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Abstract

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Efficiency may Improve when Defectors Exist[*]

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Keywords: efficiency, diversity, voluntary separation, Prisoner's Dilemma, endogenous partnerships.

JEL classification: C73.

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1 Introduction

When homogeneous players play Prisoner's Dilemma, it is commonly understood that the best outcome that players should aim for is mutual cooperation. If the game is one-shot, mutual cooperation cannot be an equilibrium outcome, and thus the "Dilemma" arises. If players can repeat the game with complete information, infinite repetition or unknown horizon admit an equilibrium with repeated mutual cooperation, under the threat of future punishment and sufficiently high discount factors (Friedman [7], Fudenberg and Maskin [8]). If players can only repeat the game for finitely many times, reputation building may sustain repeated mutual cooperation except for last periods, when there is a belief that the opponent may be of a particular type (Kreps et al. [14]).

Recently, endogenously repeated model of Prisoner's Dilemma type games is introduced (e.g., Datta [3], Ghosh and Ray [11], Kranton [13], Fujiwara-Greve and Okuno-Fujiwara [9], and McAdams [15].) This model is a natural extension of infinitely repeated Prisoner's Dilemma in a large society. It allows players to choose whether to keep the same partner or to strategically end the partnership, after each round of Prisoner's Dilemma. Players without a partner can find a new partner in a random matching process without information flow. However, because a player can defect, end the partnership, and find a new partner who cannot know the past actions of the new partner, repeated mutual cooperation from the onset of a new partnership is impossible in this model (e.g., Fujiwara-Greve and Okuno-Fujiwara [9], henceforth GO2009).

A natural question is then how much efficiency can be achieved under the endogenous repetition with no information flow to new partners. For symmetric (monomorphic) equilibria, GO2009 focused on *trust-building* strategies which play myopic D(efection) action for some periods in a new partnership and then shift to C-trigger type strategy with ending the partnership as punishment. (Since any in-game punishment can be avoided by unilateral ending of a partnership, severance is the maximal equilibrium punishment.) It was shown that sufficient length of trust-building periods would make a monomorphic equilibrium.

An alternative approach to efficiency is to consider asymmetric (polymorphic) equilibria including the most cooperative strategy, which starts with C with a new partner and continues to play C as long as the partners maintain mutual cooperation and ends the partnership otherwise. To make it a part of an equilibrium, the population must have another strategy that defects initially. We consider two types of such strategies, the one-period trust-building strategy, which is most efficient among trust-building strategies, and the most myopic strategy which plays D in any partnership and end it in one period. The myopic strategy corresponds to the irrational type players often assumed in the related incomplete information models (e.g., Ghosh and Ray [11], Kranton [13], Rob and Yang [16], and McAdams [15]).

Our main results are as follows. First, the "fundamentally asymmetric" bimorphic equilibrium consisting of the most cooperative strategy and the most myopic strategy always exists for sufficiently high survival rate of players (effective discount factors), while other combinations such as the monomorphic one-period trust-building strategy or the bimorphic combination of the most cooperative strategy and the one-period trust-building strategy may not constitute an equilibrium for any survival rate, depending on the payoff parameters.

Second, the fundamentally asymmetric equilibrium is also most efficient among **all** bimorphic equilibria involving the most cooperative strategy. Hence it is more efficient than the bimorphic equilibrium of the most cooperative strategy and the one-period trust-building strategy, if the latter exists. GO2009 has already shown that the latter is more efficient than the monomorphic equilibrium of the one-period trust-building strategy, if it exists. However, it is possible that the bimorphic equilibrium of the most cooperative strategy and the one-period trust-building strategy does not exist but the monomorphic one-period trust-building equilibrium exists. In this case we still need to compare the fundamentally asymmetric bimorphic equilibrium and the one-period trust-building equilibrium. We found that when the payoffs of the Prisoner's Dilemma satisfy the "small-stake condition", the fundamentally asymmetric equilibrium is more efficient than the monomorphic one-period trust-building equilibrium. Therefore, among the various equilibria using the three focal strategies, the one that allows defectors to occupy a part of the society forever is most efficient for a range of payoff parameters. Note that the players come from a single symmetric population and thus in any equilibrium they obtain the same long-run payoff. The small-stake condition means that the stake (difference between deviation payoff and cooperation payoff) is sufficiently smaller than the merit of cooperation (difference between mutual cooperation payoff and mutual defection payoff).

The ease of existence and the relative efficiency of the fundamentally asymmetric equilibrium of cooperators and defectors are a striking contrast to the equilibrium structure of ordinary repeated Prisoner's Dilemma, where the symmetric efficient payoff is attained by the symmetric C-trigger equilibrium. Our result may give a foundation to behavioral diversity in real markets, particularly those with small deviation gain.

There is a related work by Cho and Matsui [1], [2], in which pairs are randomly formed (from two finite populations) and the only choice of players is whether to agree to keep the relationship or to unilaterally terminate it, depending on the value created by the match. Their focus is how players settle with efficient "partnership values" in the long run. By contrast, we show that relative efficiency may result even though some players never care to establish a long-term cooperative relationship.

This paper is organized as follows. In Section 2 we formulate the Voluntarily Separable Repeated Prisoner's Dilemma, introduced by GO2009, and define the strategies of our focus. In Section 3 we give sufficient conditions for the existence of the equilibria with various combinations of the focal strategies. In Section 4 we compare efficiency of various equilibria and derive the small-stake condition. In Section 5 we give additional observations. All proofs are in Appendix.

2 Voluntarily Separable Repeated Prisoner's Dilemma

2.1 *Model*

Basic setting of the model is as in GO2009. Consider a large society of a continuum of **homogeneous** players of measure 1, over the discrete time horizon. At the beginning of each period, players without a partner enter a random matching process and form pairs.[1]

[1]For simplicity and following GO2009, we assume that a player finds a new partner for sure. This assumption makes cooperation most difficult.

	C	D
C	c, c	ℓ, g
D	g, ℓ	d, d

Table 1: Prisoner's Dilemma

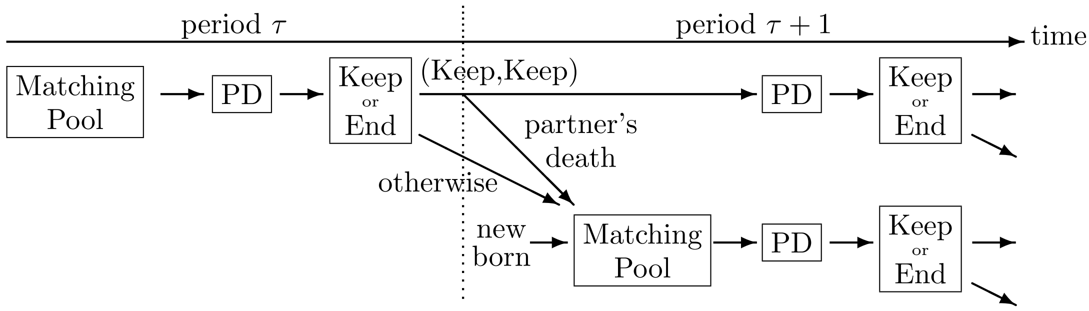


Figure 1: Outline of the VSRPD

Newly matched players have no knowledge of the past action history of each other, and they play the ordinary two-action Prisoner's Dilemma of Table 1. The actions in the Prisoner's Dilemma are observable only by the partners. After observing the actions in the Prisoner's Dilemma, the partners simultaneously choose whether to keep the partnership (action k) or to end it (action e). The partnership dissolves if at least one partner chooses action e. In addition, at the end of a period, each player may exit from the society for some exogenous reason (which we call a "death") with probability $1 - \delta$, where $0 < \delta < 1$. If a player dies, a new player enters into the society, keeping the population size constant. Players who lost the partner for some reason, as well as newly born players enter the matching pool in the next period. (This justifies the no-information-flow assumption because the players in the matching pool can have different backgrounds.) Therefore a partnership continues if and only if both partners choose action k and do not die. In this case the same partners play the Prisoner's Dilemma in the next period, skipping the matching process. The outline of VSRPD is depicted in Figure 1.

The one-shot payoffs in the Prisoner's Dilemma are shown in Table 1. Throughout the paper we maintain the assumption that the payoff parameter combination (g, c, d, ℓ) satisfies

$g > c > d > \ell$ and $2c \geqq g+\ell$. The latter is to make the symmetric action profile (C, C) efficient in one-shot. We denote by $PD = (g, c, d, \ell)$ the payoff parameter combinations satisfying these inequalities. We later specify additional conditions on the parameter combinations.

The game continues with probability δ from an individual player's point of view. Thus we focus on the expected total or average payoff, with δ being the effective discount factor of a player.

2.2 *Strategies*

Under the no-information-flow assumption, we focus on match-independent strategies[2] that only depend on the period $t = 1, 2, \ldots$ within a partnership (not the calendar time in the whole game) and the private history of actions within a partnership. Let $H_t := [\{C, D\} \times \{C, D\}]^{t-1}$ be *the set of partnership histories*[3] *at the beginning of* $t \geq 2$ and let $H_1 := \{\emptyset\}$.

Definition 1. A *pure strategy* s of VSRPD consists of $(x_t, y_t)_{t=1}^{\infty}$ where:
$x_t : H_t \to \{C, D\}$ specifies an action choice $x_t(h_t) \in \{C, D\}$ given the partnership history $h_t \in H_t$, and
$y_t : H_t \times \{C, D\}^2 \to \{k, e\}$ specifies whether to keep or end the partnership, depending on the partnership history $h_t \in H_t$ and the current period action profile.

Since any new partnership starts with a null history $\emptyset$, a pure strategy plays the same action $x_1(\emptyset) \in \{C, D\}$ at the beginning of any partnership. Subsequent actions depend on the history of actions within the current partnership only. The set of pure strategies of VSRPD is denoted as $\boldsymbol{S}$ and the set of all strategy distributions in the population is denoted as $\mathcal{P}(\boldsymbol{S})$.

We investigate the evolutionary stability of **stationary** strategy distributions **in the matching pool**. Although the strategy distribution in the matching pool may be different from the distribution in the entire society, if the former is stationary, the distribution of

[2] Since the population is a continuum, "contagious" strategies used in Kandori [12] and Ellison [6] are not useful in achieving cooperation.

[3] Note that only (k, k) throughout the past would allow players to choose actions. Hence the relevant histories on which players can condition their actions are the action combinations in the Prisoner's Dilemma only.

various states of matches is also stationary, thanks to the stationary death process.[4] Since each player is born into the random matching pool, the life-time payoff is determined by the strategy distribution in the matching pool. We assume that each player uses a pure strategy, which is natural in an evolutionary game and simplifies the analysis.

2.3 *Lifetime and Average Payoffs*

When a strategy $s \in \boldsymbol{S}$ is matched with another strategy $s' \in \boldsymbol{S}$, the *expected length* of the match is denoted as $L(s,s')$ and is computed as follows. Notice that even if s and s' intend to maintain the match, it will only continue with probability δ^2. Suppose that the planned length of the partnership of s and s' is $T(s,s')$ periods, if no death occurs. Then

$$L(s,s') := 1 + \delta^2 + \delta^4 + \cdots + \delta^{2\{T(s,s')-1\}} = \frac{1-\delta^{2T(s,s')}}{1-\delta^2}.$$

The *expected total discounted value of the payoff stream of* s *within the match with* s' is denoted as $V(s,s')$. The *average payoff* that s expects to receive within the match with s' is denoted as $v(s,s')$ and defined as follows.

$$v(s,s') := \frac{V(s,s')}{L(s,s')}, \text{ or } \ V(s,s') = L(s,s')v(s,s').$$

Next we show the structure of the lifetime and average payoff of a player endowed with strategy $s \in \boldsymbol{S}$ in the matching pool, waiting to be matched randomly with a partner. When a strategy distribution in the matching pool is $p \in \mathcal{P}(\boldsymbol{S})$ and is stationary, we write the *expected total discounted value of payoff streams* s expects to receive during his lifetime as $V(s;p)$ and the average payoff s expects to receive during his lifetime as

$$v(s;p) := \frac{V(s;p)}{L} = (1-\delta)V(s;p),$$

where $L = 1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta}$ is the expected lifetime of s. Thanks to the stationary distribution in the matching pool, we can write $V(s;p)$ as a recursive equation. If p has a

[4] See GO2009 footnote 7 for details.

finite/countable support, then we can write[5]

$$\begin{aligned} V(s;p) &= \sum_{s' \in supp(p)} p(s') \Big[V(s,s') \\ &\quad + [\delta(1-\delta)\{1+\delta^2+\cdots+\delta^{2\{T(s,s')-2\}}\} + \delta^{2\{T(s,s')-1\}}\delta] V(s;p) \Big], \end{aligned} \tag{1}$$

where $supp(p)$ is the support of the distribution p, the sum $\delta(1-\delta)\{1+\delta^2+\cdots+\delta^{2\{T(s,s')-2\}}\}$ is the probability that s loses the partner s' before $T(s,s')$, and $\delta^{2\{T(s,s')-1\}}\delta$ is the probability that the match continued until $T(s,s')$ and s survives at the end of $T(s,s')$ to go back to the matching pool. Thanks to the stationarity of p, the continuation payoff after a match ends for any reason is always $V(s;p)$. Let $L(s;p) := \sum_{s' \in supp(p)} p(s')L(s,s')$. By computation,

$$\begin{aligned} V(s;p) &= \sum_{s' \in supp(p)} p(s') \Big[V(s,s') + \{1-(1-\delta)L(s,s')\}V(s;p) \Big] \\ &= \sum_{s' \in supp(p)} p(s')V(s,s') + \{1 - \frac{L(s;p)}{L}\}V(s;p). \end{aligned} \tag{2}$$

Hence the average payoff is a nonlinear function of the strategy distribution p:

$$v(s;p) := \frac{V(s;p)}{L} = \sum_{s' \in supp(p)} \frac{p(s')V(s,s')}{L(s;p)}. \tag{3}$$

2.4 *Stability Concepts*

We now define stability concepts.

Definition 2. Given a stationary strategy distribution in the matching pool $p \in \Delta(\mathbf{S})$, $s \in \mathbf{S}$ is a *best reply against* p if, for all $s' \in \mathbf{S}$,

$$v(s;p) \geqq v(s';p),$$

and is denoted as $s \in BR(p)$.

Definition 3. A stationary strategy distribution in the matching pool $p \in \Delta(\mathbf{S})$ is a *Nash equilibrium* if, for all $s \in supp(p)$, $s \in BR(p)$.

[5]Theorem 1 and Remark 1 of Duffie and Sun [4] show that the matching probability of a particular strategy is the fraction of the strategy in the pool.

From the evolutionary perspective, a Nash equilibrium is a robust distribution against single (measure zero) entrants. Let us introduce a stronger stability concept which requires robustness against a positive measure of entrants. Different stability concepts are obtained by the difference in the potential set of entrants. The next notion restricts entrants to be incumbent strategies only.

Definition 4. A stationary strategy distribution in the matching pool $p \in \Delta(\mathbf{S})$ is a *Locally Stable Nash equilibrium* if,
(i) p is a Nash equilibrium; and
(ii) for any $s' \in supp(p)$, there exists $\bar{\epsilon} \in (0,1)$ such that, for any $\epsilon \in (0, \bar{\epsilon})$ and any $s \in supp(p) \setminus \{s'\}$,

$$v(s; (1-\epsilon)p + \epsilon s') \geqq v(s'; (1-\epsilon)p + \epsilon s'),$$

and there exists $\tilde{s} \in supp(p) \setminus \{s'\}$ such that

$$v(\tilde{s}; (1-\epsilon)p + \epsilon s') > v(s'; (1-\epsilon)p + \epsilon s').$$

As it becomes clear below, local stability selects among asymmetric Nash equilibria, consisting of multiple strategies in the population. Note, however, that for monomorphic Nash equilibria (consisting of a single strategy played by all players), local stability is vacuous. There is a stronger notion of neutrally stable distribution (GO2009), which requires the distribution be robust against any entrant (not just incumbent strategies) of a small measure. Since we focus on the payoff efficiency, not stability, we refer our readers to GO2009 and Fujiwara-Greve and Okuno-Fujiwara [10] for a further stability analysis of the strategies of our focus.

2.5 *Strategies of Focus*

We investigate the equilibrium payoffs among cooperative players and defectors. For cooperative strategies, GO2009 focused on the following *trust-building* strategies.

Definition 5. For any $T = 0, 1, 2, \ldots$, let c_T-strategy be a strategy[6] as follows:

[6] To be very precise, we are not specifying off-path actions. Thus one can think that we are dealing with a class of c_T-strategies for each $T = 0, 1, 2, \ldots$. Alternatively, we can fix a strategy in each class of c_T-strategies, without loss of generality.

$t \leqq T$: Play D and keep the partnership if and only if (D, D) is observed in the current period.

$t \geqq T+1$: Play C and keep the partnership if and only if (C, C) is observed in the current period.

The trust-building strategies start a new partnership with D, but after T periods, shift to C-trigger type behavior with ending the partnership as punishment. Ending the partnership is the maximal equilibrium punishment in VSRPD, since any in-match punishment can be avoided by unilateral severance. Note that the above definition includes the degenerate strategy c_0, which cooperates with a stranger.

GO2009 focused on this class of strategies, particularly for $T \geqq 1$, because, when players can unilaterally end a partnership without information flow, any strategy combination that starts with C is never an equilibrium in VSRPD. Against such strategy combinations, a player who plays D and end the partnership immediately would earn the highest one-shot payoff g in every partnership.

Lemma 1. (GO2009) *For any $\delta \in (0,1)$, any strategy distribution $p \in \mathcal{P}(\boldsymbol{S})$ such that all strategies in the support start with C in $t=1$ is not a Nash equilibrium.*

Lemma 1 applies to the monomorphic distribution of c_0-strategy, which is the most efficient symmetric strategy combination. This is a striking contrast to ordinary repeated Prisoner's Dilemma where the monomorphic distribution of the C-trigger strategy constitutes a subgame perfect equilibrium for sufficiently high δ. The problem of VSRPD is the lack of information flow to new partners. Because no personalized punishment is possible, the strategy combination must embed punishment. One way to do it is the initial trust-building periods. However, the initial trust-building periods is a welfare loss.[7]

Alternatively, we can construct polymorphic equilibria including c_0-strategy. To constitute an equilibrium, there must be other strategies in the society that play D initially. While

[7]Eeckhout [5] adds initial correlation stage to selectively cooperate with some strangers, which improves efficiency. In this paper we do not assume that joint randomization is possible among strangers.

any non-degenerate trust-building strategy can be a candidate, and among them c_1-strategy is most efficient, we also consider the most myopic strategy, denoted as d_0-strategy, as follows.[8]

Definition 6. Let d_0-strategy be a strategy as follows:

Play D and end the partnership regardless of the observation in this period.

The d_0-strategy is often assumed to occupy a positive fraction in the society (permanently) in incomplete information versions of VSRPD (e.g., Ghosh and Ray[11], Kranton [13], Rob and Yang [16], and McAdams [15]). A motivation of the incomplete information models is to construct a symmetric cooperative equilibrium among "rational players" from which bilateral deviation is also impossible. Incomplete information with the possibility that a randomly met opponent never plays C serves as an incentive device for rational players to establish a cooperative relationship as soon as possible. However, it has not been investigated whether the non-cooperative type players can fare as well as rational players.[9]

3 Existence

GO2009 gives a sufficient condition on the payoff parameters which warrants that c_1-strategy played by all players is a Nash equilibrium (and moreover neutrally stable). Among monomorphic trust-building strategy equilibria, clearly this is most efficient.

Remark 1. (GO2009) *For any PD= (g, c, d, ℓ) such that $g - c < c - d$, let*

$$\underline{\delta}_{c1} := \sqrt{\frac{g-c}{c-d}}.$$

Then $\underline{\delta}_{c1} \in (0, 1)$ and for any $\delta \in (\underline{\delta}_{c1}, 1)$, the monomorphic distribution of c_1-strategy is a Nash equilibrium.

Next, we turn to bimorphic equilibria involving c_0-strategy. Our first result is to establish that for any payoff parameter combination, for sufficiently high δ, there is a locally stable Nash equilibrium consisting of c_0- and d_0-strategy.

[8] In GO2009, this strategy is denoted as $\tilde{d}$-strategy.

[9] The companion paper by Fujiwara-Greve and Okuno-Fujiwara [10] considers a larger class of defecting strategies (see also Section 5) and analyzes evolutionary stability of the c_0-d_0 equilibrium.

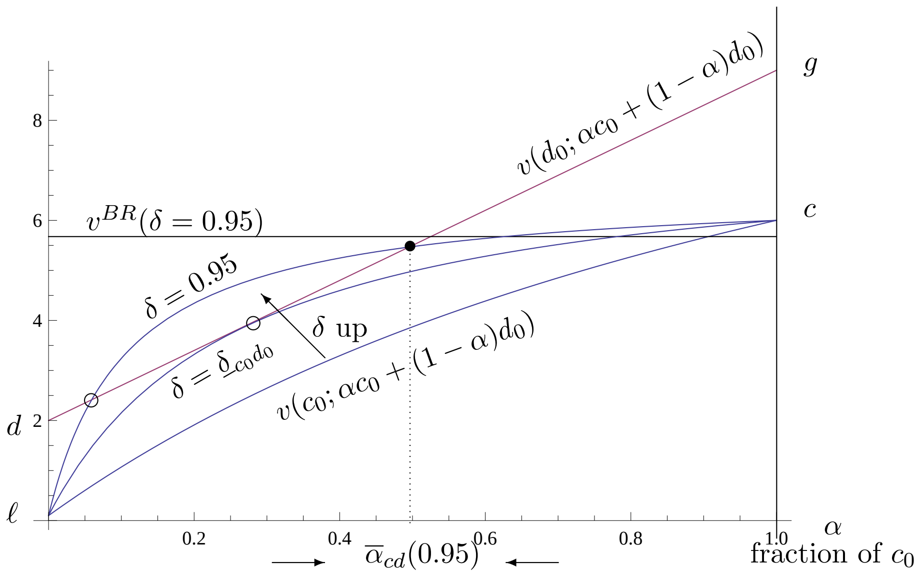


Figure 2: Existence of $\underline{\delta}_{c0d0}$ and the c_0-d_0 equilibrium

Proposition 1. *For any PD= (g, c, d, ℓ), there exists $\underline{\delta} \in (0, 1)$ such that for any $\delta \in (\underline{\delta}, 1)$, there exists $\overline{\alpha}_{cd}(\delta) \in (0, 1)$ such that $\overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0$ is a locally stable Nash equilibrium. Let the smallest such $\underline{\delta}$ be $\underline{\delta}_{c0d0}$.*

We give an intuition for the above result, which will be useful in the later analysis as well. The average payoffs of c_0- and d_0-strategies, when the stationary matching pool distribution is $\alpha c_0 + (1 - \alpha)d_0$, are as follows.

$$v(c_0; \alpha c_0 + (1 - \alpha)d_0) = \frac{\alpha \cdot \frac{c}{1-\delta^2} + (1 - \alpha)\ell}{\alpha \cdot \frac{1}{1-\delta^2} + 1 - \alpha} \tag{4}$$

$$v(d_0; \alpha c_0 + (1 - \alpha)d_0) = \alpha g + (1 - \alpha)d. \tag{5}$$

To explain, the numerator of $v(c_0; \alpha c_0 + (1 - \alpha)d_0)$ is the expected total payoff of c_0-strategy from two kinds of partnerships, and the denominator is the expected length of the two kinds of matches. The c_0-strategy ends the partnership as soon as d_0-strategy plays D in the first period. The average payoff of d_0-strategy is also the expected payoff from two kinds of partnerships, both of which end in one period.

The average payoff of c_0-strategy is concave in its share α in the matching pool, while that of d_0-strategy is linear. As δ increases, the average payoff of c_0-strategy becomes more concave and at some $\underline{\delta}_{c_0 d_0} \in (0,1)$, it touches the average payoff function of d_0-strategy from below. See Figure 2. Above $\underline{\delta}_{c_0 d_0}$, there will be two intersections between the two average payoffs. Let the intersection with a larger share of c_0-strategy be $\overline{\alpha}_{cd}(\delta)$, then the bimorphic distribution $\overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0$ is payoff-equalizing and locally stable. (See Figure 2.)

It turns out that payoff-equivalence implies the *Best Reply Condition*, which is a sufficient condition for a Nash equilibrium (GO2009). Let v^M be the common average payoff of strategies in the matching pool. Then the Best Reply Condition requires that a one-step deviation is not beneficial, that is

$$g + \delta \frac{v^M}{1-\delta} \leqq \frac{c}{1-\delta^2} + \frac{\delta(1-\delta)}{1-\delta^2} \cdot \frac{v^M}{1-\delta} \iff v^M \leqq \frac{1}{\delta^2}\{c - (1-\delta^2)g\} =: v^{BR}. \tag{6}$$

Lemma 2. *For any $\delta \in (0,1)$ and any $\alpha \in (0,1)$ such that*

$$v(d_0; \alpha c_0 + (1-\alpha)d_0) = v(c_0; \alpha c_0 + (1-\alpha)d_0),$$

the common average payoff is strictly less than v^{BR}, that is, the payoff-equalizing distribution $\alpha c_0 + (1-\alpha)d_0$ is a Nash equilibrium.

Hence the payoff-equalizing distribution $\overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0$ is a locally stable Nash equilibrium. (The other payoff-equalizing distribution with a smaller share of c_0-strategy is also a Nash equilibrium but not locally stable.)

We now show a sufficient condition of payoff parameters to warrant the existence of a locally stable Nash equilibrium consisting of c_0- and c_1-strategy. The idea is as follows. First, notice that, for any strategy s_0 which plays D in the first period of a partnership, the average payoff of c_0-strategy facing the stationary strategy distribution $\alpha c_0 + (1-\alpha)s_0$ is the same as the one for $\alpha c_0 + (1-\alpha)d_0$, since the play path of c_0-strategy is the same.

Second, the average payoff of c_1-strategy facing $\alpha c_0 + (1-\alpha)c_1$ in the matching pool is a **convex** function of α as below (see also GO2009).

$$v(c_1; \alpha c_0 + (1-\alpha)c_1) = \frac{\alpha \cdot g + (1-\alpha)(d + \delta^2 \frac{c}{1-\delta^2})}{\alpha \cdot 1 + (1-\alpha)\frac{1}{1-\delta^2}}. \tag{7}$$

To explain, the numerator is the expected in-match payoff for c_1-strategy when it is matched with a c_0-strategy (first term) and with a c_1-strategy (second term). The match with a

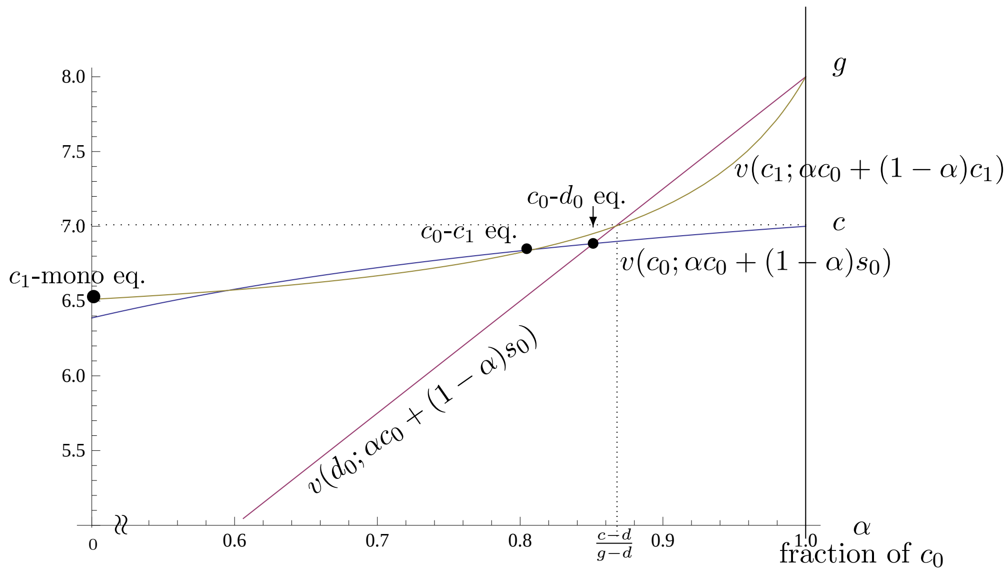


Figure 3: Average payoff functions of c_0-, d_0-, and c_1-strategy when $\delta > \underline{\delta}_{c_0 c_1}$

c_0-strategy ends immediately by c_0-strategy's severance, while the match with a c_1-strategy continues as long as the partners live. This makes the denominator.

We need to find the minimum δ which makes the average payoff function $v(c_0; \alpha c_0 + (1-\alpha)c_1)$ "touch" the average payoff function $v(c_1; \alpha c_0 + (1-\alpha)c_1)$ from below. However, the average payoff of c_1-strategy is higher than that of d_0-strategy (see Figure 3) as long as the average payoff is not more than c, which is the range of the average payoff of c_0-strategy.

Lemma 3. *For any pure strategy s_0 that plays D in the first period of a newly formed match,*

$$v(c_1; \alpha c_0 + (1-\alpha)c_1) \geqq v(d_0; \alpha c_0 + (1-\alpha)s_0) \iff c \geqq v(d_0; \alpha c_0 + (1-\alpha)s_0).$$

Therefore, even if the lower average payoff function $v(d_0; \alpha c_0 + (1-\alpha)s_0)$ intersects with $v(c_0; \alpha c_0 + (1-\alpha)s_0)$ (i.e., the c_0-d_0 equilibrium exists), the higher average payoff function $v(c_1; \alpha c_0 + (1-\alpha)c_1)$ may not intersect. This means that it is more difficult to warrant a c_0-c_1 equilibrium to exist. (See Corollary 1 below.) In order to guarantee that the average payoff functions of c_1- and c_0-strategy intersect at some $\delta \in (0, 1)$, we need to restrict payoff parameters as follows.

Proposition 2. *For any PD*$= (g, c, d, \ell)$ *such that* $g - c < \frac{(c-d)^2}{4(c-\ell)}$, *there exists* $\underline{\delta} \in (0, 1)$ *such that, for any* $\delta \in (\underline{\delta}, 1)$, *there exists a locally stable Nash equilibrium with support* $\{c_0, c_1\}$. *Let the smallest of such* $\underline{\delta}$ *be* $\underline{\delta}_{c_0 c_1}$.

Corollary 1. *For any PD*$= (g, c, d, \ell)$ *such that* $g - c < \frac{(c-d)^2}{4(c-\ell)}$, $\underline{\delta}_{c0d0} < \underline{\delta}_{c0c1}$.

We can also show that the existence of c_1-monomorphic equilibrium is easier than that of c_0-c_1 equilibrium. This is because, even if the average payoff functions of c_0- and c_1-strategy do not intersect, the latter at $\alpha = 0$ (which is the case of c_1-monomorphic distribution in the matching pool) can be below v^{BR}. (GO2009 has a similar argument. See also Figure 3.)

Corollary 2. *For any PD*$= (g, c, d, \ell)$ *such that* $g - c < \frac{(c-d)^2}{4(c-\ell)}$, $\underline{\delta}_{c1} < \underline{\delta}_{c0c1}$.

Finally, whether c_1-monomorphic equilibrium is easier to exist than c_0-d_0 equilibrium depends on the size of the "stake" $g - c$ (which must be smaller than $c - d$ for c_1-monomorphic equilibrium to exist) as follows.

Corollary 3. *For any PD* $= (g, c, d, \ell)$ *such that* $g - c < c - d$, *when* $g \downarrow c$, *then* $\underline{\delta}_{c1} < \underline{\delta}_{c0d0}$. *When* $g \uparrow (2c - d)$, *then* $\underline{\delta}_{c1} > \underline{\delta}_{c0d0}$.

In words, when the stake $g - c$ is close to $c - d$, the fundamentally asymmetric c_0-d_0 equilibrium is easier to exist.

4 Efficiency

4.1 *Small-Stake Condition*

From Lemma 3 and the fact that $v(c_0; \alpha c_0 + (1 - \alpha)s_0)$ (where s_0 is any strategy that plays D in the first period of a match) is increasing in the share α of c_0-strategy, if both of c_0-d_0 equilibrium and c_0-c_1 equilibrium exist, the one that involves d_0-strategy has a higher share of c_0-strategy. This is easy to see from Figure 3 as well. In fact, we can show a stronger result that c_0-d_0 equilibrium is **most efficient** among any bimorphic Nash equilibrium involving c_0-strategy.

Proposition 3. *For any PD= (g, c, d, ℓ) and any $\delta \in (0,1)$, let $s \in \boldsymbol{S}$ be any pure strategy such that there exists $\alpha \in (0,1)$ such that $\alpha c_0 + (1-\alpha)s$ is a Nash equilibrium. Then*

$$v(c_0; \overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0) \geqq v(c_0; \alpha c_0 + (1-\alpha)s).$$

From Proposition 3, it is immediate that even if c_0-c_1 equilibrium exists, c_0-d_0 equilibrium is more efficient.

Corollary 4. *For any PD= (g, c, d, ℓ) such that $g - c < \frac{(c-d)^2}{4(c-\ell)}$ and any $\delta \in (\underline{\delta}_{c_0 d_0}, 1)$, the bimorphic locally stable Nash equilibrium with the support $\{c_0, d_0\}$ is more efficient than the bimorphic locally stable Nash equilibrium with the support $\{c_0, c_1\}$.*

In GO2009, it is also shown that a bimorphic trust-building strategy equilibrium is more efficient than a symmetric equilibrium using the longer trust-building strategy.

Remark 2. (GO2009) *For any PD= (g, c, d, ℓ) and any $\delta \in (0,1)$ such that the c_0-c_1 bimorphic equilibrium exists, it is more efficient than the (most efficient) monomorphic equilibrium consisting of c_1-strategy.*

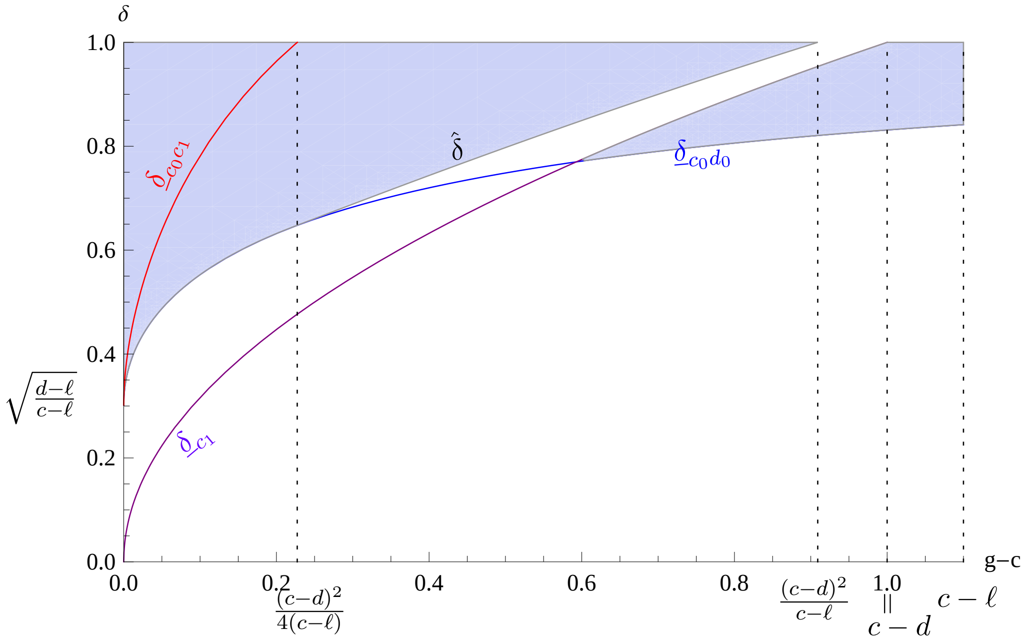
To combine the results, if the c_0-c_1 bimorphic equilibrium exists, by the payoff transitivity, the c_0-d_0 bimorphic equilibrium is more efficient than the c_1-monomorphic equilibrium. However, as we have seen, the c_0-c_1 bimorphic equilibrium may not exist, even if the other two equilibria exist. Thus, let us compare the efficiency between the c_0-d_0 bimorphic equilibrium and the c_1-monomorphic equilibrium without the payoff transitivity.

Proposition 4. *For any PD= (g, c, d, ℓ) such that*

$$g - c < \frac{(c-d)^2}{c - \ell}, \tag{8}$$

there exists $\hat{\delta} \in [\max\{\underline{\delta}_{c_0 d_0}, \underline{\delta}_{c_1}\}, 1)$ such that the bimorphic locally stable Nash equilibrium $\overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0$ is more efficient than the monomorphic equilibrium of c_1-strategy, for any $\delta \in (\hat{\delta}, 1)$

Therefore, if condition (8) is satisfied, then the fundamentally asymmetric equilibrium of c_0- and d_0-strategy is most efficient among the equilibria of our focus, for sufficiently high



Parameter values: $(c, d, \ell) = (1, 0, -.1)$

Figure 4: Comparison of $\underline{\delta}_{c_0d_0}, \underline{\delta}_{c_0c_1}, \underline{\delta}_{c_1}$, and $\hat{\delta}$

δ. Note also that among symmetric trust-building strategy equilibria, the c_1-monomorphic equilibrium is most efficient, and hence c_0-d_0 equilibrium is more efficient than any symmetric trust-building strategy equilibria.

We call (8) the *small-stake condition*. An interpretation is that the "stake" $g - c$ is sufficiently smaller (recall that $\frac{c-d}{c-\ell} < 1$) than the merit $c - d$ of cooperation. For example, when the D action means petty crimes or cheating in small value transactions, the small-stake condition is plausible.

4.2 *Synthesis of Existence and Efficiency Conditions*

The above various conditions are comparable as conditions on the size of the stake, $g - c$. Let us illustrate the sufficient minimum survival rate δ in relation to the size of the stake. In Figure 4, we marked two regions of δ in which the presence of defectors d_0-strategy gives the most efficient equilibrium.

The horizontal axis of Figure 4 represents the "stake" $g - c$ (which is bounded by $c - \ell$ under the convexity assumption of the one-shot payoffs) and the vertical axis is the survival rate δ. As the stake increases, the minimum discount factors that warrant the existence of various equilibria increases, since larger stake means bigger temptation to deviate. The efficiency-related minimum discount factor is $\hat{\delta}$ above which the fundamentally asymmetric equilibrium is most efficient among **all** equilibria consisting of trust-building strategies. The boundary of $g - c$ at which $\hat{\delta}$ hits 1 is the bound imposed by the small-stake condition. In addition, if other equilibria fail to exist, the c_0-d_0 equilibrium is trivially most efficient.

Figure 4 also illustrates that the c_0-d_0 equilibrium always exists for sufficiently large δ, while other kinds of equilibria disappear as the stake increases.

5 Concluding Remarks

We have shown that, for sufficiently high survival rates, the fundamentally asymmetric c_0-d_0 equilibrium exists (Proposition 1), and that, for a range of payoff parameters, it is most efficient among the equilibria we focus (Proposition 3). Since longer trust-building periods would reduce the payoffs, if the c_0-d_0 equilibrium is more efficient than c_1-monomorphic equilibrium, then it is more efficient than any c_T-monomorphic equilibrium with $T \geqq 1$.

An important application of Proposition 3 is that many bimorphic distributions involving c_0-strategy fail to become an equilibrium. Let us extend the notion of d_0-strategy by adding initial trust-building periods.

Definition 7. For any $T = 1, 2, \ldots$, let d_T-strategy be a strategy as follows:
$t \leqq T$: Play D and keep the partnership if and only if (D, D) is observed in the current period.
$t \geqq T + 1$: Play D and end the partnership regardless of observation.

We can show that any bimorphic distribution of the form c_0-c_T (for sufficiently large T) or c_0-d_T (for any $T \geqq 1$) cannot be a locally stable Nash equilibrium. This is because the value of the average payoff function of c_T- (for sufficiently large T) or d_T-strategy is lower than that of d_0-strategy, as functions of α, when it intersects with the average payoff function

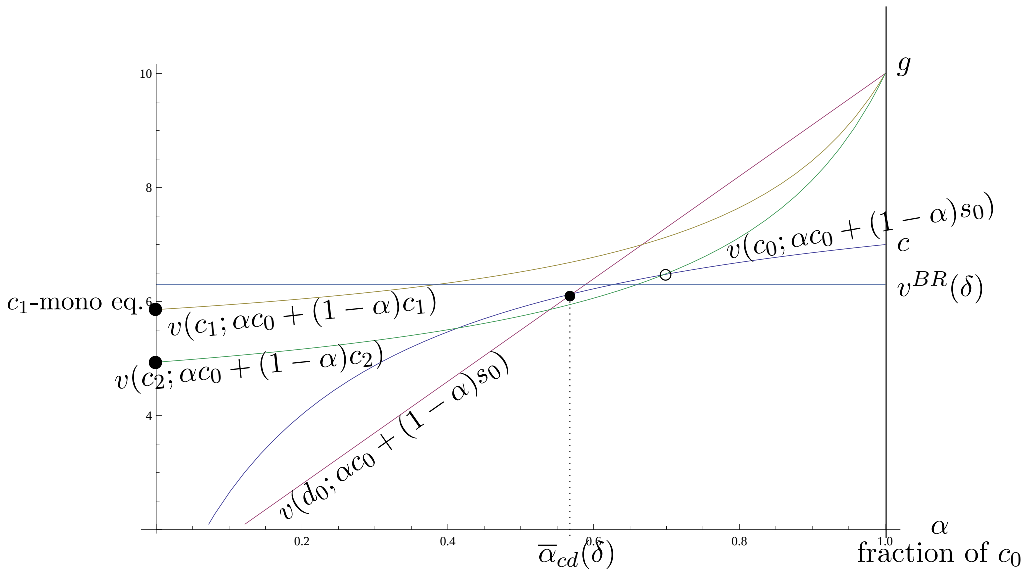


Figure 5: Average payoff functions of c_0-, d_0-, c_1-, and c_2-strategy

of c_0-strategy from below. Figure 5 illustrates this property for c_2-strategy. Hence, if there is a locally stable payoff-equalizing combination, the average payoff function of c_0-strategy intersects with that of the other strategy at a higher α than $\overline{\alpha}_{cd}(\delta)$, but by Proposition 3, such a distribution violates the Best Reply condition (6).

Corollary 5. *For any PD$= (g, c, d, \ell)$ and any $\delta \in (\underline{\delta}_{c0d0}, 1)$, there exists $\underline{T} \geqq 2$ such that for any $s \in \{c_{\underline{T}}, c_{\underline{T}+1}, \ldots, \} \cup \{d_1, d_2, \ldots\}$, any bimorphic distribution consisting of c_0- and s-strategy cannot be a locally stable Nash equilibrium.*

In general, c_1-monomorphic equilibrium may not exist so that this paper's comparison between c_0-d_0 equilibrium and c_1-equilibrium may be irrelevant. However, for sufficiently large T, c_T-monomorphic equilibrium exists, and our analysis can be extended to a comparison between c_T-monomorphic equilibrium and c_{T-1}-d_{T-1} equilibrium.

Finally, note that the bimorphic equilibrium of c_0- and d_0-strategy has a weakness that it is vulnerable to a coordinated invasion of c_1-strategy (Fujiwara-Greve and Okuno-Fujiwara [10] and Vesely and Yang [17]). In other words, it is not neutrally stable. This is because the c_1-strategy behaves like d_0-strategy when it meets one of the incumbents but cooperates from the second period on when meeting the same strategy. However, a possible problem of the positive measure of c_1-strategy entrants is that they must coordinate on the timing to shift to cooperation, but the source of such "group norm" is unclear.

Moreover, in the companion paper, Fujiwara-Greve and Okuno-Fujiwara [10], it is shown that (i) the bimorphic equilibrium is robust against a large class of "diverse" entrant distributions, and (ii) among payoff-equivalent Nash equilibria, the bimorphic equilibrium is locally stable for any $\delta \geqq \underline{\delta}_{c0d0}$, while others are not. Hence, we can conclude that the fundamentally asymmetric equilibrium is reasonably stable.

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Appendix

Proof of Lemma 2 : Fix any $\delta \in (0,1)$ and any $\alpha \in (0,1)$ such that

$$v(d_0; \alpha c_0 + (1-\alpha)d_0) = v(c_0; \alpha c_0 + (1-\alpha)d_0).$$

For notational brevity, let $p = \alpha c_0 + (1-\alpha)d_0$ and v be the common average payoff of c_0- and d_0-strategy in the matching pool;

$$v := v(d_0; \alpha c_0 + (1-\alpha)d_0) = v(c_0; \alpha c_0 + (1-\alpha)d_0).$$

We show that the Best Reply Condition (6), which is equivalent to

$$g + \delta \frac{v}{1-\delta} \leqq \frac{c}{1-\delta^2} + \frac{\delta(1-\delta)}{1-\delta^2} \cdot \frac{v}{1-\delta} \iff g - \frac{c}{1-\delta^2} + \frac{\delta^2}{1-\delta^2} v \leqq 0$$

holds with the strict inequality.

From the payoff-equivalence and the fact that $v(d_0; p) = V(d_0; p)$, we have

$$L(c_0; p)V(d_0; p) - V(c_0; p) = 0,$$

where

$$
\begin{aligned}
L(c_0;p) &= \alpha \cdot \frac{1}{1-\delta^2} + (1-\alpha) = 1 + \delta^2 \frac{\alpha}{1-\delta^2} \\
V(d_0;p) &= \alpha g + (1-\alpha)d \\
V(c_0;p) &= \alpha \frac{c}{1-\delta^2} + (1-\alpha)\ell.
\end{aligned}
$$

Hence

$$
\begin{aligned}
0 = & L(c_0;p)V(d_0;p) - V(c_0;p) \\
= & \Big\{1 + \delta^2 \frac{\alpha}{1-\delta^2}\Big\}\{\alpha g + (1-\alpha)d\} - \{\alpha \frac{c}{1-\delta^2} + (1-\alpha)\ell\} \\
= & \alpha\{g - \frac{c}{1-\delta^2} + \frac{\delta^2}{1-\delta^2} v(d_0;p)\} + (1-\alpha)(d-\ell).
\end{aligned}
$$

Since $d > \ell$ and $\alpha \in (0,1)$,

$$
g - \frac{c}{1-\delta^2} + \frac{\delta^2}{1-\delta^2} v < 0
$$

i.e., (6) holds with the strict inequality. □

Proof of Proposition 1 : In view of Lemma 2, it suffices to prove that the average payoff functions of c_0- and d_0-strategy intersect.

From (5), the average payoff of d_0-strategy against the distribution $p = \alpha c_0 + (1-\alpha)d_0$ is a linear function of α. From (4), the average payoff of c_0-strategy against p is concave in α. To see this, note that

$$
\frac{\partial v(c_0;p)}{\partial \alpha} = \frac{(1-\delta^2)(c-\ell)}{\{1-\delta^2(1-\alpha)\}^2},
$$

which is decreasing in α. Moreover, as δ increases, the concavity also increases, i.e., for any $\alpha \in (0,1)$,

$$
\frac{\partial v(c_0;p)}{\partial \delta} = \frac{2(1-\alpha)\alpha\delta(c-\ell)}{\{1-\delta^2(1-\alpha)\}^2} > 0.
$$

Hence, if the two average payoff functions intersect, the payoff-equalizing α's exist in $(0,1)$.

Let the "effective" average payoff difference be

$$
\begin{aligned}
f(\alpha;\delta) \quad &:= \quad (1-\delta^2)(\frac{\alpha}{1-\delta^2} + 1 - \alpha)\big\{v(c_0;p) - v(d_0;p)\big\} \\
&= \quad -\delta^2(g-d)\alpha^2 + \{(c+d) - (g+\ell) + \delta^2(g+\ell-2d)\}\alpha - (d-\ell)(1-\delta^2).
\end{aligned}
$$

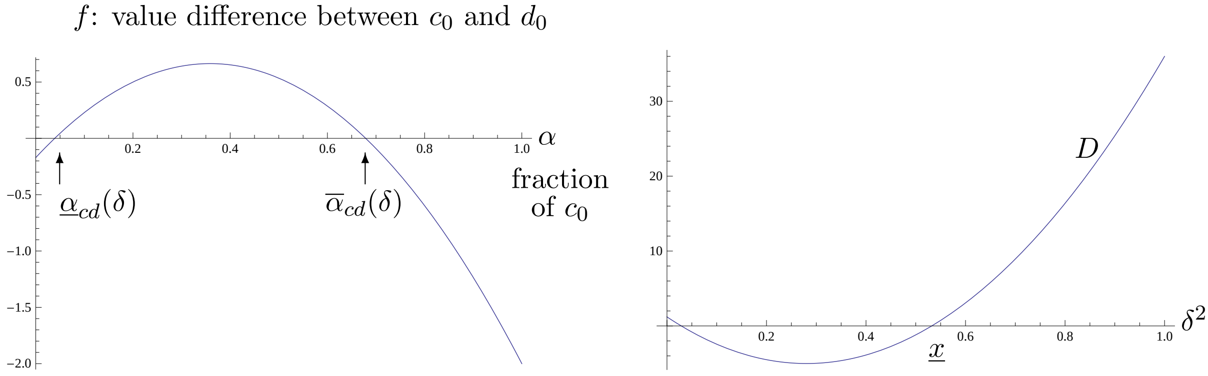

Figure 6: Existence of $\underline{\alpha}_{cd}(\delta) < \overline{\alpha}_{cd}(\delta)$ and $\underline{\delta}$[10]

We show that the quadratic equation of α, $f(\alpha;\delta) = 0$, has two real roots for sufficiently high δ, which is equivalent to the discriminant of the quadratic equation to be positive.

Let us write down the discriminant.

$$\begin{aligned} D(\delta) &= \{(c+d) - (g+\ell) + \delta^2(g+\ell-2d)\}^2 - 4\delta^2(1-\delta^2)(g-d)(d-\ell), \\ &= \{(g+\ell-2d)^2 + 4(g-d)(d-\ell)\}\delta^4 \\ &\quad +2[\{(g-d)-(d-\ell)\}\{(c+d)-(g+\ell)\} - 2(g-d)(d-\ell)]\delta^2 \\ &\quad +\{(c+d)-(g+\ell)\}^2. \end{aligned}$$

Letting $x := \delta^2$, we have

$$D(x) = A_D \cdot x^2 + B_D \cdot x + \{(c+d)-(g+\ell)\}^2, \tag{9}$$

where

$$\begin{aligned} A_D &= (g+\ell-2d)^2 + 4(g-d)(d-\ell) > 0; \\ B_D &= 2[\{(g-d)-(d-\ell)\}\{(c+d)-(g+\ell)\} - 2(g-d)(d-\ell)]. \end{aligned}$$

$D(x)$ is a quadratic and convex function of x. It is easy to see that $D(0) = \{(c+d)-(g+\ell)\}^2 > 0$ and $D(1) = (c-d)^2 > 0$ as well. By computation, the discriminant of the (again quadratic)

[10] The parameter combination is $(g, c, d, \ell) = (9, 7, 1, 0.1)$ and $\delta = 0.9$ for the left figure.

equation $D(x) = 0$ is

$$16(c-\ell)(d-\ell)(g-c)(g-d) > 0. \tag{10}$$

Therefore, there exist two solutions to $D(x) = 0$. Moreover, by computation,

$$\begin{aligned}
\frac{\partial D}{\partial x}(x) &= 2\{(c+d)-(g+\ell)+x(g+\ell-2d)\}(g+\ell-2d)-4(g-d)(d-\ell)(1-2x), \quad (11)\\
\frac{\partial D}{\partial x}(0) &= 2\{(c+d)-(g+\ell)\}\{(g-d)-(d-\ell)\}-4(g-d)(d-\ell)\\
&= 2\Big[(g-d)\{c+d-g-\ell-2(d-\ell)\}-\{c+d-(g+\ell)\}(d-\ell)\Big]\\
&= 2\Big[(g-d)\{-(g-c)-(d-\ell)\}-(d-\ell)\{-(g-c)+(d-\ell)\}\Big]\\
&= -2\{(g-d)(g-c)+(d-\ell)(c-\ell)\} < 0, \quad (12)\\
\frac{\partial D}{\partial x}(1) &= 2(c-d)\{(g-d)-(d-\ell)\}+4(g-d)(d-\ell)\\
&= 2\Big[(c-d)(g-d)-(c-d)(d-\ell)+2(g-d)(d-\ell)\Big]\\
&= 2\{(c-\ell)(g-d)+(g-c)(d-\ell)\} > 0. \quad (13)
\end{aligned}$$

Together with the fact that the two end-point values $D(0)$ and $D(1)$ are both positive, the derivative properties imply that there exists $\underline{x} \in (0,1)$ such that $D(x) > 0$ for any $x > \underline{x}$. Then, there exist two solutions to $f(\alpha;\delta) = 0$ (or, $v(c_0;p) = v(d_0;p)$) for any $\delta > \underline{\delta} = \sqrt{\underline{x}}$.

Local stability is obvious from the concavity of $v(c_0;p)$ and the linearity of $v(d_0;p)$. This can be also seen from the left figure of Figure 6 that, only at the larger solution $\overline{\alpha}_{cd}(\delta)$ to $f(\alpha;\delta) = 0$, if c_0-strategy increases the share, the value difference becomes negative. □

Proof of Lemma 3 : The equation (7) can be rearranged as

$$\begin{aligned}
v(c_1;\alpha c_0+(1-\alpha)c_1) &= \frac{\alpha g+(1-\alpha)d+(1-\alpha)\delta^2\frac{c}{1-\delta^2}}{1+(1-\alpha)\delta^2\frac{1}{1-\delta^2}}\\
&= v(d_0;\alpha c_0+(1-\alpha)d_0)\\
&\quad+\frac{(1-\alpha)\frac{\delta^2}{1-\delta^2}}{1+(1-\alpha)\delta^2\frac{1}{1-\delta^2}}\{c-v(d_0;\alpha c_0+(1-\alpha)s_0)\}.
\end{aligned}$$

Hence, $c \geq v(d_0;\alpha c_0+(1-\alpha)s_0)$ if and only if $v(c_1;\alpha c_0+(1-\alpha)c_1) \geq v(d_0;\alpha c_0+(1-\alpha)d_0)$ holds. □

Proof of Proposition 2 : First, we show that, if $v(c_0;\alpha c_0+(1-\alpha)c_1) = v(c_1;\alpha c_0+(1-\alpha)c_1)$ holds, then $\alpha c_0+(1-\alpha)c_1$ satisfies the Best Reply condition (6), i.e., it is a Nash equilibrium.

Lemma 3 showed that the average payoff function of c_1-strategy is higher than that of d_0-strategy (when facing the same fraction α of c_0-strategy) as long as the average payoff is not more than c. Hence, if $v(c_0; \tilde{\alpha}c_0 + (1-\tilde{\alpha})c_1) = v(c_1; \tilde{\alpha}c_0 + (1-\tilde{\alpha})c_1)$ holds for some $\tilde{\alpha} \in (0,1)$ (then the common average payoff is less than c), then there exists $\bar{\alpha} \in (\tilde{\alpha}, 1)$ such that $v(c_0; \bar{\alpha}c_0 + (1-\bar{\alpha})d_0) = v(d_0; \bar{\alpha}c_0 + (1-\bar{\alpha})d_0)$ holds. (See Figure 3.) By Lemma 2, the latter distribution $\bar{\alpha}c_0 + (1-\bar{\alpha})d_0$ satisfies (6). Since $v(c_0; \alpha c_0 + (1-\alpha)s_0)$ is increasing in α for any s_0 that plays D in the first period of a match, the former distribution $\tilde{\alpha}c_0 + (1-\tilde{\alpha})c_1$ also satisfies (6), i.e., it is also a Nash equilibrium.

Thus it suffices to prove that the average payoff functions of c_0- and c_1-strategy intersect within $(0,1)$. As in the Proof of Proposition 1, let the "effective" payoff difference be

$$\begin{aligned}
f_{c_0c_1}(\alpha) &:= (1-\alpha+\alpha\frac{1}{1-\delta^2})\{\alpha+(1-\alpha)\frac{1}{1-\delta^2}\}(1-\delta^2) \\
&\quad \times\{v(c_0;\alpha c_0+(1-\alpha)c_1)-v(c_1;\alpha c_0+(1-\alpha)c_1)\} \\
&= -\alpha^2\delta^2(g+c-d-\ell) \\
&\quad +\alpha\{-g+c+d-\ell+(g+2c-2d-\ell)\delta^2\} \\
&\quad -\{(1-\delta^2)d+\delta^2c-\ell\}.
\end{aligned}$$

Since $v(c_0; \alpha c_0 + (1-\alpha)c_1)$ is concave in α and $v(c_1; \alpha c_0 + (1-\alpha)c_1)$ is convex in α (this is proved in GO2009), and $v(c_1; \alpha c_0 + (1-\alpha)c_1) > v(c_0; \alpha c_0 + (1-\alpha)c_1)$ for the two end-points $\alpha = 0, 1$, if $f_{c0c1}(\alpha) = 0$ has two solutions, they are within $(0,1)$. The discriminant of $f_{c_0c_1}(\alpha) = 0$ is as follows.

$$\begin{aligned}
D_{c_0c_1}(\delta) &:= \{-g+c+d-\ell+(g+2c-2d-\ell)\delta^2\}^2 \\
&\quad -4\delta^2(g+c-d-\ell)\{(1-\delta^2)d+\delta^2c-\ell\} \\
&= \{2c^2-(g-\ell)^2+d(g+\ell)-c(2d+g+\ell)\}\delta^4+(g-\ell)^2\delta^2+(g-c-d+\ell)^2.
\end{aligned}$$

$D_{c_0c_1}(\delta)$ is a quadratic function of $x := \delta^2$. At $x = 1$, $D_{c_0c_1}(1) > 0 \iff g - c < \frac{(c-d)^2}{4(c-\ell)}$. Hence if the latter inequality is satisfied, there is $\underline{x} \in (0,1)$ (or $\underline{\delta} = \sqrt{\underline{x}} \in (0,1)$) such that for any $x > \underline{x}$ (or $\delta > \underline{\delta}$), two payoff-equivalent α's exist. The larger solution satisfies local stability. $\square$

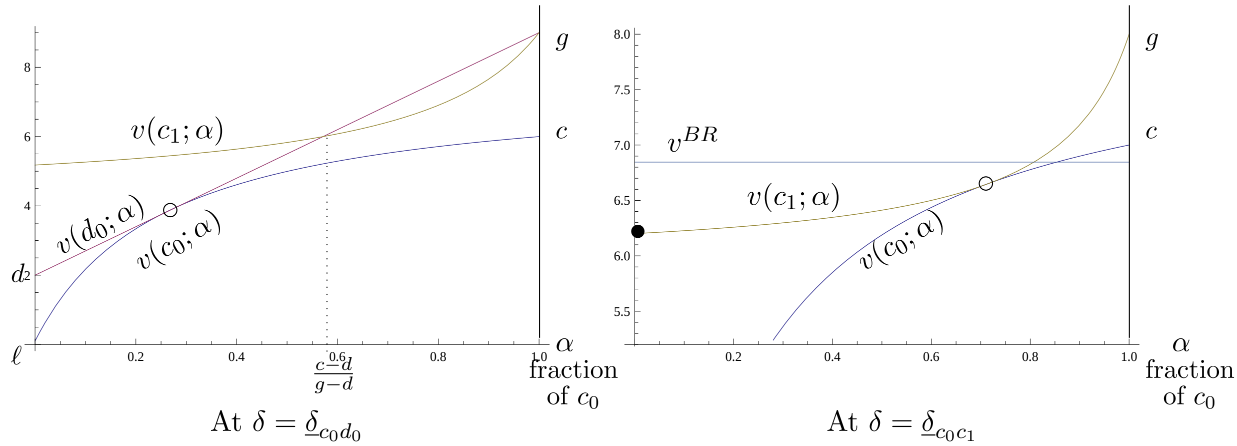

Figure 7: Intuitions for Corollaries 1 and 2

Proof of Corollary 1 : By Lemma 3, if $\alpha < \frac{c-d}{g-d}$ (where $v(d_0; \alpha c_0 + (1-\alpha)c_1) = c$), $v(c_1; \alpha c_0 + (1-\alpha)c_1) > v(d_0; \alpha c_0 + (1-\alpha)d_0)$. Consider $\delta = \underline{\delta}_{c_0 d_0}$. In this case $v(c_0; \alpha c_0 + (1-\alpha)d_0)$ "touches" $v(d_0; \alpha c_0 + (1-\alpha)d_0)$ from below (see the left figure of Figure 7). That is, $v(d_0; \alpha c_0 + (1-\alpha)d_0) \geqq v(c_0; \alpha c_0 + (1-\alpha)d_0)$ for any $\alpha \in [0,1]$. Therefore, for any $\alpha < \frac{c-d}{g-d}$ and any s_0-strategy that plays D in the first period of a match (including d_0- and c_1-strategy),

$$v(c_1; \alpha c_0 + (1-\alpha)c_1) > v(d_0; \alpha c_0 + (1-\alpha)d_0) \geqq v(c_0; \alpha c_0 + (1-\alpha)s_0).$$

This means that c_0-c_1 equilibrium does not exist at $\delta = \underline{\delta}_{c_0 d_0}$. □

Proof of Corollary 2 : Recall that $v(c_1; \alpha c_0 + (1-\alpha)c_1)$ and $v(c_0; \alpha c_0 + (1-\alpha)c_1)$ are both increasing in α. If these have intersections in $(0,1)$ and the Best Reply condition (6) is satisfied, then letting $\alpha = 0$, $v(c_1; c_1) < v^{BR}$ must also hold. Hence, if c_0-c_1-equilibrium exists, then c_1-monomorphic equilibrium exists, i.e., $\underline{\delta}_{c1} \leqq \underline{\delta}_{c_0 c_1}$. In addition, the inequality must be strict. At $\delta = \underline{\delta}_{c_0 c_1}$, the average payoff functions of c_1- and c_0-strategy give the same value at a unique $\alpha \in (0,1)$ and in this case also $v(c_1; c_1) < v^{BR}$ holds. (See the right figure of Figure 7.) □

Proof of Corollary 3 : Recall the Proof of Proposition 1. When $g \downarrow c$, (10) is 0. That is,

$D(x) = 0$ has a unique double root. From (12) and (13), even when $g \downarrow c$,

$$\frac{\partial D}{\partial x}(0) < 0, \quad \frac{\partial D}{\partial x}(1) > 0.$$

Together with $D(0) > 0$ and $D(1) > 0$, the double root $\underline{x}$ must be between 0 and 1, and hence $\underline{\delta}_{c0d0} = \sqrt{\underline{x}}$ is strictly between 0 and 1. On the other hand, when $g \downarrow c$, $\underline{\delta}_{c1} = \sqrt{\frac{g-c}{c-d}}$ converges to 0. This proves the first statement.

When $g \uparrow 2c - d$ then (10) is

$$32(c - \ell)(d - \ell)(c - d)^2 > 0.$$

This means that $D(x) = 0$ has two solutions. By the same logic as above, the larger solution is $\underline{x}$ and this is between $(0, 1)$, i.e., $\underline{\delta}_{c0d0} = \sqrt{\underline{x}}$ is strictly between 0 and 1. On the other hand, when $g \uparrow 2c - d$, $\underline{\delta}_{c1} = \sqrt{\frac{g-c}{c-d}}$ converges to 1. This proves the second statement. $\square$

Proof of Proposition 3 : By Lemma 1 of GO2009, for s to be a pure strategy such that c_0-s is a Nash equilibrium, s must play D in the first period of any partnership. Then the average payoff of c_0-strategy against any bimorphic distribution of the form $\alpha c_0 + (1 - \alpha)s$ is the same as $v(c_0; \alpha c_0 + (1 - \alpha)d_0)$. Note also that

$$v(d_0; \alpha c_0 + (1 - \alpha)s) = \alpha g + (1 - \alpha)d.$$

For a payoff-equalizing bimorphic c_0-s distribution to be more efficient than c_0-d_0 bimorphic equilibrium, α must exceed $\overline{\alpha}_{cd}(\delta)$. However, bimorphic c_0-s distribution with $\alpha > \overline{\alpha}_{cd}(\delta)$ is not a Nash equilibrium, because for $\alpha > \overline{\alpha}_{cd}(\delta)$,

$$v(d_0; \alpha c_0 + (1 - \alpha)d_0) > v(c_0; \alpha c_0 + (1 - \alpha)d_0),$$

(see Figure 2) which is equivalent to

$$v(d_0; \alpha c_0 + (1 - \alpha)s) > v(c_0; \alpha c_0 + (1 - \alpha)s),$$

i.e., c_0-strategy is not a best reply against $\alpha c_0 + (1 - \alpha)s$. $\square$

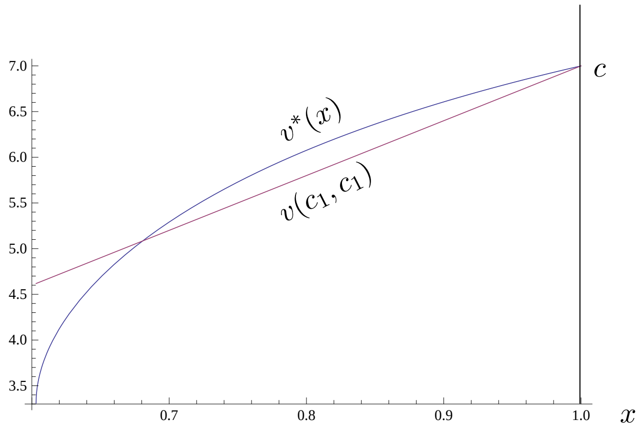

Figure 8: Average Equilibrium Payoffs

Proof of Proposition 4: Take $\delta \geqq \max\{\underline{\delta}_{c_0 d_0}, \underline{\delta}_{c_1}\}$ and let $x := \delta^2$. Then the average equilibrium payoff of the c_1-monomorphic equilibrium is a linear increasing function of x:

$$v(c_1; c_1) = \frac{d + \frac{x}{1-x}c}{\frac{1}{1-x}} = (1-x)d + xc.$$

Let the average equilibrium payoff of c_0-d_0 equilibrium be

$$v^*(x) := v(c_0; \overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0) = v(d_0; \overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0).$$

At $x = 1$, the average payoff function of c_0-strategy is so concave in α that $v^*(1) = c$. Hence the average payoffs of the two equilibria coincide;

$$v(c_1; c_1) = c = v^*(1).$$

Thus, if

$$\frac{\partial v^*}{\partial x} \mid_{x=1} < \frac{\partial v(c_1; c_1)}{\partial x} \mid_{x=1} = c - d$$

holds, then the situation is as depicted in Figure 8 so that there exists $\hat{x} \in (0, 1)$ (or $\hat{\delta} \in (0, 1)$) such that for any $x \in (\hat{x}, 1)$, $v^*(x) > v(c_1; c_1)$.

From $v^*(x) = \overline{\alpha}_{cd}(\delta)g + \{1 - \overline{\alpha}_{cd}(\delta)\}d$,

$$\frac{\partial v^*}{\partial x} = \frac{\partial \overline{\alpha}_{cd}}{\partial x} \cdot (g - d). \tag{14}$$

Thus we compute $\frac{\partial \overline{\alpha}_{cd}}{\partial x}$. To simplify the notation, let $\bar{\alpha} = \overline{\alpha}_{cd}(\delta)$. Then the definition of $\overline{\alpha}$ can be arranged as follows.

$$
\begin{aligned}
& v(c_0; \bar{\alpha}c_0 + (1-\bar{\alpha})d_0) = v(d_0; \bar{\alpha}c_0 + (1-\bar{\alpha})d_0) \\
\iff & \frac{\bar{\alpha} \cdot \frac{c}{1-x} + (1-\bar{\alpha})\ell}{\bar{\alpha} \cdot \frac{1}{1-x} + 1 - \bar{\alpha}} = \bar{\alpha}g + (1-\bar{\alpha})d \\
\iff & \bar{\alpha}c + (1-\bar{\alpha})(1-x)\ell = \{\bar{\alpha}g + (1-\bar{\alpha})d\}\{\bar{\alpha} + (1-\bar{\alpha})(1-x)\} \\
\iff & \bar{\alpha}\{c - (1-x)\ell\} + (1-x)\ell = \{\bar{\alpha}(g-d) + d\}\{\bar{\alpha}x + 1 - x\}.
\end{aligned}
$$

By differentiating both sides with respect to x, we have

$$
\frac{\partial \bar{\alpha}}{\partial x}\{c - (1-x)\ell\} - \ell(1-\bar{\alpha}) = (g-d)\frac{\partial \bar{\alpha}}{\partial x}\{\bar{\alpha}x + 1 - x\} + \{\bar{\alpha}(g-d) + d\}\{\bar{\alpha} - 1 + x \cdot \frac{\partial \bar{\alpha}}{\partial x}\}.
$$

Letting $x \to 1$ and noting that $\bar{\alpha}(1) = \frac{c-d}{g-d}$ (this is the solution to $\alpha g + (1-\alpha) = c$),

$$
\frac{\partial \bar{\alpha}}{\partial x} |_{x=1} = \frac{(g-c)(c-\ell)}{(g-d)(c-d)} > 0.
$$

Plugging this into (14), we have

$$
\frac{\partial v^*}{\partial x} |_{x=1} = \frac{(g-c)(c-\ell)}{(c-d)}.
$$

Therefore,

$$
\frac{\partial v^*}{\partial x} |_{x=1} < \frac{\partial v(c_1; c_1)}{\partial x} |_{x=1} = c - d \iff g - c < \frac{(c-d)^2}{c-\ell}.
$$

□

Proof of Corollary 5: We first consider c_T-strategies. By computation

$$
\begin{aligned}
v(c_T; \alpha c_0 + (1-\alpha)c_T) &= \frac{\alpha g + (1-\alpha)\{d + \delta^2 d + \cdots \delta^{2(T-1)}d + \frac{\delta^{2T}c}{1-\delta^2}\}}{\alpha + (1-\alpha)\frac{1}{1-\delta^2}} \\
&= \frac{v(d_0; \alpha c_0 + (1-\alpha)d_0) + (1-\alpha)\delta^2\{\frac{1-\delta^{2(T-1)}}{1-\delta^2}d + \frac{\delta^{2(T-1)} \cdot c}{1-\delta^2}\}}{1 + (1-\alpha)\frac{\delta^2}{1-\delta^2}} \\
&= v(d_0; \alpha c_0 + (1-\alpha)d_0) \\
&+ \frac{(1-\alpha)\delta^2}{1-\delta^2+\delta^2(1-\alpha)}\Big\{(1-\delta^{2(T-1)})d + \delta^{2(T-1)}c - v(d_0; \alpha c_0 + (1-\alpha)d_0)\Big\}.
\end{aligned} \tag{15}
$$

Hence, if the large bracket of (15) is negative, the average payoff of c_T-strategy is less than that of d_0-strategy. Because $(1-\delta^{2(T-1)})d+\delta^{2(T-1)}c$ is decreasing in T, for any $\delta \geqq \underline{\delta}_{c_0 d_0}$, there exists $\underline{T} \geqq 2$ such that

$$(1-\delta^{2(\underline{T}-1)})d+\delta^{2(\underline{T}-1)}c < v(d_0; \overline{\alpha}_{cd}(\delta)c_0+\{1-\overline{\alpha}_{cd}(\delta)\}d_0).$$

That is, any trust-building strategy $c_{\underline{T}+k}$ with $k \geqq 0$ has average payoff less than that of d_0-strategy and c_0-strategy, when $\alpha = \overline{\alpha}_{cd}(\delta)$ is the fraction of c_0-strategy and the rest is $c_{\underline{T}+k}$-strategy or any s-strategy that plays D in the first period of a partnership:

$$\begin{aligned} v(c_{\underline{T}+k}; \overline{\alpha}_{cd}(\delta)c_0+\{1-\overline{\alpha}_{cd}(\delta)\}c_{\underline{T}+k}) &< v(d_0; \overline{\alpha}_{cd}(\delta)c_0+\{1-\overline{\alpha}_{cd}(\delta)\}d_0) \\ &= v(c_0; \overline{\alpha}_{cd}(\delta)c_0+\{1-\overline{\alpha}_{cd}(\delta)\}s). \end{aligned}$$

Note also that, like c_1-strategy, the average payoff function of $c_{\underline{T}+k}$-strategy is convex in α. Therefore $v(c_0; \alpha c_0+(1-\alpha)c_{\underline{T}+k})$ intersects with $v(c_{\underline{T}+k}; \alpha c_0+(1-\alpha)c_{\underline{T}+k})$ from the above at some $\alpha > \overline{\alpha}_{cd}(\delta)$, as Figure 5 illustrates. This α is the only candidate of a bimorphic distribution consisting of c_0- and $c_{\underline{T}+k}$-strategy to be locally stable.

By the monotone increasing property, $\alpha > \overline{\alpha}_{cd}(\delta)$ implies that $v(c_0; \alpha c_0+(1-\alpha)c_{\underline{T}+k}) > v(c_0; \overline{\alpha}_{cd}(\delta)c_0+\{1-\overline{\alpha}_{cd}(\delta)\}d_0)$. However, Proposition 3 implies that if a bimorphic distribution $\alpha c_0+(1-\alpha)s$ is payoff-equalizing and attains the average payoff greater than that of $\overline{\alpha}_{cd}(\delta)c_0+\{1-\overline{\alpha}_{cd}(\delta)\}d_0$, then the distribution violates the Best Reply condition. Hence we conclude that there is no locally stable Nash equilibrium of the form $\alpha c_0+(1-\alpha)c_{\underline{T}+k}$.

Next, consider d_T-strategies. Again, by computation we have

$$\begin{aligned} v(d_T; \alpha c_0+(1-\alpha)d_T) &= \frac{\alpha g+(1-\alpha)\{d+\delta^2 d+\cdots\delta^{2T}d\}}{\alpha+(1-\alpha)(1+\delta^2+\cdots+\delta^{2T})} \\ &= v(d_0; \alpha c_0+(1-\alpha)d_0) \\ &\quad + \frac{(1-\alpha)\delta^2(1-\delta^{2T})}{1-\delta^2+\delta^2(1-\alpha)(1-\delta^{2T})}\Big\{d-v(d_0; \alpha c_0+(1-\alpha)d_0)\Big\}. \end{aligned} \tag{16}$$

Hence for any $\delta \in (0,1)$, any $\alpha \in (0,1)$, and any $T = 1, 2, \ldots$, $v(d_0; \alpha c_0+(1-\alpha)d_0) > v(d_T; \alpha c_0+(1-\alpha)d_T)$. (They coincide when $\alpha = 0$ or 1.) Clearly, if $v(c_0; \alpha c_0+(1-\alpha)s)$ intersects with $v(d_0; \alpha c_0+(1-\alpha)d_0)$ from the above, it also does with $v(d_T; \alpha c_0+(1-\alpha)d_T)$

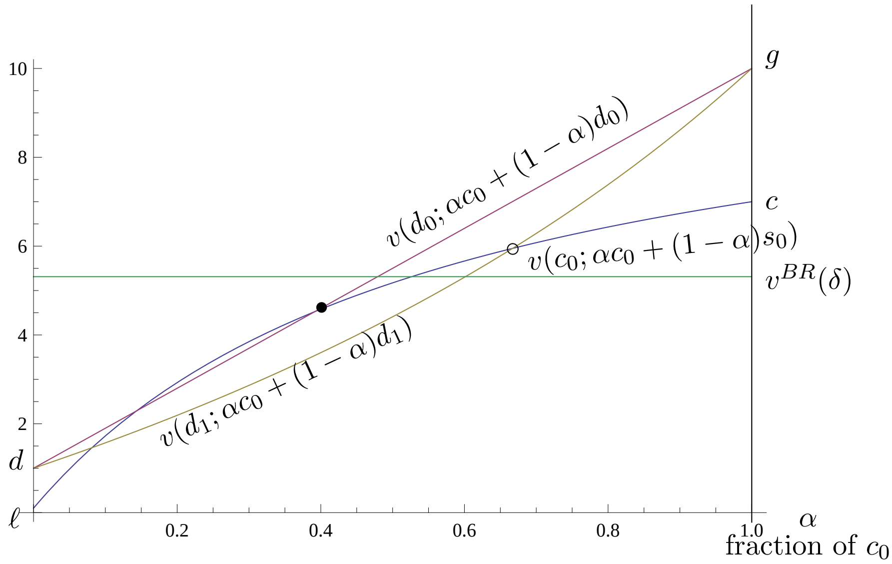

Figure 9: Average payoff functions of c_0-, d_0-, and d_1-strategy

at a higher $\alpha(> \overline{\alpha}_{cd}(\delta))$. Again such an intersection is the only candidate for locally stable equilibrium consisting of c_0- and d_T-strategy. However, $\alpha > \overline{\alpha}_{cd}(\delta)$ and Proposition 3 imply that the payoff-equalizing distribution $\alpha c_0 + (1-\alpha)d_T$ violates the Best Reply condition. See Figure 9 for an illustration. $\square$

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