TCER Working Paper Series

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June 2012

Working Paper E-47
http://tcer.or.jp/wp/pdf/e47.pdf

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Abstract

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Existence and Uniqueness of Stationary Monetary Equilibrium: A Simple Proof

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First Draft: April 1, 2012
Current Draft: June 15, 2012

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Keywords: Stationary monetary equilibrium; Conditional Pareto otimality; Stochastic overlapping generations model.

JEL Classification Numbers: E40.

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1 Introduction

Since the Lucas's (1972) pioneering work, a vast literature in economics and finance is based on the stochastic overlapping generations (SOLG) model, including studies of social security, asset pricing, the business cycle, the microeconomic foundation of banking, and the foundation of monetary theory. One of the most important properties of the SOLG model is that the competitive equilibria might be inefficient even when markets operate perfectly, as in the Arrow-Debreu abstraction. It is understood that this sort of inefficiency is caused by the lack of market clearing at infinity. Then, *money* is known for its financial intermediation role to remedy this type of inefficiency. Therefore, the stationary *monetary* equilibrium, wherein money has positive value, is of interest among the existing literature.

While the SOLG model can be studied at various level of generality, many of its basic properties and insights can be derived within the simple, but rather canonical, class consisting of one good, finite Markov states, identical agents living for two periods, and having the opportunity to trade money which is a single infinitely-lived outside asset, yielding no dividend, in constant supply. For such a class, Manuelli (1990) and Magill and Quinzii (2003) provided distinct sufficient conditions for the existence of stationary monetary equilibrium.[1] However, the Manuelli's condition is hard to interpret, whereas the Magill and Quinzii's condition can be interpreted as the suboptimality of the initial endowment allocation. It can be deduced from the argument per Aiyagari and Peled (1991) that the Magill and Quinzii's condition is necessary as well as sufficient for the existence of stationary monetary equilibrium. Therefore, the Manuelli's condition must imply the Magill and Quinzii's condition.[2]

Further, Magill and Quinzii (2003) referred to the work per Gottardi (1996) for the fixed point argument and did not provide a direct proof of the existence of stationary monetary equilibrium. However, Gottardi dealt with a more complicated SOLG model with heterogeneous agents and several securities. To our best knowledge, there is no work elsewhere in the literature, which provides the direct proof of the sufficiency as well as the necessity of the suboptimality of initial endowments for the existence of stationary monetary equilibrium in the SOLG model with identical agents. Therefore, it seems worthwhile to provide a direct and simple proof of the

[1] To be more precise, Manuelli (1990) considered the model with a general state space, not a finite state space. His proof was tailored to the model with a finite state space per Ohtaki (2011).

[2] In fact, one can easily construct an example the Manuelli's condition is *strictly* stronger than the Magill and Quinzii's.

equivalency between the existence of stationary monetary equilibrium and the suboptimality of the initial endowments, which is tailored to such a model of the monetary SOLG economy.

In this paper, we provide a direct and simple proof of the equivalency between the existence of stationary monetary equilibrium and the suboptimality of the initial endowment allocation in identical agents models. The stationary monetary equilibrium in such a model has a convenient property, i.e., it can be identified with a positive solution for a certain system of equations. Our proof finds such a solution by applying the Tarski's fixed point theorem.[3] The poof strategy is similar to Manuelli's. In fact, the construction of an appropriate mapping is nearly identical to Manuelli's. The crucial difference from his proof is the construction of an appropriate compact and convex set. We construct such a set from the suboptimality of the initial endowment allocation. Further, we demonstrate the uniqueness of stationary monetary equilibrium.

2 The Model

We consider a stationary, one-good, two-period, stochastic overlapping generations model. Time is discrete and runs from 0 to ∞. Stochastic environment is modeled by a stationary Markov process with its state space S, where S is a nonempty finite set and satisfies that $0 \notin S$. The initial state s_0 is treated as given.

After the realization of state s_t in each period $t \geq 1$, one new agent is born, lives for two periods, and dies. Her initial endowment and preference are assumed to depend only on the realizations of the Markov state during her lifetime, not on time nor on the past realizations. Thus, she is endowed with $\omega_{s_t} = (\omega^y_{s_t}, (\omega^o_{s_t s'})_{s' \in S}) \in \Re_{++} \times \Re^S_+$ as the initial endowment, where ω^y_s and $\omega^o_s = (\omega^o_{ss'})_{s' \in S}$ are endowments when young and old, and with $U^s : \Re_+ \times \Re^S_+ \to \Re$ as the lifetime utility preference. It is assumed that, for all $s \in S$, there exists a family $\{u_s, (v_{ss'})_{s' \in S}\}$ of increasing, strictly concave, continuously differentiable real-valued functions on $\Re_{++}$ such that $\lim_{x \downarrow 0} u'_s(x) = \infty$, $\lim_{x \downarrow 0} v'_{ss'}(x) = \infty$, and

$$U^s(c^y_s, (c^o_{ss'})_{s' \in S}) = u_s(c^y_s) + \sum_{s' \in S} v_{ss'}(c^o_{ss'})$$

for each $(c^y_s, (c^o_{ss'})_{s' \in S}) \in \Re_{++} \times \Re^S_{++}$, where c^y_s and $c^o_s = (c^o_{ss'})_{s' \in S}$ are consumption when young and old, respectively. We also assume that $v'_{ss'}(\omega^o_{ss'} + x) + x v''_s(\omega^o_{ss'} + x) \geq 0$ for each $s, s' \in S$

[3] As demonstrated in our proof, one can apply the Brouwer's fixed point theorem instead of the Tarski's.

and each $x > 0$.[4] Note that we do not explicitly assume the (objective/subjective) expected utility hypothesis.

After the realization of state s_1 in period 1, there also only one-period lived agent, called the *initial old*. Her endowment is given by $\omega^o_{0s_1} := \omega^o_{s_0 s_1}$ and her consumption is denoted by $c^o_{0s_1} \geq 0$.

Let $S_0 := \{0\} \times S$. Also let $\bar{\omega}_{ss'} := \omega^y_{s'} + \omega^o_{ss'}$ for all $(s, s') \in S_0 \times S$, which is the total endowment at the Markov state s. A *stationary feasible allocation* of this economy is a family of functions $c = \{c^y, c^o\}$ of $c^y : S \to \Re_+$ and $c^o : S_0 \times S \to \Re_+$ such that $c^y_{s'} + c^o_{ss'} = \bar{\omega}_{ss'}$ for all $(s, s') \in S_0 \times S$. It is *interior* if $c_s = (c^y_s, (c^o_{ss'})_{s' \in S}) \gg 0$ for all $s \in S$. Note that $\omega := (\omega^y, \omega^o)$ is an interior stationary feasible allocation. For any stationary feasible allocations b and c, we say that b *CPO-dominates* c if, for all $s \in S$, $b^o_{0s} \geq c^o_{0s}$ and $U^s(b_s) \geq U^s(c_s)$ with strict inequality somewhere. A stationary feasible allocation c is *conditionally Pareto optimal* (CPO) if there is no stationary feasible allocation b which CPO-dominates c.

As a means to intergenerational trade, we introduce *fiat money*, which is an infinitely-lived outside asset with no dividend. The stock of fiat money is constant over time and states and normalized to one. A real money balance vector $q^* \in \Re^S_{++}$ is called a *stationary monetary equilibrium* if there exist a stationary feasible allocation c^* and a nominal money holding $m^* \in \Re^S$ such that, for all $s \in S$,

$$(c^*_s, m^*_s) \in \mathop{\arg\max}_{(c_s, m_s)} \left\{ U^s(x_s) \;\middle|\; \begin{array}{c} c^y_s = \omega^y_s - q^*_s m_s \\ (\forall s' \in S)\, c^o_{ss'} = \omega^2_{ss'} + q^*_{s'} m_s \end{array} \right\}$$

and $m^*_s = 1$. Given the current restrictions on preferences, one can easily verify that $q \in \Re^S_{++}$ is a stationary monetary equilibrium if and only if it satisfies that

$$q_s u'_s(\omega^y_s - q_s) = \sum_{s' \in S} q_{s'} v'_{ss'}(\omega^o_{ss'} + q_{s'}). \tag{1}$$

for each $s \in S$. Therefore, we can identify a stationary monetary equilibrium with a positive solution of this system of equations, which belongs to $[0, \omega^y]$.[5]

3 The Result

For each stationary feasible allocation c, let $M(c) := [m_{ss'}(c)]_{s,s' \in S}$, where $m_{ss'}(c) := v'_{ss'}(c^o_{ss'})/u'_s(c^y_s)$ for all $s, s' \in S$. By strict monotonicity of preferences, $M(c)$ is positive

[4]This assumption holds if, for example, $-cv''_{ss'}(c)/v'_{ss'}(c) \leq 1$ for each $s, s' \in S$ and each $c > 0$. In fact, it follows from the facts that $v''_{ss'}(c) < 0$ and $\omega^o_{ss'} \geq 0$ that $v'_{ss'}(\omega^o_{ss'} + x) + xv''_{ss'}(\omega^o_{ss'} + x) \geq v'_{ss'}(\omega^o_{ss'} + x) + (\omega^o_{ss'} + x)v''_{ss'}(\omega^o_{ss'} + x) \geq 0$ for each $x > 0$.

[5]For each $a, b \in \Re^S$, $[a, b] := \prod_{s \in S}[a_s, b_s]$ and $[a, b) := \prod_{s \in S}[a_s, b_s)$.

square matrix for any interior stationary feasible allocations. Thus, it follows from the Perron-Frobenius' theorem that there exists a unique positive vector $x(c) \in \Re^S_{++}$ (up to normalization) such that $M(c)x(c) = \lambda(c)x(c)$ for some positive number $\lambda(c) > 0$.[6] This $\lambda(c)$ is the *dominant root* of $M(c)$, i.e., its unique dominant eigenvalue. Note that, if $\omega^o_{ss'} = 0$ for some $s, s' \in S$, then $M(c)$ is not well-defined. Therefore, in such a situation, $\lambda(c)$ and $x(c)$ are no longer well-defined.

We are now ready to state our main theorem:

Theorem.

(a) *A stationary monetary equilibrium exists if and only if either* $\omega^o_{ss'} = 0$ *for some* $s, s' \in S$ *or* $\lambda(\omega) > 1$ *holds; and*

(b) *the equilibrium is unique.*

Proof. Let $c(q) := (\omega^y_s - q_s, \omega^o_{ss'} + q_{s'})_{s,s' \in S}$ for all $q \in \Re^S$. By its definition, $c(0) = \omega$.

We first demonstrate (a). (*only if*) Suppose the contrary that $\omega^o \gg 0$ and $\lambda(\omega) \leq 1$, whereas a stationary monetary equilibrium $q \in \Re^S_{++}$ exists. Note that Eq.(1) can be rewritten as $q = M(c(q))q$. Hence, it follows from the Perron-Frobenius' theorem that $\lambda(c(q)) = 1$. Because u_s and $v_{ss'}$ are strictly concave and $q \in \Re^S_{++}$, we can obtain that $m_{ss'}(c(q)) < m_{ss'}(\omega)$ for all $s, s' \in S$. Therefore, it also follows from the Perron-Frobenius' theorem that $1 = \lambda(c(q)) < \lambda(\omega) \leq 1$, which is a contradiction. This completes the proof of the *only if* part of (a).

(*if*) For each $s \in S$, let $Q_s := [0, \omega^y_s]$ and its interior be denoted by int.Q_s. For each $s \in S$, define the function $f_s : Q_s \times \Re_+ \to \Re$ by $f_s(q, \xi) := qu'_s(\omega^y_s - q) - \xi$ for each $(q, \xi) \in Q_s \times \Re_+$. Note that, for each $s \in S$ and each $\xi > 0$, $f_s(\bullet, \xi)$ is continuous on int.Q_s, $f_s(0, \xi) = -\xi < 0$, and $\lim_{q \uparrow \omega^y_s} f_s(q, \xi) = \infty > 0$. Therefore, for each $s \in S$ and each $\xi > 0$, the intermediate value theorem ensures that there is at least one $\hat{q}_s \in$ int.Q_s such that $f_s(\hat{q}_s, \xi) = 0$. We claim that such $\hat{q}_s$ is unique. Suppose the contrary that, for some $s \in S$ and some $\xi > 0$, there are distinct $\hat{q}_s$ and $\tilde{q}_s$ such that $f_s(\hat{q}_s, \xi) = 0$ and $f_s(\tilde{q}_s, \xi) = 0$, respectively. Assume without loss of generality that $\hat{q}_s < \tilde{q}_s$. Then, it follows from the fact that $\partial(qu'_s(\omega^y_s - q))/\partial q > 0$ that $\xi = \hat{q}_s u'_s(\omega^y_s - \hat{q}_s) < \tilde{q}_s u'_s(\omega^y_s - \tilde{q}_s) = \xi$, which is a contradiction. This completes the proof of claim. By this claim, we can write $\hat{q}_s(\xi)$ rather than $\hat{q}_s$. We also claim that $\hat{q}_s(\xi) > \hat{q}_s(\xi')$ for each $s \in S$ and each $\xi, \xi' \in \Re_{++}$ with $\xi > \xi'$. Suppose the contrary that $\hat{q}_s(\xi) \leq \hat{q}_s(\xi')$ for some $s \in S$ and some $\xi, \xi' \in \Re_{++}$ with $\xi > \xi'$. Then, it follows that $\xi = \hat{q}_s(\xi)u'_s(\omega^y_s - \hat{q}_s(\xi)) \leq$

[6] See Takayama (1974) for more details on the Perron-Frobenius' theorem.

$\hat{q}_s(\xi')u'_s(\omega^y_s - \hat{q}_s(\xi')) = \xi' < \xi$, which is a contradiction. Therefore, $\hat{q}_s(\bullet)$ is increasing on $\Re_{++}$ for each $s \in S$.

Define the mapping $\phi : \Re^S_{++} \to \Re^S_{++}$ by $\phi(\xi) := (\hat{q}_s(\xi_s))_{s\in S}$ for each $\xi = (\xi_s)_{s\in S} \in \Re^S_{++}$. Also define the mapping $\psi : \Re^S_{++} \to \Re^S_{++}$ by $\psi(q) := \left(\sum_{s'\in S} q_{s'} v'_{ss'}(\omega^o_{ss'} + q_{s'})\right)_{s\in S}$ for each $q = (q_s)_{s\in S} \in \Re^S_{++}$. Note that ϕ is increasing and ϕ is nondecreasing. The strict monotonicity of ϕ is follows immediately from the fact that $\hat{q}_s$ is increasing for each $s \in S$, whereas the monotonicity of ϕ follows from the assumption that $v'_{ss'}(\omega^o_{ss'} + x) + xv''_{ss'}(\omega^o_{ss'} + x) \geq 0$. Then, define the mapping $\Phi : \Re^S_{++} \to \Re^S_{++}$ by $\Phi(\xi) := \psi(\phi(\xi))$ for each $\xi \in \Re^S_{++}$. By the monotonicity of ϕ and ψ, Φ is nondecreasing.

Suppose now that either $\omega^o_{jk} = 0$ for some $j, k \in S$ or $\lambda(\omega) > 1$ holds. One can easily observe that either of these conditions implies the existence of $x^0 \in \Re^S_{++}$ satisfying that

$$x^0_s u'_s(\omega^y_s) < \sum_{s'\in S} x^0_{s'} v'_{ss'}(\omega^o_{ss'})$$

for each $s \in S$.[7] Then, there exists a sufficiently small $\varepsilon > 0$ satisfying that

$$(\forall s \in S) \quad \varepsilon x^0_s u'_s(\omega^y_s - \varepsilon x^0_s) < \sum_{s'\in S} \varepsilon x^0_{s'} v'_{ss'}(\omega^o_{ss'} + \varepsilon x^0_{s'}), \tag{2}$$

because of strict concavity and continuity of the utility index functions. Let $\underline{\xi} := (\underline{\xi}_s)_{s\in S}$ and $\overline{\xi} := (\overline{\xi}_s)_{s\in S}$, where $\underline{\xi}_s := \varepsilon x^0_s u'_s(\omega^y_s - \varepsilon x^0_s)$ and $\overline{\xi}_s := \sum_{s'} \omega^y_{s'} v'_{ss'}(\omega^o_{ss'} + \hat{q}_{s'}(\underline{\xi}_{s'}))$ for each $s \in S$, respectively. Obviously, $\underline{\xi} \gg 0$. Note that, for each $s \in S$, $\hat{q}_s(\underline{\xi}_s)u'_s(\omega^y_s - \hat{q}_s(\underline{\xi}_s)) = \underline{\xi}_s = \varepsilon x^0_s u'_s(\omega^y_s - \varepsilon x^0_s)$ by the definitions of $\hat{q}_s$ and $\underline{\xi}_s$. Therefore, it follows from the fact that $\partial(qu'_s(\omega^y_s - q))/\partial q > 0$ that $\hat{q}_s(\underline{\xi}_s) = \varepsilon x^0_s \in \text{int}.Q_s$. For notational convenience, let $q^0 := \varepsilon x^0$.

We claim that $\underline{\xi} \ll \overline{\xi}$. By their definitions, we can obtain that, for each $s \in S$,

$$\underline{\xi}_s := q^0_s u'_s(\omega^y_s - q^0_s) < \sum_{s'\in S} q^0_{s'} v'_{ss'}(\omega^o_{ss'} + q^0_{s'}) < \sum_{s'\in S} \omega^y_{s'} v'_{ss'}(\omega^o_{ss'} + q^0_{s'}) =: \overline{\xi}_s,$$

where the first inequality follows from Eq.(2) and the second one follows from the fact that $q^0_{s'} \in \text{int}.Q_{s'}$ for each $s' \in S$. This completes the proof of the claim. We also claim that $\underline{\xi} \ll \Phi(\xi) \ll \overline{\xi}$ for each $\xi \in [\underline{\xi}, \overline{\xi}]$. Note that, for each $s \in S$ and each $\xi \in [\underline{\xi}, \infty)$, we can obtain that $\sum_{s'\in S} \hat{q}_{s'}(\xi_{s'}) v'_{ss'}(\omega^o_{ss'} + \hat{q}_{s'}(\xi_{s'})) < \sum_{s'\in S} \omega^y_{s'} v'_{ss'}(\omega^o_{ss'} + q^0_{s'}) =: \overline{\xi}_s$, where the inequality follows from the facts that $\hat{q}_{s'} < \omega^y_{s'}$, $v''_{ss'} > 0$, and $\hat{q}_{s'}(\bullet)$ is increasing for each $s \in S$. This implies that $\Phi(\xi) \ll \overline{\xi}$ for each $\xi \in [\underline{\xi}, \infty)$. Also note that $\underline{\xi} \ll \Phi(\underline{\xi})$, because $\underline{\xi}_s := q^0_s u'_s(\omega^y_s - q^0_s) <$

[7]This observation is easy to verify when $\omega^o_{jk} = 0$ for some $j, k \in S$, because $\lim_{x\downarrow 0} v'_{jk}(x) = \infty$. Therefore, suppose that $\omega^o \gg 0$ and $\lambda(\omega) > 1$. Let $x^0 := x(\omega)$. Then, the observation follows from the fact that $M(\omega)x^0 = \lambda(\omega)x^0 \gg x^0$.

$\sum_{s'\in S} q^0_{s'} v'_{ss'}(\omega^o_{ss'}+q^0_{s'})$. By the monotonicity of Φ, this implies that $\underline{\xi} \ll \Phi(\xi)$ for each $\xi \in [\underline{\xi}, \infty)$. Therefore, we can obtain that $\underline{\xi} \ll \Phi(\xi) \ll \overline{\xi}$ for each $\xi \in [\underline{\xi}, \overline{\xi}]$.

Now, let $\Xi := [\underline{\xi}, \overline{\xi}] \subset \Re^S_{++}$. Of course, Ξ is a complete partial ordered set. Further, we have demonstrated that Φ is a nondecreasing mapping from Ξ into itself. Therefore, the Tarski's fixed point theorem ensures the existence of $\xi^* \in \Xi$ such that $\Phi(\xi^*) = \xi^*$.[8] Because $\xi^* \gg 0$, $q^*_s := \hat{q}_s(\xi^*_s) \in \text{int}.Q_s$ for each $s \in S$. It is straightforward to verify that $q^* := (q^*_s)_{s\in S}$ is a stationary monetary equilibrium. This completes the proof of the *if* part of (a).

We then demonstrate (b). Define the sequence of elements of $\Re^S_{++}$, $(q^{(n)})^\infty_{n=0}$, by $q^{(0)} := \omega^y$ and, for all $n \geq 1$,

$$(\forall s \in S) \quad q^{(n)}_s u'_s(\omega^y_s - q^{(n)}_s) = \sum_{s'\in S} q^{(n-1)}_{s'} v'_{ss'}(\omega^o_{ss'} + q^{(n-1)}_{s'}),$$

inductively. We first claim that $(q^{(n)})^\infty_{n=0}$ is a well-defined positive sequence. For $n = 0$, $q^{(0)} \gg 0$ is obviously well-defined. Let $k \geq 1$ be an integer and suppose that $q^{(k)} \gg 0$ is well-defined. We then demonstrate that $q^{(k+1)} \gg 0$ is well-defined. Note that it follows from the monotonicity of $v_{ss'}$ and the hypothesis that $q^{(k)} \gg 0$ is well-defined that $\sum_{s'\in S} q^{(k)}_{s'} v'_{ss'}(\omega^o_{ss'} + q^{(k)}_{s'})$ is well-defined and positive value. Because $xu'_s(\omega^y_s - x)$ is continuous and increasing with respect to x and satisfies that $\lim_{x\downarrow 0} xu'_s(\omega^y_s - x) = 0$ and $\lim_{x\uparrow \omega^y_s} xu'_s(\omega^y_s - x) = \infty$, it follows from the intermediate value theorem that there exists a unique $q^{(k+1)}_s$ for each $s \in S$ such that $0 < q^{(k+1)}_s < \omega^y_s$ and

$$(\forall s \in S) \quad q^{(k+1)}_s u_s(\omega^y_s - q^{(k+1)}_s) = \sum_{s'\in S} q^{(k)}_{s'} v_{ss'}(\omega^o_{ss'} + q^{(k)}_{s'}).$$

This implies that $q^{(k+1)}$ is well-defined. Therefore, $(q^{(n)})^\infty_{n=0}$ is a well-defined positive sequence.

We next claim that $(q^{(n)})^\infty_{n=0}$ is nonincreasing. By the proof of the previous claim, we have $q^{(1)} \ll \omega^y = q^{(0)}$. Let $k \geq 1$ be an integer and suppose that $q^{(k)} \leq q^{(k-1)}$. We then show that $q^{(k+1)} \leq q^{(k)}$. Suppose the contrary that $q^{(k+1)}_{\tilde{s}} > q^{(k)}_{\tilde{s}}$ for some $\tilde{s} \in S$. Because $xv'_{ss'}(\omega^o_{ss'} + x)$ is nondecreasing with respect to x and $q^{(k)} \leq q^{(k-1)}$, it follows that

$$(\forall s \in S) \quad \sum_{s'\in S} q^{(k)}_{s'} v'_{ss'}(\omega^o_{ss'} + q^{(k)}_{s'}) \leq \sum_{s'\in S} q^{(k-1)}_{s'} v'_{ss'}(\omega^o_{ss'} + q^{(k-1)}_{s'}). \quad (3)$$

However, by the definition of $(q^{(n)})^\infty_{n=0}$, the monotonicity of $xu_s(\omega^y_s - x)$, and the hypothesis that $q^{(k+1)}_{\tilde{s}} > q^{(k)}_{\tilde{s}}$, we have

$$\sum_{s'\in S} q^{(k)}_{s'} v'_{\tilde{s}s'}(\omega^o_{\tilde{s}s'} + q^{(k)}_{s'}) = q^{(k+1)}_{\tilde{s}} u'_{\tilde{s}}(\omega^y_{\tilde{s}} - q^{(k+1)}_{\tilde{s}}) > q^{(k)}_{\tilde{s}} u'_{\tilde{s}}(\omega^y_{\tilde{s}} - q^{(k)}_{\tilde{s}}) = \sum_{s'\in S} q^{(k-1)}_{s'} v'_{\tilde{s}s'}(\omega^o_{\tilde{s}s'} q^{(k-1)}_{s'}),$$

[8] See Topkis (1998, Section 2.5) for more details on the Tarski's fixed point theorem. One can verify that Φ is continuous. Therefore, the Brouwer's fixed point theorem is also available for finding the fixed point.

which contradicts with Eq.(3). This implies that $q^{(k+1)} \leq q^{(k)}$. Therefore, $(q^{(n)})_{n=0}^{\infty}$ is nonincreasing.

We also claim that $(q^{(n)})_{n=0}^{\infty}$ satisfies that $q^{(n)} \geq \hat{q}$ for any $\hat{q}$ satisfying Eq.(1) and all $n \geq 0$. Note that we have demonstrated in the proof of (a) that at least one $\hat{q}$ satisfying Eq.(1) exists. Because $\hat{q} \in [0, \omega^y]$, it follows that $q^{(0)} = \omega^y \geq \hat{q}$. Let $k \geq 1$ be an integer and suppose that $q^{(k)} \geq \hat{q}$. We then demonstrate that $q^{(k+1)} \geq \hat{q}$. Suppose the contrary that $q_s^{(k+1)} < \hat{q}_s$ for some $s \in S$. Then, it follows that

$$\begin{aligned}
\sum_{s' \in S} \hat{q}_{s'} v'_{ss'}(\omega^o_{ss'} + \hat{q}_{s'}) &= \hat{q}_s u'_s(\omega^y_s - \hat{q}_s) \\
&> q_s^{(k+1)} u'_s(\omega^y_s - q_s^{(k+1)}) \\
&= \sum_{s' \in S} q_{s'}^{(k)} v'_{ss'}(\omega^o_{ss'} + q_{s'}^{(k)}) \\
&\geq \sum_{s' \in S} \hat{q}_{s'} v'_{ss'}(\omega^o_{ss'} + \hat{q}_{s'}),
\end{aligned}$$

where the first equality follows from the fact that $\hat{q}$ satisfies Eq.(1), the second inequality follows from the monotonicity of $xu'_s(\omega^y_s - x)$, the third equality follows from the definition of $(q^{(n)})_{n=0}^{\infty}$, and the last inequality follows from the monotonicity of $xv_{ss'}(\omega^o_{ss'} + x)$ and the hypothesis that $q^{(k)} \geq \hat{q}$. However, this is a contradiction. Therefore, $q^{(n)} \geq \hat{q}$ for all $n \geq 0$.

Because $(q^{(n)})_{n=0}^{\infty}$ is nonincreasing and bounded from below, $q^* := \lim_{n \uparrow \infty} q^{(n)}$ exists and satisfies that $q^* \geq \hat{q}$ for any $\hat{q}$ satisfying Eq.(1) and

$$(\forall s \in S) \quad q^*_s u'_s(\omega^y_s - q^*_s) = \sum_{s' \in S} q^*_{s'} v'_{ss'}(\omega^o_{ss'} + q^*_{s'}),$$

i.e., q^* also satisfies Eq.(1). Note that we have already shown the existence of $\hat{q} \in \prod_{s \in S} \text{int}.Q_s$, so that q^* must be also an interior solution.

Recall that $\lambda(c(q)) = 1$ for any $q \gg 0$ satisfying Eq.(1). Suppose now that there exists an interior solution $\bar{q}$ such that $q^* > \bar{q}$. Because of the strict concavity of u_s and $v_{ss'}$, $m_{ss'}(c(\bar{q})) \geq m_{ss'}(c(q^*))$ with at least one strict inequality. Then, it follows from the Peoorn-Frobenius' theorem that $\lambda(c(\bar{q})) > \lambda(c(q^*))$, which contradicts $\lambda(c(q^*)) = \lambda(c(\bar{q})) = 1$. Thus, q^* is the unique interior stationary monetary equilibrium. Q.E.D.

It is well-known that an interior stationary feasible allocation c is CPO if and only if $\lambda(c) \leq 1$.[9] Further, one can immediately verify that ω is not CPO when $\omega^o_{ss'} = 0$ for some $s, s' \in S$,

[9]This can be deduced from Ohtaki (2012, Theorem 1). To be more precise, the dominant root criterion $\lambda(c) \leq 1$

because we assume that $\lim_{x\downarrow\infty} v'_{ss'}(x) = \infty$. Therefore, as a corollary of the previous theorem, we can say that the existence of a stationary monetary equilibrium is equivalent to the conditional Pareto suboptimality of the initial endowment.

Remark that the literature imposed an assumption that $\omega^o \gg 0$.[10] This restriction plays an important role to apply the Perron-Frobenius' theorem, because $M(\omega)$ is no longer well-defined if $\omega^o_{ss'} = 0$ for some $s, s' \in S$. Note that our result includes such an anomalous situation. In this sense, the result of this paper can be applied to a broader range of models than the literature.

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is equivalent to the absence of the nonstationary as well as stationary feasible allocations that CPO-dominate c. See also Chattopadhyay and Gottardi (1999, Theorem 4).

[10] See Gottardi (1996) and Magill and Quinzii (2003) for example.

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