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Bubble cycle

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This paper analyzes the boom–bust cycle driven by rational bubbles in an overlapping-generations economy that is subject to borrowing constraints. At the heart of the analysis is the interplay among savings, investment, and the interest rate. Bubbles are more likely to crowd investment in, the stronger is the intertemporal substitution in consumption, and the more severe is the borrowing constraint. This model contradicts with Abel et al (1989)'s condition in both dimensions of dynamic efficiency and the occurrence of bubbles. We characterize the global dynamics of a stochastically bubbly economy, where emergent bubbles are followed by the investment boom, but the bursting of bubbles results in the recession. The recession is serious relative to the boom, with biased holding of bubbles.

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This paper analyzes the boom–bust cycle driven by rational bubbles in an overlapping-generations economy that is subject to borrowing constraints. At the heart of the analysis is the interplay among savings, investment, and the interest rate. Bubbles are more likely to crowd investment in, the stronger is the intertemporal substitution in consumption, and the more severe is the borrowing constraint. This model contradicts with Abel et al (1989)'s condition in both dimensions of dynamic efficiency and the occurrence of bubbles. We characterize the global dynamics of a stochastically bubbly economy, where emergent bubbles are followed by the investment boom, but the bursting of bubbles results in the recession. The recession is serious relative to the boom, with biased holding of bubbles.

Keywords: Rational bubbles, crowding in, dynamic efficiency, stochastic bubbles

JEL: E20, E32, E44

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1. Introduction

Kindleberger (1978) and Minsky (1986) addressed that the emergence of bubbles is associated with credit expansion and the investment boom, but the boom ends with the bursting of bubbles, and the aftermath is a recession or even depression. We have witnessed several episodes for boom-bust cycles of bubbles during the past three decades, in Japan, Finland, Norway, and Sweden in the late 1980s, Thailand, Malaysia, and Indonesia in the early 1990s and in the US in the past decade.

These episodes for the bubble cycle raise several challenging questions in the context of standard macroeconomics. What is the condition for bubbles to arise? When and how do bubbles create the investment boom? Then do bubbles make the economy more dynamically efficient? If bubbles create the strong boom, is also the tough of the bursting of bubbles deep?

Answering these questions is to study how the emergence of bubbles, crowding-in/out, and dynamic efficiency are related, and how their relationship differs from the basic finding developed by Diamond (1965) and Tirole (1985), where bubbles arise only if capital is overly accumulated and the economy is dynamically inefficient. The approach taken here is to construct a model of rational bubbles that is as simple as possible but useful for the investigation. Agents live for three periods, receive endowments both when young and middle-age, and have preference given by log utility over every period. Middle-aged agents have access to investment opportunities subject to borrowing constraints imposed by limited pledgeability.

At the heart of the model is the interplay among savings, investment, and the interest rate. In this environment young savings as well as middle-aged savings are increasing in the interest rate. The response of financially constrained firms on the interest rate is *not* standard when they behave first as creditors before they do as debtors. A rise in the interest rate tends to repress investment through the leverage channel, but tends to stimulate investment through the balance sheet channel. The rise in the interest rate brought about by the emergence of bubbles boost savings and thus the internal wealth of these firms, which in turn stimulates their investment. On the other hand, the increased savings fuel bubbles not to crowd investment out.

This paper has several contributions. First, bubbles may crowd investment out or in. Bubbles are more likely to crowd investment in, the stronger is the intertemporal substitution in consumption, and the more severe is the borrowing constraint. Domestic savings become the driving force of the investment boom but foreign savings are not because the former can enhance the internal wealth of financially constrained firms while the latter are in general used as outside funds.

Second, bubbles can arise if the interest rate, which diverges below from the return on investment, is less than the growth rate is, Additionally bubbles arise when the allocation is (constrained) dynamically inefficient at least if the borrowing is sufficiently limited. This model satisfies Abel et al (1989)'s condition for dynamic efficiency that is used also for testing the occurrence of bubbles, but contradicts with this condition in both dimensions of dynamic efficiency and the occurrence of bubbles.

Third, abundant bubbles as liquidity are not always associated with crowding-in. If contract enforceability is low, bubbles are abundant and crowd investment in, whereas if firms have better balance sheet, bubbles are scarce but crowd investment in.

Forth, we characterize the global dynamics of the boom-bust cycle in a stochastically bubbly economy, where emergent bubbles are followed by the investment boom, but the bursting of bubbles results in the recession. The recession is serious relative to the boom, with biased holding of bubbles.

This paper is related to excellent contributions that investigate the role of store of values in macroeconomics, including Tirole (1985), Woodford (1990), Holmstrom and Tirole (1998), Kiyotaki and Moore (1997), and Hellwig and Ranzoneri (2009). Further development in this field provides several mechanisms where bubbles are investment are complementary. The literature includes Olivier (2000), Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), Martin and Ventura (2012), and Hirano and Yanagawa (2011). Farhi and Tirole (2012) develop a three asset model of risk-neutral environment where the third non-bubbly liquidity plays a role of stimulating investment when bubbles arise. Martin and Ventura (2012) develop a model of bubble cycle where newly created bubbles are attached as part of internal wealth of financially constrained firms and contribute to enhancing investment.

The link between bubbles and dynamic efficiency is also a topic of primary concern. Grossman and Yanagawa (1993), King and Ferguson (1993), and Saint-Paul (1992) build models where there exists a wedge between private and social returns on capital, and demonstrated that bubbles can arise even when the economy is dynamically efficient. Farhi and Tirole (2012) find the same relationship in a financially constrained economy where the physical return differs from the market interest rate. Dynamic inefficiency in the bubbleless economy is similar to Diamond (1965) and Tirole (1985). But unlike theirs, dynamically inefficiency arises when investment is not excessive.

This paper shares an insight with papers looking at linking savings and the economic boom and growth. Caballero, Farhi, and Hammour (2006) build the model of rational bubbles, where savings play important roles in funding growth opportunities. Buera and Shin (2010) focus on increased domestic savings supported by a self-financing motive of financially-constrained in order to explain capital outflow of faster-growing developing countries. Song, Storesletten, and Zilibotti (2011) construct a growth model where the accumulation of savings of credit-constrained entrepreneurs is the driving force of fast growth of China. The story of saving boost is reminiscent of the growth-saving causality argued by Carroll and Weil (1994) that report that growth positively Granger-causes savings for the sample of 38 countries. Carroll, Overland, and

Weil (2000) build a growth model where habit formation on consumption explains causation from high growth to high savings.

This paper is organized as follows. Section 2 develops the model. Section 3 analyzes the model. Section 4 argues on dynamic efficiency Section 5 examines the role of foreign savings. Section 6 investigates the boom-bust cycle of bubbles. Section 7 discusses on the story of saving boost in the bubbly period.

2. The Model

Consider an overlapping-generations economy that lasts over an infinite horizon. In each period, a unit mass of agents are born, and live for three periods. There is no population growth or technological progress. Individual agents are endowed with one unit when young and ω units when middle-aged. Their preference is described by $\log c_{t-1}^{y} + \beta \log c_{t}^{m} + \beta^{2} \log c_{t+1}^{o}$, where c_{t-1}^{y} is consumption in young age, c_{t}^{m} in middle age, and c_{t+1}^{0} in old age, and $\beta (\leq 1)$ is the discount factor. Each generation is indexed by the period in which it is middle-aged. As owners of the firm, middle-aged agents have access to one linear investment project that transforms one unit of a good into R^{f} (>1) units of the good after one period. To motivate financial market imperfections, we assume that only part of the return, $R (< R^{f})$, is pledgeable to creditors. Debtors are protected by their limited liability. Assume that $R<1$, which is necessary for the borrowing constraint to be binding at the bubbly steady state, as will be obvious below.[1]

3. The Analysis

Letting r_t denote the rate of return on assets at period t, young agents born at period t-1 choose savings

(1a) $$s_{t-1}^y = \frac{1}{(1+\beta)(1+r_t)}\{s_t^m - \omega + \beta(1+r_t)\},$$

given the expected middle-aged savings s_t^m. When they become middle-aged, they choose savings

(1b) $$s_t^m = \frac{\beta}{1+\beta}\{(1+r_t)s_{t-1}^y + \omega\}.$$

Under perfect foresight, young and middle-aged savings are

(2a) $$s_t^y = \frac{1}{1+\beta+\beta^2}\{\beta(1+\beta) - \frac{\omega}{1+r_t}\} \equiv s^y(r_{t+1}), \text{ and}$$

(2b) $$s_t^m = \frac{\beta^2}{1+\beta+\beta^2}(1+r_t+\omega) \equiv s^m(r_t).$$

Middle-aged savings are increasing in the interest rate. Young savings are also increasing in the interest rate when $\omega>0$. As ω is high, the substitution effect is stronger than the income effect, and ω captures the strength of intertemporal

[1] Farhi and Tirole (2012) differ from our model in three respects. First, agents are risk neutral and consume only when old, secondly they receive endowment only when young, and thirdly, the portfolio for investors consists of three assets, securities issued by firms, bubbles, and the fundamental-backed liquidity.

substitution in consumption.[2] Additionally, young savings react to the expected interest rate, while middle-aged savings react to the past interest rate.

Young savings of generation t+1 and middle-aged savings of generation t fuel the investment of generation t and bubbles held by generation t+1. Letting b_t denote bubbles, market clearing in the capital market requires

(3) $$s^y(r_{t+1}) + s^m(r_t) = i_t + b_t .$$

The absence of arbitrage with other assets with the rate of return r_{t+1} allows bubbles to evolve as

(4) $$b_{t+1} = (1+r_{t+1})b_t .$$

Note that $b_t \geq 0$, that is, we exclude negative bubbles.

A middle-aged agent starts the firm by investing the amount i_t in the project. Letting r_{t+1} denote the interest rate prevailing between t and t+1, the agent is willing to start the firm if $R^f \geq 1+r_{t+1}$, which we call the *profitability constraint*. The firm funds the investment i_t by the internal wealth s_t^m and supplying the security $(i_t - s_t^m)$, but the issued amount is limited to the present value of the pledgeable asset $Ri_t/(1+r_{t+1})$;

(5) $$i_t - s_t^m \leq Ri_t/(1+r_{t+1}) .$$

Equation (5) states that the debtor can borrow up to the pledgeable asset. We will call this inequality the *borrowing constraint*.

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[2] We have $\dfrac{\partial \log s^y}{\partial \log(1+r)} = \dfrac{\omega}{\beta(1+\beta)(1+r)-\omega}$, which is increasing in ω for any $1+r$.

When the borrowing constraint (5) is not binding, the profitability constraint should be binding with equality, where the interest rate is $1+r_{t+1} = R^f$, which is greater than the growth rate. Bubbles never arise. The investment realizes the first best $i^{FB} \equiv s^y(R^f - 1) + s^m(R^f - 1)$. Bubbles never arise because $1 < R^f$.

We turn to the case when the borrowing constraint (5) is binding with equality. Investment is written as a multiple of the internal wealth s_t^m and the leverage $1/\{1 - R/(1+r_{t+1})\}$;

(6) $$i_t = \frac{s^m(r_t)}{1 - R/(1+r_{t+1})}.$$

This equation reveals two opposing effects of the interest rate on investment. On the one hand, a rise in the period t+1 interest rate decreases leverage $1/\{1 - R/(1+r_{t+1})\}$ and represses investment of the financially constrained firms. On the other hand, a rise in the period t interest rate increases the internal wealth and stimulates investment of these firms. We call the former *the leverage effect,* and the latter *the balance sheet effect*

Combined with (3) and (5), savings (1b) are rewritten as

$$s_t^m = \frac{\beta}{1+\beta}(Ri_{t-1} + \omega + b_t).$$[3]

Combining (2b) with the latter equation yields the demand function of the security issued by firms;

(7) $$\frac{\beta+\beta^2}{1+\beta+\beta^2}(1+r_{t+1}+\omega) = Ri_t + \omega + b_{t+1}.$$

[3] Hereafter whenever we refer to equation (5), the borrowing constraint is binding with equality.

Investment is constrained by the security secured by the pledgeable asset $Ri_t/(1+r_{t+1})$ and the internal wealth $\beta(Ri_{t-1}+\omega+b_t)/(1+\beta)$, which constitutes the supply function of the security;

(8) $$i_t = \frac{Ri_t}{1+r_{t+1}} + \frac{\beta}{1+\beta}(Ri_{t-1}+\omega+b_t),$$

The demand and supply functions and the equation for the bubble evolution fully characterize the dynamic system. We define the competitive equilibrium of a bubbly economy as a sequence $\{i_t, b_t, r_t\}_{t=0}^{\infty}$ that satisfies (4), (7), and (8).

Before going to the bubbly economy, we can describe the competitive equilibrium of a bubbleless economy by imposing $b_t = 0$ on (7) and (8);

(9) $$\frac{\beta+\beta^2}{1+\beta+\beta^2}(1+r_{t+1}+\omega) = Ri_t+\omega, \text{ and}$$

(10) $$i_t = \frac{Ri_t}{1+r_{t+1}} + \frac{\beta}{1+\beta}(Ri_{t-1}+\omega).$$

Proposition 1: Suppose that

(#1) $$\frac{\beta^2 R(R^f+\omega)}{R^f-R} - \{\beta(1+\beta) - \frac{\omega}{R^f}\} < 0$$

holds. The competitive equilibrium of a bubbleless economy that satisfies the binding borrowing constraint is dynamically stable. The steady state interest rate denoted $r^D(R,\omega)$ is increasing in R and ω.

The improvement in the pledgeability tends to strengthen the effective demand for loans, pushing the interest rate up. The rise in the middle-aged wealth ω also increases their internal wealth, and thus raises the interest rate up. Strictly speaking, what matters is not the middle-aged wealth but the relative middle-aged wealth to the young's wealth.[4] The rise in the relative wealth tends to improve the balance sheet of debtors, and pushes the interest rate up even it is followed by the increase in the aggregate savings. In this model the variable ω plays joint roles of boosting savings and improving the balance sheet.

We turn to the competitive equilibrium of a bubbly economy that satisfies the binding borrowing constraint. The steady state is represented as $\{i^B, b^B, r^B\}$, satisfying

(11) $$i^B = \frac{s^m(0)}{1-R}, \quad b^B = s^y(0) - \frac{Rs^m(0)}{1-R}, \text{ and } r^B = 0.$$

where $s^y(0) = \dfrac{\beta(1+\beta)-\omega}{1+\beta+\beta^2}$ and $s^m(0) = \dfrac{\beta^2(1+\omega)}{1+\beta+\beta^2}$.

Bubbles are positively valued only if

(#2) $$\beta(1+\beta) - \omega - \frac{\beta^2 R(1+\omega)}{1-R} > 0$$

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4 Extending the model to assume that young agents are endowed with $\omega_0 (\neq 1)$, saving functions are replaced by $s_{t-1}^y = \dfrac{\omega_0}{1+\beta+\beta^2}\{\beta(1+\beta) - \dfrac{\omega/\omega_0}{1+r_t}\}$, and $s_t^m = \dfrac{\beta^2\omega_0}{1+\beta+\beta^2}(1+r_t+\dfrac{\omega}{\omega_0})$, respectively. Then the interest rate is increasing in ω/ω_0.

The lower is either R or ω, the lower is the interest rate in the bubbleless economy and the more likely the bubbly economy arises. Condition (#2) is equivalent to the property that the bubbleless interest rate is negative.

Property 2: Bubbles are positively valued if and only if (#2) holds, in other words, the bubbleless interest rate is negative, that is, $r^D(R,\omega) < 0$.

Proof: Defining the excess savings function in the bubbleless economy as $ES(r) \equiv \beta(1+\beta) - \frac{\omega}{1+r} - \frac{\beta^2 R(1+r+\omega)}{1+r-R}$, we see that $ES(r)$ is increasing. Thus we see $r^D(R,\omega) < 0$ if and only if $ES(0) > 0$, which is equivalent to (#2).Q.E.D.

This condition is general. Santos and Woodford (1997) show that, for a large variety of economic environments, a necessary condition for bubbles to occur is that the net present value of the endowment be infinite. Their condition for bubbles applies to our economy. Bubbles arise only if the interest rate is below the economic growth rate, i.e., zero in this model, a condition equivalent to the net present value of the endowment being infinite.

Limited pledgeability yields the wedge between the return on investment and the interest rate, and thus bubbles can arise even when $R^f > 1$, that is, the return on investment is greater than the growth rate. Abel et al (1989) evaluates dynamic efficiency by comparing the capital income and investment, and this condition holds for our model with $R^f > 1$ around the steady state, but bubbles can arise.

Proposition 2: Suppose that (#2) holds.[5] There exists a bubbly steady state of the competitive equilibrium that satisfies the binding borrowing constraint. When R or ω is low (high), investment is small (large) and bubbles are abundant (scarce).

The latter part directly comes from checking (11). When either R or ω is low, the borrowing constraint is so severe that savings fuel bubbles more than investment. This, in turn, results in more abundant bubbles.

The next concern is under what conditions bubbles crowd investment in or out. The rise in the interest rate rises tends to repress investment through the leverage channel, while it tends to stimulate investment through the balance sheet channel. To check which is stronger between the two channels, it is useful to rewrite investments in more general form. We rewrite the savings of the middle-aged as $s^m(r_t) = (1+r_t)A^y + A^m$, where $A^y = \frac{\beta^2}{1+\beta+\beta^2}$, and $A^m = \frac{\beta^2\omega}{1+\beta+\beta^2}$. The steady state of the bubbleless

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[5] Strictly, the binding borrowing constraint requires three inequalities; $R<1+r$ (binding borrowing constraint), $\omega/\beta(1+\beta)<1+r$ (positive borrowing), and $1+r<R^f$ (nonbinding profitability constraint). The first and third conditions always hold because of $R<1<R^f$ by assumption. The second condition implies $s^y(0)>0$, and requires $\omega<\beta(1+\beta)$, which holds whenever Condition (#2) holds.

economy satisfies $s^y(r^D)+(1+r^D)A^y+A^m=i^D$ and $1+r^D=\dfrac{Ri^D}{s^y(r^D)}$, and thus $i^D=\dfrac{s^y(r^D)+A^m}{1-RA^y/s^y(r^D)}$. On the other hand, the steady state of the bubbly economy satisfies $s^y(0)+A^y+A^m=i^B+b^B$ and $s^y(0)-b^B=Ri^B$, and thus $i^B=\dfrac{A^y+A^m}{1-R}$. We have

(12) $$i^B-i^D=\frac{\{A^y-s^y(r^D)\}\{(1-R)s^y(r^D)-R(A^y+A^m)\}}{(1-R)\{s^y(r^D)-RA^y\}}.$$

From the above argument we have $1+r^D=\dfrac{R\{s^y(r^D)+A^m\}}{s^y(r^D)-RA^y}$. If the bubbly economy exists, we should have $r^D<0$ from Property 2, and thus $(1-R)s^y(r^D)>R(A^y+A^m)$. Next the positively valued investment guarantees $s^y(r^D)>RA^y$. Therefore, crowding-in occurs if and only if $A^y>s^y(r^D)$, or equivalently $\beta(1+r^D)<\omega$.

Proposition 3: Suppose that (#2) holds.[6] Bubbles crowd investment in if and only if

(*) $$\beta\{1+r^D(R,\omega)\}<\omega,$$

which has the following properties:

(i) There is a threshold $R^*(>0)$, below which crowding-in occurs and above which crowding-out occurs, for any $\omega>0$.

(ii) There is a threshold $\omega^*>0$, above which crowding-in occurs and below which crowding-out occurs, for any $R>0$.

[6] Condition (#1) is sufficient if (#2) holds.

(iii) Bubbles crowd investment out when $\omega = 0$, for any $R > 0$.

Proof: See the Appendix.

Bubbles are more likely to crowd investment in, the lower is the pledgeable return R, and the higher is the middle-aged wealth ω. In Figure 2 we depict the parameter space under which crowding-in (or –out) occurs (see the Appendix for the derivation).

As ω, the measure of intertemporal substitution, is high, the rise in the interest rate boosts young's savings and thus the internal wealth of financially constrained firms, which in turn stimulates investment strongly (Proposition 3(i)).[7] Low pledgeability implies the low interest rate in the bubbleless economy and the large jump in the interest rate when bubbles arise. As R is low, bubbles eventually push up the debtor's internal wealth, and stimulate investment (Proposition 3(ii)). When the intertemporal substitution is low ($\omega=0$), bubbles crowd investment out (Proposition 3(iii)). Bubbles tend to crowd investment in if intertemporal substitution in consumption is strong, and enforceability in financial contracts is low.

An interesting question is whether bubbles are more likely to crowd investment in when bubbles are abundant or scarce. Propositions 2 and 3 state that in terms of R bubbles and abundant and crowd investment in, while in terms of ω, bubbles are

[7] Two offsetting effects virtually operate in terms of the change in ω. For an increase in ω, the channel of the saving boost is more likely to make Condition (*) hold, while the channel through the endogenous change in the interest rate is less likely to make the condition hold. We prove that the former effect always dominates the latter one.

scarce and crowd investment in. This contrasting result comes from the feature that in the model ω plays the joint roles of measuring intertemporal substitution and of lessening the agency cost of financially constrained firms. This finding that crowding-in is not always associated with abundant bubbles is the source of several boom-bust cycle of bubbles as studied below.

Remark 1: Crowding-in does not arise in the risk neutral environment. If the preference is changed to assume that agents consume only in the final period, the internal wealth of their middle-age is $A^y(1+r_t)+A^m=1+r_t+\omega$. Since then $A^y=1$, and also $s^y(r)=1$, Condition (*) then states that bubbles are neutral to investment. In Farhi and Tirole (2012) construct a risk-neutral environment where bubbles crowd investment in by introducing the non-bubbly liquidity as the third liquidity.

Remark 2: The intuition from Proposition 3 does not directly carry over to the no-borrowing case. At $R=0$, bubbleless and bubbly investments are $i^D=\beta\omega/(1+\beta)$ and $i^B=\beta^2(1+\omega)/(1+\beta+\beta^2)$. Crowding-in occurs whenever Condition (#2) (with $R=0$) holds, and Condition (*) is irrelevant.[8] While $\omega>0$ is necessary for crowding-in to occur for $R>0$, $\omega=0$ allows crowding-in to occurs for $R=0$.

[8] When $R=0$, the region for crowding-in is $\omega\in[0,\beta(1+\beta))$ in Figure 2.

Remark 3: Woodford (1990) is related to this model in that both share the insight on intertemporal substitution in consumption.[9] In his model crowding-in occurs for any finite elasticity of intertemporal substitution, while in our model crowding-in occurs if intertemporal substitution is strong.[10]

Remark 4: With heterogeneity among firms, crowding-in and crowding-out coexist. Suppose that a half of firms are not almost allowed to borrow ($R \to 0$), and the reaming half has $\omega = 0$. Bubbles crowd investment of the former firms, and investment of the latter out.

We turn to the dynamics. We impose a technical assumption.

($) $$\omega < \beta R(1+\omega) .$$

This assumption is a sufficient condition for the dynamic behavior to be well defined.

Proposition 4: Suppose that (#2) holds, and additionally that either (*) holds or ($) holds unless (*) holds. Given $i_{t-1} > 0$, there exist maximum feasible bubbles $b(i_{t-1})$, for which the competitive equilibrium converges to the bubbly steady state. On the convergent path, investment is monotone increasing (decreasing) and bubbles are

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[9] Woodford's model differs from this model in that the production function is concave, agents are infinitely lived, and the private borrowing is not allowed.

[10] In Woodford's model intertemporal substitution will have a quantitative implication on crowding-in.

monotone decreasing (increasing) for $i_{t-1} < i^B$ $(i_{t-1} > i^B)$. For $b_t < b(i_{t-1})$, the economy is asymptotically bubbleless.

Proof: See the Appendix.

Figure 3A illustrates a typical phase diagram when bubbles crowd investment in. Given i^D, there exist bubbles $b(i^D)$, for which the economy approaches monotonically the bubbly steady state. Along the saddle-path, emergent bubbles are followed by the investment boom. In contrast, Figure 3B illustrates a typical case when bubbles crowd investment out.

The $i_t = i_{t-1}$ locus may be positively or negatively sloped, depending on whether bubbles crowd investment in or out. When $R \to 0$, it is upwardly sloped, given by $b_t = (1 - R + 1/\beta)i_{t-1} - \omega$, while when $\omega = 0$ (that satisfies ($)), it is downwardly sloped, given by $b_t = \{\beta R - (1+\beta)\}i_{t-1} + \beta/(1+\beta+\beta^2)$.

4. Some Result on Dynamic Efficiency

Standard models of rational bubbles (e.g., Diamond 1965, Ihori 1978, and Tirole 1985) state that bubbles can occur only if the investment level exceeds the first best and the allocation is dynamically inefficient. The subsequent contributions, including Grossman and Yanagawa (1993), King and Ferguson (1993), Saint-Paul (1992), and Farhi and Tirole (2012), state that bubbles can occur when the investment level is less than the first best and the allocation is dynamically efficient.[11] We investigate how investment, bubbles, and dynamic efficiency are related.

We define an allocation to be dynamically efficient if there is no other resource-feasible allocation that increases the lifetime utility of some agent without reducing that of others. We use the Pareto-optimality and dynamic efficiency interchangeably. First of all, the bubbleless competitive equilibrium that does not satisfy the binding borrowing constraint is dynamically efficient. Since the return to capital is higher than the growth rate $(R^f > 1)$, decreasing investment could reduce the aggregate consumption, and would make it impossible to raise someone's utility without reducing others' one.

The next concern is whether the bubbleless competitive equilibrium that satisfies the binding borrowing constraint is dynamically efficient when $R^f > 1$. Here we define an allocation to be *constrained* dynamically efficient if there is no other resource-feasible allocation that increases the lifetime utility of some agent without reducing that of others, and that satisfies the binding borrowing constraint.

When the competitive equilibrium satisfies the binding borrowing constraint, the lifetime utility of agents of generation t is

(13) $$(1+\beta+\beta^2)\log(1+r_t+\omega)-\log(1+r_t)-\beta^2\log\{1-R/(1+r_{t+1})\}+const$$.[12]

[11] The first three papers build models where there exists a wedge between private and social rates of return on capital, and demonstrate that bubbles arise in dynamically efficient economies.

The sum of the first two terms is first decreasing and later increasing with a minimum $\omega/\beta(1+\beta)-1$, which is equal to the interest rate of the bubbleless economy with $R \to 0$. Since the interest rate is increasing in R (Proposition 1), the region where it is increasing is only relevant. When bubbles arise, the interest rate tends to be increasing over time. The former two terms capture the efficiency gain of introducing bubbles, while the third term, which is decreasing in r_{t+1}, captures the efficiency loss of the borrowing constraint.

The direction of the welfare is in general unclear, but when the effect of the third term is small, we can provide some insight.

Property 3: Suppose that (*) holds and that R is sufficiently small. When bubbles arise at T, the bubbly equilibrium runs the higher interest rate than the bubbleless steady state for any $t \geq T+1$.

Proof: We have $s^y(r^D) - \dfrac{Rs^m(0)}{1-R} < s^y(0) - \dfrac{Rs^m(0)}{1-R} = b^B \leq b_T \leq s^y(r_{T+1})$, where the first equality follows from (1a) and $r^D < 0$, the equality follows from (11), and the second inequality comes from Proposition 4 stating that bubbles are monotone decreasing for $i^D < i^B$, and the third inequality says that young savings finance all

[12] We use (2b), (6), and two optimality conditions on consumption, $\beta(1+r_t)c_{t-1}^y = c_t^m$ and $\dfrac{\beta(R^f - R)}{1-R/(1+r_{t+1})}c_t^m = c_{t+1}^o$, where $\dfrac{R^f - R}{1-R/(1+r_{t+1})}$ is the return on capital faced by the firm owner from (6).

bubbles. Since $s^y(.)$ is increasing, there exists a $\hat{R}(T+1)$ below which $s^y(r^D) <$ $s^y(r_{T+1})$ and $r^D < r_{T+1}$. This argument holds for any $t \geq T+1$. Q.E.D.

Proposition 5: Suppose that (#2) and (*) hold. Bubbles realize a more efficient allocation of the competitive equilibrium if R is sufficiently small.

Proof: Let us start from the bubbleless steady state with (i^D, r^D). Suppose that at date T, middle-aged agents of generation T receive transfer b_T from the young of generation T+1. As of date T, agents of generation T maximize $\log c_T^m + \beta \log i_T + \beta \log(R^f - R)$, where $c_{t+1}^o = (R^f - R)i_t$ is used.

We show that the budget constraint is relaxed by introducing bubbles. When bubbles emerge, the savings less the transfer $s^y(r_{T+1}) - b_T$, the transfer b_T, and the wealth of the middle-aged $s^y(r^D)(1+r^D)+\omega$ are used for c_T^m and i_T. The middle-aged budget constraint is $s^y(r_{T+1}) + s^y(r^D)(1+r^D)+\omega = c_T^m + i_T$.

On the other hand, when bubbles do not emerge, the young's savings $s^y(r^D)$ and the wealth of the middle-aged $s^y(r^D)(1+r^D)+\omega$ finance c_T^m and i_T. The budget constraint would be $s^y(r^D) + s^y(r^D)(1+r^D)+\omega = c_T^m + i_T$. We have $r^D < r_{T+1}$ from Property 3, and thus $s^y(r^D) < s^y(r_{T+1})$. Bubbles relax the budget constraint. Agents of generation T are better off.

We next examine the utility of agents of generation t for all $t \geq T+1$, which is described by (13). Consider first the case for $R \to 0$. The interest rate of the bubbleless economy is $\omega/\beta(1+\beta)-1$, at which rate the sum of the first two term attains the minimum. The third term approaches zero. When bubbles arise, Property 3 states that $r_t > \omega/\beta(1+\beta)-1$ for all $t \geq T+1$. They are strictly better off.

Next since the interest rate r_t is continuous in R, every three term is continuous in R. Property 3 says that for sufficiently small R, the sum of the first two terms is strictly higher when bubbles emerge. On the other hand, we can lessen the third term arbitrarily as R is close to zero. There exists an $\widetilde{R}$ below which agents are strictly better off when bubbles arise. Q.E.D.

The allocation of the bubbleless competitive equilibrium is (constrained) dynamically inefficient at least if the borrowing is sufficiently limited. Bubbles give the efficiency gain to participants to the market, but exert the negative externality to the credit market subject to the borrowing constraint. The efficiency loss is larger as firms borrow more.

This finding is similar to Diamond (1965) and Tirole (1985) in that bubbles can arise if the allocation is dynamically inefficient, but distinguishable in that investment is less than the first best in this model, while it exceeds the first best in theirs. This model satisfies the Abel et al's (1989) condition for dynamic efficiency, but the allocation can be dynamically inefficient.

5. Stochastic Bubbles and the Boom–Bust Cycle

Emerging bubbles are very often followed by the investment boom, but the boom ends with the bursting of bubbles, and the aftermath of the crash is a severe recession. To describe the boom–bust cycle of bubbles, as in Weil (1987), we allow bubbles to burst stochastically depending on the realization of a sunspot. Suppose that in each period bubbles burst with probability $1-\lambda$. Once bubbles burst, the economy returns to the bubbleless economy forever.

Let $\tilde{r}_{t+1}$ denote the rate of return on bubbles in period t+1. Bubbles evolve as

(14) $$b_{t+1} = (1+\tilde{r}_{t+1})b_t ,$$

where $\tilde{r}_{t+1} > -1$ when bubbles persist, while $\tilde{r}_{t+1} = -1$ when bubbles burst. So long as bubbles persist, young agents hold both securities issued by firms $s_{t-1}^y - b_{t-1}$ and bubbles b_{t-1}. Letting c_t^m $(c_t^{m,c})$ denote consumption when bubbles persist (burst), first-order conditions on risky bubbles and safe securities are

(15) $$(c_{t-1}^y)^{-1} = \lambda\beta(1+\tilde{r}_t)(c_t^m)^{-1}, \text{ and}$$

(16) $$(c_{t-1}^y)^{-1} = \lambda\beta(1+r_t)(c_t^m)^{-1} + (1-\lambda)\beta(1+r_t)(c_t^{m,c})^{-1},$$

When bubbles persist, the beginning-of-period income of the middle-aged is $(1+\tilde{r}_t)b_{t-1} + (1+r_t)(s_{t-1}^y - b_{t-1}) + \omega$, which is replaced, using (3), (5), and (14), by $b_t + Ri_{t-1} + \omega$, while when bubbles burst, it declines to $Ri_{t-1} + \omega$. When the preference is the log utility, the middle-aged consumption is a constant of the income. Accordingly, we define the measure of risk premium as

(17) $$\delta_t \equiv \frac{1+\tilde{r}_t}{1+r_t} = \frac{1}{\lambda}\{1+(1-\lambda)\frac{b_t}{Ri_{t-1}+\omega}\},$$

which exceeds $1/\lambda$, capturing that the expected return on bubbles should be higher than the return on securities, i.e., $\lambda(1+\tilde{r}_t) > 1+r_t$.[13] The risk-premium is positively related to the investor's holding of risky bubbles relative to safe assets, $(b_t/Ri_{t-1}+\omega)$.

The equation for bubble evolution (14) is accordingly replaced by

(18) $$b_{t+1} = \delta_{t+1}(1+r_{t+1})b_t .$$

Arranging equations yield middle-aged savings as

$$s_t^m = \frac{\beta^2}{1/(\lambda\delta_t)+\beta+\beta^2}\{1+r_t+\omega+(\delta_t-1)(1+r_t)b_{t-1}\},$$

See Appendix for the derivation. The introduction of riskiness changes savings in two ways. First, it increases the saving rate, and secondly it increases the beginning-of-period income given r_t.

The demand function is written as

(19) $$\frac{\beta}{1/(\lambda\delta_{t+1})+\beta+\beta^2}\{1+r_{t+1}+\omega+(\delta_{t+1}-1)(1+r_{t+1})b_t\} = \frac{1}{1+\beta}(Ri_t+\omega+b_{t+1}),$$

while the supply function is not affected by the risk concern, and remains as (8).

13 In general, the return on bubbles could incorporate the liquidity premium that arises from the fact that the future entrepreneur's return differs according to whether bubbles burst or not (e.g., Farhi and Tirole 2012). However, in an environment of the log-utility, the entrepreneur's return evaluated in terms of utility does not affect the return on bubbles, and the return on bubbles does not reflect the liquidity premium. This property specific to the log-utility drastically simplifies the analysis. Particularly, dynamics are described as a two-dimensional system as in the deterministic model.

The competitive equilibrium of a stochastically bubbly economy that satisfies the binding borrowing constraint is defined as a sequence $\{i_t, b_t, r_t, \delta_t\}_{t=0}^{\infty}$ that satisfies four equations (8), (17)-(19).

Proposition 6: Suppose that

(#3) $$(\lambda - R)(1+\beta)\{1-\omega(\frac{1}{\lambda\beta^2}-1)\frac{\beta}{1+\beta}\} - \lambda\beta(\omega+R) > 0$$

There exists the unique steady state of the stochastically bubbly economy that satisfies the borrowing constraint. Furthermore, as $\lambda \to 1$, the allocation $\{i(\lambda), b(\lambda), r(\lambda), \delta(\lambda)\}$ converges to the one of the deterministic bubbly model.

Proof: See Appendix.

Condition (#3) states that the greater is λ, and the lower is either R or ω, the more likely the stochastically bubbly economy arises. As $\lambda \to 1$, this condition reduces to (#2). At the steady state the rate of return on bubbles is zero, and the rate of return on securities is negative so that bubbles grow faster than the safe interest rate. Simulations in Table 1 show that as the probability of bursting 1- λ is high, the higher risk premium leads to the lower interest rate, smaller bubbles, and smaller investment. We turn to the dynamics (we take $\beta = 0.5$).

Proposition 7: Suppose that (#3) and (*) hold and that 1-λ is sufficiently small. There exists a unique equilibrium that converges to the bubbly steady of the competitive equilibrium, given i_{t-1}.

Proof: See Appendix.

The stochastic economy has the similar dynamic properties as the deterministic economy at least for sufficiently small 1-λ. The bubbly economy goes on the stable path to the bubbly steady state so long as bubbles persist. However, once bubbles burst, the economy returns back to the bubbleless economy. How the bursting of bubbles will result in the recession is a topic of concern.

Suppose that bubbles burst at the timing when the middle-aged hold bubbles as assets at date T. Once bubbles burst, they lose part of assets. Their savings declines from $\frac{\beta}{1+\beta}(Ri_{T-1}+b_T+\omega)$ to $\frac{\beta}{1+\beta}(Ri_{T-1}+\omega)$. Thus the supply and demand functions are written as

(20) $$i_T = \frac{Ri_T}{1+r_{T+1}} + \frac{\beta}{1+\beta}(Ri_{T-1}+\omega), \text{ and}$$

(21) $$\frac{\beta+\beta^2}{1+\beta+\beta^2}(1+r_{T+1}+\omega) = Ri_T+\omega,$$

which are the same as the bubbleless equilibrium, (9) and (10). Once i_{T-1} has been determined, (20) and (21) determine i_T and r_{T+1}. When there is crowding-in and $i^D < i_{T-1} < i^B$, i_T falls less than i_{T-1}, that is, the bursting of bubbles results in recession, Afterwards investment follows the dynamic law of motion of the bubbleless equilibrium, and converges monotonically to i^D. On the other hand, when there is crowding-out and $i^B < i_{T-1} < i^D$, the bursting of bubbles can rather stimulate investment.

Figure 4A illustrates the typical boom-bust cycle that arises when bubbles crowd investment in. Starting from i^D, the emergence of bubbles is followed by the investment boom, but the bursting of bubbles represses investment at i^C, and involves the monotone convergence to i^D. Investment and GDP are higher than the bubbleless economy over the cycle. A subject of interest is if the efficiency is higher.

Proposition 8: Suppose that (*) and (#3) hold, and that 1-λ is sufficiently small. There exists a stochastically bubbly equilibrium, where a boom-bust cycle realizes the more efficient allocation than the bubbleless equilibrium.

Proof: Let $U(r_t, r_{t+1})$ denote the lifetime utility of agents of generation t, given by (13) when bubbles persist, $U^C(r_t, r_{t+1})$ denote the one when bubbles burst at date t, and $U^D(r_t, r_{t+1})$ denote the one when there are no bubbles. We prove

(A) $\lambda U(r_t, r_{t+1}) + (1-\lambda) U^C(r_t, r_{t+1}) \geq U^D(r^D, r^D)$, and

(B) $U^D(r_t, r_{t+1}) \geq U^D(r^D, r^D)$, for any $t \geq T+1$,

where bubbles burst at date T. Condition (A) says that agents are better off in the bubbly equilibrium than the bubbleless steady state, and (B) says that agents are better off after the bursting of bubbles than the bubbleless steady state.

We first show that (B) holds when $i^D < i_t < i^B$. When the bursting of bubbles occurs at date T, $i_T > i_{T+1} > \ldots > i^D$ and $r_{T+1} > r_{T+2} > \ldots > r^D$ follows from (21). Since

$U^D(.)$ is increasing in r_t for $r_t > \omega/\beta(1+\beta)-1$ and decreasing in r_{t+1}, we see

$$U^D(r_{T+1}, r_{T+2}) > U^D(r_{T+2}, r_{T+2}) > U^D(r_{T+2}, r_{T+3}) > \ldots > U^D(r_{T+N}, r_{T+N+1})$$

for any N. Consider a small ε and an integer $N(\varepsilon)$, satisfying $r_{T+N(\varepsilon)+1} = r^D + \varepsilon$, such that $U^D(r_{T+N(\varepsilon)}, r^D + \varepsilon) > U^D(r^D + \varepsilon, r^D + \varepsilon)$. This inequality holds for any $\varepsilon > 0$. By taking the limit for both sides, we have

$$\lim_{\varepsilon \to 0} U^D(r_{T+N(\varepsilon)}, r^D + \varepsilon) > \lim_{\varepsilon \to 0} U^D(r^D + \varepsilon, r^D + \varepsilon) = U^D(r^D, r^D).$$

(B) holds.

We next show that $U(r_t, r_{t+1}) \geq U^D(r^D, r^D)$ if R is sufficiently small. It is obvious from Proposition 5.

We finally show that $U^C(r_t, r_{t+1}) \geq U^D(r^D, r^D)$. If $r_t < r_{t+1}$, we see $U^C(r_t, r_{t+1}) > U(r_t, r_{t+1})$. Combined with the previous argument, we have $U^C(r_t, r_{t+1}) \geq U^D(r^D, r^D)$. If $r_t > r_{t+1}$, from the first argument, $U^C(r_t, r_{t+1}) > U^D(r_{t+1}, r_{t+2})$ since $r_t > r_{t+1} > r_{t+2}$. Thus $U^C(r_t, r_{t+1}) \geq U^D(r^D, r^D)$ holds. The latter two arguments imply that (A) holds. Q.E.D.

Biased holding of bubbles among investors

The recession at the onset of the bursting of bubbles is expected to be serious when financially constrained firms hold large bubbles. Consider a situation where bubbles burst with probability $1-\lambda \to 0$. We introduce two dimensions of heterogeneity into the model. Assume that a fraction $\gamma(1-\gamma)$ of agents have (not) access to investment opportunities, Assume additionally that the second type of agents are *Knightian* in the sense that they maximize their utility given that the worst scenario happens, that is, bubbles burst. Thus agents of the first type have access to investment and hold all bubbles. Finally assume that the first type can absorb all bubbles at the steady state, which is parameterized by $\gamma(1-R)\{\beta(1+\beta)-\omega\} < R\beta^2(1+\omega)$. When bubbles burst at period T, the asset demand function remains unchanged, but the asset supply function is replaced by

(22) $$i_T = \frac{Ri_T}{1+r_{T+1}} + \frac{\beta}{1+\beta}\{\gamma(Ri_{T-1}+\omega)-(1-\gamma)b_T\}.$$

See the Appendix for the derivation. Equation (22) captures that the repression of investment hinges directly on the size of bubbles. The recession is more serious as the holding of bubbles concentrates on financially constrained firms.

We calculate how large investment declines relative to the gain from crowding-in given that the bursting of bubbles arises at the bubbly steady state. Table 2A illustrates the boom-bust ratio when $\gamma = 1$, that is, there is no biased holding of bubbles (we take $\beta = 0.5$). The ratios are less than unity in all cases, implying that investment declines at the bursting of bubbles, but is still above the one of the bubbleless steady state. The boom-bust ratios are relatively insensitive to parameters. The situation is quite different if there is the biased holding. Table 2B depicts the case for $\gamma = 0.5$. The ratios are above unity in all cases, implying that investment falls below the level of the bubbleless steady state. The recession is more serious as R is high or ω is low. Figure 4B illustrates a typical case of the cycle with the serious recession.

6. Domestic versus Foreign Savings

Thus far we have highlighted domestic savings in fueling investment. Historical evidence has witnessed many episodes of boosting bubbles that are driven by foreign capital. Here we examine the role of foreign savings that do not contribute much to the enhancement of the internal wealth to entrepreneurs, and focus on distinguishable features of foreign from domestic savings in the bubbly economy.

Let us consider a foreign country that has a simple structure. Agents live for two periods subject to the preference $\log c_t^y + \beta^f \log c_{t+1}^o$, receive one unit when young, and have access to a decreasing-returns-to-scale technology that transforms i_t^f units into $f(i_t^f)$ after one period, with $f'>0$ and $f''<0$.

The investment of the foreign country i_t^f is expressed as a decreasing function of the world interest rate, $i^f(r_{t+1})$, an inverse function of the first-order condition $f'(i_t^f)=1+r_{t+1}$. The competitive equilibrium in the global economy is the same as before except for clearing in the capital market;

(23) $$s^y(r_{t+1}) + s^m(r_t) + s^f = i_t + b_t + i^f(r_{t+1}),$$

where $s^f\{\equiv 1/(1+\beta^f)\}$ is foreign savings. We define the competitive equilibrium of a bubbly global economy that satisfies the binding borrowing constraint in home country, as a sequence $\{i_t, b_t, r_t\}_{t=0}^{\infty}$ that satisfies (1a), (1b), (4), (5), and (23).

Proposition 9: Suppose that

$$\text{(\#4)} \qquad \{\beta(1+\beta)-\omega\} - \frac{\beta^2 R(1+\omega)}{1-R} - (1+\beta+\beta)^2\{i^f(0)-s^m\} > 0$$

holds. There exists a bubbly steady state in the global competitive equilibrium that satisfies the binding borrowing constraint in home country, where the steady state is described as

$$\text{(24)} \qquad i^B = \frac{s^m(0)}{1-R}, \qquad b^B = s^y(0) + s^f - \frac{Rs^m(0)}{1-R} - i^f(0), \quad \text{and} \quad r^B = 0.$$

Compared to (11), we easily see that capital inflows (or outflows), $s^f - i^f(0)$ contribute only to appreciating (depreciating) bubbles, but not to promoting investment. Savings stimulate investment only when they enhance the internal wealth of financially constrained firms.[14]

7. Discussion: Bubbles and Saving Boost

The theoretical finding states that domestic savings are the driving force of the investment boom in the bubbly economy, but foreign savings are not. This finding is

[14] Strictly, foreign savings may assist the investment boom indirectly by enlarging the parameter space where bubbles crowd investment in. See the Appendix for the detailed analysis.

consistent with some evidence that reports that the current account deficit is correlated with asset bubbles, but not with the investment boom (e.g., Laibson and Mollerstrom 2010). On the other hand, we have few evidence that private saving rates have boost on the emergence of bubbles except for the recent China.[15]

The following figures shed an insight on the story of saving boost. Figures 5A and 5B illustrate consumption as a ratio of net worth in the US and Japan. Noteworthy, the ratios are below trend during the bubbly periods, 2003-2007 in the US and 1986-1991 in Japan, suggesting that asset prices appreciated faster than consumption increased.

Eisner (1980) and Peek (1983) proposed to define savings in terms of the income that includes capital gains of assets as well as the flow of the value added. According to theirs, since the savings from the flow of income is fairly stable, people should have saved the large proportion of the appreciation in the value of their asset portfolios.

Horioka (1996) finds that when the saving rate is defined in terms of income that includes land capital gains, households' saving rate rose by about 30% in the late 1980s during the land-price bubble in Japan. Sinai and Souleles (2005) emphasize the specificity of housing (and land) as assets, arguing that the aggregate wealth effect from house price fluctuations is relatively small, taking into account the risk hedge of home ownership.

[15] Song, Storesletten, and Zilibotti (2011) report that the household's saving rate has risen toward this century in China. On the other hand, according to the national account, the household's savings rates are roughly constant over the bubbly periods in the US and Japan.

This story of saving boost is related closely to a large body of empirical evidence reporting the small wealth effect on consumption because the boost in savings is strong when consumption reacts little to appreciations in wealth.[19] Peek (1983) and Juster et al. (2005) investigate the impact of capital gains on household savings in the US, and report that savings increase significantly when capital gains are large. Case et al. (2011) investigate wealth effects using data covering a panel of US states over 1978-2009, and report that the elasticity of housing wealth on consumption is significant, but systematically far smaller than the elasticity of income.[20] Ogawa et al. (1996) investigate wealth effects using data covering a pool of Japanese prefectures on three years, 1979, 1984, 1989, and report the significant effect of financial wealth, but virtually no impact of land wealth on consumption. The story of saving boost is reminiscent of the growth-saving causality argued by Carroll and Weil (1994) that report that growth positively Granger-causes savings for the sample of 38 countries. Carroll, Overland, and Weil (2000) build a growth model where habit formation on consumption explains causation from high growth to high savings.

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[19] Poterba (2000) provides the survey on the stock market wealth effect and, on the real estate wealth effect.

[20] Lettau and Ludvigson (2004) investigate the consumption-wealth link using US time-series data, finding that the majority of fluctuations in asset values are attributable to transitory innovations that display virtually no association with consumption.

Conclusion

We have analyzed the mechanism of the boom-bust cycle of bubbles. I hope this paper adds something to the understanding of bubbles in macroeconomics. This model is tractable, and further analysis is developed in several directions.

Whether the monetary policy influences the boom-bust cycle is an interesting topic. Introducing public bond in the stochastically bubbly model is one direction in the real model without price rigidity. Risky bubbles and safe public debt are imperfect substitutes in the investor's portfolio, and thus there will be room for the open market operation to work.

Also, this model is extended to the global economy to see how the global imbalance, the boom-bust cycle of the global economy, and the current account surplus of growing emerging countries are related. The saving boost mechanism of this model could describe well the behavior of saving glut countries and the increasing global imbalance.

Introducing explicitly financial institutions will allow us to investigate the relation between bubbles and banks' prudential policies. Government's policy response to bubbles may create an added channel through which regulations affect the economy. For example, inappropriately tightening banks' leverage may repress investment and appreciate bubbles, rather destabilizing the financial markets.

Finally, investigating the saving behavior in the bubbly economy is an important empirical research issue. The research in this direction is expected to give a hint on the relation among small wealth effect, the investment boom, and saving boost.

Appendix

Proof of Proposition 1

It follows from (9) and (10) that

(A1) $$\Gamma(i_t) \equiv i_t[1 - \frac{(\beta+\beta^2)R}{(1+\beta+\beta^2)Ri_t+\omega}] = \frac{\beta}{1+\beta}(Ri_{t-1}+\omega).$$

The RHS is positive and increasing for $i_{t-1} > 0$. Define a threshold investment by $\underline{i} \equiv \frac{\beta(1+\beta)R-\omega}{R(1+\beta+\beta^2)}$. The LHS is convex and satisfies $\Gamma(0)=0$. When $\underline{i}<0$, it is increasing for $i_t > 0$, while when $\underline{i} > 0$, it is increasing for $i_t > \underline{i}$, with $\Gamma(\underline{i}) = 0$. There exists the unique steady state $i^D (< i^{FB})$ if $\Gamma(i^{FB}) > \frac{\beta}{1+\beta}(Ri^{FB}+\omega)$, or equivalently

(#1) $$\frac{\beta^2 R(R^f+\omega)}{R^f - R} - \{\beta(1+\beta) - \frac{\omega}{R^f}\} < 0,$$

while $i^D = i^{FB}$ otherwise. The inequality $i^D < i^{FB}$ holds if and only if $1+r < R^f$ from (3), which implies that when (#1) holds, the borrowing constraint should be binding. For $i_{t-1} < i^D$, $\Gamma(i_t) = \frac{\beta}{1+\beta}(Ri_{t-1}+\omega) > \Gamma(i_{t-1})$, and i_t is increasing, while for $i_{t-1} > i^D$, $\Gamma(i_t) = \frac{\beta}{1+\beta}(Ri_{t-1}+\omega) < \Gamma(i_{t-1})$, and i_t is decreasing. Investment is dynamically stable.

Combining (2a), (2b), (6) with (3) yields

(A2) $\beta^2 R(1+r_t+\omega) = (1+r_{t+1}-R)\ \{\beta(1+\beta) - \frac{\omega}{1+r_{t+1}}\} \equiv \Lambda(1+r_{t+1})$.

The LHS is positive and increasing, while the RHS approaches infinity as $1+r_{t+1} \to 0$,

satisfies $\lim_{1+r\to 0} \Lambda(1+r_{t+1}) = \infty$, $\Lambda(R) = \Lambda(\frac{\omega}{\beta(1+\beta)}) = 0$, and is increasing and convex

for $\max\{R, \frac{\omega}{\beta(1+\beta)}\} < 1+r_{t+1}$. There exists two solutions, $1+r^-$ and $1+r^+$, that

satisfy $\beta^2 R(1+r+\omega) = \Lambda(1+r)$, and satisfies $0 < 1+r^- < R < 1+r^+$. Only $1+r^+$

satisfies the binding borrowing constraint. Around $r = r^+$, $\partial r^+ / \partial R > 0$ and

$\partial r^+ / \partial \omega < 0$ hold . Q.E.D.

Proof of Proposition 4

We use (4) to rewrite (7) as

(B1) $$1+r_{t+1} = \frac{(1+\beta+\beta^2)Ri_t + \omega}{\beta(1+\beta) - (1+\beta+\beta^2)b_t} \equiv \Phi(i_t, b_t),$$

where $\Phi_i \equiv \frac{\partial \Phi}{\partial i_t} = \frac{(1+\beta+\beta^2)R}{\beta(1+\beta)-(1+\beta+\beta^2)b_t} > 0$ and $\Phi_b \equiv \frac{\partial \Phi}{\partial b_t} =$

$\frac{(1+\beta+\beta^2)\Phi(.)}{\beta(1+\beta)-(1+\beta+\beta^2)b_t} > 0$. Note that the denominator is positive from (11).

Incorporating (B1) into (4) yields,

(B2) $$b_{t+1} = \Phi(i_t, b_t) b_t$$

On the other hand, incorporating (B1) into (8) yields

(B3) $$i_t\{1 - \frac{R}{\Phi(i_t, b_t)}\} = \frac{\beta}{1+\beta}(Ri_{t-1} + b_t + \omega),$$

Then we implicitly derive

(B4) $$i_t = \Psi(i_{t-1}, b_t),$$

with $\Psi_i \equiv \dfrac{\partial \Psi}{\partial i_t} = \dfrac{\beta R/(1+\beta)}{1 - R/\Phi + i_t R \Phi_i/\Phi^2} > 0$ and $\Psi_b \equiv \dfrac{\partial \Psi}{\partial b_t} = \dfrac{\beta/(1+\beta) - R i_t \Phi_b}{1 - R/\Phi + i_t R \Phi_i/\Phi^2}$.

Two equations (B2) and (B4) constitute the two dimensional system consisting of (i_{t-1}, b_t). We first show the local stability of the bubbly steady state. Letting ρ denote an eigenvalue and $M(\rho) = 0$ denote the characteristic equation, we can write $M(\rho) = \rho^2 - \{\Psi_i + \Phi_i \Psi_b b + \Phi_b b + 1\}\rho + \Psi_i (\Phi_b b + 1)$. This function has the following properties; $M(1) = -b(\Phi_i \Psi_b + \Phi_b - \Psi_i \Phi_b) = -\dfrac{(1+\beta+\beta^2)(1-R)(1+\beta)b}{\beta(1+\beta) - (1+\beta+\beta^2)b} < 0$, and $M(0) = \Psi_i (\Phi_b b + 1) > 0$. Both eigenvalues are positive, and one is larger than unity and the other is less than unity. Therefore there exists the local stable manifold that is saddle-path stable. Furthermore, there exists a unique global stable manifold obtained through backward iteration of the local stable manifold that converges to the bubble steady state.

We turn to the analysis using phase diagram. We derive $b_{t+1} = b_t$ from (B1);

(B5) $$b_t = -R i_{t-1} + \frac{\beta(1+\beta) - \omega}{1+\beta+\beta^2},$$

We next derive the $i_t = i_{t-1}$ locus. We use (B3) to write

(B6) $$\Sigma(i_{t-1}, b_t) \equiv i_{t-1}\{1 - \frac{R\{\beta(1+\beta) - (1+\beta+\beta^2)b_t\}}{(1+\beta+\beta^2)R i_{t-1} + \omega}\} = \frac{\beta}{1+\beta}(R i_{t-1} + b_t + \omega),$$

with $\partial\Sigma/\partial i_{t-1} > 0$ and $\partial\Sigma/\partial b_t > 0$. Then there exists a continuous function,

(B7) $$b_t = \Omega(i_{t-1}), \quad \text{with} \quad \frac{d\Omega}{di_{t-1}} = -\frac{\partial\Sigma/\partial i_{t-1} - \beta R/(1+\beta)}{\partial\Sigma/\partial b_t - \beta/(1+\beta)},$$

that satisfies (B6) except for $i_{t-1} = i^+ = \beta\omega/(1+\beta+\beta^2)R$ that satisfies $\partial\Sigma/\partial i_{t-1} = \beta R/(1+\beta)$. We check two properties to characterize the $i_t = i_{t-1}$ locus.

Lemma 1: Bubbles crowd investment in if and only if $i^D < i^+$.

Proof: Combining (9) and (*) yield $i^D < i^+$.

Lemma 2: There exists some threshold investment i^-, only below which $\partial\Sigma/\partial i_{t-1} < \beta R/(1+\beta)$ is satisfied. In addition, i^- is less than i^D.

Proof: The inequality $\partial\Sigma/\partial i_{t-1} > \beta R/(1+\beta)$ holds if and only if

(!) $$(1+\beta+\beta^2)b_t \geq \beta(1+\beta) - (1-\frac{\beta R}{1+\beta})\frac{\{(1+\beta+\beta)Ri_{t-1}+\omega\}^2}{\omega R}.$$

There exists a threshold i^- above which $\partial\Sigma/\partial i_{t-1} > \beta R/(1+\beta)$ holds for any $b_t > 0$. At $(i^D, 0)$, the bubbleless economy is stable, and satisfies $\partial\Sigma/\partial i_{t-1} > \beta R/(1+\beta)$. We should have $i^- < i^D$. Q.E.D.

Lemma 1 and 2 imply that the $i_t = i_{t-1}$ locus is well defined when $i^D < i^+$ and thus crowd-in occurs. On the other hand, when $i^+ < i^D$, and thus crowding-out occurs. the $i_t = i_{t-1}$ locus is well defined when $i^+ < i^S < i^D$, but may not when $i^B < i^+ < i^D$. The inequality $i^+ < i^S$ implies $\omega < \beta R(1+\omega)$. For the latter, the curve is not continuous around i^+, and thus the global analysis is complicated to analyze.

When $i^D < i^+$, at $(i^D, 0)$, the bubbleless economy is stable, and $\partial\Sigma/\partial i_{t-1} > \beta R/(1+\beta)$. For $[i^D, i^+)$, $\partial\Sigma/\partial b_t < \beta/(1+\beta)$ holds, and from Lemma 2 $\partial\Sigma/\partial i_{t-1} > \beta R/(1+\beta)$ also holds. Thus i_{t-1} is increasing, with $\lim_{i_{t-1}\to i^+} \Omega(i_{t-1}) = +\infty$ for $[i^D, i^+)$. On the other hand, when $i^+ < i^S < i^D$, $\partial\Sigma/\partial b_t > \beta/(1+\beta)$ holds, and from

Lemma 2 $\partial\Sigma/\partial i_{t-1} > \beta R/(1+\beta)$ also holds. Thus i_{t-1} is decreasing, with

$\lim_{i_{t-1}\to i^+} \Omega(i_{t-1}) = +\infty$ for $(i^+ < i^S < i^D]$.

Any point below (above) the $b_{t+1} = b_t$ locus satisfies $b_t < (>) - Ri_{t-1} + \frac{\beta(1+\beta)-\omega}{1+\beta+\beta^2}$, or $1 > (<) 1 + r_{t+1} = b_{t+1}/b_t$; b_t is *decreasing* (*increasing*). On the other hand, any point in the left side of the $i_t = i_{t-1}$ locus satisfies $i_{t-1} < (>) \Omega(b_t)$, or $\Sigma(i_{t-1}, b_t) < (>) \frac{\beta}{1+\beta}(Ri_{t-1} + b_t + \omega) = \Sigma(i_t, b_t)$; i_t is *increasing* (*decreasing*).

We divide the whole space into four regions. We call the set $\{(i_{t-1}, b_t) | i_{t-1} < i_t, b_t > b_{t+1}\}$ the region I. We call the set $\{(i_{t-1}, b_t) | i_{t-1} < i_t, b_t < b_{t+1}\}$ the region II. We call the set $\{(i_{t-1}, b_t) | i_{t-1} > i_t, b_t < b_{t+1}\}$ the region III. We call the set $\{(i_{t-1}, b_t) | i_{t-1} > i_t, b_t > b_{t+1}\}$ the region IV. See Figure 3A or 3B. Hereafter we can focus on the case for crowding depicted in Figure 3A.

We show that the stable manifold that converges to (i^B, b^B) lies in the regions I and III, not II and IV. Suppose by contradiction that it lies in the region II, any point on the manifold (i_{t-1}, b_t) satisfies $b_t < .... < b_{t+s}$ for all $s \geq 1$, but on the other hand, it has $b^B < b_t$, which contradicts with the uniqueness of the bubbly steady state. Suppose by contradiction that it lies in the region IV, any point on the manifold (i_{t-1}, b_t) satisfies $b_t > .... > b_{t+s}$ for all $s \geq 1$, but on the other hand, it has $b^B > b_t$, a contradiction.

We show that bubbles are monotone increasing or decreasing along the convergent path. If the economy converges to the bubbly steady state from the left, it lies in the region I, implying that $1 + r_{t+s} < 1$ for all $s \geq 1$, and using (3), $b_t > ... > b_{t+s}$. Bubbles are monotonically decreasing. If the economy converges to the bubbly steady state from the right, it lies in the region III, implying that $1+r_{t+s}>1$ for all $s\geq 1$, and using (3), $b_t<...<b_{t+s}$. Bubbles are monotonically increasing.

We show that for $b_t<b(i_{t-1})$, the economy is asymptotically bubbleless. Consider $i_{t-1}=i'_{t-1}$ and $b_t>b'_t$, where (i_{t-1},b_t) lies on the stable manifold. Using (B4), we have $i_t>i'_t$. Using (B2), we have $b_{t+1}>b'_{t+1}$, and in the long-run, $\lim_{t\to\infty}b_t=b^B>\lim_{t\to\infty}b'_t$. If $i_{t-1}<i^B$, the economy lies in the region I or IV. We have $1+r_{t+s}<0$ for all $s\geq 1$, and thus $b_t>...>b_{t+s}>...\to 0$. The economy is asymptotically bubbleless. If $i_{t-1}>i^B$, the economy initially lies in the region III or IV. We have $\lim_{t\to\infty}b_t=b^B>\lim_{t\to\infty}b'_t$, and i_{t-1} is decreasing, and thus if the economy lies in the region III it should go into the region IV after some finite periods. Once the economy lies in the region IV, it remains the region II or IV. We have $1+r_{t+s}<0$ for all $s\geq 1$, and thus $b_t>...>b_{t+s}>...\to 0$. The economy is asymptotically bubbleless.

We show that for $b_t>b(i_{t-1})$, the economy is not part of equilibrium. Consider $i_{t-1}=i'_{t-1}$ and $b_t<b'_t$, where (i_{t-1},b_t) lies on the stable manifold. Using (B4), we have $i_t<i'_t$. Using (B2), we have $b_{t+1}<b'_{t+1}$, and $\lim_{t\to\infty}b_t=b^B<\lim_{t\to\infty}b'_t$. If $i_{t-1}<i^B$, the economy lies in the region I or II or III. The economy that initially lies in the region I should go into the region II or region III after some finite periods. We have $1+r_{t+s}>0$ for all $s\geq 1$, and thus $b_t<...<b_{t+s}$. However, there is no any bubbly steady state than the one with b^B, a contradiction. The economy that initially lies in the region

II or III remains the region II or III. We have $1+r_{t+s}>0$ for all $s\geq 1$, and thus $b_t<...<b_{t+s}$., a contradiction. If $i_{t-1}>i^B$, the economy initially lies in the region III, and later goes into the region II or the III. We have $1+r_{t+s}>0$ for all $s\geq 1$, and thus $b_t<...<b_{t+s}$, a contradiction. Q.E.D.

Proof of Proposition 3

(i) is straightforward from the property that $r^D(.)$ is increasing in R. In order to prove (ii) and (iii), we use (A2) to solve the interest rate in terms of ω; $1+r(\omega)=$ $\dfrac{B(\omega)+\sqrt{D(\omega)}}{2\beta\{1+\beta(1-R)\}}$, where $B(\omega)\equiv R\beta(1+\beta)+\omega-R\beta^2\omega$ and $D(\omega)\equiv\{R\beta(1+\beta)+\omega(1-R\beta^2)\}^2-4R\omega\{1+\beta(1-R)\}$. We define a new function, $1+\hat{r}(\omega)=\dfrac{R\beta(1+\beta)+\omega(1-R\beta^2)}{\beta\{1+\beta(1-R)\}}$, which is greater than $1+r(\omega)$ for any $\omega>0$ and equal only at $\omega=0$. $\beta\{1+\hat{r}(\omega)\}$ is increasing, with the slope being less than unity, and satisfies $\beta\{1+\hat{r}(0)\}=\dfrac{R(1+\beta)}{1+\beta(1-R)}$. On the other hand, $\beta\{1+r(\omega)\}$ is also increasing and less than $\beta\{1+\hat{r}(\omega)\}$ for any $\omega>0$. In Figure 1 there exists a threshold $\omega^*(>0)$ only below which $\omega<\beta\{1+r(\omega)\}$. This proves (ii) and (iii). Q.E.D.

Proof of characterizing the parameter space under which crowding-in and crowding-out occur

We define a function, $\Psi(R,\omega)\equiv\omega-\beta\{1+r^D(R,\omega)\}$, where $\Psi(R,\omega)$ is decreasing in R and increasing in ω from Proposition 3(i), and thus the locus $\omega=\phi(R)$ satisfying $\Psi(R,\phi(R))=0$ is upwardly sloping. In addition, we have $\omega\to 0$ as $R\to 0$. Q.E.D.

Derivation of saving functions in the stochastically bubbly economy

In this environment, so long as bubbles survive, young agents choose savings to satisfy

(C1) $$(1+r_t)(s_{t-1}^y - b_{t-1}) + (1+\tilde{r}_t)b_{t-1} + \omega - s_t^m = \lambda\beta(1+\tilde{r}_t)(1-s_{t-1}^y),$$

given their expected middle-aged savings. (C1) is an alternative expression for (15). On the other hand, middle-aged agents choose savings

(C2) $$s_t^m = \frac{\beta}{1+\beta}\{(1+r_t)(s_{t-1}^y - b_{t-1}) + (1+\tilde{r}_t)b_{t-1} + \omega\},$$

given their young savings. We incorporate s_t^m in (C2) into (C1) and rearrange terms to obtain young's savings;

(C3) $$s_{t-1}^y = \{(1+\beta)\beta + \frac{1+r_t}{\lambda(1+\tilde{r}_t)}\}^{-1}\{(1+\beta)\beta - \frac{\omega + (\tilde{r}_t - r_t)b_{t-1}}{\lambda(1+\tilde{r}_t)}\},$$

We replace s_{t-1}^y in (C1) by (C2) and rearrange it to obtain middle-aged savings;

(C4) $$s_t^m = \frac{\beta^2}{1/\lambda\delta_t + \beta + \beta^2}\{1+r_t + \omega + (\tilde{r}_t - r_t)b_{t-1}\},$$ Q.E.D

As $\lambda \to 1$, $\delta_t \to 1$, and $r_t \to \tilde{r}_t$, and (C3) and (C4) reduces to the non-stochastic counterparts, respectively.

Proof of Proposition 6

We first prove the necessity and sufficiency for Condition (#3). We combine (17) and $\tilde{r} = 0$ with the asset supply function (8) to write the asset supply function as a continuously increasing function $b = \Theta^s(i)$ that is implicitly derived from

$$1 - \frac{\beta R}{1+\beta} - \frac{\beta(b+\omega)}{(1+\beta)i} = \frac{R}{\lambda}\{1 + \frac{(1-\lambda)b}{Ri+\omega}\}.$$

We turn to the asset demand function. We rewrite middle-aged savings (C2) by combining (3), (5), and $\tilde{r}=0$ as

(D1) $$s^m = \frac{\beta}{1+\beta}(b+Ri+\omega),$$

Combined with (C1), (D1), and $\tilde{r}=0$, young savings are linked to middle-aged savings by

(D2) $$s^y = 1 - \frac{1}{\beta^2\lambda}s^m,$$

We combine (D1), (D2), and $\tilde{r}=0$ with the market clearing (3) to have

(D3) $$1+(1-\frac{1}{\lambda\beta^2})\frac{\beta}{1+\beta}(b+Ri+\omega) = i+b$$

We rearrange (D3) to write the asset supply function as a linear and decreasing function, $b=\Theta^d(i)$, where $\lambda\beta^2<1$. The intersection of $b=\Theta^s(i)$ and $b=\Theta^d(i)$ determines the bubbly steady state. The function $b=\Theta^s(i)$ satisfies $\Theta^s(0)=-\omega$. Letting $i^{\#}$ denote the investment that satisfies $0=\Theta^s(i^{\#})$, the necessary and sufficient condition for the existence of the bubbly steady state with positively valued bubbles is $b=\Theta^d(i^{\#})>0$, where $i^{\#}=\dfrac{\lambda\beta\omega}{(\lambda-R)(1+\beta)-\lambda\beta R}$. We use (D3) to write the latter condition as

$$(\lambda-R)(1+\beta)\{1-\omega(\frac{1}{\lambda\beta^2}-1)\frac{\beta}{1+\beta}\} - \lambda\beta(\omega+R) > 0 .$$

We turn to the latter property. As $\lambda \to 1$, $\delta_t \to 1$ and $r_t \to \tilde{r}_t$ from (17). Accordingly, (18) reduces to (4), and (19) reduces to (7). Q.E.D.

Proof of Proposition 7

The stochastic system that is composed of (8), (17)-(19) is continuous in λ at least when 1-λ is sufficiently small, and approaches the deterministic case as λ →1. Thus it is sufficient to prove that the two-dimensional system of (i_{t-1}, b_t) remains to describes fully the stochastic equilibrium. The following lemma is useful for later analysis.

Lemma 3 $(1+\beta)\beta > b_{t-1}\{(1+\beta)\beta + \frac{1}{\lambda\delta_t}\}$,

Proof: Young investors only hold bubbles and from (C-3), we have

$s^y_{t-1} = \{(1+\beta)\beta + \frac{1+r_t}{\lambda(1+\tilde{r}_t)}\}^{-1}\{(1+\beta)\beta - \frac{\omega + (\tilde{r}_t - r_t)b_{t-1}}{\lambda(1+\tilde{r}_t)}\} > b_{t-1}$, Rearranging yields

$(1+\beta)\beta - b_{t-1}\{(1+\beta)\beta + \frac{1}{\lambda\delta_t}\} > \frac{\omega + (\tilde{r}_t - r_t)b_{t-1}}{\lambda(1+\tilde{r}_t)}$ >0. Q.E.D.

Using (17) and (18), totally differentiation of (19) yields

$$\delta_{t+1}\left(\frac{\lambda\beta}{1+\lambda\beta\delta_{t+1}+\lambda\beta^2\delta_{t+1}} - \frac{b_t}{1+\beta}\right)d(1+r_{t+1}) = \frac{R}{1+\beta}di_t + \frac{\delta_{t+1}(1+r_{t+1})}{1+\beta}db_t$$

$$+\Delta(1+r_{t+1}, i_t, b_t)\{\delta_r d(1+r_t) + \delta_i di_{t-1} + \delta_b db_{t-1}\}$$

$$-\frac{(\delta_{t+1}-1)\lambda\beta\delta_{t+1}\{b_t d(1+r_{t+1}) + (1+r_{t+1})db_t\}}{1+\lambda\beta\delta_{t+1}+\lambda\beta^2\delta_{t+1}},$$

where $\Delta(.)$ is a continuous function, $\delta_r \equiv \frac{\partial\delta}{\delta(1+r_{t+1})} = (1-\lambda)b_t\ \mathrm{I}(1+r_{t+1,}i_t, b_t)$,

$\delta_b \equiv \frac{\partial\delta}{\delta b_t} = (1-\lambda)(1+r_{t+1})\ \mathrm{I}(1+r_{t+1,}i_t, b_t)$, and $\delta_i \equiv \frac{\partial\delta}{\delta i_t} = \frac{(1-\lambda)R(1+r_{t+1})b_t}{(Ri_t+\omega)}\ \mathrm{I}(1+r_{t+1,}i_t, b_t)$,

with $\mathrm{I}(1+r_{t+1,}i_t, b_t) \equiv \frac{\delta_{t+1}}{\lambda(Ri_t+\omega) - (1-\lambda)(1+r_{t+1})b_t}$.

As $1-\lambda \to 0$, $\delta_r \to 0$, $\delta_i \to 0$, $\delta_b \to 0$, and $\delta_{t+1} \to 1$ so that for sufficiently small $1-\lambda$, under Lemma 3, there exists a well-defined function,

(E1) $$1+r_{t+1} = \hat{\Phi}^r(i_t, b_t),$$

where $\hat{\Phi}_i^r \equiv \partial\hat{\Phi}^r / \partial i_t > 0$ and $\hat{\Phi}_b^r \equiv \partial\hat{\Phi}^r / \partial b_t > 0$. Incorporating (17) and (E1) into (18) and rearranging yield

(E2) $$b_{t+1} = [\frac{\lambda}{\hat{\Phi}^r(i_t, b_t) b_t} - \frac{1-\lambda}{Ri_t + \omega}]^{-1} \equiv \tilde{\Phi}^r(i_t, b_t),$$

with $\tilde{\Phi}_i^r \equiv \partial\tilde{\Phi}^r / \partial i_t$ and $\tilde{\Phi}_b^r \equiv \partial\tilde{\Phi}^r / \partial b_t > 0$.

On the other hand, incorporating (E1) into (8) yields a well-defined function,

(E3) $$i_t = \tilde{\Phi}^i(i_{t-1}, b_t),$$

for sufficiently small $1-\lambda$, with $\tilde{\Phi}_i^i \equiv \partial\tilde{\Phi}^i / \partial i_t > 0$ and $\tilde{\Phi}_b^i \equiv \partial\tilde{\Phi}^i / \partial b_t$, which is implicitly derived from

$$i_t = \frac{Ri_t}{\hat{\Phi}^r(i_t, b_t)} + \frac{\beta}{1+\beta}(Ri_{t-1} + \omega + b_t).$$

Two equations (E2) and (E3) constitute the dynamic system of (i_{t-1}, b_t). Q.E.D.

Derivation of (22)

Assume that a fraction $\gamma(1-\gamma)$ of agents have (not) access to investment opportunities, save $s_t^{yA}(s_t^{yN})$ and $s_t^{mA}(s_t^{mN})$, hold security $x_t^A(x_t^N)$ and bubbles $b_t^A(b_t^N)$, and invest $i_t^A(i_t^N)$ $(i_t^N = 0)$. The latter type are Knightian and hold no bubbles $(b_t^N = 0)$.

We have $\gamma x_{T-1}^A + (1-\gamma) x_{T-1}^N = \frac{Ri_{T-1}}{1+r_T}$ from (5). Since $s_{T-1}^{yA} = s_{T-1}^{yN}$, we have $x_{T-1}^A + b_{T-1}^A = x_{T-1}^N$. Combining latter two equations yields $x_{T-1}^A = \frac{Ri_{T-1}}{1+r_T} - (1-\gamma) b_{T-1}^A$

and thus $s_T^{mA} = \frac{\beta}{1+\beta}\{Ri_{T-1} + \omega - (1-\gamma)b_T^A\}$. We write the supply function as

$i_T^A = \frac{Ri_T^A}{1+r_{T+1}} + \frac{\beta}{1+\beta}\{Ri_{T-1} + \omega - (1-\gamma)b_T^A\}$. With $i_T = \gamma i_T^A$ and $b_T = \gamma b_T^A$, we rewrite

as $i_T = \frac{Ri_T}{1+r_{T+1}} + \frac{\beta}{1+\beta}\gamma(Ri_{T-1} + \omega) - \frac{\beta}{1+\beta}(1-\gamma)b_T$. Q.E.D.

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Figure 1 Threshold ω for Crowding-In

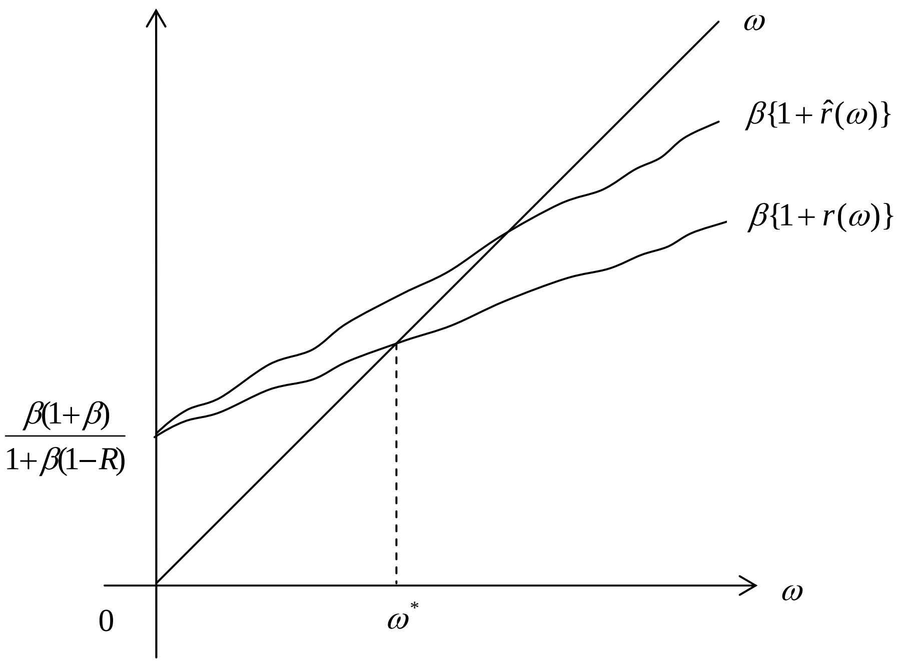


Figure 2 Parameter Space for Crowding-In and -Out

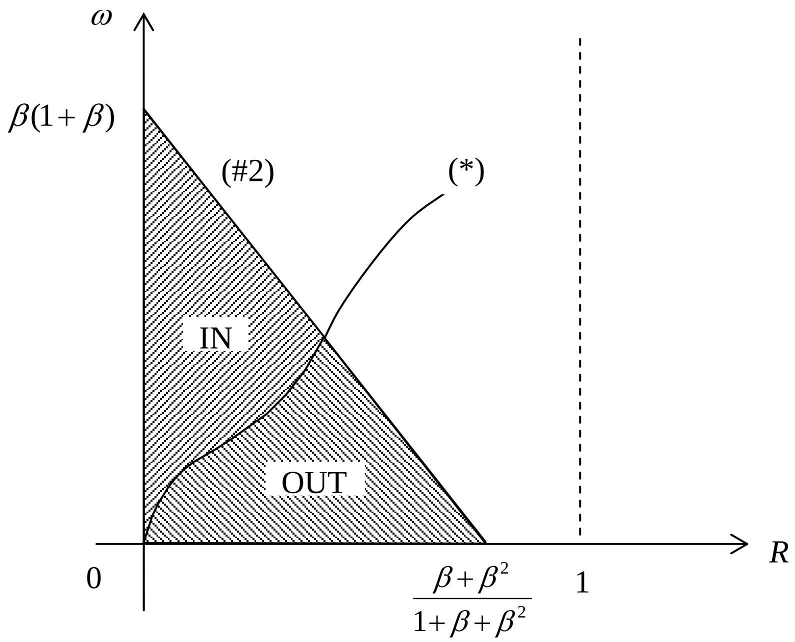


Figure 3A Phase Diagram for Crowding-In

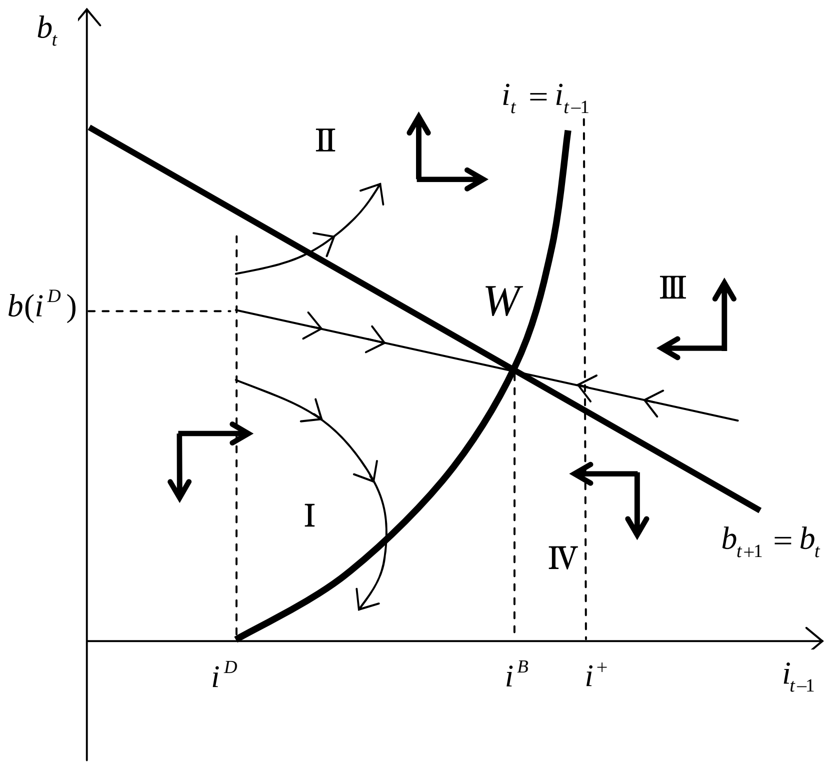


Figure 3B Phase Diagram for Crowding-Out

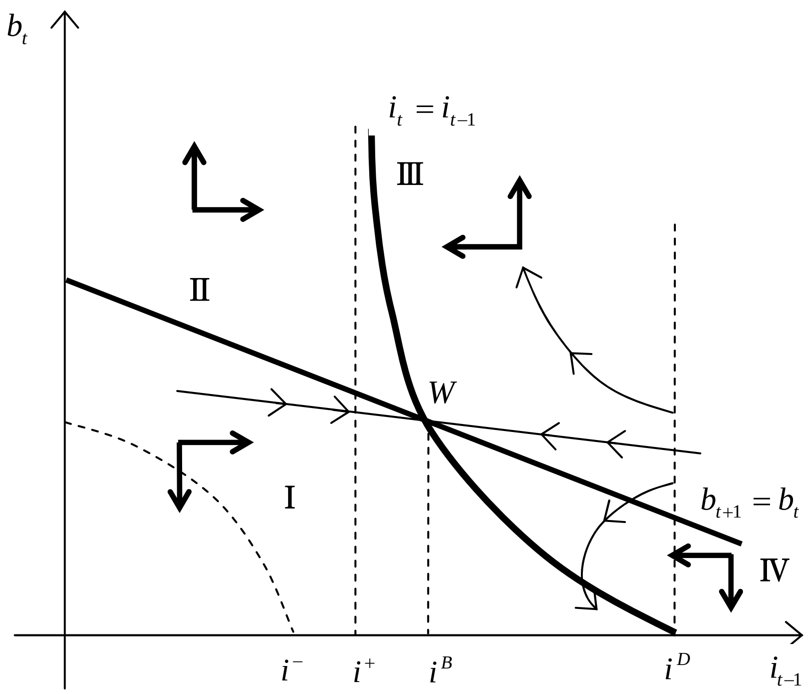


Figure 4A: Mild Recession

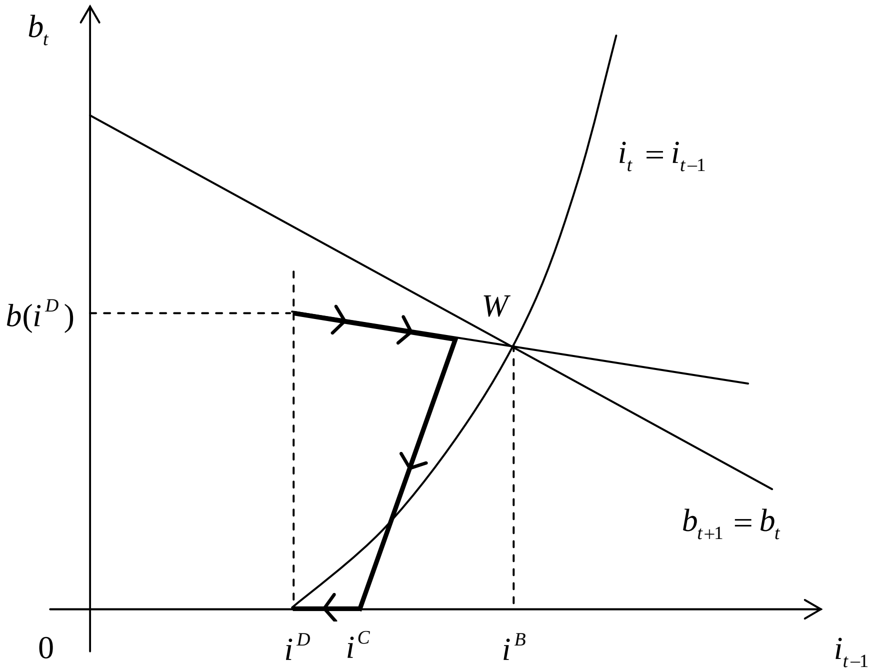


Figure 4B: Severe Recession

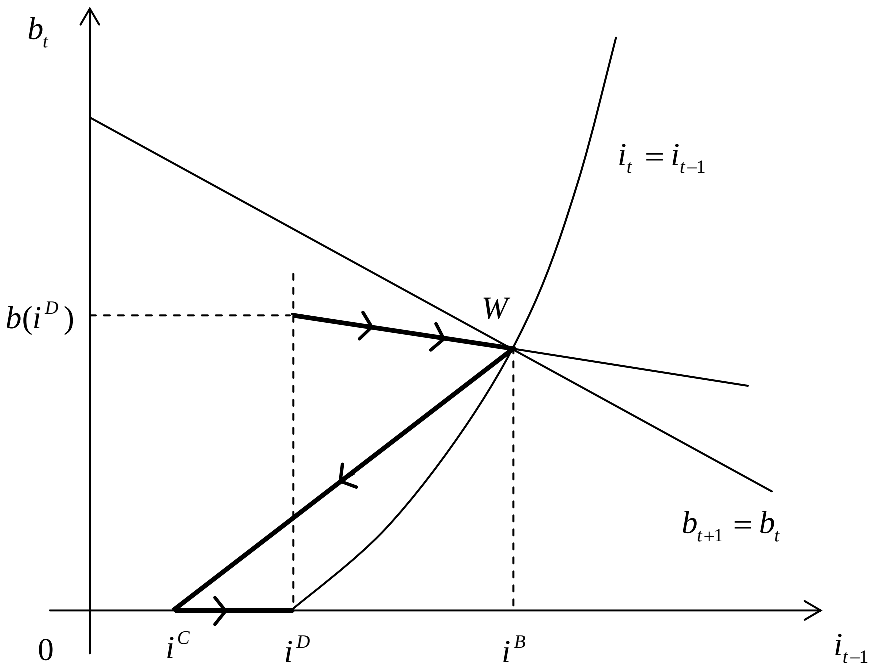
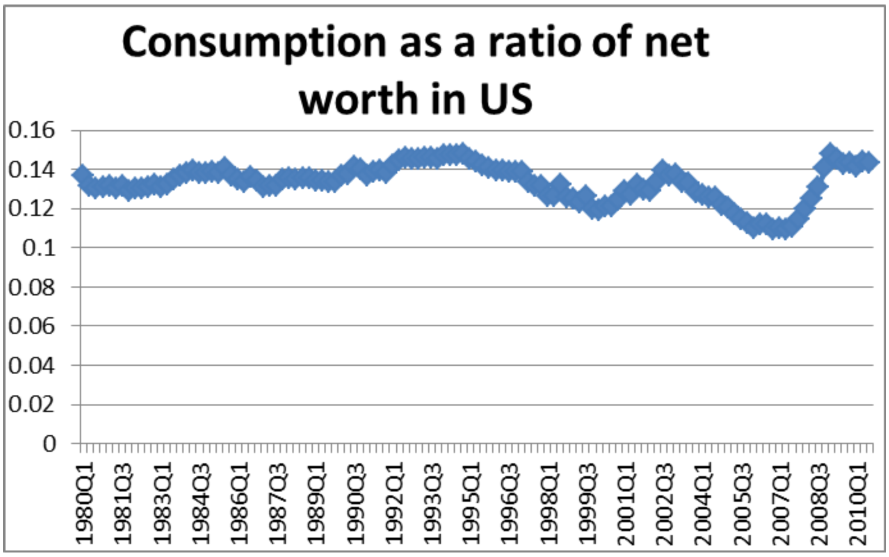
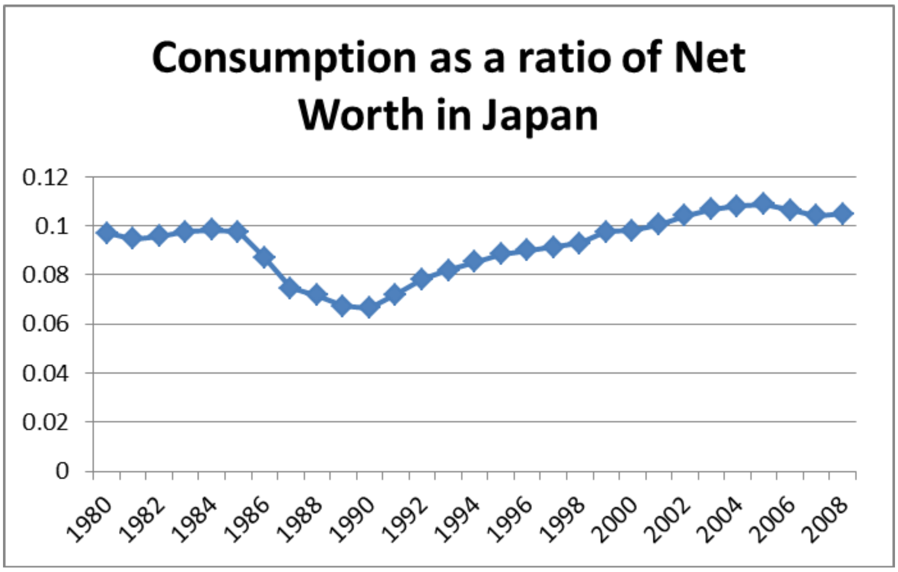


Figure 5A



Note: The source of data is the flow of funds accounts of the US (FRB's). Net worth is defined as total assets less total liabilities of three sectors, households and nonprofit organizations, nonfarm nonfinancial corporate business, and nonfarm nonfinancial noncorporate business. Total assets include tangible assets and financial assets. Note that tangible assets include real estate at market value.

Figure 5B



Note: The source of the data is the National Income Accounts (Cabinet Office). Net worth is available from "Closing Stocks of Assets/Liabilities for the Nation" in the section "Supporting Tables".

Table 1 Simulation in Stochastically Bubbly Economy

β	λ	**R**	ω	**b**	**1+r**	i^B	i^D
0.5	1	0.35	0.45	0.091	1	0.34	0.28
	0.95			0.064	0.944	0.33	
	0.9			0.034	0.894	0.32	
0.5	1	0.3	0.4	0.288	1	0.32	0.24
	0.95			0.149	0.935	0.32	
	0.9			0.123	0.877	0.31	
0.5	1	0.2	0.3	0.508	1	0.29	0.16
	0.95			0.326	0.908	0.29	
	0.9			0.298	0.830	0.29	

Table 2A Crash ratio when there is no biased holding of bubbles

R ω	0.05	0.1	0.15	0.2	0.25	0.3
0.25	0.981	0.955	0.925	0.893	0.864	0.840
0.3	0.981	0.958	0.930	0.900	0.871	0.845
0.35	0.981	0.959	0.933	0.905	0.877	0.851
0.4	0.982	0.960	0.936	0.909	0.882	0.856
0.45	0.982	0.961	0.938	0.912	0.886	0.861
0.5	0.982	0.962	0.939	0.915	0.890	0.865

Table 2B Crash ratio when there is biased holding of bubbles

R ω	0.05	0.1	0.15	0.2	0.25	0.3
0.25	1.380	1.426	1.491	1.578	1.664	1.701
0.3	1.331	1.350	1.373	1.395	1.407	1.395
0.35	1.287	1.290	1.291	1.286	1.271	1.238
0.4	1.246	1.239	1.227	1.208	1.179	1.137
0.45	1.206	1.191	1.172	1.145	1.110	1.064
0.5	1.167	1.147	1.122	1.091	1.053	1.006