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Abstract

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Diverse Behavior Patterns in a Symmetric Society with Voluntary Partnerships*

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October 21, 2013

Abstract: In the literature of voluntarily repeated Prisoner's Dilemma, the focus is on how long-term cooperation is established, when newly matched partners cannot know the past actions of each other. In this paper we investigate how non-cooperative and cooperative players co-exist. In many incomplete information versions of a similar model, inherently non-cooperative players are assumed to exist in the society, but their long-run fitness has not been analyzed. In reality and in experiments, we also observe that some people are cooperative, while others never cooperate. We show that a bimorphic equilibrium of the most cooperative strategy and the most myopic strategy exists for sufficiently high survival rate of players, and that it is evolutionarily stable under uncoordinated mutations. For lower survival rates, adding initial periods of defection makes similar bimorphic equilibria. Both types of equilibria confirm persistence of defectors.

Key words: Diversity, evolution, voluntary separation, repeated Prisoner's Dilemma.
JEL classification: C 73.

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1. INTRODUCTION

Society is not uniform in behavior. In particular, even though the situation makes it clear that mutual cooperation is efficient, still some people may behave differently. In many Prisoner's Dilemma and Trust Game experiments, there are different behaviors among subjects; some are cooperative, while others are non-cooperative.[1] In real-life transactions also, there is persistent presence of cheaters, even though cheating is detected and punished. It is too easy to attribute such diversity (co-existence of contrasting patterns of behavior) to external causes such as mistakes, framing, or incomplete learning. We can alternatively postulate that behavioral diversity has its own merit and thus will survive in the long run.

Theoretically, it is also important to investigate how fundamentally asymmetric strategy combinations fare in a symmetric model. In ordinary repeated or random matching game of Prisoner's Dilemma, co-existence of cooperative and non-cooperative strategies are not an equilibrium. In ordinary infinitely repeated Prisoner's Dilemma[2], the C-trigger strategy and the strategy that defects after any history both constitute a symmetric equilibrium on its own, but together they do not constitute an equilibrium.[3] Namely, the C-trigger strategy is not a best reply to the D-always strategy. In the random matching game with the Prisoner's Dilemma as the stage game[4], it is possible to construct a cooperative equilibrium with a more complex strategy than the C-trigger, but if some players in the society always defect, then starting the game with cooperation is not a best reply. Some

[1]For surveys, see Camerer (2003) and Ledyard (1995). Recent experiments of infinitely repeated Prisoner's Dilemma include Dal Bó and Fréchette (2011) and Fudenberg et al. (2012). The latter found co-existence of cooperative and non-cooperative strategies when monitoring is imperfect. Gächter et al. (2011) gives experimental results of Trust game (gift exchange) situations with diverse behaviors. Biologists also find such behavioral diversity, e.g., Dobata et al. (2009). Genetically not so different L (cheater)-type ants seem to move from one colony to another to exploit S (normal)-type ants. Izquierdo et al. (2010, 2013) and references therein show simulation results which we can interpret as co-existence of cooperators and defectors (although in a restricted set of strategies).

[2]For a "perfect folk theorem", see Fudenberg and Maskin (1986). A good survey of various repeating mechanisms is given in Mailath and Samelson (2006).

[3]To be precise, there is no pure-strategy equilibrium in which some players use the C-trigger and others use the D-always strategy.

[4]See Kandori (1992), Ellison (1994), and Harrington (1995) for a finite population model, and Okuno-Fujiwara and Postlewaite (1995) and Takahashi (2010) for a continuum population model. Recently, Deb (2012) provides a folk theorem with general stage games and cheap talk. All these are non-evolutionary, rational player models.

incomplete information versions of voluntarily repeated Prisoner's Dilemma assumed that inherently non-cooperative players exist in the society (e.g., Ghosh and Ray, 1996, and Kranton, 1996), but their long-run fitness has not been analyzed.[5]

Fujiwara-Greve and Okuno-Fujiwara (2009), henceforth Greve-Okuno, showed that the Voluntarily Separable Repeated Prisoner's Dilemma framework admits many polymorphic (asymmetric) equilibria in a symmetric single population model.[6] The key is the endogenous length of repeated interactions. Cooperative players get exploited by defectors but such a partnership is terminated quickly, while a match with another cooperative player will last a long time. Therefore defection against a cooperator may not give a high payoff in the long-run. Greve-Okuno (2009) focused on *trust-building* strategies, which cooperate after some periods of defection, and showed the existence of polymorphic equilibria among them by the above logic.

In this paper, we include fundamentally non-cooperative strategies in the analysis. The polymorphic trust-building equilibria in Greve-Okuno (2009) emerge due to mis-coordination of the initial trust-building periods, but the underlying norm is the same for all players, to eventually find someone to cooperate with each other for a long time. Here, we investigate a more fundamentally bimorphic equilibrium, in which some players never intend to cooperate, while others try to establish a long-term cooperative relationship with a stranger.

Although the contrasting bimorphic distribution is vulnerable to a coordinated invasion of mutants/entrants (Vesely and Yang, 2012), we show that it is robust against a class of "diverse" polymorphic entrants. The class includes strategy distributions resulting when every player randomly and independently experiments with various trust-building strategies as well as strategy distributions with only defection and escape at some point. This stability concept is related to generalized dynamics of selection and directed mutation (e.g., Samuelson and Zhang, 1992, Weibull, 1995, and Samuelson, 1997). Our bimorphic equi-

[5]There are other kinds of incomplete information models of endogenous partnership formation, e.g., Cho and Matsui (2009, 2013) and McAdams (2011). In those models, randomly matched partners discover the match quality after matching. Hence there is no fixed "types" in such models.

[6]There are also infinitely many symmetric trust-building equilibria, which is one of the main findings of Greve-Okuno (2009).

librium thus justifies the existence of inherently non-cooperative players in evolutionary setting and gives a foundation to well-documented behavioral diversity.

We also show that the contrasting-strategy equilibrium is payoff-equivalent to countably many polymorphic equilibria involving various lengths of trust-building strategies. This is due to the same play path being induced on the most cooperative strategy, and the recursive structure of the dynamic game, i.e., the continuation payoff after ending a partnership is the same as the lifetime payoff, because all new partnerships start with a null history. Hence, if two strategies give the same lifetime payoff at the null history, then breaching into either of them at a later period in the matching pool also gives the same continuation payoff. Interestingly, only the contrasting-bimorphic distribution is locally stable whenever it exists. Other equivalent distributions are locally stable only in a smaller range of survival rates (discount factors) of players. Thus, the simple but fundamentally contrasting strategy combination is quite stable.

When the survival rate is not as high as the level that sustains the contrasting-strategy bimorphic equilibrium, adding trust-building periods to both strategies makes similar equilibria. Therefore, in a wide range of survival rates, fundamentally different behavior patterns are persistent.

Our game can be called a *large social game*. Jackson and Watts (2010) formulated a *social game* in which players not only choose strategies but also with whom to play the game. While their model is finite (one-shot or finitely repeated game by finite populations) and assumes a lot of information among players, ours is infinite in both horizon and the number of players and assumes minimal information. However, our purpose of the study is in accordance with one of theirs: we analyze how endogeneity of partnerships affects the play of the game.

Infinite horizon social games are also studied by Cho and Matsui (2012, 2013).[7] In their models, players are not homogeneous. Pairs are randomly formed from two finite populations, and the only strategic decisions are whether to keep the relationship or unilaterally

[7]Random matching games, on which the usual evolutionary game theory (e.g., Maynard Smith, 1982) is based, are infinite horizon games as well, but the players do not strategically choose/terminate partnerships.

terminate it, depending on the realized value of a match. Thus their focus is how players settle with a “partnership value”. By contrast, we show that some players end up in long-term cooperative partnerships, while others never settle down, even though all players have the same characteristics (the set of strategies and the payoff function).

This paper is organized as follows. In Section 2 we describe the Voluntarily Separable Repeated Prisoner’s Dilemma (VSRPD) model, introduced by Greve-Okuno (2009), and define focal strategies. In Section 3 we apply standard evolutionary stability concepts to the contrasting strategy combination of the most cooperative and the most myopic strategy. In Section 4, we define an evolutionary stability concept with respect to a set of entrant distributions and derive a sufficient set of diverse entrant distributions against which the contrasting strategy equilibrium is robust. In Section 5 we show payoff-equivalent distributions to the focal equilibrium and show local instability of the former. In Section 6, we look at lower survival rates than the one for the contrasting-bimorphic equilibrium to exist and show that our analysis can be extended. Section 7 gives concluding remarks. All proofs are in Appendix.

2. MODEL

2.1. Voluntarily Separable Repeated Prisoner’s Dilemma

In this section we describe the model of *Voluntarily Separable Repeated Prisoner’s Dilemma* (VSRPD) introduced by Greve-Okuno (2009). Consider a large society of a continuum of *homogeneous* players of measure 1, over the infinite, discrete time horizon. At the beginning of each period, players are either matched with a partner from the previous period or without a partner. Those without a partner enter a random matching process and form pairs[8] to play the following extensive form game.

Newly matched players have no knowledge of the past action history of each other, and they play the ordinary two-action Prisoner’s Dilemma of Table 1. The actions in

[8]For simplicity and following Greve-Okuno (2009), we assume that a player finds a new partner for sure. This assumption makes cooperation most difficult.

	C	D
C	c, c	ℓ, g
D	g, ℓ	d, d

Table 1: Prisoner's Dilemma: $g > c > d > \ell$ and $2c \geqq g + \ell$.

the Prisoner's Dilemma are observable only by the current partners. After observing the actions in the Prisoner's Dilemma, the partners simultaneously choose whether to keep the partnership (action k) or to end it (action e). The partnership dissolves if at least one partner chooses action e. In addition, at the end of a period, each player may exit from the society for some exogenous reason (which we call a "death") with probability $1 - \delta$, where $0 < \delta < 1$. If a player dies, a new player enters into the society, keeping the population size constant. Players who lost the partner for some reason, as well as newly born players enter the matching pool in the next period. (This justifies the no-information-flow assumption because the players in the matching pool can have different backgrounds.) Therefore a partnership continues if and only if both partners choose action k and do not die. In this case the same partners play the Prisoner's Dilemma in the next period, skipping the matching process. At the beginning of the next period, unmatched players are matched into pairs to play the Prisoner's Dilemma afresh. The game continues this way *ad infinitum*. The outline of VSRPD is depicted in Figure 1.

The one-shot payoffs in the Prisoner's Dilemma are in Table 1, where $g > c > d > \ell$ and $2c \geqq g + \ell$. The latter makes the symmetric pure-action profile (C, C) efficient. The game continues with probability δ from an individual player's point of view. Thus we focus on the expected total/average payoff, with δ being the effective discount factor of a player.

Under the no-information-flow assumption, we focus on match-independent strategies[9] that only depend on the period $t = 1, 2, \ldots$ within a partnership (not the calendar time in the whole game) and the private history of actions within a partnership. Let $H_t := [\{C, D\} \times \{C, D\}]^{t-1}$ be *the set of partnership histories*[10] *at the beginning of* $t \geqq 2$ and let

[9]Since the population is a continuum, "contagious" strategies used in Kandori (1992) and Ellison (1994) cannot achieve cooperation.

[10]The relevant histories on which partners can condition their actions are the action combinations in the

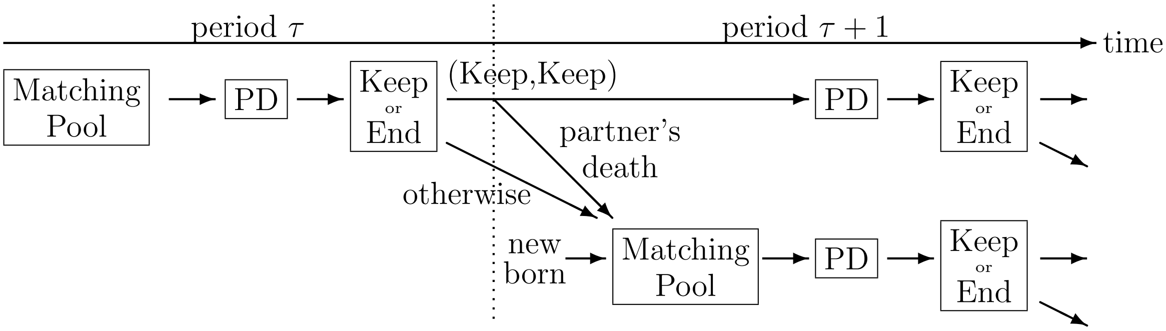

Figure 1: Outline of the VSRPD

$H_1 := \{\emptyset\}$.

DEFINITION 1 : A *pure strategy* s of VSRPD consists of $(x_t, y_t)_{t=1}^{\infty}$ where:
$x_t : H_t \to \{C, D\}$ specifies an action choice $x_t(h_t) \in \{C, D\}$ given the partnership history $h_t \in H_t$, and
$y_t : H_t \times \{C, D\}^2 \to \{k, e\}$ specifies whether to keep or to end the partnership, depending on the partnership history $h_t \in H_t$ and the current period action profile.

The set of pure strategies of VSRPD is denoted as $\mathbf{S}$ and the set of all strategy distributions in the population is denoted as $\Delta(\mathbf{S})$. A pure strategy can be viewed as a degenerate strategy distribution and thus belongs to $\Delta(\mathbf{S})$ as well. Hence we can write a strategy combination of a strategy distribution and a pure strategy as $\alpha p + (1-\alpha)s$.

We assume that each player uses a pure strategy, which is natural in an evolutionary game and simplifies the analysis. We allow entrants/mutants to be a distribution of pure strategies. (See Section 4 below.)

We investigate evolutionary stability of **stationary** strategy distributions **in the matching pool**. Although the strategy distribution in the matching pool may be different from the distribution in the entire society, if the former is stationary, the distribution of various states of matches is also stationary, thanks to the stationary death process.[11] Since each

Prisoner's Dilemma only, because the continuation decision history must be (k, k) throughout.

[11]See Greve-Okuno (2009) footnote 7 for details. For specific strategies, e.g., c_T- and d_T-strategies, we can prove that any stationary distribution in the matching pool exists consistently with the model.

player is born into the random matching pool, the life-time payoff is determined by the strategy distribution in the matching pool.

2.2. *Average and Lifetime Payoffs*

When a strategy $s \in \mathbf{S}$ is matched with another strategy $s' \in \mathbf{S}$, the *expected length* of the match is denoted as $L(s, s')$ and is computed as follows. Notice that even if s and s' intend to maintain the match, it will only continue with probability δ^2. Suppose that the planned length of the partnership of s and s' is $T(s, s')$ periods, if no death occurs. Then

$$L(s, s') := 1 + \delta^2 + \delta^4 + \cdots + \delta^{2\{T(s,s')-1\}} = \frac{1 - \delta^{2T(s,s')}}{1 - \delta^2}.$$

The *expected total discounted value of the payoff stream of s within the match with s'* is denoted as $V(s, s')$. The *average per period payoff* that s expects to receive within the match with s' is defined as

$$v(s, s') := \frac{V(s, s')}{L(s, s')}.$$

Next, consider a player endowed with strategy $s \in \mathbf{S}$ in the matching pool, waiting to be matched randomly with a partner. When the stationary strategy distribution in the matching pool is $p \in \Delta(\mathbf{S})$, we write the *expected total discounted value of payoff streams s* expects to receive during his lifetime as $V(s; p)$ and the average per period payoff s expects to receive during his lifetime as

$$v(s; p) := \frac{V(s; p)}{L} = (1 - \delta)V(s; p),$$

where $L = 1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta}$ is the expected lifetime of s.

Thanks to the stationary distribution in the matching pool, we can write $V(s; p)$ as a recursive equation. If p has a finite/countable support, then we can write[12]

[12]Theorem 1 and Remark 1 of Duffie and Sun (2012) show that the matching probability of a particular strategy is the fraction of the strategy in the pool. We thank Yeneng Sun for helping us to find these details.

$$V(s;p) = \sum_{s' \in supp(p)} p(s') \Big[V(s,s')$$
$$+ [\delta(1-\delta)\{1 + \delta^2 + \cdots + \delta^{2\{T(s,s')-2\}}\} + \delta^{2\{T(s,s')-1\}}\delta] V(s;p) \Big],$$

where $supp(p)$ is the support of the distribution p, the sum $\delta(1-\delta)\{1+\delta^2+\cdots+\delta^{2\{T(s,s')-2\}}\}$ is the probability that s loses the partner s' before $T(s,s')$, and $\delta^{2\{T(s,s')-1\}}\delta$ is the probability that the match continued until $T(s,s')$ and s survives at the end of $T(s,s')$ to go back to the matching pool. Stationarity of p implies that the continuation payoff after a match ends for any reason is always $V(s;p)$.

Let $L(s;p) := \sum_{s' \in supp(p)} p(s') L(s,s')$. By computation,

$$V(s;p) = \sum_{s' \in supp(p)} p(s') \Big[V(s,s') + \{1 - (1-\delta) L(s,s')\} V(s;p) \Big]$$
$$= \sum_{s' \in supp(p)} p(s') V(s,s') + \{1 - \frac{L(s;p)}{L}\} V(s;p).$$

Hence the average payoff is a nonlinear function of the strategy distribution p:

$$v(s;p) := \frac{V(s;p)}{L} = \sum_{s' \in supp(p)} \frac{p(s') V(s,s')}{L(s;p)}.$$

2.3. Cooperating and Non-cooperating Strategies

We investigate stability of fundamental behavioral diversity, in the sense that some players are cooperative, while others never cooperate. For cooperative strategies, Greve-Okuno (2009) focused on the following *trust-building* strategies.

DEFINITION 2 : For any $T = 0, 1, 2, \ldots$, let c_T-strategy be a strategy as follows:
$t \leqq T$: Play D and keep the partnership if and only if (D,D) is observed in the current period.
$t \geqq T+1$: Play C and keep the partnership if and only if (C,C) is observed in the current period.

This class of trust-building strategies initially plays D (trust-building phase) before starting a C-trigger type strategy (cooperation phase) where ending the partnership is the punishment. Since a player can avoid in-match punishment by ending the partnership unilaterally, severance is the maximal equilibrium punishment. The reason that Greve-Okuno (2009) focused on this class of strategies is that in VSRPD, there is no Nash equilibrium[13] in which all players play C in the first period of a partnership. (Lemma 1 of Greve-Okuno (2009).) This is due to the lack of information flow across partnerships. Therefore, to consider cooperative *monomorphic* equilibria, it is natural to focus on the above trust-building strategies, with initial D play.

In this paper, we turn to **polymorphic** equilibria, with as much cooperation as possible. Thus, we investigate how equilibria including c_0-strategy, which starts the cooperation phase immediately with a stranger, can be sustained. In order to constitute an equilibrium, some players must play D in a new partnership. Since Greve-Okuno (2009) had already considered equilibria with different length trust-building strategies, and trust-building strategies all have the same idea to establish a long-term cooperative relationship eventually, we look at a completely opposite type strategy, namely to defect and run away immediately, to be matched with c_0-strategy.

Definition 3 : Let d_0-strategy be as follows: At $t = 1$, play D and end the partnership regardless of the action combination in that period.

The defect-and-run d_0-players are often observed in experiments and real markets, and often assumed to occupy a positive fraction in the society (permanently) in incomplete information versions of VSRPD (e.g., Ghosh and Ray, 1996, Kranton, 1996, and Rob and Yang, 2010).[14] However, it has not been investigated whether such myopic type players can fare as well as "rational" players, such as c_0-strategy.

[13]For the precise definition, see Section 3.

[14]Often, the motivation of incomplete information models is different from ours. Our idea is that the myopic types are plausible and may fare well, while Ghosh and Ray (1996) and Kranton (1996) introduced myopic types in order to induce rational types to play a symmetric, cooperative equilibrium.

3. STABILITY UNDER MONOMORPHIC ENTRANTS

We investigate evolutionary stability of the most **contrasting** strategy combination, consisting of c_0- and d_0-strategy. These strategies are polar types in behavior: c_0-players cooperate with any stranger and d_0-players never cooperate with anyone and change partners every period. Economic examples of such diversity can be found in many markets. For example, in the internet markets, most of the sellers and buyers would play c_0-strategy to do honest transactions even when they met for the first time, while some try to cheat and run away, that is, to play d_0-strategy. The abundance of incomplete information models with d_0-strategy as the irrational type suggests how plausible they are.

In this section we consider some standard stability concepts. First, we define Nash equilibrium in VSRPD model.

Definition 4 : A stationary strategy distribution in the matching pool $p \in \Delta(\mathbf{S})$ is a *Nash equilibrium* if, for all $s \in supp(p)$ and all $s' \in \mathbf{S}$,

$$v(s;p) \geqq v(s';p).$$

From the evolutionary perspective, a Nash equilibrium is a robust distribution against single (measure zero) entrants/mutants. Let us introduce stronger stability concepts which require robustness against a positive measure of entrants/mutants. Different stability concepts are obtained by the difference in the potential set of entrants.

Definition 5 : A stationary strategy distribution in the matching pool $p \in \Delta(\mathbf{S})$ is a *locally stable Nash equilibrium* if,
(i) p is a Nash equilibrium; and
(ii) for any $s' \in supp(p)$, there exists $\bar{\epsilon} \in (0,1)$ such that, for any $\epsilon \in (0, \bar{\epsilon})$,

$$\forall\ s \in supp(p) \setminus \{s'\}, \quad v(s;(1-\epsilon)p + \epsilon s') \geqq v(s';(1-\epsilon)p + \epsilon s'), \quad \text{and}$$
$$\exists\ \tilde{s} \in supp(p) \setminus \{s'\}; \quad v(\tilde{s};(1-\epsilon)p + \epsilon s') > v(s';(1-\epsilon)p + \epsilon s').$$

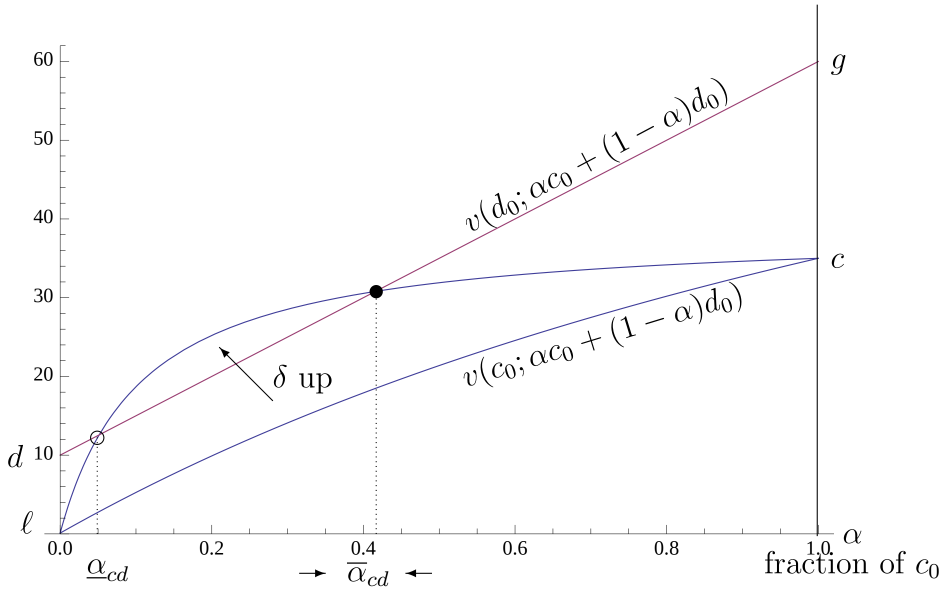

Figure 2: locally stable c_0-d_0 Equilibrium

The local stability requires that, if one of the incumbent strategies increases its share by a small fraction ϵ, then all other incumbent strategies fare at least as well as the increased strategy and some fare strictly better, so that the evolutionary pressure restores the share balance. In other words, local stability requires that the distribution is stable against a positive measure of entrants using one of the incumbent strategies. (See also Figure 2.)

The underlying dynamic we assume is as follows. Occasionally, a newborn player is endowed with a different strategy than her predecessor's, when entering the matching pool. Let ϵ be the measure of such "entrants/mutants" and their strategy[15] be s'. In the "medium"-run the population adjusts to yield a stationary post-entry distribution $(1-\epsilon)p + \epsilon s'$ in the matching pool (where p is the incumbent distribution). After that, in the "long"-run, the selection pressure works according to the post-entry average fitness $v(s; (1-\epsilon)p + \epsilon s')$ of each strategy s.

In our companion paper (Fujiwara-Greve et al., 2013), it is shown that for sufficiently large δ, there is a locally stable Nash equilibrium consisting of c_0- and d_0-strategy.

Remark 1 *(Fujiwara-Greve et al., 2013) There exists* $\underline{\delta} \in (0,1)$ *such that for any* $\delta \in$

[15] In the later analysis we allow entrants to have a distribution of strategies.

$(\underline{\delta}, 1)$*, there is* $\overline{\alpha}_{cd}(\delta) \in (0,1)$ *such that the bimorphic distribution,* $\overline{\alpha}_{cd}(\delta)c_0 + \{1-\overline{\alpha}_{cd}(\delta)\}d_0$*, is the unique locally stable Nash equilibrium with the support* $\{c_0, d_0\}$*. Let the smallest such* $\underline{\delta}$ *be* $\underline{\delta}_{c_0 d_0}$.

The intuition is as follows. Let α be the (stationary) fraction of c_0-strategy and the rest be d_0-strategy in the matching pool. The average payoff of the two strategies are as follows.

$$v(c_0; \alpha c_0 + (1-\alpha)d_0) = \frac{\alpha \frac{c}{1-\delta^2} + (1-\alpha)\ell}{\alpha \cdot \frac{1}{1-\delta^2} + (1-\alpha)} \tag{1}$$

$$v(d_0; \alpha c_0 + (1-\alpha)d_0) = \alpha g + (1-\alpha)d. \tag{2}$$

To explain (1), if c_0-strategy meets another c_0-strategy, the match lasts $1/(1-\delta^2)$ periods in expectation and the in-match long-run payoff is $c/(1-\delta^2)$. This happens with probability α. With probability $1-\alpha$, c_0-strategy meets d_0-strategy, which gives ℓ but the match is ended after one period. The average payoff of c_0-strategy is the expected long-run payoff of the two kinds of matches divided by the expected length of the two kinds of matches. For d_0-strategy, it earns g against c_0-strategy and d against d_0-strategy and any match lasts only one period, yielding (2) as the average payoff.

For any $\delta \in (0,1)$, the average payoff function (2) of d_0-strategy is linear in the share α of c_0-strategy and does not depend on δ, while the average payoff function (1) of c_0-strategy is concave in α and increases (becomes more concave) as the exogenous rate of partnership dissolution declines, or δ increases. At some δ, it must have two intersections with the average payoff of d_0-strategy. See Figure 2. Payoff-equivalence of c_0- and d_0-strategy is in fact sufficient for the strategy combination to become a Nash equilibrium, which is shown in Lemma 2 of Fujiwara-Greve et al. (2013).[16] Only the larger intersection satisfies the local stability as Figure 2 shows. In sum, the **assumption** of the existence of myopic players in the incomplete information models can be endogenized. The key is the assortative match among cooperative players, which leads to the concavity of their average payoffs.

However, the locally stable bimorphic Nash equilibrium $\overline{\alpha}_{cd}(\delta)c_0 + \{1-\overline{\alpha}_{cd}(\delta)\}d_0$ does not satisfy neutral stability, which requires stability against **any** entrant strategy.

[16] A general version is proved in Lemma 2 in Appendix of this paper.

Definition 6 : (Greve-Okuno, 2009) A stationary strategy distribution in the matching pool $p \in \Delta(\mathbf{S})$ is a *Neutrally Stable Distribution* (NSD) if, for any $s' \in \mathbf{S}$, there exists $\bar{\epsilon} \in (0,1)$ such that for any $\epsilon \in (0, \bar{\epsilon})$,

$$\forall\ s \in supp(p), \quad v(s; (1-\epsilon)p + \epsilon s') \geqq v(s'; (1-\epsilon)p + \epsilon s').$$

Remark 2 *For any $\delta \in (\underline{\delta}_{c_0 d_0}, 1)$, the locally stable bimorphic Nash equilibrium $\overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0$ is not neutrally stable.*

The proof of Remark 2 (in Appendix) is essentially the same as the instability of d_0-monomorphic Nash equilibrium (Lemma 2 in Greve-Okuno (2009)). A "secret-handshake" c_1-strategy imitates d_0-strategy and hence earns the same payoff when meeting the incumbents, but does not end the partnership after (D, D) in the first period and earns cooperation payoff afterwards when meeting another c_1-strategy.[17]

Notice that, this is a "coordinated" invasion that all entrants/mutants play the same strategy, c_1. Alternatively, entry/mutation may be uncoordinated, consisting of multiple strategies. In the next section we consider stability under uncoordinated invasions.

4. STABILITY UNDER DIVERSE ENTRANTS

We extend the analysis to allow entrants with different strategies appearing simultaneously. There are many scenarios that make this happen. If the players are humans, they may experiment with (or make mistakes to play) different strategies at the same time. If the players are not so conscious decision-makers, still it is possible that multiple genes mutate simultaneously to a variety of behavior patterns. Monomorphic entrants require precise coordination among them to play the same strategy, while polymorphic entrants do not need coordination. We thus think that polymorphic entrants/mutations are more likely to evolve spontaneously.

For example, when some entrants play c_1-strategy, other entrants may imitate it to play one period of trust-building (i.e., defect but keep the partnership if (D, D) is observed)

[17] Vesely and Yang (2012) has a general result that any strategy distribution with on-path separation can be invaded by a secret-handshake type strategy.

and, if the partnership continued to the second period, they defect and run away. In this case, the post-entry payoff of c_1-strategy is reduced and moreover its "exploiter" may not fare that well against the incumbents, either.

In general, we formulate a class of exploiters against c_T-strategies as follows.

Definition 7 : For any $T = 1, 2, \ldots$, let d_T-strategy be a strategy as follows:
$t \leqq T$: Play D and keep the partnership regardless of the partnership history:
$t \geqq T+1$: Play D and end the partnership regardless of the observation in this period.

The exploiter against c_1-strategy we mentioned above is d_1-strategy. Let us show that, although the monomorphic entrant c_1-strategy can invade the c_0-d_0-equilibrium, a class of c_1-d_1 entrants cannot. To see this, consider a post-entry distribution

$$p^{PE} = (1-\epsilon)\{\overline{\alpha}_{cd}(\delta)c_0 + (1-\overline{\alpha}_{cd}(\delta))d_0\} + \epsilon\{q_{c_1}c_1 + (1-q_{c_1})d_1\}.$$

Notice that the post-entry payoff of c_0- and d_0-strategy depends only on the post-entry share of c_0-strategy. To simplify the notation, let $x_{c_0} := (1-\epsilon)\overline{\alpha}_{cd}(\delta)$ and the post-entry payoffs of c_0- and d_0-strategies as follows.

$$v(c_0; p^{PE}) = \frac{x_{c_0}\frac{c}{1-\delta^2} + (1-x_{c_0})\ell}{x_{c_0}\cdot\frac{1}{1-\delta^2} + 1 - x_{c_0}} =: v_{cd}(c_0; x_{c_0}) \tag{3}$$

$$v(d_0; p^{PE}) = x_{c_0}g + (1-x_{c_0})d =: v_{cd}(d_0; x_{c_0}). \tag{4}$$

The average payoff of c_1-strategy can be arranged as a weighted sum of $v_{cd}(d_0; x_{c_0})$ and $v_{cd}(c_0; x_{c_0})$ as follows.

$$\begin{aligned} v(c_1; p^{PE}) &= \frac{x_{c_0}\cdot g + (1-x_{c_0})d + \delta^2\epsilon\{q_{c_1}\cdot\frac{c}{1-\delta^2} + (1-q_{c_1})\ell\}}{1+\delta^2\epsilon\{\frac{q_{c_1}}{1-\delta^2} + 1 - q_{c_1}\}} \\ &= v_{cd}(d_0; x_{c_0}) + \frac{\delta^2\epsilon L(c_0; q_{c_1})}{1+\delta^2\epsilon L(c_0; q_{c_1})}\Big[v_{cd}(c_0; q_{c_1}) - v_{cd}(d_0; x_{c_0})\Big], \end{aligned} \tag{5}$$

where $L(c_0; q_{c_1}) := q_{c_1}\cdot\frac{1}{1-\delta^2} + (1-q_{c_1})$.

Similarly

$$\begin{aligned} v(d_1; p^{PE}) &= \frac{x_{c_0}\cdot g + (1-x_{c_0})d + \delta^2\epsilon\{q_{c_1}\cdot g + (1-q_{c_1})d\}}{1+\delta^2\epsilon} \\ &= v_{cd}(d_0; x_{c_0}) + \frac{\delta^2\epsilon}{1+\delta^2\epsilon}\Big[v_{cd}(d_0; q_{c_1}) - v_{cd}(d_0; x_{c_0})\Big]. \end{aligned} \tag{6}$$

From (5) and (6), the signs of the second terms of $v(c_1; p^{PE})$ and $v(d_1; p^{PE})$ determine whether they are more or less than $v(d_0; p^{PE})$. Sufficiently small q_{c_1} implies that the second terms are negative, so that the incumbent d_0-strategy has higher average payoff than that of **both** entrant strategies. Moreover, $x_{c_0} = (1-\epsilon)\overline{\alpha}_{cd}(\delta) < \overline{\alpha}_{cd}(\delta)$ implies that c_0-strategy earns even higher average payoff than d_0-strategy does (see Figure 2). Therefore, all incumbents can have higher post-entry average payoff than all entrants, when the relative share of c_1 against d_1 is sufficiently small.

The above example indicates that if entrants/mutants are diverse in such a way that c_T-strategies are sufficiently less than its exploiters, then the c_0-d_0-equilibrium cannot be invaded. Thus we consider a stability concept with respect to a set of possible (polymorphic) entrant/mutant distributions.

Definition 8 : Given $\mathcal{M} \subset \Delta(\mathbf{S})$, a stationary strategy distribution p in the matching pool is *Evolutionarily Stable against (Polymorphic) Entrants within* $\mathcal{M}$ if,
(i) p is a locally stable Nash equilibrium,
(ii) for any $q \in \mathcal{M}$ and any $s' \in supp(q) \setminus supp(p)$, there exists $\bar{\epsilon} \in (0,1)$ such that, for any $\epsilon \in (0, \bar{\epsilon})$ and any $s \in supp(p)$,

$$v(s; (1-\epsilon)p + \epsilon \cdot q) > v(s'; (1-\epsilon)p + \epsilon \cdot q).$$

In words, any new (polymorphic) entrants/mutants from $\mathcal{M}$ cannot thrive, and after selection, local stability restores the balance among incumbents. The standard concept of Evolutionarily Stable Strategy corresponds to the most stringent case such that $\mathcal{M} = \Delta(\mathbf{S})$.

Our stability concept can probably be connected to limits of some monotone dynamic processes with (a direction of) mutation, similar to the one considered in Samuelson and Zhang (1992), as ESS can be connected to stable points of monotone dynamic processes (e.g., replicator dynamic). As Samuelson (1997) surveys, however, even the latter connection is weak, i.e., for general games, the stable points of monotone dynamic processes do not coincide with ESS. In addition, the medium-run and long-run process, which seems to be most appropriate for the dynamic process of matching pool strategy distribution formation and selection of strategies, cannot be characterized by a single differential equation.

Izquierdo, Izquierdo, and Vega-Redondo (2013) restricted attention to stationary Markov strategies and obtained a connection between Nash distributions and limit stationary points of a monotone dynamic with completely mixed mutations. By contrast, in order to include as many strategies as possible in the analysis, we adopt the “static” stability.

We now specify a sufficient set of polymorphic entrants/mutants that makes the locally stable equilibrium $p^* = \overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0$ evolutionarily stable. But before doing that, we claim that the following set of c_T- and d_T-strategies is sufficient for the stability analysis of the c_0-d_0 equilibrium:

$$S_{cd}^{\infty} = \{c_0, d_0, c_1, d_1, \ldots\}.$$

The c_T-strategies are important for their “secret handshake” property to earn high payoffs among themselves, after imitating d_t-strategies with $t \leqq T-1$, while d_T-strategies are important for their “short run” property to exploit c_t-strategies with $t \leqq T$.[18] Therefore, if a strategy in S_{cd}^{∞} is worth experimenting with, then others (with longer T's) are also worth trying. Other strategies that differ off the play path are not relevant for such arguments. Therefore, we focus on S_{cd}^{∞}.

Let $\Delta(S_{cd}^{\infty})$ be the set of all probability distributions over S_{cd}^{∞} with a generic element $\mathbf{x} = (x_{c_0}, x_{d_0}, x_{c_1}, x_{d_1}, \ldots)$, where $x_s \in [0,1]$ is the share of strategy $s \in S_{cd}^{\infty}$ so that $\sum_{s \in S_{cd}^{\infty}} x_s = 1$. For any $T = 0, 1, \ldots$, d_t-strategy with $t \geqq T$ and c_t-strategy with $t \geqq T+1$ behave the same way against c_T-strategy, and hence we sometimes combine their shares in a distribution $\mathbf{x}$ as $x_{d_T+} := \sum_{t=T}^{\infty} x_{d_t} + \sum_{t=T+1}^{\infty} x_{c_t}$.

Proposition 1 *For any $\delta \in (\underline{\delta}_{c_0 d_0}, 1)$, define a set of diverse entrant distributions by*

$$E(\delta) := \{q = (q_{c_0}, q_{d_0}, q_{c_1}, \ldots) \in \Delta(S_{cd}^{\infty}) \mid \; q_{c_0} \leqq \overline{\alpha}_{cd}(\delta), \quad \frac{q_{c_T}}{q_{c_T} + q_{d_T+}} < \overline{\alpha}_{cd}(\delta), \quad \forall T \geqq 1\}.$$

If $\mathcal{M} \subseteq E(\delta)$, then $p^* = (\overline{\alpha}_{cd}(\delta), 1 - \overline{\alpha}_{cd}(\delta), 0, \ldots) \in \Delta(S_{cd}^{\infty})$ is Evolutionarily Stable against Polymorphic Entrants within $\mathcal{M}$.

[18] Of course, alternating action strategies that play (C, D) and (D, C) on the play path among themselves, considered in Section 4.2 of Greve-Okuno (2009) are another important class. However, this class also requires a specific coordination and therefore we do not focus on it as entrants.

To interpret $E(\delta)$, if some c_T-strategy is present in the entrant distribution, the entrants also include sufficiently many exploiters of c_T-strategy, so that the ratio[19] $\frac{q_{c_T}}{q_{c_T}+q_{d_T+}}$ is below $\overline{\alpha}_{cd}(\delta)$. Specifically, if c_1-strategy is present in the entrant distribution, then sufficiently large share of the entrant distribution has d_T ($T \geqq 1$) or c_T-strategies ($T \geqq 2$) to reduce the payoff of c_1-strategy. Alternatively, c_T-strategies ($T \geqq 1$) can be absent in the entrant distribution, while some of d_T-strategies must be present.

Such diversity among entrants can arise in natural extensions of existing mutation processes which put a positive probability on all strategies every period, including the one considered in Kandori, Mailath, and Rob (1993). In a finite population, finite strategy model as that of Kandori, Mailath, and Rob (1993), it would eventually allow a coordination on a strategy, but, in our infinite population and infinite strategy model, no eventual coordination is warranted and, instead, the diversity of entrants is likely to induce a distribution in $E(\delta)$. An alternative justification of diverse entrants is that newly entered players do not have a common norm and end up with various strategies.

Let us give a simple example of a class of mutation processes which can put a positive probability on each strategy in S^{∞}_{cd}. Take any real number $\gamma \in (0, 1]$ and a "base" strategy c_T, for some $T \in \{0, 1, \ldots\}$. Consider a "branching" mutation process, which randomizes between strategies in a particular order but with a fixed relative probability. The probability to mutate/experiment to play c_T-strategy is γ, and the rest of the probability $1 - \gamma$ is concentrated on the set $\{d_T, c_{T+1}, d_{T+1}, \ldots\}$. Among these, the mutation process chooses d_T with relative probability γ (the absolute probability is then $(1 - \gamma)\gamma$), and the rest is concentrated on the set $\{c_{T+1}, d_{T+1}, c_{T+2}, \ldots\}$, in which c_{T+1} is chosen with the relative probability γ, and so on. The resulting entrant/mutant distribution takes the following geometric form:

$$q_{c_T} = \gamma, \quad q_{d_T} = (1 - \gamma)\gamma, \quad q_{c_{T+1}} = (1 - \gamma)^2\gamma, \ldots,$$
$$q_{c_{T+t}} = (1 - \gamma)^{2(T+t-1)}\gamma, \; q_{d_{T+t}} = (1 - \gamma)^{2(T+t-1)+1}\gamma, \ldots.$$

[19]Note that $q_{d_T+} = \sum_{t=T}^{\infty} q_{d_t} + \sum_{t=T+1}^{\infty} q_{c_t}$ is similarly defined as x_{d_T+} in the text.

Then for any $t = 0, 1, 2, \ldots$,

$$\frac{q_{c_{T+t}}}{q_{c_{T+t}} + q_{d_{T+t+}}} = \frac{(1-\gamma)^{2(T+t-1)}\gamma}{(1-\gamma)^{2(T+t-1)}\gamma + \frac{(1-\gamma)^{2(T+t-1)+1}}{1-(1-\gamma)} \cdot \gamma} = \gamma.$$

When $\gamma = 1$, the branching process generates a coordinated entrant, and as $\gamma \to 0$, it generates "uniform distribution" entrants. Any branching mutation process with $\gamma < \overline{\alpha}_{cd}(\delta)$ generates an entrant distribution in $E(\delta)$.

Other possible mutation processes that yield an entrant distribution in $E(\delta)$ include ones that only generate d_T-strategies (singleton or any mixture) and ones that generate finitely many c_T-, c_{T+1}-, ..., c_{T+k}-strategies and the exploiter d_{T+k}-strategy (for some $T \in \{0, 1, 2, \ldots\}$), with sufficiently large probability of d_{T+k}-strategy. This class includes the example at the beginning of this section.

Weibull (1995), Example 2.4, shows that in ordinary evolutionary games (random matching with one-shot game), an ESS, which is robust against a single strategy mutation, is not necessarily resistant against simultaneous multiple mutations. We can interpret Proposition 1 as giving an "opposite" example such that, although a distribution is vulnerable to entry of a single strategy, it is robust against a class of mixed strategy entrants including the successful pure strategy. This is thanks to the recursive structure of VSRPD: If c_1-strategy can exploit d_0-strategy, then d_1- or c_2-strategy can exploit c_1-strategy, and so on, even in a symmetric society.

Finally, we show that the above logic does not hold for d_T-monomorphic Nash equilibrium for any T. That is, d_T-monomorphic Nash equilibrium in **not robust** against not only coordinated c_{T+1}-strategy entrant (an analogue of Lemma 2 of Greve-Okuno, 2009), but also mixed entrants of c_{T+1}-strategy and d_{T+1}-strategy.

Remark 3 : For any $T < \infty$, let $p^{PE} = (1-\epsilon)d_T + \epsilon\{xc_{T+1} + (1-x)d_{T+1}\}$ where $x \in (0, 1)$. Then $v(d_T; p^{PE}) < v(d_{T+1}; p^{PE})$.

For $T = \infty$, let $p^{PE} = (1-\epsilon)d_\infty + \epsilon\{xc_0 + (1-x)d_0\}$. Then $v(d_\infty; p^{PE}) < v(d_0; p^{PE})$.

Hence the "terminating" equilibria of d_T-monomorphic distributions are unstable[20] with

[20] Schumacher (2013) gives a dynamic instability of a D-always-strategy (keep if and only if the partner cooperates) when the only alternative strategy is c_0.

respect to the same set of entrants which makes the c_0-d_0 equilibrium stable.

5. INSTABILITY OF PAYOFF-EQUIVALENT DISTRIBUTIONS

The special feature of the bimorphic equilibrium of c_0-d_0 distribution is not only that it has very contrasting strategies. The bimorphic distribution has countably many payoff-equivalent polymorphic distributions, which is shown below. The payoff-equivalence is due to the recursive structure of VSRPD. However, all payoff-equivalent distributions turned out to be locally unstable near $\underline{\delta}_{c_0d_0}$, because c_0-strategy can increase its share and its average payoff. This is an additional support to the significance of the c_0-d_0 equilibrium.

First, we show that the bimorphic equilibrium of c_0-d_0 distribution is payoff-equivalent to the following form of "geometric" distributions

$$p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T) = \alpha_0 \cdot c_0 + (1-\alpha_0)\alpha_1 \cdot c_1 + \cdots + \{\times_{t=0}^{T-1}(1-\alpha_t)\}\alpha_T \cdot c_T + \{\times_{t=0}^{T}(1-\alpha_t)\}d_T,$$

for any $T = 1, 2, \ldots$, provided that $\alpha_0 = \alpha_1 = \cdots = \overline{\alpha}_{cd}(\delta)$. The α_t's are the relative ratio of c_t-strategy against strategies that play D in the first t periods of a match.

Lemma 1 *For any $\delta \in (\underline{\delta}_{c_0d_0}, 1)$, and any $T = 1, 2, \ldots$, let $\overline{p} = p_{c0}^{d_T}(\overline{\alpha}_{cd}(\delta), \overline{\alpha}_{cd}(\delta), \ldots, \overline{\alpha}_{cd}(\delta))$, i.e., all relative ratios of c_t-strategies ($t = 1, 2, \ldots, T$) are $\overline{\alpha}_{cd}(\delta)$. Then, for any $t = 1, 2, \ldots, T$,*

$$v(c_0; \overline{p}) = v(d_0; \overline{p}) = v(c_t; \overline{p}) = v(d_T; \overline{p}).$$

The point is that the average payoff of a c_t-strategy ($t = 1, 2, \ldots, T$) and d_T-strategy can be decomposed as a weighted sum, as in the proof of Proposition 1. For example, the average payoff of c_1-strategy under the distribution $p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T)$ is

$$v(c_1; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T)) = v_{cd}(d_0; \alpha_0) + \frac{(1-\alpha_0)\delta^2 L(c_0; \alpha_1)}{1 + (1-\alpha_0)\delta^2 L(c_0; \alpha_1)} \Big\{ v_{cd}(c_0; \alpha_1) - v_{cd}(d_0; \alpha_0) \Big\}.$$

Hence, if $\alpha_t = \overline{\alpha}_{cd}(\delta)$ for all $t = 0, 1, \ldots, T$, then $v_{cd}(d_0; \alpha_t) = v_{cd}(c_0; \alpha_t) = v_{cd}(d_0; \alpha_0)$ so that the average payoff of c_1- and c_0-strategy coincide. An illustration of payoff decomposition and equivalence is given in Figure 3 for the case of $T = 2$, where v^M stands for the average payoff starting in the matching pool.

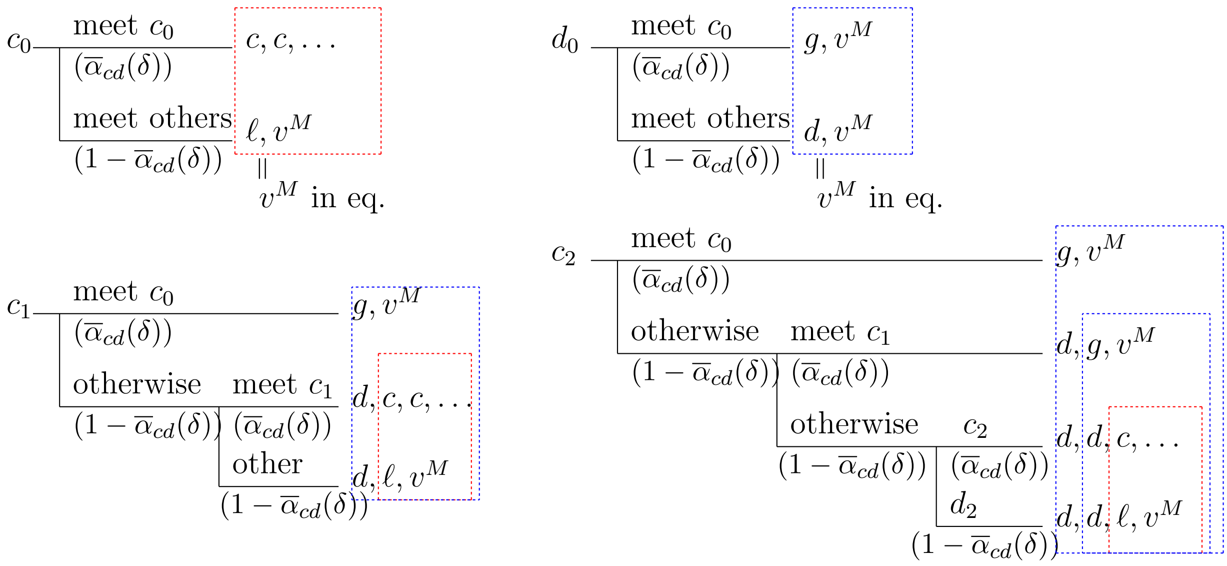

Figure 3: Equivalence of $c_0 - d_0$ and p_{c0}^{d2} distribution

For δ near $\underline{\delta}_{c_0d_0}$, any payoff-equivalent distribution of the above form (for $T \geqq 1$) is not locally stable, because a small increase of c_0-strategy **increases** its post-entry average payoff.

Proposition 2 *For each $T = 1, 2, \ldots$, there exists $\overline{\delta} > \underline{\delta}_{c_0d_0}$ and $\overline{\epsilon} > 0$ such that for any $\delta \in (\underline{\delta}_{c_0d_0}, \overline{\delta})$ and any $\epsilon \in (0, \overline{\epsilon})$, $v(c_0; p^{PE}) > v(s; p^{PE})$ for any $s \in \{c_1, c_2, \ldots, c_T, d_T\}$, where $p^{PE} := (1-\epsilon) \cdot p_{c0}^{d_T}(\overline{\alpha}_{cd}(\delta), \overline{\alpha}_{cd}(\delta), \ldots, \overline{\alpha}_{cd}(\delta)) + \epsilon \cdot c_0$.*

Since c_0-d_0 equilibrium is locally stable for any $\delta > \underline{\delta}_{c_0d_0}$, this is an additional support for the significance of the contrasting strategy distribution. The idea of the proof is to show that at $\delta = \underline{\delta}_{c_0d_0}$,

$$\frac{\partial v(c_0; p^{PE})}{\partial \epsilon} |_{\epsilon=0} > \frac{\partial v(s; p^{PE})}{\partial \epsilon} |_{\epsilon=0}$$

for any $s \in \{c_1, c_2, \ldots, c_T, d_T\}$.

As Figure 4 illustrates for the case of $T = 1$ (i.e., the incumbent distribution is $\overline{\alpha}_{cd}(\delta)c_0 + \{1-\overline{\alpha}_{cd}(\delta)\}\overline{\alpha}_{cd}(\delta)c_1 + \{1-\overline{\alpha}_{cd}(\delta)\}^2 d_1$), the post-entry average payoff of c_0-strategy is concave in ϵ, while others are not. Hence the above inequality warrants that for a range of ϵ near

[21] The parameter value combination is $(g, c, d, \ell, \delta) = (60, 31, 10, 0.1, 0.95)$.

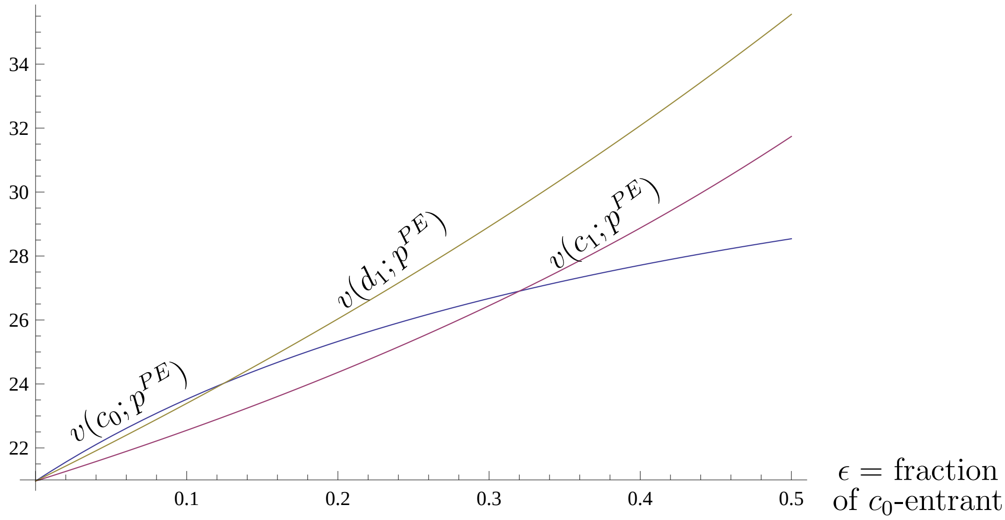

Figure 4: Instability of p_{c0}^{d1} against c_0-entrant[21]

0, c_0-strategy has strictly higher average payoff than others. However, as δ increases, the derivative $\frac{\partial v(c_0;p^{PE})}{\partial \epsilon} |_{\epsilon=0}$ declines, so that only near $\underline{\delta}_{c0d0}$, the above inequality holds.

Notice that the limit case ($T = \infty$) of the equivalent distribution is the infinite-polymorphic distribution of trust-building strategies

$$p_0^{\infty}(\overline{\alpha}_{cd}(\delta)) := \sum_{\tau=0}^{\infty} \overline{\alpha}_{cd}(\delta)\{1 - \overline{\alpha}_{cd}(\delta)\}^{\tau} c_{\tau}$$

considered in Greve-Okuno (2009). In Greve-Okuno (2009), stability of the infinite-polymorphic distribution was shown by changing the common relative fraction of all incumbent strategies **simultaneously**. This stability was not exactly the local stability, and the above analysis clarifies that the distribution is not locally stable for a range of δ.

6. GENERAL BIMORPHIC EQUILIBRIA AND EQUIVALENT DISTRIBUTIONS

For $\delta < \underline{\delta}_{c_0 d_0}$, we can extend the above analysis by adding initial T periods of (D, D) sequence to c_0- and d_0-strategy as in the monomorphic equilibrium analysis of Greve-Okuno (2009). The existence of c_T-d_T-equilibria also shows persistence of diverse behavior patterns for a wide range of survival rates.

Greve-Okuno (2009) showed that a monomorphic distribution of c_T-strategy is a Nash equilibrium if and only if playing D in the cooperation phase is not better than following the cooperation phase, i.e.,

$$g + \delta V(c_T, c_T) \leqq \frac{c}{1-\delta^2} + \Big\{1 - \frac{L(c_T, c_T)}{L}\Big\} V(c_T, c_T)$$

$$\text{(7)} \qquad \Longleftrightarrow v(c_T, c_T) = (1-\delta^{2T})d + \delta^{2T} c \leqq \frac{1}{\delta^2}\{c - (1-\delta^2)g\} =: v^{BR}.$$

(7) is called the Best Reply Condition. Clearly when $T = 0$, it is not satisfied. Since $v(c_T, c_T)$ is decreasing in T but v^{BR} is constant, there is a lower bound to T above which the Best Reply Condition is satisfied. Specifically, for any[22] $\delta \in (\sqrt{\frac{g-c}{g-d}}, 1)$, there exists $\underline{\tau}(\delta) \in \Re_{++}$ such that

$$\text{(8)} \qquad \{1 - \delta^{2\underline{\tau}(\delta)}\}d + \delta^{2\underline{\tau}(\delta)} c = v^{BR}.$$

Then for any $T \geqq \underline{\tau}(\delta)$, c_T-strategy played by all players is a Nash equilibrium. Now, for T slightly **less than** $\underline{\tau}(\delta)$, we have a c_T-d_T equilibrium as follows.

Proposition 3 *There exists $\delta_* \in (0,1)$ such that, for any $\delta \in (\delta_*, 1)$, there exists $\underline{\tau}_{cd}(\delta) \in \Re$ such that for any positive integer $T \in [\underline{\tau}_{cd}(\delta), \underline{\tau}(\delta))$, there is a unique $\overline{\alpha}_T(\delta) \in (0,1)$ such that $\overline{\alpha}_T(\delta)c_T + \{1 - \overline{\alpha}_T(\delta)\}d_T$ constitutes a locally stable Nash equilibrium.*

Let us give an intuition of the proof (see also Figure 5). Since d_T-strategy can be interpreted as a one-step deviation from c_T-strategy, at $T = \underline{\tau}(\delta)$, $v(c_{\underline{\tau}(\delta)}; c_{\underline{\tau}(\delta)}) = v(d_{\underline{\tau}(\delta)}; c_{\underline{\tau}(\delta)}) = v^{BR}$, that is, when the share of $c_{\underline{\tau}(\delta)}$-strategy α is 1, the average payoffs of $c_{\underline{\tau}(\delta)}$- and $d_{\underline{\tau}(\delta)}$-strategy coincide. If the average payoff function of $c_{\underline{\tau}(\delta)}$-strategy intersect with that of $d_{\underline{\tau}(\delta)}$-strategy from the above at $\alpha = 1$, as in Figure 5, they have another intersection at $\alpha < 1$. Then, by continuity, the average payoff functions of c_T-strategy and d_T-strategy have two intersections for T slightly less than $\underline{\tau}(\delta)$ as well. The larger intersection corresponds to a locally stable Nash equilibrium by the same logic as that of Remark 1.

[22]For the derivation of $\sqrt{\frac{g-c}{g-d}}$, see Greve-Okuno (2009).

[23]The parameter value combination is $(g, c, d, \ell, \delta) = (600, 302, 100, 0.1, 0.9019)$.

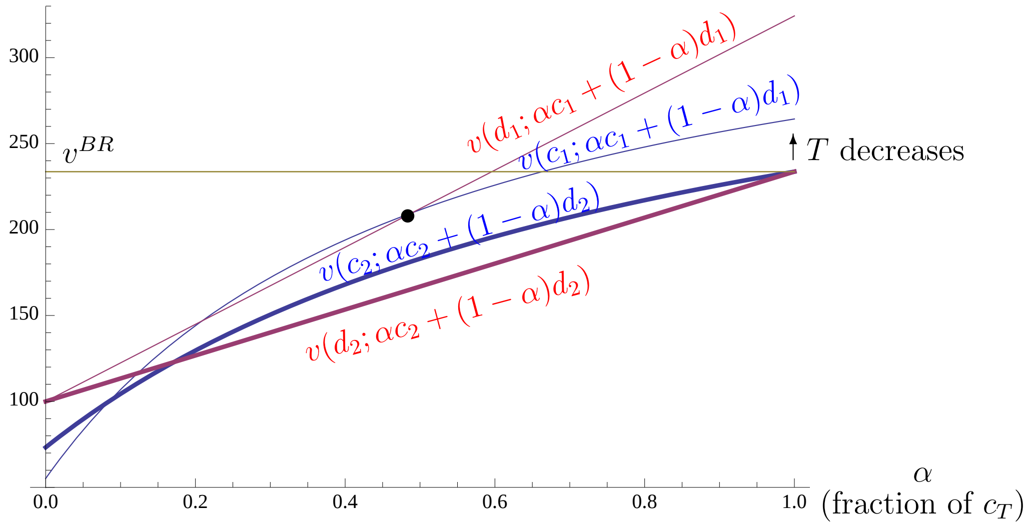

Figure 5: Existence of c_T-d_T equilibrium when $\underline{\tau}(\delta) = 2^{23}$

Figure 6 shows a parametric summary of equilibrium existence.[24] We also have a similar payoff-equivalence result to Lemma 1 for c_T-d_T-equilibrium.

Corollary 1 *For any $\delta \in (\delta_*, 1)$, the locally stable Nash equilibrium $\overline{\alpha}_T(\delta)c_T + \{1 - \overline{\alpha}_T(\delta)\}d_T$, if exists, is payoff-equivalent to the infinitely-many trust-building strategy distribution of the form*

$$\sum_{n=0}^{\infty} \overline{\alpha}_T(\delta)\{1 - \overline{\alpha}_T(\delta)\}^n c_{T+n(T+1)}.$$

In summary, for a wide range of δ, we can extend the analyses in Sections 4-5 to bimorphic equilibria consisting of a cooperative c_T-strategy and non-cooperative d_T-strategy. Hence co-existence of contrasting behavior patterns is persistent. As in the case of c_0-d_0 equilibrium (and as Vesely and Yang, 2012, points out), c_{T+1}-strategy can invade the c_T-d_T equilibrium, but the pure-strategy invasion is a coordinated entrant. There should be also "self-destructing" set of polymorphic entrants against which c_T-d_T equilibrium is robust.

[24]For some parameter combinations, e.g., $(g, c, d, \ell, \delta) = (8.7, 4.5, 1, 0.1, 0.895)$, both c_0-d_0 equilibrium and c_1-d_1 equilibrium exist for the same δ.

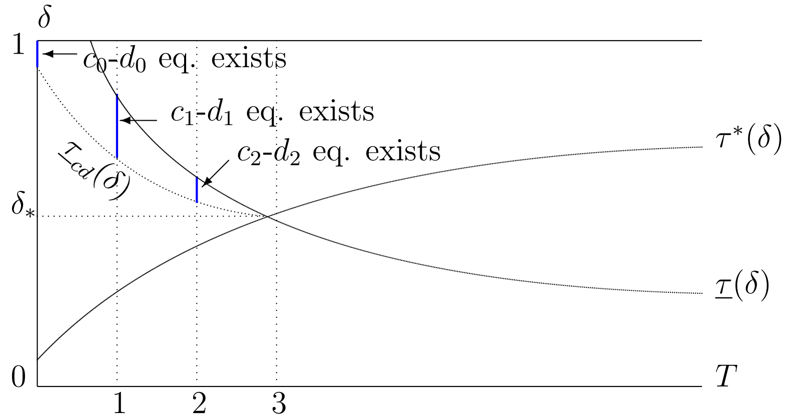

Figure 6: Parametric summary of $c_T - d_T$ equilibrium existence

7. CONCLUDING REMARKS

We have shown various stability properties[25] of a pair of contrasting behavior patterns: c_0-strategy which cooperates even with a stranger to try to establish a long-term cooperative relationship, and d_0-strategy which always defects and runs away. Although the model is symmetric, such fundamentally different norms of behavior can co-exist because their average payoffs can be equalized at a high level. Cooperators have non-linear average payoff in their share because of assortative matching among themselves. As the rate of exogenous dissolution of a match declines, cooperators' payoff increases so that there are payoff-equalizing share balances between cooperators and defectors. One of them corresponds to a locally stable Nash equilibrium. Moreover, the bimorphic equilibrium of c_0- and d_0-strategy is robust against a class of polymorphic entrants. Among them, there is a wide variety of "uncoordinated" entrant distributions, because entrants exploit each other. Since there are infinitely many strategies to potentially emerge, such mis-coordination is plausible.

Seemingly different equilibria are not fundamentally different. The contrasting bimorphic equilibrium is payoff equivalent to countably many distributions including trust-building strategies. The point is the "recursive" structure of the VSRPD model. Players

[25]Unlike many of the related literature, we did not restrict the stability analysis within Markov strategies.

entering the matching pool nullify the past and thus the continuation payoff after ending a partnership is the same as the life time payoff. Therefore ending the partnership or keeping (renegotiating) it to shift to cooperation can be payoff equivalent. This is a key feature of the VSRPD model.

Among the payoff-equivalent distributions, the contrasting bimorphic equilibrium is the only locally stable equilibrium whenever it exists. Near the boundary of $\underline{\delta}_{c0d0}$, which admits payoff equivalence of c_0- and d_0-strategy, c_0-strategy is more advantageous than other strategies unless the only other strategy present is d_0-strategy.

Let us also mention other advantages of the c_0-d_0 equilibrium. Non-degenerate ($T > 0$) trust-building strategies require long memory and many states to implement them, while c_0- and d_0-strategies are very simple. Hence, if we consider complexity cost of playing a strategy, then c_1-strategy is no longer an alternative best reply to the c_0-d_0 equilibrium and the latter becomes evolutionary stable even against coordinated entrants. Moreover, non-degenerate trust-building strategies require coordination on the exact timing to shift to cooperation. The length of the trust-building periods should be common knowledge or a group norm among the entrants, but the source of such knowledge or norm is unclear (this also applies to the trust-building equilibria analyzed in Greve-Okuno, 2009). For this reason, coordinated entrants are also not so plausible.

Coordinated invasion has problems not only of difficulty in coordination of the timing, but also of "psychological cost" in implementation. We can interpret that a c_0-strategy is based on a norm of long-term cooperation, while d_0-strategy is based on a norm of non-cooperation. In terms of such norms, c_1-strategy forces players to play according to the d_0-norm in period 1 and then to switch to the c_0-norm from period 2 on. Playing according to two opposite norms can generate psychological cost, compared to playing a single norm strategy. Just like complexity cost makes a c_1-strategy not an alternative best reply to the c_0-d_0 equilibrium, psychological cost of c_1-strategy makes the c_0-d_0 equilibrium stable against costly c_1-entrants.

In our companion paper, Fujiwara-Greve et al. (2013), we also show that, among various equilibrium combinations of c_0-, d_0-, and c_1-strategy, the c_0-d_0 equilibrium can be most efficient over a broad range of parameter values, even without the consideration of the complexity or psychological cost to implement a c_1-strategy. The idea is that if the size of the "stake" $g - c$ is not so large, even if players can coordinate on the monomorphic c_1-equilibrium, one-period trust building by the whole society is costly. Therefore, in this case, the c_0-d_0 equilibrium is not only stable but also (informationally constrained) efficient.

Finally, we note two future research directions. An important extension is a two population model of firms and workers, to make a closed model of efficiency wage theory (e.g., Okuno, 1981, Shapiro and Stiglitz, 1984). If there is an equilibrium with a contrasting strategy distribution on the worker side, e.g., cooperative workers and non-cooperative workers, it gives a further rationale to equilibrium unemployment in a homogeneous worker population.

We placed our model of VSRPD as a *large social game*, in which players not only choose actions but also with whom to play the game in a large society. It can be a first step towards the research of endogenous network formation with consideration of within-network strategic behavior. There is a large literature of network formation researches (see for example, Jackson, 2008) but they usually omit the strategic behavior within a network. We showed that pairwise cooperative networks (between c_0-players) and non-networking players can co-exist in the society. This also implies that it is not guaranteed that all agents in the society end up in a (long-term) network.

APPENDIX: PROOFS

Proof of Remark 2: Consider entry of c_1-strategy. For any $\epsilon > 0$, let $p^{PE}(\epsilon) = (1-\epsilon)[\overline{\alpha}_{cd}(\delta)c_0 + \{1-\overline{\alpha}_{cd}(\delta)\}d_0] + \epsilon \cdot c_1$ be the post-entry distribution.

$$\begin{aligned} v(d_0; p^{PE}(\epsilon)) &= (1-\epsilon)[\overline{\alpha}_{cd}(\delta)g + \{1-\overline{\alpha}_{cd}(\delta)\}d] + \epsilon d \\ v(c_1; p^{PE}(\epsilon)) &= \frac{(1-\epsilon)[\overline{\alpha}_{cd}(\delta)g + \{1-\overline{\alpha}_{cd}(\delta)\}d] + \epsilon(d + \delta^2 \frac{c}{1-\delta^2})}{(1-\epsilon)\cdot 1 + \epsilon \frac{1}{1-\delta^2}} \\ &= v(d_0; p^{PE}(\epsilon)) + \frac{\epsilon \frac{\delta^2}{1-\delta^2}}{1-\epsilon+\frac{\epsilon}{1-\delta^2}} \{c - v(d_0; p^{PE}(\epsilon))\}. \end{aligned}$$

At $\alpha = \overline{\alpha}_{cd}(\delta)(< 1)$,

$$v(d_0; \overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0) = v(c_0; \overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0) < v(c_0, c_0) = c.$$

Hence

$$v(d_0; p^{PE}(\epsilon)) = (1 - \epsilon)v(d_0; \overline{\alpha}_{cd}(\delta)c_0 + \{1 - \overline{\alpha}_{cd}(\delta)\}d_0) + \epsilon d < c.$$

That is, c_1-strategy can invade the bimorphic distribution. *Q.E.D.*

Proof of Proposition 1: Since p^* is a locally stable Nash equilibrium, it suffices to prove the condition (ii) for any $q = (q_{c_0}, q_{d_0}, q_{c_1}, q_{d_1}, \ldots) \in E(\delta)$.

For any $\epsilon \in (0, 1)$, let the post-entry distribution be

$$\mathbf{x}(\epsilon) := (1 - \epsilon)p^* + \epsilon \cdot q = (x_{c_0}(\epsilon), x_{d_0}(\epsilon), x_{c_1}(\epsilon), x_{d_1}(\epsilon), \ldots).$$

Note that, $q \in E(\delta)$ implies that $x_{c_0}(\epsilon) \leqq \overline{\alpha}_{cd}(\delta)$, for any $\epsilon \in (0, 1)$, and $x_s(\epsilon) = \epsilon q_s$ for all $s \in \{c_1, d_1, \ldots\}$.

For $s = c_0, d_0$, define $v_{cd}(s; \alpha) := v(s; \alpha c_0 + (1 - \alpha)d_0)$ as the average payoff under a bimorphic distribution consisting of α of c_0-strategy and $1 - \alpha$ of d_0-strategy. Then

$$\begin{aligned} v(c_0; \mathbf{x}(\epsilon)) &= \frac{x_{c_0}(\epsilon)\frac{c}{1-\delta^2} + \{1 - x_{c_0}(\epsilon)\}\ell}{x_{c_0}(\epsilon)\frac{1}{1-\delta^2} + \{1 - x_{c_0}(\epsilon)\}} = v_{cd}(c_0; x_{c_0}(\epsilon)); \\ v(d_0; \mathbf{x}(\epsilon)) &= x_{c_0}(\epsilon) \cdot g + \{1 - x_{c_0}(\epsilon)\}d = v_{cd}(d_0; x_{c_0}(\epsilon)). \end{aligned}$$

For c_1- and d_1- strategy, the post-entry average payoff depends only on $x_{c_0}(\epsilon)$, $x_{c_1}(\epsilon)$, and $x_{d_T+}(\epsilon) := \sum_{t=T}^{\infty} x_{d_t}(\epsilon) + \sum_{t=T+1}^{\infty} x_{c_t}(\epsilon)$. To see this, take c_1-strategy. Its in-match payoff is g against c_0-strategy, d against d_0-strategy, $d + \delta^2 \frac{c}{1-\delta^2}$ against c_1-strategy, and $d + \delta^2 \ell$ against any other strategy. Therefore the post-entry average payoff is

$$v(c_1; \mathbf{x}(\epsilon)) = \frac{x_{c_0}(\epsilon) \cdot g + \{1 - x_{c_0}(\epsilon)\}d + \delta^2\{x_{c_1}(\epsilon) \cdot \frac{c}{1-\delta^2} + x_{d_1+}(\epsilon) \cdot \ell\}}{1 + \delta^2\{x_{c_1}(\epsilon) \cdot \frac{1}{1-\delta^2} + x_{d_1+}(\epsilon)\}}.$$

Similarly,

$$v(d_1; \mathbf{x}(\epsilon)) = \frac{x_{c_0}(\epsilon) \cdot g + \{1 - x_{c_0}(\epsilon)\}d + \delta^2\{x_{c_1}(\epsilon) \cdot g + x_{d_1+}(\epsilon) \cdot d\}}{1 + \delta^2\{x_{c_1}(\epsilon) + x_{d_1+}(\epsilon)\}}.$$

We can simplify these as follows. (To conserve space, we omit (ϵ) except for $x_{c_0}(\epsilon)$ in the following.) Let $\overline{x}_1 := x_{c_1} + x_{d1+}$ and $\alpha_1 := x_{c_1}/\overline{x}_1$ (this is independent of ϵ). Then $x_{c1} = \alpha_1\overline{x}_1$ and $x_{d1+} = (1-\alpha_1)\overline{x}_1$. Also, let $L(c_0;\alpha) := \alpha \cdot \frac{1}{1-\delta^2} + (1-\alpha)$, which is the expected length of partnerships for c_0, under a distribution with its share α. Using these, we have that

$$
\begin{aligned}
\text{(9)} \qquad v(c_1;\mathbf{x}(\epsilon)) &= \frac{v_{cd}(d_0;x_{c_0}(\epsilon)) + \delta^2\overline{x}_1\{\alpha_1 \cdot \frac{c}{1-\delta^2} + (1-\alpha_1)\ell\}}{1+\delta^2\overline{x}_1 L(c_0;\alpha_1)} \\
&= v_{cd}(d_0;x_{c_0}(\epsilon)) + \frac{\delta^2\overline{x}_1 L(c_0;\alpha_1)}{1+\delta^2\overline{x}_1 L(c_0;\alpha_1)}\{v_{cd}(c_0;\alpha_1) - v_{cd}(d_0;x_{c_0}(\epsilon))\};\\
\text{(10)} \qquad v(d_1;\mathbf{x}(\epsilon)) &= \frac{v_{cd}(d_0;x_{c_0}(\epsilon)) + \delta^2\overline{x}_1\{\alpha_1 g + (1-\alpha_1)d\}}{1+\delta^2\overline{x}_1} \\
&= v_{cd}(d_0;x_{c_0}(\epsilon)) + \frac{\delta^2\overline{x}_1}{1+\delta^2\overline{x}_1}\{v_{cd}(d_0;\alpha_1) - v_{cd}(d_0;x_{c_0}(\epsilon))\}.
\end{aligned}
$$

For c_T- and d_T-strategy with $T \geqq 2$, it is now clear that the post-entry average payoff depends only on $x_{c_0}(\epsilon), x_{d_0}(\epsilon), \ldots, x_{c_T}(\epsilon)$, and $x_{d_T+}(\epsilon)$. For $t = 1, 2, \ldots, T$, let $\overline{x}_t = x_{ct} + x_{dt+}$ and $\alpha_t = x_{ct}/\overline{x}_t$. Then

$$
\begin{aligned}
v(c_T;\mathbf{x}(\epsilon)) &= \frac{x_{c_0}(\epsilon)\cdot g + \{1-x_{c_0}(\epsilon)\}d + \delta^2\{x_{c_1}\cdot g + x_{d_1+}\cdot d\} + \cdots + \delta^{2T}\{x_{c_T}\cdot \frac{c}{1-\delta^2} + x_{d_T+}\ell\}}{1+\delta^2\{x_{c_1}+x_{d_1+}\} + \cdots + \delta^{2T}\{x_{c_T}\cdot\frac{1}{1-\delta^2} + x_{d_T+}\}} \\
&= \frac{v_{cd}(d_0;x_{c_0}(\epsilon)) + \delta^2\overline{x}_1\{\alpha_1 g + (1-\alpha_1)d\} + \cdots + \delta^{2T}\overline{x}_T\{\alpha_T\frac{c}{1-\delta^2} + (1-\alpha_T)\ell\}}{1+\delta^2\overline{x}_1 + \cdots + \delta^{2T}\overline{x}_T L(c_0;\alpha_T)},
\end{aligned}
$$

so that

$$
\begin{aligned}
\text{(11)} \quad v(c_T;\mathbf{x}(\epsilon)) &= v_{cd}(d_0;x_{c_0}(\epsilon)) \\
&+ \frac{\delta^2\overline{x}_1}{1+\sum_{t=1}^{T-1}\delta^{2t}\overline{x}_t + \delta^{2T}\overline{x}_T L(c_0;\alpha_T)}\Big\{v_{cd}(d_0;\alpha_1) - v_{cd}(d_0;x_{c_0}(\epsilon))\Big\} \\
&+ \frac{\delta^4\overline{x}_2}{1+\sum_{t=1}^{T-1}\delta^{2t}\overline{x}_t + \delta^{2T}\overline{x}_T L(c_0;\alpha_T)}\Big\{v_{cd}(d_0;\alpha_2) - v_{cd}(d_0;x_{c_0}(\epsilon))\Big\} \\
&+ \cdots + \frac{\delta^{2T}\overline{x}_T L(c_0;\alpha_T)}{1+\sum_{t=1}^{T-1}\delta^{2t}\overline{x}_t + \delta^{2T}\overline{x}_T L(c_0;\alpha_T)}\Big\{v_{cd}(c_0;\alpha_T) - v_{cd}(d_0;x_{c_0}(\epsilon))\Big\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
v(d_T;\mathbf{x}(\epsilon)) &= \frac{x_{c_0}(\epsilon)\cdot g + \{1-x_{c_0}(\epsilon)\}d + \delta^2\{x_{c_1}\cdot g + x_{d_1+}\cdot d\} + \cdots + \delta^{2T}\{x_{c_T}\cdot g + x_{d_T+}\cdot d\}}{1+\delta^2\{x_{c_1}+x_{d_1+}\} + \cdots + \delta^{2T}\{x_{c_T} + x_{d_T+}\}} \\
&= \frac{v_{cd}(d_0;x_{c_0}(\epsilon)) + \delta^2\overline{x}_1\{\alpha_1 g + (1-\alpha_1)d\} + \cdots + \delta^{2T}\overline{x}_T\{\alpha_T g + (1-\alpha_T)d\}}{1+\delta^2\overline{x}_1 + \cdots + \delta^{2T}\overline{x}_T}
\end{aligned}
$$

implies that

$$(12) \qquad v(d_T;\mathbf{x}(\epsilon)) = v_{cd}(d_0;x_{c_0}(\epsilon)) + \frac{\delta^2\overline{x}_1}{1+\sum_{t=1}^T \delta^{2t}\overline{x}_t}\Big\{v_{cd}(d_0;\alpha_1) - v_{cd}(d_0;x_{c_0}(\epsilon))\Big\}$$
$$+ \frac{\delta^4\overline{x}_2}{1+\sum_{t=1}^T \delta^{2t}\overline{x}_t}\Big\{v_{cd}(d_0;\alpha_2) - v_{cd}(d_0;x_{c_0}(\epsilon))\Big\}$$
$$+\cdots+ \frac{\delta^{2T}\overline{x}_T}{1+\sum_{t=1}^T \delta^{2t}\overline{x}_t}\Big\{v_{cd}(d_0;\alpha_T) - v_{cd}(d_0;x_{c_0}(\epsilon))\Big\}.$$

There exists $\bar{\epsilon}_0 \in (0,1)$ such that for any $\epsilon \in (0,\bar{\epsilon}_0]$, $x_{c_0}(\epsilon)$ is in the region of $[\underline{\alpha}_{cd}(\delta), \overline{\alpha}_{cd}(\delta)]$;

$$\underline{\alpha}_{cd}(\delta) \leqq (1-\epsilon)\overline{\alpha}_{cd}(\delta) + \epsilon q_{c_0} = x_{c_0}(\epsilon) \leqq \overline{\alpha}_{cd}(\delta).$$

In this region, d_0-strategy has lower average payoff than that of c_0-strategy.

$$(13) \qquad v(d_0;\mathbf{x}(\epsilon)) = v_{cd}(d_0;x_{c_0}(\epsilon)) \leqq v_{cd}(c_0;x_{c_0}(\epsilon)) = v(c_0;\mathbf{x}(\epsilon)), \quad \forall \epsilon \in (0,\bar{\epsilon}_0].$$

(See Figure 2.) (13) means that it suffices to prove that for each $T = 1,2,\dots$ and sufficiently small ϵ, entrant strategies d_T and c_T earn less than d_0-strategy does;

$$v(d_T;\mathbf{x}(\epsilon)) < v(d_0;\mathbf{x}(\epsilon)) \;\; \text{and} \;\; v(c_T;\mathbf{x}(\epsilon)) < v(d_0;\mathbf{x}(\epsilon)).$$

Step 1: For each $T = 1,2,\dots$, there exists $\bar{\epsilon}_T^d \in (0,1)$ such that

$$(14) \qquad v(d_T;\mathbf{x}(\epsilon)) < v(d_0;\mathbf{x}(\epsilon)), \quad \forall \epsilon \in (0,\bar{\epsilon}_T^d).$$

Proof of Step 1: To show (14), it suffices to prove that the second to the last terms of (12) are all negative, that is $v_{cd}(d_0;a_t) < v_{cd}(d_0;x_{c_0}(\epsilon))$ for all $t = 1,2,\dots,T$.

For each $t \geqq 1$, $q \in E(\delta)$ implies that

$$\alpha_t = \frac{q_{c_T}}{q_{c_T} + q_{d_T+}} < \overline{\alpha}_{cd}(\delta).$$

Hence, for each $t \geqq 1$, there exists $\bar{\epsilon}_t \in (0,1)$ such that for any $\epsilon \in (0,\bar{\epsilon}_t)$,

$$\alpha_t < (1-\epsilon)\overline{\alpha}_{cd}(\delta) + \epsilon q_{c_0} = x_{c_0}(\epsilon).$$

Since $v_{cd}(d_0;\alpha)$ is increasing in α,

$$(15) \qquad v_{cd}(d_0;\alpha_t) < v_{cd}(d_0;x_{c_0}(\epsilon)) \;\; \forall t = 1,2,\dots, \quad \forall \epsilon \in (0,\bar{\epsilon}_t).$$

Let $\bar{\epsilon}_T^d = \min_{t \leqq T} \bar{\epsilon}_t \in (0,1)$. Then (12) and (15) imply that for any $T = 1, 2, \ldots,$

$$v(d_T; \mathbf{x}(\epsilon)) < v_{cd}(d_0; x_{c_0}(\epsilon)), \quad \forall \epsilon \in (0, \bar{\epsilon}_T^d).$$

This completes the proof of Step 1. //

Step 2: For any $T = 1, 2, \ldots$, there exists $\bar{\epsilon}_T^c \in (0,1)$ such that

$$v(c_T; \mathbf{x}(\epsilon)) < v_{cd}(d_0; x_{c_0}(\epsilon)), \quad \forall \epsilon \in (0, \bar{\epsilon}_T^c).$$

Proof of Step 2: Again, it suffices to prove that second to the $T+1$-th terms of (11) are all negative. But we have already shown in (15) that the second to T-th terms are all negative for any $\epsilon \in (0, \min_{t \leqq T-1} \bar{\epsilon}_t)$. It remains to prove that the last term is negative for some range of ϵ, that is, there exists $\hat{\epsilon}_T \in (0,1)$ such that

$$v_{cd}(c_0; \alpha_T) < v_{cd}(d_0; x_{c_0}(\epsilon)), \; \forall \epsilon \in (0, \hat{\epsilon}_T).$$

The assumption $q \in E(\delta)$ only implies that $\alpha_T < \overline{\alpha}_{cd}(\delta)$, hence we have two cases.
Case 1: $\alpha_T < \underline{\alpha}_{cd}(\delta)$.

In this case (see Figure 2), we have $v_{cd}(c_0; \alpha_T) < v_{cd}(d_0; \underline{\alpha}_{cd}(\delta))$. Recall that for any $\epsilon \in (0, \bar{\epsilon}_0)$, $\underline{\alpha}_{cd}(\delta) \leqq x_{c_0}(\epsilon)$ so that $v_{cd}(d_0; \underline{\alpha}_{cd}(\delta)) \leqq v_{cd}(d_0; x_{c_0}(\epsilon))$. Therefore,

$$v_{cd}(c_0; \alpha_T) < v_{cd}(d_0; x_{c_0}(\epsilon)), \; \forall \epsilon \in (0, \bar{\epsilon}_0).$$

Case 2: $\underline{\alpha}_{cd}(\delta) \leqq \alpha_T < \overline{\alpha}_{cd}(\delta)$.

Note that $v_{cd}(d_0; \alpha)$ is continuous and increasing in α. If $\underline{\alpha}_{cd}(\delta) < \alpha_T < \overline{\alpha}_{cd}(\delta)$, then

$$v_{cd}(d_0; \alpha_T) < v_{cd}(c_0; \alpha_T) \;\text{ and }\; v_{cd}(d_0; \overline{\alpha}_{cd}(\delta)) = v_{cd}(c_0; \overline{\alpha}_{cd}(\delta)).$$

By the Intermediate Value Theorem, there exists $\hat{\alpha}(\alpha_T) \in (\underline{\alpha}_{cd}(\delta), \overline{\alpha}_{cd}(\delta))$ such that

$$v_{cd}(c_0; \alpha_T) = v_{cd}(d_0; \hat{\alpha}(\alpha_T)).$$

See also Figure 7. If $\underline{\alpha}_{cd}(\delta) = \alpha_T$, then clearly there exists $\hat{\alpha}(\alpha_T) = \underline{\alpha}_{cd}(\delta) < \overline{\alpha}_{cd}(\delta)$ such that

$$v_{cd}(c_0; \alpha_T) = v_{cd}(d_0; \hat{\alpha}(\alpha_T)).$$

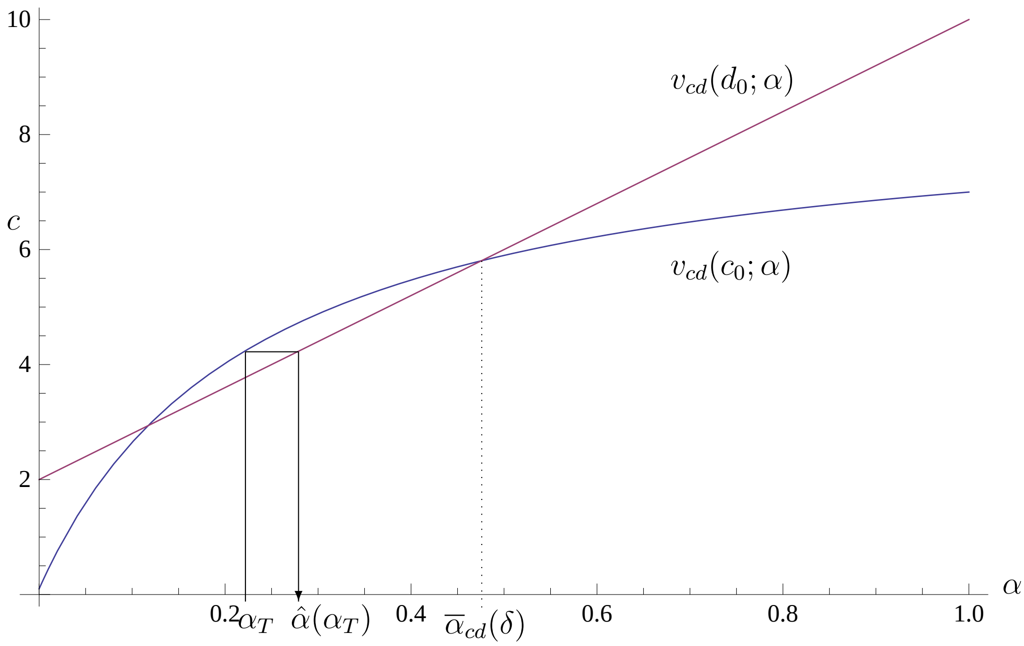

Figure 7: Existence of $\hat{\alpha}(\alpha_T)$

In either case, $\hat{\alpha}(\alpha_T) < \overline{\alpha}_{cd}(\delta)$ holds. Hence there exists $\tilde{\epsilon}_T \in (0,1)$ such that for any $\epsilon \in (0, \tilde{\epsilon}_T)$, $x_{c_0}(\epsilon)$ exceeds $\hat{\alpha}(\alpha_T)$;

$$\hat{\alpha}(\alpha_T) < (1-\epsilon)\overline{\alpha}_{cd}(\delta) + \epsilon q_{c_0} = x_{c_0}(\epsilon),$$

hence

$$v_{cd}(c_0;\alpha_T) = v_{cd}(d_0;\hat{\alpha}(\alpha_T)) < v_{cd}(d_0;x_{c_0}(\epsilon)), \quad \forall \epsilon \in (0,\tilde{\epsilon}_T).$$

In summary, for any $\alpha_T < \overline{\alpha}_{cd}(\delta)$, there exists $\hat{\epsilon}_T = \min\{\bar{\epsilon}_0, \tilde{\epsilon}_T\} \in (0,1)$ such that

$$v_{cd}(c_0;\alpha_T) < v_{cd}(d_0;x_{c_0}(\epsilon)), \quad \forall \epsilon \in (0,\hat{\epsilon}_T). \tag{16}$$

Let $\bar{\epsilon}^c_T = \min\{\min_{t \leqq T-1} \bar{\epsilon}_t, \hat{\epsilon}_T\}$, Then (11), (15), and (16) imply that for any $T = 1, 2, \ldots,$

$$v(c_T;\mathbf{x}(\epsilon)) < v_{cd}(d_0;x_{c_0}(\epsilon)), \quad \forall \epsilon \in (0,\bar{\epsilon}^c_T). \tag{17}$$

This completes the proof of Step 2 and the Proposition as well. *Q.E.D.*

Proof of Remark 3: First, take an arbitrary $T < \infty$ and let $p^{PE} = (1-\epsilon)d_T + \epsilon(xc_{T+1} + (1-x)d_{T+1})$. Clearly, $v(d_T; p^{PE}) = d$, while

$$v(d_{T+1}; p^{PE}) = \frac{(1+\delta^2+\cdots+\delta^{2T})d + \epsilon\delta^{2(T+1)}\{xg + (1-x)d\}}{(1+\delta^2+\cdots+\delta^{2T}) + \epsilon\delta^{2(T+1)}}$$
$$= d + \frac{\epsilon\delta^{2(T+1)}\{xg + (1-x)d - d\}}{(1+\delta^2+\cdots+\delta^{2T}) + \epsilon\delta^{2(T+1)}} > d, \quad \text{because } x > 0.$$

Next, let $p^{PE} = (1-\epsilon)d_\infty + \epsilon\{xc_0 + (1-x)d_0\}$. Then

$$v(d_\infty; p^{PE}) = \frac{(1-\epsilon)\frac{d}{1-\delta^2} + \epsilon\{xg + (1-x)d\}}{(1-\epsilon)\frac{1}{1-\delta^2} + \epsilon}$$
$$= d + \frac{\epsilon}{(1-\epsilon)\frac{1}{1-\delta^2} + \epsilon}\{xg + (1-x)d - d\}$$
$$= d + \frac{\epsilon}{(1-\epsilon)\frac{1}{1-\delta^2} + \epsilon}x(g-d).$$

On the other hand,

$$v(d_0; p^{PE}) = (1-\epsilon)d + \epsilon\{xg + (1-x)d\} = d + \epsilon x(g-d).$$

Note that the denominator of the coefficient of the second term of $v(d_\infty; p^{PE})$ is

$$(1-\epsilon)\frac{1}{1-\delta^2} + \epsilon = (1-\epsilon) + (1-\epsilon)\frac{\delta^2}{1-\delta^2} + \epsilon > 1.$$

Hence $v(d_0; p^{PE}) > v(d_\infty; p^{PE})$. *Q.E.D.*

Proof of Lemma 1: Recall that for any $\alpha \in (0,1)$, $L(c_0; \alpha) = \alpha \cdot \frac{1}{1-\delta^2} + 1 - \alpha$ and

$$v_{cd}(c_0; \alpha) := v(c_0; \alpha c_0 + (1-\alpha)d_0) = \frac{\alpha \cdot \frac{c}{1-\delta^2} + (1-\alpha)\ell}{L(c_0; \alpha)}$$
$$v_{cd}(d_0; \alpha) := v(d_0; \alpha c_0 + (1-\alpha)d_0) = \alpha g + (1-\alpha)d.$$

For any $(\alpha_0, \alpha_1, \ldots, \alpha_T)$, the average payoff functions are formulated as follows.

$$v(c_0; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T)) = \frac{\alpha_0 \cdot \frac{c}{1-\delta^2} + (1-\alpha_0)\ell}{\alpha_0 \cdot \frac{1}{1-\delta^2} + 1 - \alpha_0} = v_{cd}(c_0; \alpha_0)$$
$$v(c_1; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T)) = \frac{\alpha_0 \cdot g + (1-\alpha_0)d + (1-\alpha_0)\delta^2\{\alpha_1 \cdot \frac{c}{1-\delta^2} + (1-\alpha_1)\ell\}}{1 + (1-\alpha_0)\delta^2 L(c_0; \alpha_1)}$$
$$= v_{cd}(d_0; \alpha_0) + \frac{(1-\alpha_0)\delta^2 L(c_0; \alpha_1)}{1 + (1-\alpha_0)\delta^2 L(c_0; \alpha_1)}\Big\{v_{cd}(c_0; \alpha_1) - v_{cd}(d_0; \alpha_0)\Big\}$$

$$
\begin{aligned}
v(c_2; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T)) &= \frac{1}{1 + (1-\alpha_0)\delta^2 + (1-\alpha_0)(1-\alpha_1)\delta^4 L(c_0;\alpha_2)} \Big[\alpha_0 \cdot g + (1-\alpha_0)d \\
&+ (1-\alpha_0)\delta^2\{\alpha_1 g + (1-\alpha_1)d\} \\
&+ (1-\alpha_0)(1-\alpha_1)\delta^4\{\alpha_2 \cdot \frac{c}{1-\delta^2} + (1-\alpha_2)\ell\}\Big] \\
&= v_{cd}(d_0;\alpha_0) \\
&+ \frac{(1-\alpha_0)\delta^2}{L(c_2; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T))}\Big\{v_{cd}(d_0;\alpha_1) - v_{cd}(d_0;\alpha_0)\Big\} \\
&+ \frac{(1-\alpha_0)(1-\alpha_1)\delta^4 L(c_0;\alpha_2)}{L(c_2; p_{c0}^{d_T}(\alpha_0, \alpha_1, \alpha_2))}\Big\{v_{cd}(c_0;\alpha_2) - v_{cd}(d_0;\alpha_0)\Big\},
\end{aligned}
$$

where $L(c_2; p_{c0}^{d_T}(\alpha_0, \alpha_1, \alpha_2)) = 1 + (1-\alpha_0)\delta^2 + (1-\alpha_0)(1-\alpha_1)\delta^4 L(c_0;\alpha_2)$. To generalize, for each $t = 1, 2, \ldots, T$, the expected length of partnerships that c_t-strategy experiences is

$$
L(c_t; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_t)) := 1 + (1-\alpha_0)\delta^2 + \cdots + \{\times_{\tau=0}^{t-2}(1-\alpha_\tau)\}\delta^{2(t-1)} + \{\times_{\tau=0}^{t-1}(1-\alpha_\tau)\}\delta^{2t} L(c_0;\alpha_t)
$$

and

$$
L(d_T; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T)) := 1 + (1-\alpha_0)\delta^2 + \cdots + \{\times_{\tau=0}^{T-1}(1-\alpha_\tau)\}\delta^{2T}
$$

is that of d_T-strategy. Using these, we have

(18)

$$
\begin{aligned}
v(c_T; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T)) &= v_{cd}(d_0;\alpha_0) \\
&+ \frac{(1-\alpha_0)\delta^2}{L(c_T; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T))}\Big\{v_{cd}(d_0;\alpha_1) - v_{cd}(d_0;\alpha_0)\Big\} \\
&+ \cdots + \frac{\times_{\tau=0}^{T-2}(1-\alpha_\tau)\cdot\delta^{2(T-1)}}{L(c_T; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T))}\Big\{v_{cd}(d_0;\alpha_{T-1}) - v_{cd}(d_0;\alpha_0)\Big\} \\
&+ \frac{\times_{\tau=0}^{T-1}(1-\alpha_\tau)\cdot\delta^{2T} L(c_0;\alpha_T)}{L(c_T; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T))}\Big\{v_{cd}(c_0;\alpha_T) - v_{cd}(d_0;\alpha_0)\Big\};
\end{aligned}
$$

(19)

$$
\begin{aligned}
v(d_T; p_{c0}^{d_T}(\alpha_0, \alpha_1, \ldots, \alpha_T)) &= \frac{1}{1 + (1-\alpha_0)\delta^2 + \cdots + \{\times_{\tau=0}^{T-1}(1-\alpha_\tau)\}\delta^{2T}}\Big[\alpha_0 \cdot g + (1-\alpha_0)d \\
&+ (1-\alpha_0)\delta^2\{\alpha_1 g + (1-\alpha_1)d\} \\
&+ \cdots + \{\times_{\tau=0}^{T-1}(1-\alpha_\tau)\}\delta^{2T}\{\alpha_T g + (1-\alpha_T)d\}\Big]
\end{aligned}
$$

$$= v_{cd}(d_0;\alpha_0)$$
$$+ \frac{(1-\alpha_0)\delta^2}{L(d_T; p_{c0}^{d_T}(\alpha_0,\alpha_1,\ldots,\alpha_T))}\Big\{v_{cd}(d_0;\alpha_1) - v_{cd}(d_0;\alpha_0)\Big\}$$
$$+ \cdots + \frac{\times_{\tau=0}^{T-1}(1-\alpha_\tau)\cdot\delta^{2T}}{L(d_T; p_{c0}^{d_T}(\alpha_0,\alpha_1,\ldots,\alpha_T))}\Big\{v_{cd}(d_0;\alpha_T) - v_{cd}(d_0;\alpha_0)\Big\}.$$

Therefore, if $\alpha_t = \overline{\alpha}_{cd}(\delta)$ for all $t = 0, 1, \ldots, T$, then $v_{cd}(d_0;\alpha_t) = v_{cd}(c_0;\alpha_t) = v_{cd}(d_0;\alpha_0)$ so that the average payoffs coincide. *Q.E.D.*

Proof of Proposition 2: We show that

$$\frac{\partial v(c_0;p^{PE})}{\partial \epsilon}\mid_{\epsilon=0} > \frac{\partial v(s;p^{PE})}{\partial \epsilon}\mid_{\epsilon=0} \quad \forall s \in \{c_1, c_2, \ldots, c_T, d_T\}$$

when $\delta = \underline{\delta}_{c_0 d_0}$.

We first arrange the post-entry distribution into the relative ratio form. (For notational simplicity, we write $\overline{\alpha}_{cd} = \overline{\alpha}_{cd}(\delta)$.) Notice that the post-entry distribution is

$$\begin{aligned} p^{PE} = {} & \{(1-\epsilon)\overline{\alpha}_{cd} + \epsilon\}\cdot c_0 + (1-\epsilon)(1-\overline{\alpha}_{cd})\overline{\alpha}_{cd}\cdot c_1 \\ & + (1-\epsilon)(1-\overline{\alpha}_{cd})^2\,\overline{\alpha}_{cd}\cdot c_2 + \cdots \\ & + (1-\epsilon)(1-\overline{\alpha}_{cd})^T\,\overline{\alpha}_{cd}\cdot c_T \\ & + (1-\epsilon)(1-\overline{\alpha}_{cd})^{T+1} d_T. \end{aligned} \tag{20}$$

We want to arrange this as the relative ratio form such that

$$\begin{aligned} p^{PE} = {} & \alpha_0^{PE}\cdot c_0 + (1-\alpha_0^{PE})\alpha_1^{PE}\cdot c_1 \\ & + (1-\alpha_0^{PE})(1-\alpha_1^{PE})\alpha_2^{PE}\cdot c_2 + \cdots + \\ & + \{\times_{\tau=0}^{T-1}(1-\alpha_\tau^{PE})\}\alpha_T^{PE}\cdot c_T + \{\times_{\tau=0}^{T}(1-\alpha_\tau^{PE})\}d_T, \end{aligned} \tag{21}$$

where α_t^{PE} is the relative post-entry ratio of c_t-strategy against the total share of $c_{t+1}, \ldots, c_T$, and d_T-strategy. Hence $\alpha_0^{PE} = \{(1-\epsilon)\overline{\alpha}_{cd} + \epsilon\}$. For other α_t^{PE}, (20) can be arranged iteratively to cancel out numerators and denominators as

$$p^{PE} = \alpha_0^{PE} \cdot c_0 + (1-\alpha_0^{PE})\{\frac{(1-\epsilon)(1-\overline{\alpha}_{cd})\overline{\alpha}_{cd}}{1-\alpha_0^{PE}}\} \cdot c_1$$
$$+ (1-\alpha_0^{PE})\big[1 - \{\frac{(1-\epsilon)(1-\overline{\alpha}_{cd})\overline{\alpha}_{cd}}{1-\alpha_0^{PE}}\}\big]\{\frac{(1-\epsilon)(1-\overline{\alpha}_{cd})^2\ \overline{\alpha}_{cd}}{(1-\alpha_0^{PE})-(1-\epsilon)(1-\overline{\alpha}_{cd})\overline{\alpha}_{cd}}\} \cdot c_2$$
$$+ (1-\alpha_0^{PE})\big[1 - \{\frac{(1-\epsilon)(1-\overline{\alpha}_{cd})\overline{\alpha}_{cd}}{1-\alpha_0^{PE}}\}\big]\Big[1 - \{\frac{(1-\epsilon)(1-\overline{\alpha}_{cd})^2\ \overline{\alpha}_{cd}}{(1-\alpha_0^{PE})-(1-\epsilon)(1-\overline{\alpha}_{cd})\overline{\alpha}_{cd}}\}\Big] \times$$
$$\Big\{\frac{(1-\epsilon)(1-\overline{\alpha}_{cd})^3\ \overline{\alpha}_{cd}}{(1-\alpha_0^{PE})-(1-\epsilon)\overline{\alpha}_{cd}\sum_{\tau=1}^{2}(1-\overline{\alpha}_{cd})^\tau}\Big\} \cdot c_3 + \cdots .$$

Therefore, for each $t = 1, 2, \ldots, T$, let

$$\alpha_t^{PE} := \frac{(1-\epsilon)(1-\overline{\alpha}_{cd})^t\overline{\alpha}_{cd}}{(1-\alpha_0^{PE})-(1-\epsilon)\overline{\alpha}_{cd}\sum_{\tau=1}^{t-1}(1-\overline{\alpha}_{cd})^\tau}.$$

Then (20) and (21) coincide. Note also that when $\epsilon = 0$,

$$\alpha_t^{PE}\ |_{\epsilon=0} = \overline{\alpha}_{cd} \quad \forall t = 0, 1, 2, \ldots . \tag{22}$$

Using these relative ratios, we can compute the average post-entry payoffs of the strategies and differentiate them. As a preparation, note that

$$\frac{\partial \alpha_0^{PE}}{\partial \epsilon} = 1 - \overline{\alpha}_{cd}; \tag{23}$$

and, for any $t = 1, 2, \ldots, T$, by computation we have

$$\frac{\partial \alpha_t^{PE}}{\partial \epsilon}\ |_{\epsilon=0} = \frac{1}{(1-\overline{\alpha}_{cd})^2}\Big[-(1-\overline{\alpha}_{cd})^t\ \overline{\alpha}_{cd}\Big\{(1-\overline{\alpha}_{cd}) - \overline{\alpha}_{cd}\sum_{\tau=1}^{t-1}(1-\overline{\alpha}_{cd})^\tau\Big\}$$
$$+ (1-\overline{\alpha}_{cd})^t\overline{\alpha}_{cd}\Big\{(1-\overline{\alpha}_{cd}) - \overline{\alpha}_{cd}\sum_{\tau=1}^{t-1}(1-\overline{\alpha}_{cd})^\tau\Big\}\Big] = 0. \tag{24}$$

From (18) and (19) in the Proof of Lemma 1, for any $t = 1, 2, \ldots, T$,

$$v(c_t; p^{PE}) = v_{cd}(d_0; \alpha_0^{PE})$$
$$+ \sum_{\tau=1}^{t-1} A_\tau\{v_{cd}(d_0; \alpha_\tau^{PE}) - v_{cd}(d_0; \alpha_0^{PE})\} + B_t\{v_{cd}(c_0; \alpha_t^{PE}) - v_{cd}(d_0; \alpha_0^{PE})\};$$
$$v(d_t; p^{PE}) = v_{cd}(d_0; \alpha_0^{PE}) + \sum_{\tau=1}^{t} \tilde{A}_\tau\{v_{cd}(d_0; \alpha_\tau^{PE}) - v_{cd}(d_0; \alpha_0^{PE})\},$$

where

$$A_\tau = \frac{\{\times_{k=0}^{\tau-1}(1-\alpha_k^{PE})\}\delta^{2\tau}}{L(c_t;p^{PE})}, \quad \forall\tau = 1,2,\dots,t-1,$$
$$B_t = \frac{\{\times_{k=0}^{t-1}(1-\alpha_k^{PE})\}\delta^{2t}L(c_0;\alpha_t^{PE})}{L(c_t;p^{PE})},$$
$$\tilde{A}_\tau = \frac{\{\times_{k=0}^{\tau-1}(1-\alpha_k^{PE})\}\delta^{2\tau}}{L(d_t;p^{PE})}, \quad \forall\tau = 1,2,\dots,T,$$
$$L(c_t;p^{PE}) = 1+(1-\alpha_0^{PE})\delta^2+(1-\alpha_0^{PE})(1-\alpha_1^{PE})\delta^4+\cdots+\{\times_{k=0}^{t-1}(1-\alpha_k^{PE})\}\delta^{2t}L(c_0;\alpha_t^{PE}),$$
$$L(d_t;p^{PE}) = 1+(1-\alpha_0^{PE})\delta^2+(1-\alpha_0^{PE})(1-\alpha_1^{PE})\delta^4+\cdots+\{\times_{k=0}^{t-1}(1-\alpha_k^{PE})\}\delta^{2t}.$$

This also shows that for c_t- and d_t-strategy, only $\alpha_0^{PE},\dots,\alpha_t^{PE}$ matter. Let us differentiate the average payoff of c_t-strategy ($t \geqq 1$) with respect to ϵ via $\alpha_0^{PE},\dots,\alpha_t^{PE}$ and using (23).

$$\begin{aligned}\frac{\partial v(c_t;p^{PE})}{\partial\epsilon} &= (1-\overline{\alpha}_{cd})\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}\\ &+\sum_{\tau=1}^{t-1}A_\tau\Big\{\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}\cdot\frac{\partial\alpha_\tau^{PE}}{\partial\epsilon}-(1-\overline{\alpha}_{cd})\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}\Big\}\\ &+\sum_{\tau=1}^{t-1}\Big[\Big\{v_{cd}(d_0;\alpha_\tau^{PE})-v_{cd}(d_0;\alpha_0^{PE})\Big\}\times\sum_{k=0}^{\tau-1}\frac{\partial A_\tau}{\partial\alpha_k^{PE}}\cdot\frac{\partial\alpha_k^{PE}}{\partial\epsilon}\Big]\\ &+B_t\Big\{\frac{\partial v_{cd}(c_0;\alpha)}{\partial\alpha}\cdot\frac{\partial\alpha_t^{PE}}{\partial\epsilon}-(1-\overline{\alpha}_{cd})\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}\Big\}\\ &+\Big\{v_{cd}(c_0;\alpha_t^{PE})-v_{cd}(d_0;\alpha_0^{PE})\Big\}\times\sum_{k=0}^{t-1}\cdot\frac{\partial B_t}{\partial\alpha_k^{PE}}\cdot\frac{\partial\alpha_k^{PE}}{\partial\epsilon}].\end{aligned}$$

From (22) and (24), at $\epsilon = 0$, many terms disappear.

$$\begin{aligned}\text{(25)}\quad \frac{\partial v(c_t;p^{PE})}{\partial\epsilon}\Big|_{\epsilon=0} &= (1-\overline{\alpha}_{cd})\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}\\ &-(1-\overline{\alpha}_{cd})\Big\{\sum_{\tau=1}^{t-1}A_\tau\Big|_{\epsilon=0}\Big\}\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}-(1-\overline{\alpha}_{cd})\Big\{B_t\Big|_{\epsilon=0}\Big\}\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}\\ &= (1-\overline{\alpha}_{cd})\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}\cdot\frac{1}{L(c_t;p_{c0}^{d_T}(\overline{\alpha}_{cd},\dots,\overline{\alpha}_{cd}))},\end{aligned}$$

where the last equality comes from (22) and the computation as follows.

$$\frac{L(c_t;p^{PE})}{L(c_t;p^{PE})} = \frac{1}{L(c_t;p^{PE})}+A_1+\cdots+A_{t-1}+B_t$$
$$\iff 1-\sum_{\tau=1}^{t-1}A_\tau-B_t = \frac{1}{L(c_t;p^{PE})}.$$

Similarly,

$$\frac{\partial v(d_T;p^{PE})}{\partial\epsilon}\mid_{\epsilon=0} = (1-\overline{\alpha}_{cd})\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}\cdot\frac{1}{L(d_T;p_{c0}^{d_T}(\overline{\alpha}_{cd},\ldots,\overline{\alpha}_{cd}))}. \tag{26}$$

By the definition, at $\delta=\underline{\delta}_{c_0d_0}$ and $\alpha=\overline{\alpha}_{cd}$, the average payoff of c_0- and d_0-strategy are tangent; $\frac{\partial v_{cd}(c_0;\alpha)}{\partial\alpha}=\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}$. Hence the derivative of the average payoff of c_0-strategy is

$$\frac{\partial v_{cd}(c_0;p^{PE})}{\partial\epsilon}\mid_{\epsilon=0}=\left[\frac{\partial v_{cd}(c_0;\alpha)}{\partial\alpha}\cdot\frac{\partial\alpha_0^{PE}}{\partial\epsilon}\right]\mid_{\epsilon=0}=(1-\overline{\alpha}_{cd})\frac{\partial v_{cd}(c_0;\alpha)}{\partial\alpha}=(1-\overline{\alpha}_{cd})\frac{\partial v_{cd}(d_0;\alpha)}{\partial\alpha}.$$

Since $L(c_t;p_{c0}^{d_T}(\overline{\alpha}_{cd},\ldots,\overline{\alpha}_{cd}))>1$ and $L(d_T;p_{c0}^{d_T}(\overline{\alpha}_{cd},\ldots,\overline{\alpha}_{cd}))>1$, when $\delta=\underline{\delta}_{c_0d_0}$, (25) and (26) imply that

$$\frac{\partial v_{cd}(c_0;p^{PE})}{\partial\epsilon}\mid_{\epsilon=0}>\frac{\partial v_{cd}(s;p^{PE})}{\partial\epsilon}\mid_{\epsilon=0}\quad\forall s\in\{c_1,\ldots,c_T,d_T\}.$$

Q.E.D.

PROOF OF PROPOSITION 3: We first show a Lemma which generalizes Lemma 2 of Fujiwara-Greve, Okuno-Fujiwara, and Suzuki (2013).

LEMMA 2 *For any $T=0,1,2,\ldots$ and any $\alpha\in(0,1)$,*

$$v(d_T;\alpha c_T+(1-\alpha)d_T)=v(c_T;\alpha c_T+(1-\alpha)d_T)=:v\Rightarrow v<v^{BR}.$$

That is, if d_T- and c_T-strategies are payoff-equivalent, the bimorphic distribution is a Nash equilibrium.

PROOF OF LEMMA 2: From the derivation of the Best Reply Condition (7),

$$\begin{aligned} v<v^{BR} &\iff g+\delta\frac{v}{1-\delta}<\frac{c}{1-\delta^2}+\frac{\delta(1-\delta)}{1-\delta^2}\cdot\frac{v}{1-\delta}\\ &\iff g-\frac{c}{1-\delta^2}+\frac{\delta^2}{1-\delta^2}v<0.\end{aligned}$$

Hence we show the last inequality.

For notational brevity, let $p=\alpha c_T+(1-\alpha)d_T$. By the assumption,

$$v(d_T;p)=v(c_T;p)\iff L(c_T;p)V(d_T;p)-L(d_T;p)V(c_T;p)=0,$$

where

$$\begin{aligned}
L(d_T;p) &= 1+\delta^2+\cdots+\delta^{2T}\\
L(c_T;p) &= 1+\delta^2+\cdots+\delta^{2(T-1)}+\delta^{2T}\{\alpha\cdot\frac{1}{1-\delta^2}+(1-\alpha)\} = 1+\delta^2+\cdots+\delta^{2T}+\delta^{2(T+1)}\frac{\alpha}{1-\delta^2}\\
&= L(d_T;p)+\delta^{2(T+1)}\frac{\alpha}{1-\delta^2}\\
V(d_T;p) &= (1+\delta^2+\cdots+\delta^{2(T-1)})d+\delta^{2T}\{\alpha g+(1-\alpha)d\}\\
V(c_T;p) &= (1+\delta^2+\cdots+\delta^{2(T-1)})d+\delta^{2T}\{\alpha\frac{c}{1-\delta^2}+(1-\alpha)\ell\}.
\end{aligned}$$

Hence

$$\begin{aligned}
0 &= L(c_T;p)V(d_T;p)-L(d_T;p)V(c_T;p)\\
&=\Big\{L(d_T;p)+\delta^{2(T+1)}\frac{\alpha}{1-\delta^2}\Big\}\Big[(1+\delta^2+\cdots+\delta^{2(T-1)})d+\delta^{2T}\{\alpha g+(1-\alpha)d\}\Big]\\
&\quad -L(d_T;p)\Big[(1+\delta^2+\cdots+\delta^{2(T-1)})d+\delta^{2T}\{\alpha\frac{c}{1-\delta^2}+(1-\alpha)\ell\}\Big]\\
&=L(d_T;p)\delta^{2T}\{\alpha g+(1-\alpha)d\}+\frac{\delta^{2(T+1)}\alpha}{1-\delta^2}V(d_T;p)\\
&\quad -L(d_T;p)\delta^{2T}\{\alpha\frac{c}{1-\delta^2}+(1-\alpha)\ell\}\\
&=L(d_T;p)\delta^{2T}\Big[\alpha\{g-\frac{c}{1-\delta^2}\}+(1-\alpha)(d-\ell)\Big]+\delta^{2T}\alpha\frac{\delta^2}{1-\delta^2}V(d_T;p).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\Big[\alpha\{g-\frac{c}{1-\delta^2}\}+(1-\alpha)(d-\ell)\Big]+\alpha\frac{\delta^2}{1-\delta^2}\cdot\frac{V(d_T;p)}{L(d_T;p)}=0\\
\iff\ &\alpha\Big\{g-\frac{c}{1-\delta^2}+\frac{\delta^2}{1-\delta^2}v(d_T;p)\Big\}+(1-\alpha)(d-\ell)=0.
\end{aligned}$$

Since $(1-\alpha)(d-\ell)>0$, we have that $g-\frac{c}{1-\delta^2}+\frac{\delta^2}{1-\delta^2}v<0$. □

Therefore, it suffices to prove that for some T and $\alpha\in(0,1)$, the average payoff of c_T-strategy intersects with that of d_T-strategy from the above (see Figure 5), which warrants local stability at the payoff-equivalent distribution and Lemma 2 implies that it is a Nash equilibrium.

Step 1: For any $\delta\in(0,1)$, there exists $\tau^*(\delta)$ such that if $\underline{\tau}(\delta)<\tau^*(\delta)$, then the average payoff of $c_{\underline{\tau}(\delta)}$-strategy intersects with that of $d_{\underline{\tau}(\delta)}$-strategy from the above;

$$\frac{\partial v(c_{\underline{\tau}(\delta)};p)}{\partial\alpha}\Big|_{\alpha=1}<\frac{\partial v(d_{\underline{\tau}(\delta)};p)}{\partial\alpha}\Big|_{\alpha=1}.$$

Proof of Step 1: Let $T = \underline{\tau}(\delta)$ and $p = \alpha c_T + (1-\alpha)d_T$ for notational brevity.

By computation, for any $\alpha \in (0,1)$,

$$\frac{\partial v(d_T;p)}{\partial \alpha} = \frac{(1-\delta^2)\delta^{2T}}{1-\delta^{2(T+1)}}(g-d). \tag{27}$$

For c_T-strategy,

$$\begin{aligned}\frac{\partial v(c_T;p)}{\partial \alpha} &= \frac{1}{L(c_T;p)^2}\Big[\delta^{2T}\big(\frac{c}{1-\delta^2}-\ell\big)L(c_T;p) - \frac{\delta^{2(T+1)}}{1-\delta^2}V(c_T;p)\Big] \\ &= \frac{\delta^{2T}}{L(c_T;p)(1-\delta^2)}\Big[\{c-(1-\delta^2)\ell\} - \delta^2 v(c_T;p)\Big]\end{aligned}$$

At $\alpha = 1$, $L(c_T;p) = 1/(1-\delta^2)$ and $v(c_T;p) = v(c_T,c_T) = (1-\delta^{2T})d + \delta^{2T}c$, hence

$$\frac{\partial v(c_T;p)}{\partial \alpha}\Big|_{\alpha=1} = \delta^{2T}\big[\{c-(1-\delta^2)\ell\} - \delta^2\{(1-\delta^{2T})d + \delta^{2T}c\}\big]. \tag{28}$$

From (27) and (28), we have

$$\begin{aligned}&\frac{\partial v(d_T;p)}{\partial \alpha}\Big|_{\alpha=1} > \frac{\partial v(c_T;p)}{\partial \alpha}\Big|_{\alpha=1} \\ \iff & \frac{(1-\delta^2)}{1-\delta^{2(T+1)}}(g-d) > \{c-(1-\delta^2)\ell\} - \delta^2\{(1-\delta^{2T})d+\delta^{2T}c\} \\ \iff & \delta^{2(T+1)}(c-d) > (1-\delta^2)(d-\ell) + \{c-d-\frac{1-\delta^2}{1-\delta^{2(T+1)}}(g-d)\}\end{aligned} \tag{29}$$

At $T = \underline{\tau}(\delta)$, $v(c_{\underline{\tau}(\delta)}; c_{\underline{\tau}(\delta)}) = v^{BR}$ holds, by the definition in (8). Therefore

$$\begin{aligned}&(1-\delta^{2T})d + \delta^{2T}c = \frac{1}{\delta^2}\{c-(1-\delta^2)g\} \\ \iff & c-(1-\delta)^2 g - \{\delta^2(1-\delta^{2T})d + \delta^{2(T+1)}c\} = 0 \\ \iff & (1-\delta^{2(T+1)})(c-d) - (1-\delta^2)(g-d) = 0.\end{aligned}$$

This implies that the last bracket of (29) is 0. In sum,

$$\frac{\partial v(d_T;p)}{\partial \alpha}\Big|_{\alpha=1} > \frac{\partial v(c_T;p)}{\partial \alpha}\Big|_{\alpha=1} \iff \delta^{2(T+1)}(c-d) > (1-\delta^2)(d-\ell). \tag{30}$$

Let $\tau^*(\delta)$ be the solution to $\delta^{2(\tau+1)}(c-d) = (1-\delta^2)(d-\ell)$. Since $\delta^{2(\tau+1)}(c-d)$ is decreasing in τ, if $T = \underline{\tau}(\delta) < \tau^*(\delta)$, (30) is satisfied. (See Figure 6.) //

Step 2: There exists $\delta_* \in (0,1)$ such that $\underline{\tau}(\delta) < \tau^*(\delta)$ for any $\delta \in (\delta_*, 1)$.

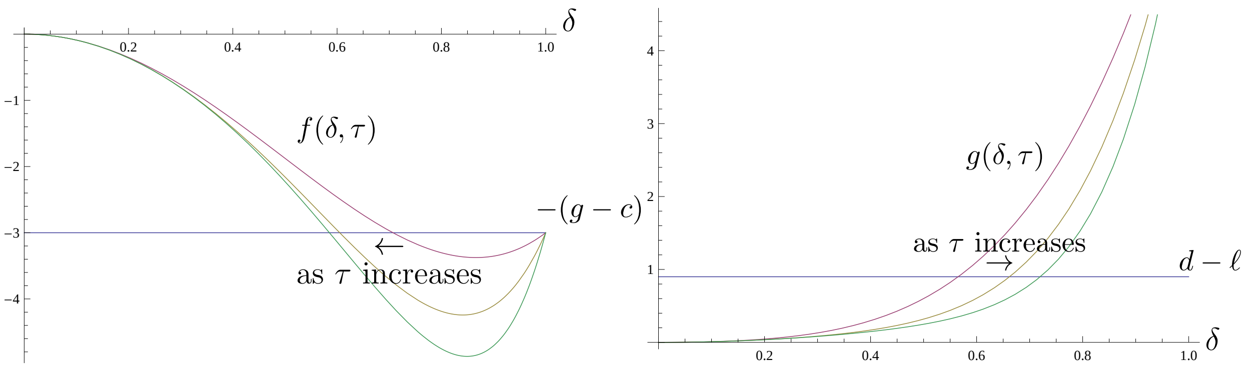

Figure 8: Properties of $\underline{\tau}(\delta)$ and $\tau^*(\delta)$

Proof of Step 2: We prove that $\underline{\tau}(\delta)$ is decreasing in δ and $\tau^*(\delta)$ is increasing in δ.

Recall that $\underline{\tau}(\delta)(>0)$ is the solution to

$$(1-\delta^{2\tau})d+\delta^{2\tau}c=\frac{1}{\delta^2}\{c-(1-\delta^2)g\} \iff f(\delta,\tau):=\delta^2\{-(g-d)+\delta^{2\tau}(c-d)\}=-(g-c).$$

For any τ, $f(0,\tau)=0>-(g-c)$, $f(1,\tau)=-(g-c)$, but f is not a monotone function of δ. By differentiation,

$$\frac{\partial f}{\partial \delta}=2\delta\{(\tau+1)\delta^{2\tau}(c-d)-(g-d)\}.$$

This means that f decreases (resp. increases) in δ if and only if $\delta<\left[\frac{g-d}{(\tau+1)(c-d)}\right]^{\frac{1}{2\tau}}$ (resp. $\delta>\left[\frac{g-d}{(\tau+1)(c-d)}\right]^{\frac{1}{2\tau}}$), so that f has a unique bottom. Hence $f(\delta,\tau)$ hits $-(g-c)$ at a unique $\delta\in(0,1)$ (see the left figure of Figure 8) if and only if the slope of f is positive at $\delta=1$, i.e., $\frac{\partial f}{\partial \delta}(1,\tau)=2\{(\tau+1)(c-d)-(g-d)\}>0$, or $\tau>\frac{g-c}{c-d}$. Because $f(\delta,\tau)$ is decreasing in τ, for any $\tau>\frac{g-c}{c-d}$, the δ that makes $f(\delta,\tau)=-(g-c)$ shifts to the left, as τ increases. In other words, $\underline{\tau}(\delta)$ is decreasing in δ. Similarly, recall that $\tau^*(\delta)$ is the solution to

$$\delta^{2(\tau+1)}(c-d)=(1-\delta^2)(d-\ell) \iff g(\delta,\tau):=\delta^2(d-\ell)+\delta^{2(\tau+1)}(c-d)=d-\ell.$$

By computation, $g(0,\tau)=0<d-\ell$, $g(1,\tau)=c-\ell>d-\ell$ for any τ, and g is monotonically increasing in δ and monotonically decreasing in τ. Hence the δ that makes $g(\delta,\tau)=d-\ell$ increases as τ increases. See the right figure of Figure 8.

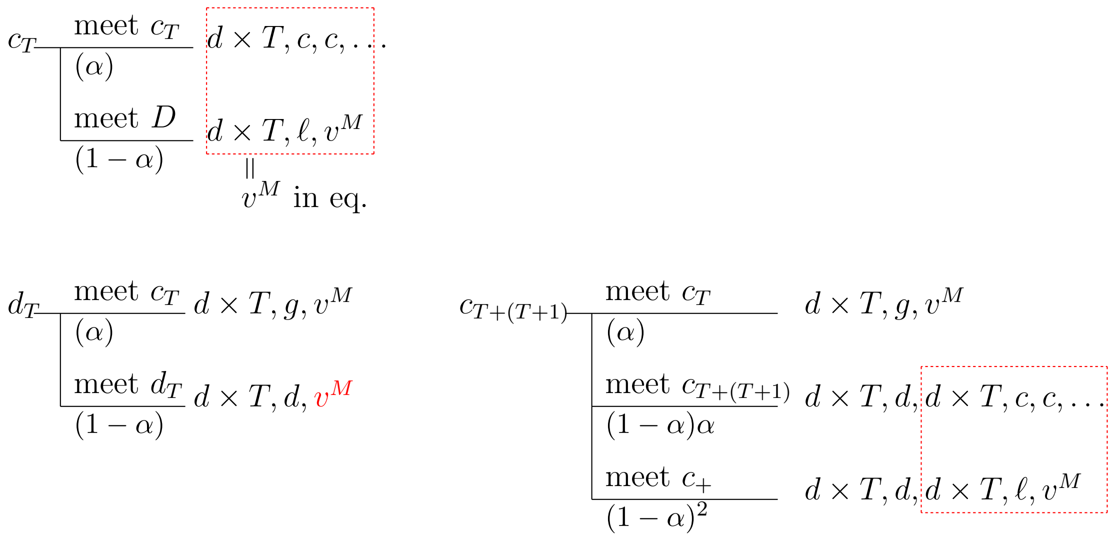

Figure 9: Equivalence of $c_T - d_T$ and $c_{T+n(T+1)}$ distribution

Finally, as $\delta \to 1$, $g(1,\infty) = d - \ell$ implies that $\tau^*(1) = \infty$, while $f(1,0) = -(g-d)$ implies that $\underline{\tau}(1) = 0$. Hence, there exists $\delta_* \in (0,1)$ such that (see Figure 6)

$$\delta \gtreqless \delta_* \iff \tau^*(\delta) \gtreqless \underline{\tau}(\delta).$$

//

Step 3: For any $\delta > \delta_*$, there exists $\underline{\tau}_{cd} < \underline{\tau}(\delta)(< \tau^*(\delta))$ such that for any integer $T \in [\underline{\tau}_{cd}, \underline{\tau}(\delta))$, there is a unique $\overline{\alpha}_T \in (0,1)$ such that $\overline{\alpha}_T c_T + (1-\overline{\alpha}_T)d_T$ is the unique locally stable Nash equilibrium with the support $\{c_T, d_T\}$.

Proof of Step 3: By the continuity of the average payoff functions and Steps 1 and 2, for T slightly less than $\underline{\tau}(\delta)$, the two intersections of $v(c_T; \alpha c_T + (1-\alpha)d_T) = v(d_T; \alpha c_T + (1-\alpha)d_T)$ still exist and within $(0,1)$ (see Figure 5), and the larger one satisfies local stability. *Q.E.D.*

Proof of Crollary 1: We show that a geometric distribution of the form

$$\sum_{n=0}^{\infty} \overline{\alpha}_T(\delta)\{1-\overline{\alpha}_T(\delta)\}^n c_{T+n(T+1)}$$

is payoff-equivalent to $\overline{\alpha}_T(\delta)c_T + \{1-\overline{\alpha}_T(\delta)\}d_T$.

Suppose that c_T- and c_{T+k}-strategy are in a payoff-equivalent distribution to $\overline{\alpha}_T(\delta)c_T + \{1-\overline{\alpha}_T(\delta)\}d_T$, where k is the smallest positive integer. As Figure 9 illustrates, c_{T+k}-strategy has the play path such that (D, D) for the first T periods regardless of the partner's strategy, then at $T+1$-th period, either (D, C) (if the partner was c_T-strategy, where the first coordinate is the focal strategy's action) or (D, D) (if the partner was either c_{T+k}-strategy or a longer trust-building strategy). To have the same payoff as that of c_T-strategy, the continuation payoff from $T+2$-th period on in a partnership must be the same as that of c_T-strategy starting in a matching pool, denoted as v^M. (This corresponds to red boxes in Figure 9.) Therefore, the continuation path for c_{T+k}-strategy must be (D, D) for T periods (in total, $T+1+T$ periods from the beginning) and after that (C, C) with probability α and (C, D) with probability $(1-\alpha)$. Hence k must be $T+1$, and thus the "next" trust-building strategy to be included should be $c_{T+2(T+1)}$-strategy and so on. The payoff equivalence at $\alpha = \overline{\alpha}_T(\delta)$ is analogous to Lemma 1. *Q.E.D.*

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