorems

In the previous subsection, we have studied optimal allocations. As a device of implementing such allocations, this subsection considers stationary monetary equilibrium. Before introducing such an equilibrium, we suppose the private ownerships on the endowment. For any $s = \alpha, \beta$, let $(\omega_s^1, \omega_\alpha^2, \omega_\beta^2) \in \Re_{++}^3$ be the initial endowment stream of the agent born at s. We denote by ω the initial allocation associated with the initial endowments. Of course, it must satisfy that $\omega_s^1 + \omega_s^2 = \bar{\omega}_s$ for all $s \in S$. An infinitely-lived outside asset, which yields no dividend and is often called fiat money, is also introduced. We normalize its supply to one. We can then define a stationary monetary equilibrium: A pair (q, c) of real price vector $q \in \Re_+^S$ of money and a stationary feasible allocation $c = (c^1, c^2)$ is a *stationary equilibrium* if there exists some $m \in \Re^S$ such that

- for any $s \in S$, (c_s, m_s) belongs to the set

$$\operatorname*{arg\,max}_{(d_s, z) \in (\Re_+ \times \Re_+^S) \times \Re} \left\{ U^s(d_s) : \begin{array}{l} d_s^1 = \omega_s^1 - q_s z \\ (\forall s' \in S)\, d_{ss'}^2 = \omega_{s'}^2 + q_{s'} z \end{array} \right\},$$

- for all $s \in S$, $m_s = 1$.

It is called a *stationary monetary equilibrium* if $q \gg 0$.

As shown by Aiyagari and Peled [1], a stationary monetary equilibrium exists if the initial allocation is conditionally Pareto suboptimal. In the box diagram, this condition means that the initial allocation ω, which is depicted by W in Figure 6, does not belong to the Pareto set.

By the sequential budget constraints of the agent born at $s \in S$, we can obtain that

$$c_s^1 + c_{ss}^2 = \bar{\omega}_s$$

[6]More precisely, one can verify that a stationary feasible allocation c is (a) CPO iff it holds that

$$0 < \frac{U_\alpha^\beta(c_\beta^1, c_{\beta\alpha}^2, c_{\beta\beta}^2)}{U_1^\beta(c_\beta^1, c_{\beta\alpha}^2, c_{\beta\beta}^2) - U_\beta^\beta(c_\beta^1, c_{\beta\alpha}^2, c_{\beta\beta}^2)} \leq \frac{U_1^\alpha(c_\alpha^1, c_{\alpha\alpha}^2, c_{\alpha\beta}^2) - U_\alpha^\alpha(c_\alpha^1, c_{\alpha\alpha}^2, c_{\alpha\beta}^2)}{U_\beta^\alpha(c_\alpha^1, c_{\alpha\alpha}^2, c_{\alpha\beta}^2)}$$

and (b) CGRO iff it holds that

$$\frac{U_1^\alpha(c_\alpha^1, c_{\alpha\alpha}^2, c_{\alpha\beta}^2) - U_\alpha^\alpha(c_\alpha^1, c_{\alpha\alpha}^2, c_{\alpha\beta}^2)}{U_\beta^\alpha(c_\alpha^1, c_{\alpha\alpha}^2, c_{\alpha\beta}^2)} = \frac{U_\alpha^\beta(c_\beta^1, c_{\beta\alpha}^2, c_{\beta\beta}^2)}{U_1^\beta(c_\beta^1, c_{\beta\alpha}^2, c_{\beta\beta}^2) - U_\beta^\beta(c_\beta^1, c_{\beta\alpha}^2, c_{\beta\beta}^2)}.$$

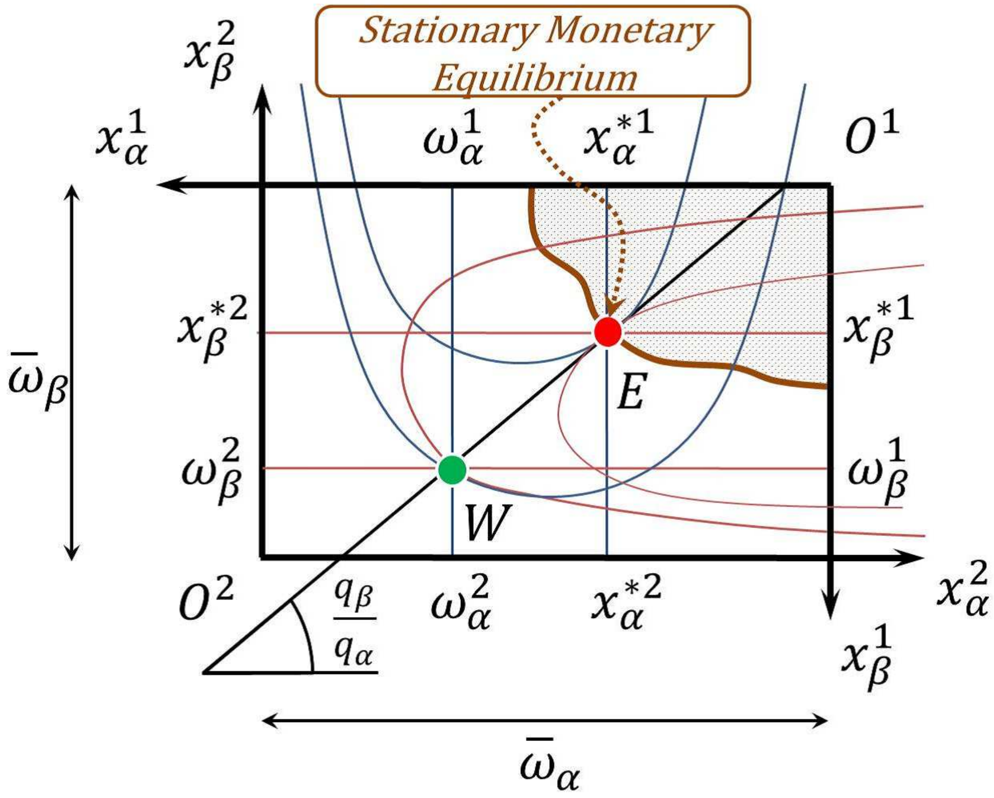


Figure 6: Stationary Monetary Equilibrium and the First Welfare Theorem

and

$$c_{s\beta}^2 = \frac{q_\beta}{q_\alpha} c_{s\alpha}^2 - \frac{q_\beta}{q_\alpha} \omega_\alpha^2 + \omega_\beta^2.$$

The last equation is something like the budget line between two commodities contingent upon states α and β. One of possible "budget lines" is depicted by the line through W with slope q_β/q_α in Figure 6. Each agent born in and after period 1 must choose her consumption stream in this "budget line" to maximize her lifetime utility. Then, a stationary monetary equilibrium allocation can be drawn as the point E in Figure 6. At a stationary monetary equilibrium allocation, the "budget line" must be tangent to indifference curves of both agent α and β.[7] One should note that the point E lies on the golden rule curve, i.e., that a stationary monetary equilibrium achieves CGRO. This can be interpreted as the first welfare theorem in SOLG models with money.

[7] To be more precise, one can obtain that

$$\frac{q_\beta}{q_\alpha} = \frac{U_1^\alpha(c_\alpha^1, c_{\alpha\alpha}^2, c_{\alpha\beta}^2) - U_\alpha^\alpha(c_\alpha^1, c_{\alpha\alpha}^2, c_{\alpha\beta}^2)}{U_\beta^\alpha(c_\alpha^1, c_{\alpha\alpha}^2, c_{\alpha\beta}^2)} = \frac{U_\alpha^\beta(c_\beta^1, c_{\beta\alpha}^2, c_{\beta\beta}^2)}{U_1^\beta(c_\beta^1, c_{\beta\alpha}^2, c_{\beta\beta}^2) - U_\beta^\beta(c_\beta^1, c_{\beta\alpha}^2, c_{\beta\beta}^2)}$$

at any stationary monetary equilibrium (q, c).

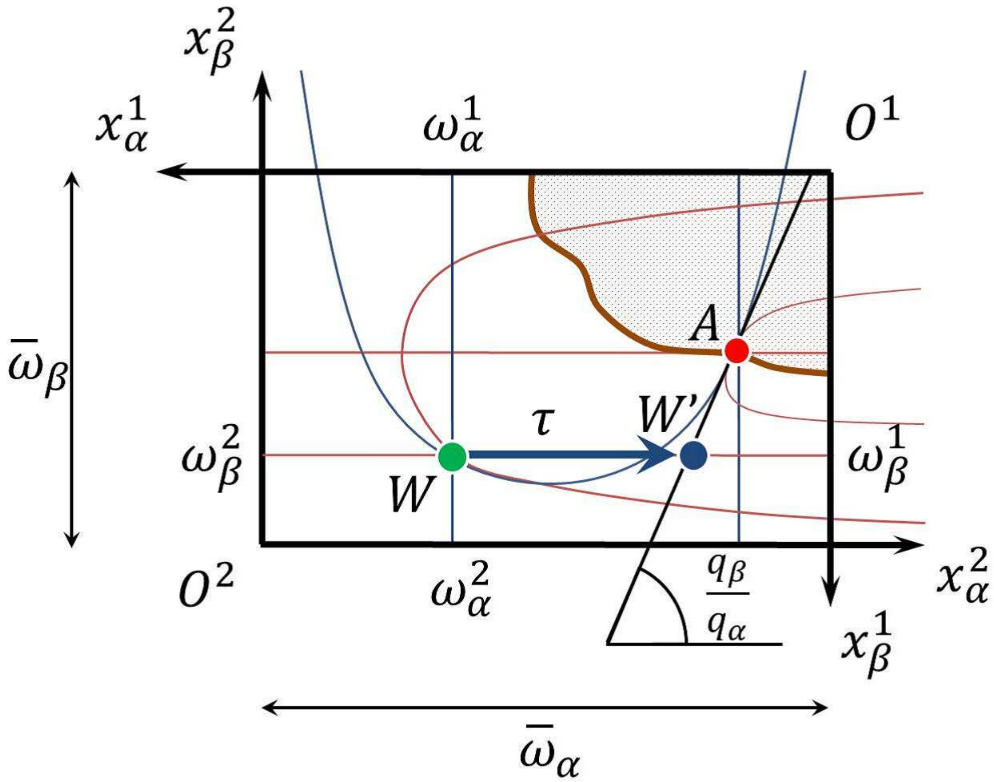


Figure 7: The Second Welfare Theorem

Similar to the first welfare theorem, we can also argue on the second welfare theorem in the box diagram. To do so, we extend the definition of stationary equilibrium by allowing some lump-sum transfer: Given $\tau \in \Re^S$, a pair (q, c) of real price vector of money $q : \Re^S_+$ and a stationary feasible allocation $c = (c^1, c^2)$ is a *stationary equilibrium with transfer* τ if there exists some $m : S \to \Re$ such that

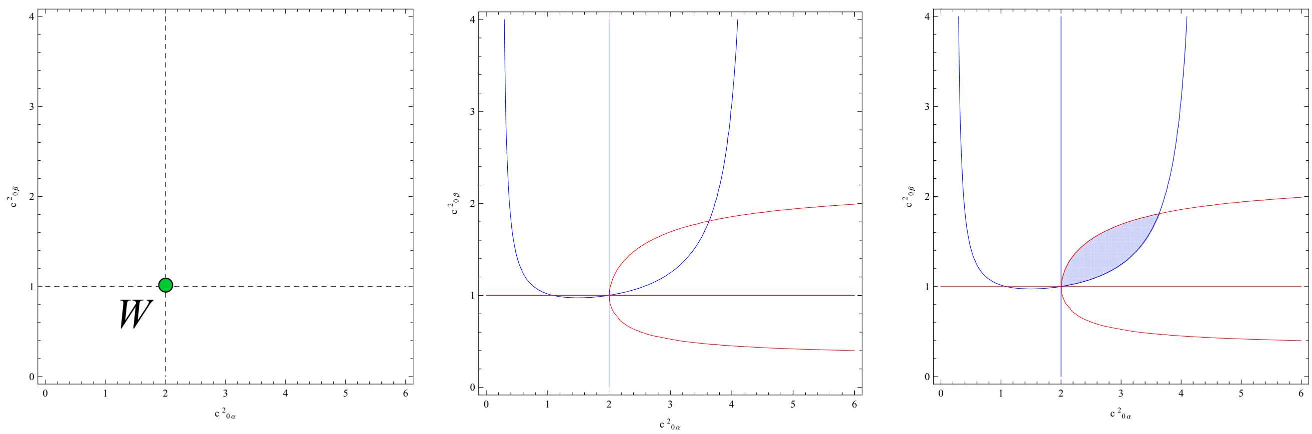
- for any $s \in S$, (c_s, m_s) belongs to the set

$$\operatorname*{arg\,max}_{(d_s,z)\in(\Re_+\times\Re^S_+)\times\Re} \left\{ U^s(d_s) : \begin{array}{l} d^1_s = \omega^1_s - \tau_s - q_s z \\ (\forall s' \in S)\, d^2_{ss'} = \omega^2_{s'} + \tau_{s'} + q_{s'} z \end{array} \right\},$$

- for all $s \in S$, $m_s = 1$.

Suppose now that an authority is willing to implement the CGRO allocation A in Figure 7 as an allocation at stationary monetary equilibrium with some transfer. In such a situation, the authority can adopt the transfer rule such as τ in Figure 7. Then, allocation A can be achieved by a stationary monetary equilibrium with τ.[8]

[8]See also Theorem 4 of [16].



(a) (b) (c)

Figure 8: (a) Range of Stationary Feasible Allocations, (b) Box Diagram, (c) Pareto-improving Allocations

2.5 Numerical Example

In this subsection, we provide an numerical example by specifying the economy.[9] Let the total endowment be $(\bar{\omega}_\alpha, \bar{\omega}_\beta) = (6,4)$ and the private endowment be $(\omega^1_\alpha, \omega^1_\beta, \omega^2_\alpha, \omega^2_\beta) = (4,3,2,1)$. The space of stationary feasible allocations is then depicted by Figure 8 (a).

Preferences are assumed to have the form:

$$U^s(c^1_s, c^2_{s\alpha}, c^2_{s\beta}) := \frac{(c^1_s)^{1-\gamma}}{1-\gamma} + \delta\left(\frac{(c^2_{s\alpha})^{1-\gamma}}{1-\gamma}\pi_{s\alpha} + \frac{(c^2_{s\beta})^{1-\gamma}}{1-\gamma}\pi_{s\beta}\right),$$

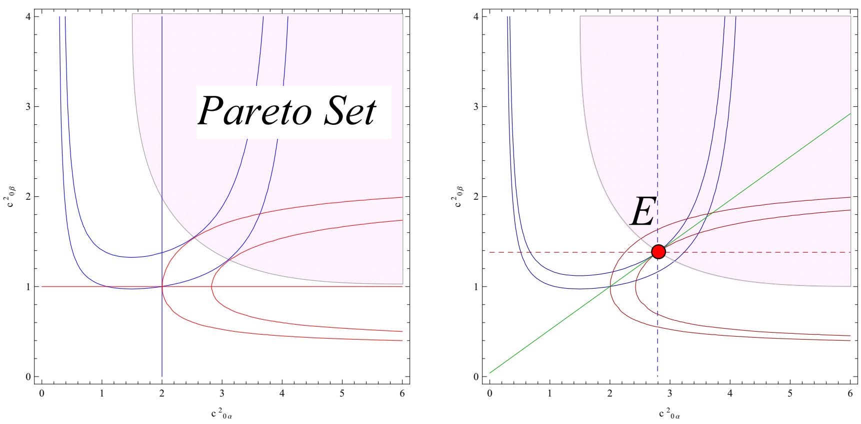
where γ, δ, and $\pi := [\pi_{ss'}]_{s,s'\in S}$ are the index of relative risk aversion, the time preference, and the transition probability matrix, respectively. While the current economy is too simple to properly calibrate the model to match historic prices and quantities as similar to [10], we want to consider a specification of preferences, which is 'roughly consistent' with those of SOLG models in the existing literature which takes a unit of period to be 20–40 years.[10] In this paper, we choose $\gamma = 2$ and $\delta = 0.44$. Moreover, the transition probability matrix is assumed to be given by

$$\begin{bmatrix} \pi_{\alpha\alpha} & \pi_{\alpha\beta} \\ \pi_{\beta\alpha} & \pi_{\beta\beta} \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{bmatrix}.$$

By adding agents' indifference curves through the initial allocation to Figure 8 (a), we can obtain Figure 8 (b). Stationary feasible allocations, each of which improves the initial allocation, are depicted by the shaded area in Figure 8 (c).

[9] Figures in this subsection are drawn by *Mathematica 8.0*.

[10] See, for example, [2].



(a) (b)

Figure 9: (a) Pareto Set, (b) Stationary Monetary Equilibrium

At the current specification, CPO allocations can be depicted as the shaded area in Figure 9 (a). As shown in Subsection 2.3, the Pareto set spreads to the northeast area over the golden rule curve. Moreover, a stationary monetary equilibrium allocation can be numerically calculated as $(c^1_\alpha, c^1_\beta, c^2_{0\alpha}, c^2_{0\beta}) \approx (3.207, 2.619, 2.793, 1.381)$. This is drawn in Figure 9 (b). One should notice that the equilibrium lies on the golden rule curve.

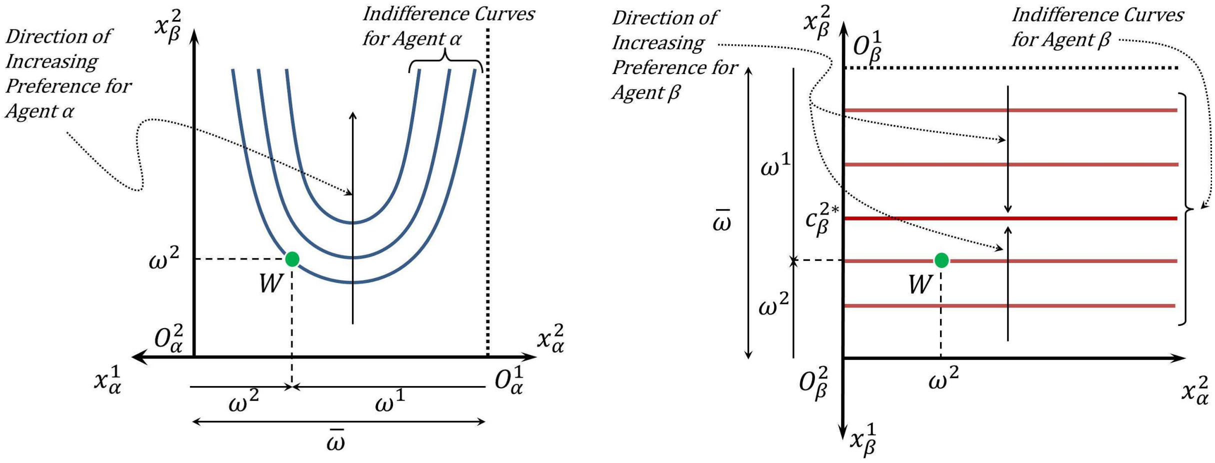
3 An Application

The box diagram has great possibility of applications. We will be able to have new insights to lots of economic phenomena. As an example, this section applies our box diagram to the issue on the stochastic bubbles theoretically studied per Weil [20]. According to his setting, we assume the separability of preferences: there exist some $u_1, u_2 : \Re_+ \to \Re$ and some transition probability matrix $\pi := [\pi_{ss'}]_{s,s' \in S}$ such that

$$U^s(c^1_s, c^2_{s\alpha}, c^2_{s\beta}) = u_1(c^1_s) + u_2(c^2_{s\alpha})\pi_{s\alpha} + u_2(c^2_{s\beta})\pi_{s\beta}$$

for each $s \in S$, where u_1 and u_2 are strictly monotone increasing, strictly concave, and continuously differentiable and satisfy $\lim_{x\downarrow 0} u'_i(x) = \infty$ for $i = 1, 2$. The total and private endowments are given by $\bar{\omega} \in \Re_{++}$ and $(\omega^1, \omega^2) \in \Re^2_{++}$, which are independent of realizations of states. Uncertainty in this setting is often called *extrinsic uncertainty* or *sunspot*, since it has no effect on both endowments and von Neumann-Morgenstern index functions. A stationary equilibrium (q, c) is then called a *stationary sunspot equilibrium* if $q_\alpha \neq q_\beta$.

We now explore a stationary sunspot equilibrium (q, c) such that $q_\alpha > 0$ and $q_\beta = 0$, provided that $\pi_{\alpha s} > 0$ for $s \in S$, $\pi_{\beta\alpha} = 0$, and $\pi_{\beta\beta} = 1$. Such an equilibrium is interpreted as *stochastic*



(a) (b)

Figure 10: Indifference Curves: (a) For agent α, (b) For agent β

bubbles, since the bubbles on money burst with probability $\pi_{\alpha\beta}$. Before exploring a stationary sunspot equilibrium with stochastic bubbles, one should remark the shapes of indifference curves in the box diagram. Now let c_β^{2*} be a unique solution of the optimization problem:

$$\max_{c_\beta^2 \in [0,\bar{\omega}]} u_1(\bar{\omega} - c_\beta^2) + u_2(c_\beta^2).$$

Since belief of the agent born at state α is such that $\pi_{\alpha s} > 0$ for each $s \in S$, she has U-shaped indifference curves as in Figure 10 (a). On the other hand, indifference curves of the agent born at state β are depicted by straight lines as in Figure 10 (b), since her belief is such that $\pi_{\beta\alpha} = 0$ and thus she is no longer interested in the second-period consumption at state α.

Under extrinsic uncertainty, the space of stationary feasible allocations is drawn by a square as in Figure 11. Since we explore a stationary sunspot equilibrium with $q_\alpha > 0$ and $q_\beta = 0$, the "budget lines" which agents face at the equilibrium are drawn by the straight line parallel to the horizontal axis, which passes through the point W. Adding agents' indifference curves to the box, we can find a stationary equilibrium with stochastic bubbles as the point E in Figure 11.

One can visually find that a sunspot equilibrium with stochastic bubbles exists when and only when the slope of agent α's indifference curve at the initial endowment is negative. To be more precise, a necessary and sufficient condition for the existence of sunspot equilibrium with stochastic bubbles is that

$$\frac{\Delta c_\beta^2}{\Delta c_{0\alpha}^2} = \frac{U_1^\alpha(\omega^1, \omega^2, \omega^2) - U_\alpha^\alpha(\omega^1, \omega^2, \omega^2)}{U_\beta^\alpha(\omega^1, \omega^2, \omega^2)} = \frac{u_1'(\omega^1) - u_2'(\omega^2)\pi_{\alpha\alpha}}{u_2'(\omega^2)\pi_{\alpha\beta}}$$

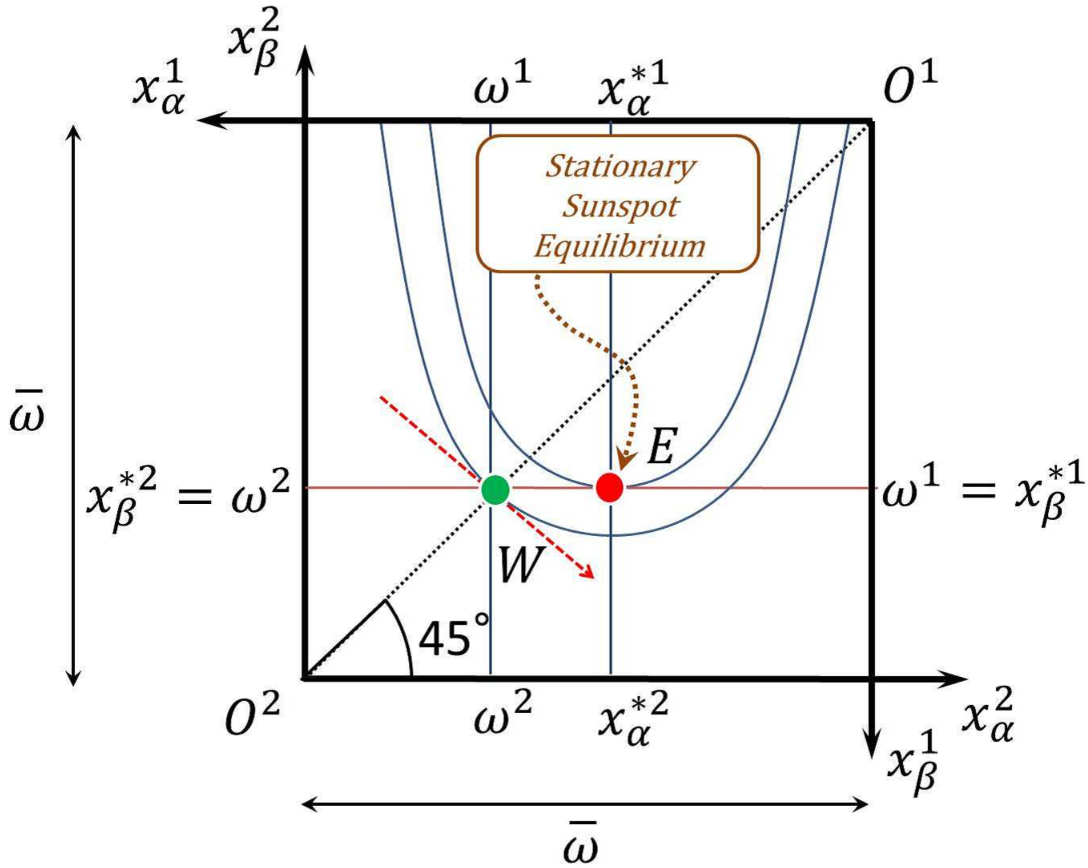


Figure 11: Stochastic Bubbles

is negative. This condition is equivalent to the necessary and sufficient condition provided by [20, Proposition 1]: $u_1'(\omega^1)/u_2'(\omega^2) < \pi_{\alpha\alpha}$. Also note that our condition ensures that $c_\beta^{2*} > \omega^2$.

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