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Abstract

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Optimality in a Stochastic OLG Model with Ambiguity

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Keywords: Maxmin expected utility; Conditional Pareto optimality; Dominant root criterion; Stochastic overlapping generations model.

JEL Classification Numbers: D60; D81; E40.

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1 Introduction

In the overlapping generations (OLG) model, competitive equilibrium might not achieve an optimal allocation, even when markets operate perfectly, as in the Arrow-Debreu abstraction. It is now understood that this sort of inefficiency is caused by the lack of market clearing at infinity (Geanakoplos, 1987). In order to design active policies (such as social security) which remedy this type of inefficiency, it is important to identify optimality with easily verifiable conditions. A cornerstone of the literature about characterizations of optimality in the OLG model is the work per Balasko and Shell (1980). It contributed to the literature by demonstrating that optimality of an equilibrium allocation in a deterministic OLG environment is characterized by conditions on the equilibrium price corresponding to the allocation. One of implications of this result is that, in a deterministic OLG model, we can examine whether the equilibrium allocation is optimal by *observing* the associated equilibrium price and therefore the policy maker no longer needs to examine the allocation itself (nor to know preferences).

Thanks to previous studies, we now know that the Balasko-Shell type of criteria of optimality in a deterministic environment can be naturally extended to a stochastic environment. For stationary feasible allocations, Peled (1984), Aiyagari and Peled (1991), Manuelli (1990), Chattopadhyay (2001), and Ohtaki (2013a) found that optimality can be characterized by a certain condition on the dominant root of the contingent price matrix related to a stationary equilibrium. For general feasible allocations, on the other hand, Chattopadhyay and Gottardi (1999), Chattopadhyay (2006), and Bloise and Calciano (2008) founded in a various level of generality that the Balask-Shell type of optimality criteria is still applicable to equilibrium contingent price processes.[1] Therefore, even in a stochastic OLG environment, we might be able to examine optimality of equilibrium allocations by observing the associated price (contingent upon date-events).

Although one of important restrictions to obtain these results is a pair of convexity and smoothness (such as the Gaussian curvature condition) of preferences, we aim to reexamine a characterization of optimality in a stochastic OLG model with the maxmin expected utility (MMEU) preference à la Gilboa and Schmeidler (1989) and Casadesus-Masanell, Klibanoff, and Ozdenoren (2000). Differently from the standard expected utility hypothesis, a decision maker

[1] The Balasko-Shell type of optimality criteria can be also extended to the production economy. Interested readers might be found, for example, Demange and Laroque (1999, 2000), Barbie, Hagedorn, and Kaul (2007), and Gottardi and Kübler (2011).

endowed with an MMEU preference assigns a *set* of probability measures, not a *single* probability measure, to uncertainty and behaves as if she maximizes the minimum of expected utilities over the set of measures. This multiplicity of priors is often called *ambiguity*, which is a case of *true* uncertainty in the sense of Knight (1921).[2] The MMEU preference is known as one of reasonable ways to explain several anomalies such as the Ellsberg paradox (1961). It can be convex when its von Neumann-Morgenstern utility index function is concave (Ohtaki, 2013b) but might not be differentiable at some point as a result of the minimization of expected utilities. Therefore, one of remarkable features of this article is to provide a characterization of optimality in a stochastic OLG model with convex but *non*smooth preferences due to MMEU.

In order to capture the role of the MMEU preference, we consider a simple, but rather canonical, stationary pure-endowment stochastic OLG model consisting of infinite horizon with discrete time periods running from $-\infty$ to ∞, finite Markov states, one perishable commodity per period, and one two-period-lived agent per generation. Furthermore, we restrict our attention on optimality of stationary feasible allocations and adopt conditional Pareto optimality (CPO) as an optimality criterion.[3] According to this criterion, agents' welfare is evaluated by conditioning their utility on the state at the date of their birth. In such a framework with the MMEU preference, four observations are provided. First, CPO of a stationary *feasible* allocation is characterized by the *set* of dominant roots of matrices of marginal rates of substitution at the allocation, which contains one. Second, CPO of a stationary *equilibrium* allocation is characterized by the set of dominant roots of matrices of *supporting prices*, not the equilibrium price matrix itself, which contains one. Third, the introduction of money in constant supply achieves CPO, provided that a stationary equilibrium with circulating money exists. Fourth, the introduction of equity cannot achieve CPO.

This article contributes to the literature by providing a characterization of optimality in a stochastic OLG model with convex but nonsmooth preferences due to MMEU. Under smooth preferences, CPO is characterized by the dominant root of the matrix of the rates of marginal rates of substitution, being equal to one (Demange and Laroque, 1999; Ohtaki, 2013a). The first observation is therefore a natural extension of the existing result. On the other hand, the

[2]In the tradition of Knight (1921), "uncertainty" is *risk* if it is reducible to a single probability measure and otherwise true *uncertainty*.

[3]The concept of CPO is first introduced by Muench (1977). In this article, CPO can be identified with conditional golden rule optimality, studied in Ohtaki (2013a) for example. This is because it is assumes that time runs from $-\infty$ to ∞ and therefore no initial old exists.

second observation has a remarkable implication, i.e.: observing the equilibrium price *does not necessarily* tell us whether the associated allocation is optimal. This is because we should now consider the set of *supporting* price matrices, not the *observed* equilibrium price matrix itself, to examine optimality. Although we have found such a violation of the Balasko-Shell type of criteria of optimality, the third observation implies that intergenerational trade through money ensures optimality, in the sense of CPO, of equilibrium allocations.

Finally, we should mention that this article also contributes to the literature about the application of decision making under ambiguity to dynamic economics and finance. Decision making under ambiguity is already applied to a wide range of intertemporal macroeconomic models: asset pricing as in Epstein and Wang (1994, 1995), search theory as in Nishimura and Ozaki (2004), real option as in Nishimura and Ozaki (2007), learning as in Epstein and Schneider (2007), and growth theory as in Fukuda (2008) are such examples but these does not necessarily addressed to the issue about optimality of allocations. Actually, there seems few work characterizing optimality of allocations in a dynamic general equilibrium setting with ambiguity. One of exception is the work per Dana and Riedel (2013).[4] However, differently from ours, their results are obtained in a finite-horizontal economy with the incomplete preference à la Bewley (2002) and without overlapping of generations. To our best knowledge, therefore, this article is the first of characterizing optimality under ambiguity in an infinite-horizontal general equilibrium setting with overlapping of generations.

The organization of this paper is as follows: Section 2 presents details of the model. Section 3 defines the concept of stationary feasibility of allocations and argues its basic property. Section 4 introduces the concept of CGRO and characterizes it for stationary feasible allocations. Section 5 applies results given in the previous section to stationary equilibrium allocations. Proofs of main results are provided in Section 6. The Appendix provides some of mathematical tools using this article.

[4]We can find a lot of studies characterizing optimality of allocations in a static, not dynamic, general equilibrium environment: Billot, Chateauneuf, Gilboa, and Tallon (2000), Chateauneuf, Dana, and Tallon (2000), Dana (2004), Kajii and Ui (2009), Rigotti and Shannon (2005, 2012), Rigotti, Shannon, and Strzalecki (2008), Dana and Le Van (2010), Strzalecki and Werner (2011), and Carlier and Dana (2013) are such examples. Interested readers might be able to find other applications of ambiguity to the static economic environment including Dow and Werlang (1992), Mukerji and Tallon (1998, 2001, 2004a,b), Kajii and Ui (2005), Karni (2009), Rinaldi (2009), Condie and Ganguli (2011), Gollier (2011), Lopomo, Rigotti, and Shannon (2011), and Mandler (2013).

2 Ingredients of the Economy

We consider a stationary pure-endowment stochastic overlapping generations model with the maxmin expected utility (MMEU) preference. Time is discrete and runs from $-\infty$ to ∞. The stochastic environment is modeled by a stationary Markov process with its state space $\mathcal{S} = \{1, \ldots, S\}$. Each element of $\mathcal{S}$ is called a *state*. The set of all probability measures on $\mathcal{S}$ is denoted by $\Delta_{\mathcal{S}}$.

After the realization of state $s_t \in \mathcal{S}$ in each period $t \geq 1$, one new agent is born, lives for two periods, and dies. In this article, she is often called an *agent* s_t. Her initial endowment stream and preference are assumed to depend only on the realizations of states during her lifetime, not on time nor on the past realizations. Thus, she is endowed with $\omega_{s_t} = (\omega^y_{s_t}, (\omega^o_{s_t s_{t+1}})_{s_{t+1}\in\mathcal{S}}) \in \Re_{++} \times \Re^{\mathcal{S}}_{+}$ as the initial endowment stream, where $\omega^y_{s_t}$ and $\omega^o_{s_t} = (\omega^o_{s_t s_{t+1}})_{s_{t+1}\in\mathcal{S}}$ are endowments when young and old, respectively. Also she ranks her consumption streams $c_{s_t} = (c^y_{s_t}, (c^o_{s_t s_{t+1}})_{s_{t+1}\in\mathcal{S}}) \in \Re_{+} \times \Re^{\mathcal{S}}_{+}$ according to her lifetime utility function $U^{s_t} : \Re_{+} \times \Re^{\mathcal{S}}_{+} \to \Re$, where $c^y_{s_t}$ and $c^o_{s_t} = (c^o_{s_t s_{t+1}})_{s_{t+1}\in\mathcal{S}}$ are consumption when young and old, respectively. We assume that the lifetime utility function belongs to the class of MMEU preferences, i.e.: for each $s \in \mathcal{S}$, there exist a nonempty, compact, and convex subset Π_s of $\Delta_{\mathcal{S}}$ and a family $\{u^{ss'} : s' \in \mathcal{S}\}$ of real-valued functions on $\Re_{+} \times \Re_{+}$, each of which is strictly monotone increasing, concave, and continuously differentiable on the interior of its domain, such that

$$U^s(c^y_s, (c^o_{ss'})_{s'\in\mathcal{S}}) = \min_{\pi_s\in\Pi_s} \sum_{s'\in\mathcal{S}} u^{ss'}(c^y_s, c^o_{ss'})\pi_{ss'}$$

for each $(c^y_s, (c^o_{ss'})_{s'\in\mathcal{S}}) \in \Re_{+} \times \Re^{\mathcal{S}}_{+}$. Furthermore, assume that $\pi_{ss'} > 0$ for each $s, s' \in \mathcal{S}$ and each $\pi_s \in \Pi_s$. Because $u^{ss'}$ is strictly monotone increasing, concave, and continuous for each $s, s' \in \mathcal{S}$, it follows that, for each $s \in \mathcal{S}$, U^s is strictly monotone increasing, concave, and continuous.[5] For notational convenience, let $\Pi := \prod_{s\in\mathcal{S}} \Pi_s = \{(\pi_s)_{s\in\mathcal{S}} : (\forall s \in \mathcal{S})\ \pi_s \in \Pi_s\}$, the set of transition probability matrices induced by the family $(\Pi_s)_{s\in\mathcal{S}}$.

In order to close this section, we introduce some notations. For each $s \in \mathcal{S}$ and each $c = (c^y, (c^o_{s'})_{s'\in\mathcal{S}}) \in \Re_{+} \times \Re^{\mathcal{S}}_{+}$, let

$$\mathcal{B}_s(c) := \arg\min_{\pi_s\in\Pi_s} \sum_{s'\in\mathcal{S}} u^{ss'}(c^y, c^o_{s'})\pi_{ss'},$$

[5]See also Ohtaki (2013b).

which is the set of probability measures in Π_s minimizing the expected utility given c. Also, for each $c = (c_s)_{s\in\mathcal{S}} = (c^y_s, (c^o_{ss'})_{s'\in\mathcal{S}})_{s\in\mathcal{S}} \in (\Re_+ \times \Re^{\mathcal{S}}_+)^{\mathcal{S}}$, let

$$\mathcal{B}(c) = \prod_{s\in\mathcal{S}} \mathcal{B}_s(c_s),$$

the product of $\mathcal{B}_s(c_s)$ over $\mathcal{S}$. Note that $\mathcal{B}_s(c_s) \subset \Pi_s$ for each $s \in \mathcal{S}$ and $\mathcal{B}(c) \subset \Pi$. Also note that if c is fully-insured, it holds that $\mathcal{B}_s(c_s) = \Pi_s$ for each $s \in \mathcal{S}$ and therefore $\mathcal{B}(c) = \Pi$. Furthermore, for each $s, s' \in \mathcal{S}$, each $c = (c^y, (c^o_{s'})_{s'\in\mathcal{S}}) \in \Re_+ \times \Re^{\mathcal{S}}_+$, and each $\pi \in \Delta_{\mathcal{S}}$, let

$$m^c_{ss'}(\pi) := \frac{u^{ss'}_2(c^y, c^o_{s'})\pi_{s'}}{\sum_{\tau\in\mathcal{S}} u^{s\tau}_1(c^y, c^o_\tau)\pi_\tau}$$

Finally, for each $c = (c_s)_{s\in\mathcal{S}} = (c^y_s, (c^o_{ss'})_{s'\in\mathcal{S}})_{s\in\mathcal{S}} \in (\Re_+ \times \Re^{\mathcal{S}}_+)^{\mathcal{S}}$ and $\pi = [\pi_{ss'}] \in \Delta^{\mathcal{S}}_{\mathcal{S}}$, let $M_c(\pi) := [m^{c_s}_{ss'}(\pi_s)]_{s,s'\in\mathcal{S}}$, which is a matrix of marginal rates of substitution given c and π. The current restrictions on preferences imply that $M_c(\pi)$ is a positive square matrix. By the Perron-Frobenius theorem,[6] any positive square matrix M has a unique dominant root, denoted by $\lambda^f(M)$, and it holds that $My(M) = \lambda^f(M)y(M)$ for some positive vector unique up to normalization, $y(M)$.

3 Stationary Feasible Allocations

Let $\bar{\omega}_{ss'} := \omega^y_{s'} + \omega^o_{ss'}$ for each $s, s' \in \mathcal{S}$, which is the total endowment when the current and preceding states are s' and s, respectively. A *stationary feasible allocation* of this economy is a pair $c = (c^y, c^o)$ of functions $c^y : \mathcal{S} \to \Re_+$ and $c^o : \mathcal{S} \times \mathcal{S} \to \Re_+$ such that $c^y_{s'} + c^o_{ss'} = \bar{\omega}_{ss'}$ for all $s, s' \in \mathcal{S}$. It is *interior* if $c_s = (c^y_s, (c^o_{ss'})_{s'\in\mathcal{S}}) \gg 0$ for all $s \in \mathcal{S}$ and *fully-insured* (with respect to the second-period consumption) if $c^o_{ss'} = c^o_{ss''}$ for each $s, s', s'' \in \mathcal{S}$. Note that $\omega := (\omega^y, \omega^o)$ is one of stationary feasible allocations. We denote by $\mathcal{A}$ the set of all stationary feasible allocations.

In order to provide shaper argument, we impose a further restriction on the total endowment. Throughout the rest of this paper, it is assumed that $\bar{\omega}_{ss'} \equiv \bar{\omega}_{s'}$ for each $s, s' \in \mathcal{S}$, i.e.: the total endowment depends only on the realization of the current state s', not on the preceding state s. Given this restriction, we can obtain a useful property of stationary feasible allocations.

Proposition 1 *For any stationary feasible allocation $c = (c^y, c^o)$, it holds that:*

A. *for any $s, s' \in \mathcal{S}$, $c^y_{s'} = \bar{\omega}_{s'} - c^o_{ss'}$; and*

B. *for any $s, s', s'' \in \mathcal{S}$, $c^o_{ss''} = c^o_{s's''}$.*

[6]See, for example, Debreu and Herstein (1953) and Takayama (1974) for more details on the Perron-Frobenius theorem.

The proof of this proposition is very preliminary and therefore it is omitted. This proposition says that: at a stationary feasible allocation, (A) the first-period consumption is uniquely determined by the second-period consumption and (B) the second-period consumption is independent of the state realized in the period when the agent is born. Therefore, the set of stationary feasible allocations is identifiable with the set of pairs (x^y, x^o) of functions of $\mathcal{S}$ to $\Re_+$ such that $x^y_s + x^o_s = \bar{\omega}_s$ for each $s \in \mathcal{S}$ or, more simply, with $X := \{x^o \in \Re^{\mathcal{S}}_+ : (\forall s' \in \mathcal{S})\ 0 \leq x^o_{s'} \leq \bar{\omega}_{s'}\}$.[7] Similarly, an interior stationary feasible allocation is related to an element of the interior of X, int.X. Note that, since ω is also a stationary feasible allocation, one can ignore the subscript s of $\omega^o_{ss'}$ for each $s, s' \in \mathcal{S}$.[8]

Because the set X, elements of which are second-period consumptions, can be identified with the set of stationary feasible allocations, we can derive another utility function on X, denoted by $\hat{U}^s$, from U^s. To be more precise, for each $s \in \mathcal{S}$, define the function $\hat{U}^s : X \to \Re$ by

$$(\forall x^o \in X) \quad \hat{U}^s(x^o) := U^s(\bar{\omega}_s - x^o_s, (x^o_{s'})_{s' \in \mathcal{S}}).$$

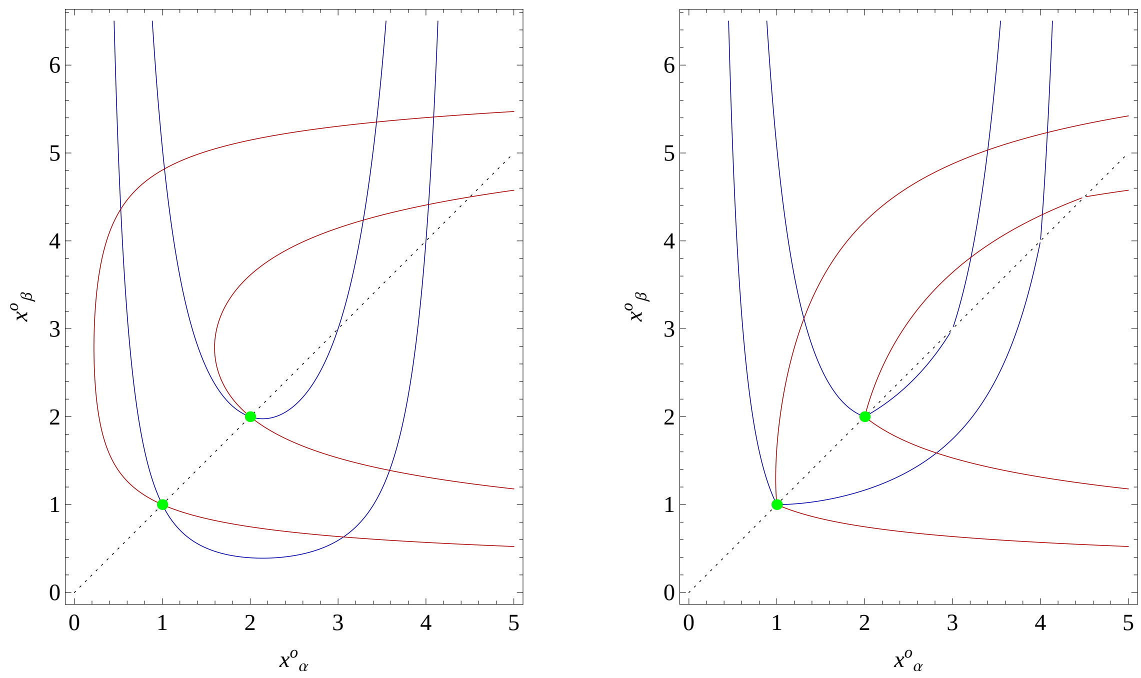
One should note that, by this derived utility function $\hat{U}^s$, we can draw indifferent curves in X. To close this section, we illustrate this fact in the following two-state example.

Example 1 Suppose that $\mathcal{S} = \{\alpha, \beta\}$ and there exist some real-valued functions v_y and v_o on $\Re_+$ such that $u^{ss'}(c^y, c^o) = v_y(c^y) + v_o(c^o)$ for each $(c^y, c^o) \in \Re^o_+$. Because there are only two states, α and β, we can write, for each $s \in \mathcal{S}$, $\Pi_s = \{\pi_s \in \Delta_{\mathcal{S}} : \underline{\pi}_{s\alpha} \leq \pi_{s\alpha} \leq \overline{\pi}_{s\alpha}\}$ for some $\underline{\pi}_s, \overline{\pi}_s \in \Delta_{\mathcal{S}}$ such that $0 < \underline{\pi}_{s\alpha} \leq \overline{\pi}_{s\alpha} < 1$. Furthermore, specify the economy by $(\bar{\omega}_\alpha, \bar{\omega}_\beta) = (5, 6.5)$, and $v_y(x) = v_o(x) = \ln x$ for each $x > 0$.[9] Note that, when $\underline{\pi}_{s\alpha} = \overline{\pi}_{s\alpha}$ for each $s \in \mathcal{S}$, there exists no ambiguity and the model degenerates into one with the standard expected hypothesis. The set X identifiable with the set of stationary feasible allocations is then given by $[0, 5] \times [0, 6.5]$. Therefore, it can be depicted by the box as in Figure 1, where $(\underline{\pi}_{\alpha\alpha}, \overline{\pi}_{\alpha\alpha}, \underline{\pi}_{\beta\alpha}, \overline{\pi}_{\beta\alpha}) = (0.75, 0.75, 0.25, 0.25)$ in Panel A of Figure 1 and $(\underline{\pi}_{\alpha\alpha}, \overline{\pi}_{\alpha\alpha}, \underline{\pi}_{\beta\alpha}, \overline{\pi}_{\beta\alpha}) = (0.25, 0.75, 0.25, 0.75)$ in Panel B of Figure 1. In Figure 1, we also draw indifferent curves, derived from $\hat{U}^s$ for each $s \in \mathcal{S}$, through the points $(1, 1)$ and $(2, 2)$. The blue and red curves are related to $\hat{U}^\alpha$ and $\hat{U}^\beta$, respectively. Note that $\hat{U}^s(1, 1) < \hat{U}^s(2, 2)$ for each $s \in \mathcal{S}$. Also note that, in the

[7] We can identify a stationary feasible allocation c with an element x^o of X when it holds that $c^o_{ss'} = x^o_{s'}$ for each $s, s' \in \mathcal{S}$.

[8] Therefore, the current model is close to one considered per Labadie (2004) rather than one considered per Magill and Quinzii (2003).

[9] Interested readers can find in the work per Faro (2013) an axiomatization of the maxmin expected utility preference with logarithmic index functions.



A. $(\underline{\pi}_{\alpha\alpha}, \overline{\pi}_{\alpha\alpha}, \underline{\pi}_{\beta\alpha}, \overline{\pi}_{\beta\alpha}) = (0.75, 0.75, 0.25, 0.25)$ B. $(\underline{\pi}_{\alpha\alpha}, \overline{\pi}_{\alpha\alpha}, \underline{\pi}_{\beta\alpha}, \overline{\pi}_{\beta\alpha}) = (0.25, 0.75, 0.25, 0.75)$

Figure 1: Box Diagrams

presence of ambiguity (Panel B of Figure 1), indifferent curves have *kinks* on the 45 degree line. This is because $\hat{U}^s$ might not be differentiable at $(x^o_\alpha, x^o_\beta) \in X$ such that $x^o_\alpha = x^o_\beta$. In fact, for each $s \in \mathcal{S}$ and each $x^o \in X$,

$$\begin{aligned}
\hat{U}^s(x^o) &= \min_{\pi_s \in \Pi_s} \left(\ln(\bar{\omega}_s - x^o_s) + \sum_{\tau \in \mathcal{S}} \pi_{s\tau} \ln x^o_\tau \right) \\
&= \ln(\bar{\omega}_s - x^o_s) + \ln x^o_\beta + \min_{\pi_s \in \Pi_s} \left[\left(\ln x^o_\alpha - \ln x^o_\beta \right) \pi_{s\alpha} \right] \\
&= \begin{cases} \ln(\bar{\omega}_s - x^o_s) + \underline{\pi}_{s\alpha} \ln x^o_\alpha + (1 - \underline{\pi}_{s\alpha}) \ln x^o_\beta & \text{if} \quad x^o_\alpha > x^o_\beta, \\ \ln(\bar{\omega}_s - x^o_s) + \ln x^o_s & \text{if} \quad x^o_\alpha = x^o_\beta, \quad \text{and} \\ \ln(\bar{\omega}_s - x^o_s) + \overline{\pi}_{s\alpha} \ln x^o_\alpha + (1 - \overline{\pi}_{s\alpha}) \ln x^o_\beta & \text{if} \quad x^o_\alpha < x^o_\beta. \end{cases}
\end{aligned}$$

Therefore, the slope of the agent s's indifferent curve through $x^o \in X$, denoted by $\widehat{MRS}_s(x^o)$ if any, can be calculated as:[10]

$$\widehat{MRS}_\alpha(x^o) = -\frac{\hat{U}^\alpha_1(x^o)}{\hat{U}^\alpha_2(x^o)} = \begin{cases} \dfrac{[(1 + \underline{\pi}_{\alpha\alpha})x^o_\alpha - \underline{\pi}_{\alpha\alpha}\bar{\omega}_\alpha]x^o_\beta}{(1 - \underline{\pi}_{\alpha\alpha})(\bar{\omega}_\alpha - x^o_\alpha)x^o_\alpha} & \text{if} \quad x^o_\alpha > x^o_\beta, \\ \dfrac{[(1 + \overline{\pi}_{\alpha\alpha})x^o_\alpha - \overline{\pi}_{\alpha\alpha}\bar{\omega}_\alpha]x^o_\beta}{(1 - \overline{\pi}_{\alpha\alpha})(\bar{\omega}_\alpha - x^o_\alpha)x^o_\alpha} & \text{if} \quad x^o_\alpha < x^o_\beta, \end{cases}$$

[10]For each $s \in \mathcal{S}$, if $\hat{U}^s$ is differentiable at $x^o \in X$, then we can obtain that $0 = \hat{U}^s_1(x^o)dx^o_\alpha + \hat{U}^s_2(x^o)dx^o_\beta$. Therefore, in such a case, $\widehat{MRS}_s(x^o) = -dx^o_\beta/dx^o_\alpha = -\hat{U}^s_1(x^o)/\hat{U}^s_2(x^o)$.

and

$$\widehat{MRS}_\beta(x^o) = -\frac{\hat{U}_1^\beta(x^o)}{\hat{U}_2^\beta(x^o)} = \begin{cases} \dfrac{\underline{\pi}_{\beta\alpha}(\bar{\omega}_\beta - x_\beta^o)x_\beta^o}{[(2-\underline{\pi}_{\beta\alpha})x_\alpha^o - (1-\underline{\pi}_{\beta\alpha})\bar{\omega}_\beta]x_\alpha^o} & \text{if} \quad x_\alpha^o > x_\beta^o, \\ \dfrac{\overline{\pi}_{\beta\alpha}(\bar{\omega}_\beta - x_\beta^o)x_\beta^o}{[(2-\overline{\pi}_{\beta\alpha})x_\alpha^o - (1-\overline{\pi}_{\beta\alpha})\bar{\omega}_\beta]x_\alpha^o} & \text{if} \quad x_\alpha^o < x_\beta^o \end{cases}$$

if $x_\alpha^o \neq x_\beta^o$ but might not be calculated if $x_\alpha^o = x_\beta^o$ (This is true when $\underline{\pi}_{s\alpha} < \overline{\pi}_{s\alpha}$). ■

We often use the box diagram as in Example 1 to present examples. One can find the work provided per Ohtaki (2012) for more details on the box diagram with differentiable lifetime utility functions.[11]

4 Conditional Pareto Optimality

In this section, we define the concept of conditional Pareto optimality (CPO) and characterize it. For any two stationary feasible allocations b and c, we say that b *CPO-dominates* c if

$$(\forall s \in \mathcal{S}) \quad U^s(b_s) \geq U^s(c_s)$$

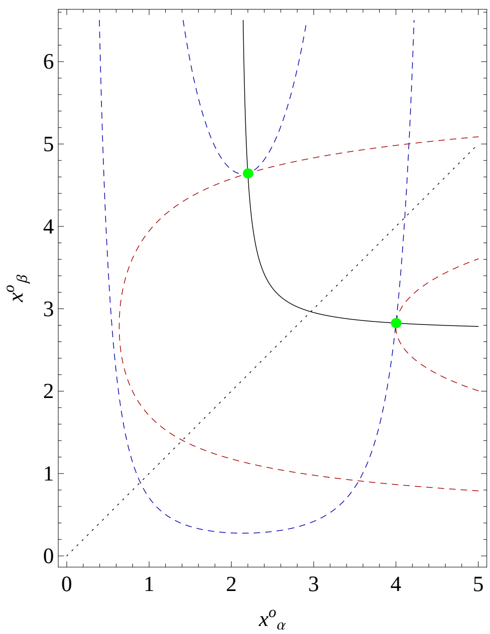
with strict inequality somewhere. The concept of CPO is then defined as follows:

Definition 1 A stationary feasible allocation c is said to be *conditionally Pareto optimal* if there exists no other stationary feasible allocation b that CPO-dominates c.

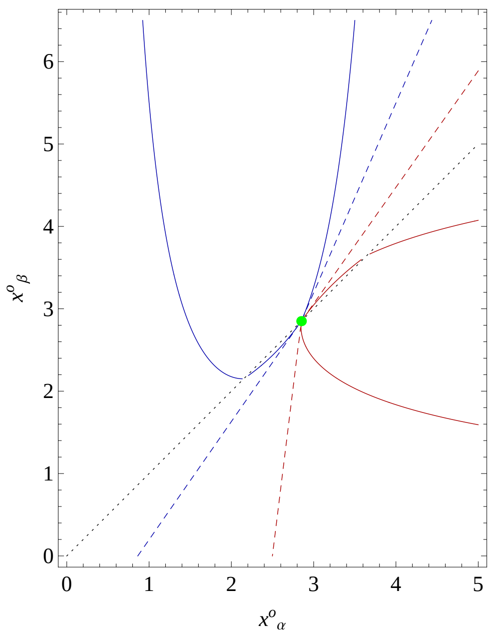
In this definition, "conditionally" means the fact that agents' welfare is evaluated by *conditioning* their lifetime utility on the state at the date of birth. Our definition of CPO closely related to conditional golden rule optimality (CGRO) in the stochastic OLG model with the initial old.

Remark 1 Note in our model that there exists no *initial old*, who is a one-period-lived agent born in the initial period, because time runs from $-\infty$ to ∞ and therefore there is no initial period. When the initial period and the initial old exist, CPO should be the optimality criterion considering the initial old's welfare adding to other newly born agents' welfare and it differs from our definition of CPO. In such a case, our CPO is identifiable with CGRO, which is an optimality criterion evaluating welfare ignoring the initial old, rather than CPO with the initial old. Ohtaki (2013a) argued in detail on the relationship between CPO and CGRO in the smooth preference model with the initial old.

[11] The authors welcome future generations (researchers) who call the box diagram presented in this article the *Ohtaki box diagram*.



A. $(\underline{\pi}_{\alpha\alpha}, \overline{\pi}_{\alpha\alpha}, \underline{\pi}_{\beta\alpha}, \overline{\pi}_{\beta\alpha}) = (0.75, 0.75, 0.25, 0.25)$



B. $(\underline{\pi}_{\alpha\alpha}, \overline{\pi}_{\alpha\alpha}, \underline{\pi}_{\beta\alpha}, \overline{\pi}_{\beta\alpha}) = (0.25, 0.75, 0.25, 0.75)$

Figure 2: Conditional Golden Rule Optimality

In the absence of ambiguity, we can apply the results per Ohtaki (2013a) to our model and provide a characterization of interior CPO allocations. To be more precise, in such a case, i.e.: when there exists some transition probability matrix $\pi = (\pi_s)_{s\in\mathcal{S}}$ such that $\Pi_s = \{\pi_s\}$ for each $s \in \mathcal{S}$, an interior CPO allocation can be characterized by the dominant root of the unique matrix of marginal rates of substitution, being equal to *one*. This type of characterizations through the dominant roots is often called the *dominant root criterion*. However, the existing result is obtained under smoothness of preferences. We have yet no idea on characterizations of interior CPO allocations in the presence of ambiguity. In order to obtain some intuition about characterizations of interior CPO allocations in the presence of ambiguity, we reconsider a two-state example (without ambiguity) as in Example 1.

Example 2 Consider the same economy as Example 1. In the absence of ambiguity, there exists some transition probability matrix $p = (\pi_s)_{s\in\mathcal{S}}$ such that $\Pi_s = \{\pi_s\}$ for each $s \in \mathcal{S}$. Let c be a stationary feasible allocation. The dominant root criterion, $\lambda^f(M_c(\pi)) = 1$, is equivalent to the condition that $M_c(\pi)y(M_c(\pi)) = \lambda^f(M_c(\pi))y(M_c(\pi)) = y(M_c(\pi))$ or equivalently

$$\pi_{\alpha\alpha}v_o'(c^o_{\alpha\alpha})y_\alpha(M_c(\pi))) + \pi_{\alpha\beta}v_o'(c^o_{\alpha\beta})y_\beta(M_c(\pi)) = v_y'(c^y_\alpha)y_\alpha(M_c(\pi))$$

and

$$\pi_{\beta\alpha} v_o'(c^o_{\beta\alpha}) y_\alpha(M_c(\pi))) + \pi_{\beta\beta} v_o'(c^o_{\beta\beta}) y_\beta(M_c(\pi)) = v_y'(c^y_\beta) y_\beta(M_c(\pi)).$$

These equation imply that

$$\frac{v_y'(c^y_\alpha) - \pi_{\alpha\alpha} v_o'(c^o_{\alpha\alpha})}{\pi_{\alpha\beta} v_o'(c^o_{\alpha\beta})} = \frac{y_\beta(M_c(\pi))}{y_\alpha(M_c(\pi))} = \frac{\pi_{\beta\alpha} v_o'(c^o_{\beta\alpha})}{v_y'(c^y_\beta) - \pi_{\beta\beta} v_o'(c^o_{\beta\beta})},$$

which is equivalent to

$$\widehat{MRS}_\alpha(c^o_{\alpha\alpha}, c^o_{\alpha\beta}) = \widehat{MRS}_\beta(c^o_{\beta\alpha}, c^o_{\beta\beta}) > 0,$$

where the last strict inequality corresponds to the fact that $y(M_c(\pi))$ is a positive vector. One can also verify that the last condition implies the dominant root criterion. Therefore, we can say that, at a CPO allocation, indifferent curves of agents α and β are tangent to each other (and the slope of them must be positive). With the same specification as Figure 1, i.e.: $\underline{\pi}_{\alpha\alpha} = \overline{\pi}_{\alpha\alpha} = \pi_{\alpha\alpha} = 0.75$ and $\underline{\pi}_{\beta\alpha} = \overline{\pi}_{\beta\alpha} = \pi_{\beta\alpha} = 0.25$, Panel A of Figure 2 depicts by the solid curve the set of CPO allocations in the absence of ambiguitiy. ■

Intuition for Characterization. If the tangency argument as in Example 2 is still applicable in the presence of ambiguity, the stationary feasible allocation corresponding to $(x^o_\alpha, x^o_\beta) = (2.85, 2.85)$ in Panel B of Figure 2, for example, might be CPO. Denote by c the stationary allocation corresponding to $(x^o_\alpha, x^o_\beta) = (2.85, 2.85)$. At $(x^o_\alpha, x^o_\beta) = (2.85, 2.85)$, $\widehat{MRS}_\alpha|_{\pi_\alpha = \underline{\pi}_\alpha}(x^o) \approx 1.43$, $\widehat{MRS}_\alpha|_{\pi_\alpha = \overline{\pi}_\alpha}(x^o) \approx 2.30$, $\widehat{MRS}_\beta|_{\pi_\beta = \underline{\pi}_\beta}(x^o) \approx 8.11$, and $\widehat{MRS}_\beta|_{\pi_\beta = \overline{\pi}_\beta}(x^o) \approx 1.41$, where $\widehat{MRS}_s|_{\pi_s = p}(x^o)$ is the value of $\widehat{MRS}_s(x^o)$ calculated at $\pi_s = p$ for each $p \in \Delta_{\mathbb{S}}$. Therefore, we can observe that there might exists some $a > 0$ such that

$$\frac{v_y'(c^y_\alpha) - \underline{\pi}_{\alpha\alpha} v_o'(c^o_{\alpha\alpha})}{\underline{\pi}_{\alpha\beta} v_o'(c^o_{\alpha\beta})} = \widehat{MRS}_\alpha|_{\pi_\alpha = \underline{\pi}_\alpha}(x^o) \le a \le \widehat{MRS}_\alpha|_{\pi_\alpha = \overline{\pi}_\alpha}(x^o) = \frac{v_y'(c^y_\alpha) - \overline{\pi}_{\alpha\alpha} v_o'(c^o_{\alpha\alpha})}{\overline{\pi}_{\alpha\beta} v_o'(c^o_{\alpha\beta})}$$

and

$$\frac{\overline{\pi}_{\beta\alpha} v_o'(c^o_{\beta\alpha})}{v_y'(c^y_\beta) - \overline{\pi}_{\beta\beta} v_o'(c^o_{\beta\beta})} = \widehat{MRS}_\beta|_{\pi_\beta = \overline{\pi}_\beta}(x^o) \le a \le \widehat{MRS}_\beta|_{\pi_\beta = \underline{\pi}_\beta}(x^o) = \frac{\underline{\pi}_{\beta\alpha} v_o'(c^o_{\beta\alpha})}{v_y'(c^y_\beta) - \underline{\pi}_{\beta\beta} v_o'(c^o_{\beta\beta})}.$$

These inequalities can be rewritten as

$$\frac{v_y'(c^y_\alpha) - \underline{\pi}_{\alpha\alpha} v_o'(c^o_{\alpha\alpha})}{\underline{\pi}_{\alpha\beta} v_o'(c^o_{\alpha\beta})} \le a \le \frac{\underline{\pi}_{\beta\alpha} v_o'(c^o_{\beta\alpha})}{v_y'(c^y_\beta) - \underline{\pi}_{\beta\beta} v_o'(c^o_{\beta\beta})}$$

and

$$\frac{\overline{\pi}_{\beta\alpha} v_o'(c^o_{\beta\alpha})}{v_y'(c^y_\beta) - \overline{\pi}_{\beta\beta} v_o'(c^o_{\beta\beta})} \le a \le \frac{v_y'(c^y_\alpha) - \overline{\pi}_{\alpha\alpha} v_o'(c^o_{\alpha\alpha})}{\overline{\pi}_{\alpha\beta} v_o'(c^o_{\alpha\beta})}.$$

Then, we can verify that

$$M_c(\underline{\pi})\begin{bmatrix}1\\a\end{bmatrix} \leq \begin{bmatrix}1\\a\end{bmatrix} \quad \text{and} \quad M_c(\overline{\pi})\begin{bmatrix}1\\a\end{bmatrix} \geq \begin{bmatrix}1\\a\end{bmatrix}.$$

By applying the Perron-Frobenius theorem, therefore, we can obtain that

$$\lambda^f(M_c(\underline{\pi})) \leq 1 \leq \lambda^f(M_c(\overline{\pi})),$$

which might be equivalent to

$$1 \in \left\{\lambda^f(M) : M \in M_c(\Pi)\right\} =: D,$$

where D is the set of dominant roots of the matrices of marginal rates of substitution given $\Pi = (\Pi_s)_{s\in\mathcal{S}}$. Here, we have obtained a new dominant root criterion.

The last inclusion seems a natural extension of the standard dominant root criterion. However, we should remark that we have found the last inclusion for a *fully-insured* stationary feasible allocation in a *two*-state economy. Taking care of cases for multi-state stationary feasible allocations not necessarily being fully-insured, we can obtain the following characterization of CPO.

Theorem 1 *An interior stationary feasible allocation c is conditionally Pareto optimal if and only if* $1 \in \left\{\lambda^f(M) : M \in (M_c \circ \mathcal{B})(c)\right\}$.

Note that the set of matrices of marginal rates of substitution is calculated given $\mathcal{B}(c)$, not Π. Therefore, even when Π_s has multiple elements for each $s \in \mathcal{S}$, the equivalent condition degenerates into the standard one if $\mathcal{B}(c)$ is singleton, i.e.: c is CPO if and only if the dominant root of a unique matrix of marginal rates of substitution is equal to one. As a corollary of the previous theorem, we can characterize CPO for fully-insured stationary feasible allocations.

Corollary 1 *An interior fully-insured stationary feasible allocation c is conditionally Pareto optimal if and only if* $1 \in \left\{\lambda^f(M) : M \in M_c(\Pi)\right\}$.

When there exists no ambiguity, the model degenerates into one with standard expected utility model and the set of dominant roots of the matrices of the marginal rates of substitutions become singleton. Therefore, we can also obtain the following corollary.

Corollary 2 *Suppose that there exists some transition probability matrix* $p = (\pi_s)_{s\in\mathcal{S}}$ *such that* $\Pi_s = \{\pi_s\}$ *for each* $s \in \mathcal{S}$*. Then, an interior stationary feasible allocation* c *is conditionally Pareto optimal if and only if* $1 = \lambda^f(M_c(\pi))$*.*

This is consistent with Theorem 2 of Ohtaki (2013b), which characterizes CPO when lifetime utility functions are differentiable.

5 Optimality of Stationary Equilibrium Allocations

The previous section characterized CPO of stationary "feasible" allocations. The results also correspond to welfare analysis of stationary equilibrium. This section examines the relationship between CPO and stationary "equilibrium" allocations.

5.1 Supporting Price Matrix

We first define the concept of supporting price matrices, which can be interpreted as candidates of the equilibrium prices given an allocation:

Definition 2 Let c be a stationary feasible allocation. A positive matrix $P = [p_{ss'}]_{s,s'\in\mathcal{S}}$ is a *supporting price matrix* of c if

$$U^s(b_s) > U^s(c_s) \quad \text{implies} \quad b^y_s + \sum_{s'\in\mathcal{S}} b^o_{ss'} p_{ss'} > c^y_s + \sum_{s'\in\mathcal{S}} c^o_{ss'} p_{ss'}$$

for each stationary feasible allocation b and each $s \in \mathcal{S}$. Moreover, we denote by $\mathcal{P}(c)$ the set of all supporting price matrix.

The set of supporting price matrices has a closed representation:

Proposition 2 *For each interior stationary feasible allocation* c*,* $\mathcal{P}(c) = (M_c \circ \mathcal{B})(c)$*.*

Remark 2 Bloise and Calciano (2008) characterized optimality of feasible allocations, which are allowed not to be stationary, by examining its supporting prices. Their results required a supporting price of a feasible allocation *smoothly supports* the allocation. However, in our model, a stationary feasible allocations might not be smoothly supported due to indifferentiability of lifetime MMEU functions. As shown by Proposition 2, we can observe that $\mathcal{P}(c) = (M_c \circ \mathcal{B})(c)$ for each stationary feasible allocation c, i.e.: the set of support prices for c coincides with the set of matrices of marginal rates of substitution induced by c. One of important implications of this observation is that for each fully-insured stationary feasible allocation c, the set of its supporting prices, $\mathcal{P}(c)$, has multiple elements. Actually, it must be equal to the set $M_c(\Pi)$. In such a case, one should remark that a fully-insured stationary feasible allocation is not smoothly supported.

As remarked above, we cannot necessarily apply characterizations per Bloise and Calciano (2008) to our model. Combining Theorem 1 with Proposition 2, we can obtain the following characterization:

Theorem 2 *An interior stationary feasible allocation c is conditionally Pareto optimal if and only if* $1 \in \lambda(\mathcal{P}(c))$.

As a corollary, one can find that an interior stationary feasible allocation might be CPO even when the dominant root of its given supporting price matrix is not equal to one. This is a remarkable difference from the standard argument with smooth preferences.

5.2 Complete Market

We now define a stationary equilibrium with complete market, i.e.: a stationary equilibrium at which agents can buy and sell all contingent commodities in a centralized market.

Definition 3 A pair (P^*, c^*) of a positive price matrix $P^* = [p^*_{ss'}]_{s,s' \in \mathcal{S}}$ of contingent commodities and a stationary feasible allocation $c^* = (c^*_s)_{s \in \mathcal{S}}$ is called a *stationary equilibrium* if

- for all $s \in \mathcal{S}$, c^*_s belongs to the set

$$\underset{(c^y_s, c^o_s) \in \Re_+ \times \Re^{\mathcal{S}}_+}{\arg\max} \left\{ U^s(c_s) : \, c^y_s + \sum_{s' \in \mathcal{S}} c^o_{ss'} p^*_{ss'} \leq \omega^y_s + \sum_{s' \in \mathcal{S}} \omega^o_{ss'} p^*_{ss'} \right\}$$

  given p^*_s; and

- for all $s, s' \in \mathcal{S}$, $c^{*y}_{s'} + c^{*o}_{ss'} = \bar{\omega}_{ss'}$.

In this definition, the former condition is the optimization problem of each agent $s \in \mathcal{S}$ subject to a lifetime budget constraint, and the latter is the market clearing conditions. Moreover, for each stationary feasible allocation c, we denote by $\mathcal{P}^*(c)$ the set of all positive price matrix $P = [p_{ss'}]_{s,s' \in \mathcal{S}}$ such that (P, c) is a stationary equilibrium. Because it can be easily verify that $\mathcal{P}^*(c) \subset \mathcal{P}(c)$, we can obtain the following proposition:

Proposition 3 *An interior stationary feasible allocation c is conditionally Pareto optimal if $1 \in \lambda(\mathcal{P}^*(c))$.*

In the existing literature with smooth preferences, one of advantages of the dominant root criterion for an equilibrium allocation is that we can examine optimality of the allocation by examining the dominant roots of the *observed* equilibrium price and the policy maker does not need information about the allocation nor preferences. On the other hand, in the presence of ambiguity, we should remark the fact that we may not say anything about optimality of an observed stationary equilibrium (P, c) because optimality of its allocation is examined by the *set* of supporting prices, $\mathcal{P}(c)$, not an observed equilibrium contingent price matrix P. Exceptionally, however, when the observed price matrix has the dominant root being unity, we can say that the correspondence equilibrium allocation is optimal.

Corollary 3 *For each stationary equilibrium (P, c) with $c_s \gg 0$ for each $s \in \mathcal{S}$, c is conditionally Pareto optimal if $\lambda^f(P) = 1$.*

As noted above, one should note that, even when $\lambda^f(P) \neq 1$, the correspondence equilibrium allocation may be CPO.

We close this subsection with a result in the absence of ambiguity. When there is no ambiguity, the model degenerates into one with standard expected utility model and we can obtain the well-known characterization of CPO for stationary equilibrium allocations.

Corollary 4 *Suppose that there exists some transition probability matrix $\pi = (\pi_s)_{s \in \mathcal{S}}$ such that $\Pi_s = \{\pi_s\}$ for each $s \in \mathcal{S}$. Then, for each stationary equilibrium (P, c) with $c_s \gg 0$ for each $s \in \mathcal{S}$, c is conditionally Pareto optimal if and only if $\lambda^f(P) = 1$.*

This is consistent with Proposition 2 of Ohtaki (2013b), which characterizes CPO for stationary equilibrium allocations when lifetime utility functions are differentiable.

5.3 Sequentially Complete Markets with Money

As shown in the previous proposition, a stationary equilibrium itself might not be CPO even when markets operate perfectly. However, we can construct a market mechanism which generates a CPO allocation by introducing an infinitely-lived outside asset, which yields no dividend, money. Suppose in the this subsection that there exists one unit of money. Also suppose that spot markets of one-period contingent claims exist and are complete.

Definition 4 A triplet (q^*, P^*, c^*) of a positive money price vector $q^* \in \Re^{\mathcal{S}}_{++}$, a positive price matrix $P^* = [p^*_{ss'}]_{s,s' \in \mathcal{S}}$ of contingent claims, and a stationary feasible allocation $c^* = (c^*_s)_{s \in \mathcal{S}}$ is called a *stationary equilibrium with circulating money* if there exists some money holding vector $m^* \in \Re^{\mathcal{S}}$ and some contingent claim portfolio matrix $\theta^* \in \Re^{\mathcal{S} \times \mathcal{S}}$ such that

- for all $s \in \mathcal{S}$, $(c^*_s, m^*_s, \theta^*_s)$ belongs to the set

$$\underset{(c^y_s, c^o_s, m_s, \theta_s)}{\arg\max} \left\{ U^s(c_s) : \begin{array}{l} c^y_s = \omega^y_s - q^*_s m_s - \sum_{s' \in \mathcal{S}} \theta_{ss'} p_{ss'} \\ (\forall s' \in \mathcal{S})\, c^o_{ss'} = \omega^o_{ss'} + q^*_{s'} m_s + \theta_{ss'} \end{array} \right\}$$

  given q^* and p^*_s; and
- for all $s, s' \in \mathcal{S}$, $m^*_s = 1$ and $\theta^*_{ss'} = 0$.

In this definition, the former condition is the optimization problem of each agent $s \in \mathcal{S}$ subject to sequential budget constraints, and the latter is the asset market clearing conditions. One can easily verify that the good market equilibrium condition also holds at a stationary equilibrium with circulating money.

We can then find that an introduction of money may generate a CPO allocation:

Theorem 3 *An interior stationary feasible allocation of a stationary equilibrium with circulating money, if any, is always conditionally Pareto optimal.*

In other words, when a stationary equilibrium with circulating money exists, it always generates a CPO allocation. This financial intermediate role of money for remedying inefficiency in the OLG model is a well-known result in the literature and the last theorem showed that the result still holds even in the presence of ambiguity.

Remark 3 Gottardi (1996) considered a stochastic OLG model, wherein each generation consists of heterogeneous agents with differentiable lifetime utility functions and several securities exist, and showed that a stationary monetary equilibrium generically exists and is locally isolated.[12] Applying his result, we can show generic existence of stationary equilibrium with circulating money. This is because his proof of generic existence itself is independent of differentiability of lifetime utility functions. We should remark, however, that stationary monetary equilibrium might not be locally isolated because lifetime utility functions in our model are not

[12] To impose not only smoothness but also additive separability and some elasticity condition on the lifetime utility function, one can observe uniqueness of stationary equilibrium circulating money. See for example Ohtaki (2013c).

differentiable in the presence of ambiguity. Indeterminacy and its robustness are therefore left to open.[13]

5.4 Sequentially Complete Markets with Equity

We finally consider an economy with "equity" instead of "money." Suppose in this subsection that there exists one unit of an infinitely-lived asset yielding a dividend of $d_s \geq 0$ units of the consumption good at state $s \in \mathcal{S}$, where $d \in \Re^{\mathcal{S}}_{+} \setminus \{0\}$.[14] Also suppose that spot markets of one-period contingent claims exist and are complete.

Definition 5 A triplet (q^*, P^*, c^*) of a positive equity price vector $q^* \in \Re^{\mathcal{S}}_{++}$, a positive price matrix $P^* = [p^*_{ss'}]_{s,s' \in \mathcal{S}}$ of contingent claims, and a stationary feasible allocation $c^* = (c^*_s)_{s \in \mathcal{S}}$ is called a *stationary equilibrium with equity* if there exists some equity holding vector $z^* \in \Re^{\mathcal{S}}$ and some contingent claim portfolio matrix $\theta^* \in \Re^{\mathcal{S} \times \mathcal{S}}$ such that

- for all $s \in \mathcal{S}$, $(c^*_s, z^*_s, \theta^*_s)$ belongs to the set

$$\operatorname*{arg\,max}_{(c^y_s, c^o_s, z_s, \theta_s)} \left\{ U^s(c_s) : \begin{array}{l} c^y_s = \omega^y_s - q^*_s z_s - \sum_{s' \in \mathcal{S}} \theta_{ss'} p_{ss'} \\ (\forall s' \in \mathcal{S})\, c^o_{ss'} = \omega^o_{ss'} + (q^*_{s'} + d_{s'}) z_s + \theta_{ss'} \end{array} \right\}$$

  given q^* and p^*_s; and
- for all $s, s' \in \mathcal{S}$, $z^*_s = 1$ and $\theta^*_{ss'} = 0$.

In this definition, the former condition is the optimization problem of each agent $s \in \mathcal{S}$ subject to sequential budget constraints, and the latter is the asset market clearing conditions. One can easily verify that the good market equilibrium condition also holds at a stationary equilibrium circulating equity.

We should note that, even in the economy with equity, CPO of an interior stationary *feasible* allocation can be characterized as Theorem 1, by redefining the total endowment as $\bar{\omega}_{s'} = \omega^y_{s'} + \omega^o_{s'} + d_{s'}$ for each $s' \in \mathcal{S}$. Therefore, our task is to examine optimality of stationary equilibrium allocations with equity. The following statement is the last theorem of this article:

Theorem 4 *An interior stationary feasible allocation of a stationary equilibrium with equity, if any, is never conditionally Pareto optimal.*

[13]The issue about indeterminacy and its robustness in a stochastic OLG model under ambiguity has been studied per Ohtaki and Ozaki (2013).

[14]Therefore, the asset can be the equity of any productive asset like "land" or a Lucas "tree." It can be identified with money if $d_s = 0$ for each $s \in \mathcal{S}$.

In other words, when a stationary equilibrium with equity exists, it does never generate a CPO allocation.

Remark 4 Suboptimality of the equity market in the deterministic OLG model has been argued per Dow and Gorton (1993). Theorem 4 extends their result to the stochastic environment. This observation on suboptimality of the equity market follows from the lack of the initial olds. Actually, consider a case that the initial old exists and redefine CPO as treating the initial old's welfare. In such a case, we can verify CPO of stationary equilibrium with equity by applying Sakai (1988).

6 Proofs

Proof of Theorem 1. Let $c = (c^y_s, (c^o_{ss'})_{s' \in \mathcal{S}})_{s \in \mathcal{S}}$ be an interior stationary feasible allocation and let $x^y_s := c^y_s$ and $x^o_{s'} := c^o_{ss'}$ for each $s, s' \in \mathcal{S}$. By Proposition 1.B, (x^y, x^o) is well defined. It is easy to verify that c is a CPO allocation if and only if there exists a Pareto weight $\gamma : \mathcal{S} \to \Re_{++}$ such that

$$c \in \arg\max_{b \in \mathcal{A}} \sum_{s \in \mathcal{S}} \gamma^s U^s(b_s),$$

where $\mathcal{A}$ is the set of stationary feasible allocations. Because c is a stationary feasible allocation, we can obtain from Proposition 1.A that $x^y_{s'} = c^y_{s'} = \bar{\omega}_{s'} - c^o_{ss'} = \bar{\omega}_{s'} - x^o_{s'}$ for each $s, s' \in \mathcal{S}$. Then, it follows that c is CPO if and only if there exists $\gamma : \mathcal{S} \to \Re_{++}$ such that

$$\begin{aligned} 0 &\in \partial\left(\sum_{s \in \mathcal{S}} \gamma^s \hat{U}^s(x^o)\right) && (1) \\ &= \sum_{s \in \mathcal{S}} \partial\left(\gamma^s \min_{\pi_s \in \Pi_s} \sum_{s' \in \mathcal{S}} u^{ss'}(\bar{\omega}_s - x^o_s, x^o_{s'})\pi_{ss'}\right) && (2) \\ &= \sum_{s \in \mathcal{S}} \mathrm{co}\left\{\gamma^s \begin{pmatrix} u_2^{s1}(\bar{\omega}_1 - x^o_1, x^o_1)\pi_{s1} \\ \vdots \\ -\sum_{\tau \in \mathcal{S}} u_1^{s\tau}(\bar{\omega}_s - x^o_s, x^o_\tau)\pi_{s\tau} + u_2^{ss}(\bar{\omega}_s - x^o_s, x^o_s)\pi_{ss} \\ \vdots \\ u_2^{sS}(\bar{\omega}_1 - x^o_S, x^o_S)\pi_{sS} \end{pmatrix} : \pi_s \in \mathcal{B}_s(c_s)\right\} && (3) \\ &= \left\{\left(-\gamma^s \sum_{\tau \in \mathcal{S}} u_1^{s\tau}(\bar{\omega}_s - x^o_s, x^o_\tau)\pi_{s\tau} + \sum_{s'} \gamma^{s'} u_2^{s's}(\bar{\omega}_{s'} - x^o_{s'}, x^o_s)\pi_{s's}\right)_{s \in \mathcal{S}} : \pi \in \mathcal{B}(c)\right\} && (4) \end{aligned}$$

or equivalently,

$$\gamma \in \{\gamma M_c(\pi) : \pi \in \mathcal{B}(c)\},$$

where ∂f represents the supperdifferential of f (See Definition A in the Appendix), Eq.(1) follows from Theorem C in the Appendix, Eq.(2) follows from Theorem A in the Appendix, Eq.(3) follows from Theorem B in the Appendix, and Eq.(4) follows from easy calculation and the fact that $\mathcal{B}_s(c_s)$ is convex for each $s \in S$. The last inclusion is equivalent to $\gamma \min_{\pi \in \mathcal{B}(c)} M_c(\pi) \leq \gamma \leq \gamma \max_{\pi \in \mathcal{B}(c)} M_c(\pi)$ for each $M \in (M_c \circ \mathcal{B})(c)$. By the Perron-Frobenius theorem, therefore, the existence of $\gamma \in \Re^{\mathcal{S}}_{++}$ such that $\gamma \in \{\gamma M_c(\pi) : \pi \in \mathcal{B}(c)\}$ is equivalent to $\lambda^f(\min_{\pi \in \mathcal{B}(c)} M_c(\pi)) \leq 1 \leq \lambda^f(\max_{\pi \in \mathcal{B}(c)} M_c(\pi))$, or equivalently, $1 \in \left\{\lambda^f(M) : M \in (M_c \circ \mathcal{B})(c)\right\}$. This completes the proof. Q.E.D.

Proof of Proposition 2. Let c be an interior stationary feasible allocation. We first show $\mathcal{P}(c) \subset (M_c \circ \mathcal{B})(c)$. Let $P = [p_{ss'}]_{s,s' \in \mathcal{S}} \in \mathcal{P}(c)$ be a supporting price matrix. By their definitions, (P, c) satisfies that $U^s(c_s) \geq U^s(b_s)$ for each stationary feasible allocation b satisfying that $b^y_s + \sum_{s' \in S} b^o_{ss'} p_{ss'} \leq c^y_s + \sum_{s' \in S} c^o_{ss'} p_{ss'}$ and each $s \in \mathcal{S}$. Therefore, $c^o_s = (c^o_{ss'})_{s' \in \mathcal{S}}$ belongs to the set

$$\arg\max_{b^o_s \in \Re^{\mathcal{S}}_+} U^s \left(c^y_s + \sum_{s' \in \mathcal{S}} (c^o_{ss'} - b^o_{ss'}) p_{ss'}, b^o_s \right).$$

Then, for each $s \in \mathcal{S}$, $c^o_s = (c^o_{ss'})_{s' \in \mathcal{S}}$ must be characterized by

$$\begin{aligned} 0 \quad &\in \quad \partial \left(\min_{\pi_s \in \Pi_s} \sum_{s' \in \mathcal{S}} u^{ss'} \left(c^y_s + \sum_{\tau \in \mathcal{S}} (c^o_{s\tau} - c^o_{s\tau}) p_{s\tau}, c^o_{ss'} \right) \pi_{ss'} \right) \\ &= \quad \left\{ \left(-p_{ss'} \sum_{\tau \in \mathcal{S}} u^{s\tau}_1 (c^y_s, c^o_{s\tau}) \pi_{s\tau} + u^{ss'}_2 (c^y_s, c^o_{ss'}) \pi_{ss'} \right)_{s' \in \mathcal{S}} : \pi_s \in \mathcal{B}_s(c_s) \right\} \end{aligned} \tag{5}$$

because of Theorem B and C in the Appendix. This implies that

$$P \in \{M_c(\pi) : \pi \in \mathcal{B}(c)\}$$

and therefore $\mathcal{P}(c) \subset (M_c \circ \mathcal{B})(c)$.

Conversely, let $M = [m_{ss'}]_{s,s' \in \mathcal{S}} \in (M_c \circ \mathcal{B})(c)$. By its definition, it follows that there exists some $\pi \in \mathcal{B}(c)$ such that, for each $s, s' \in \mathcal{S}$,

$$m_{ss'} = \frac{u^{ss'}_2 (c^y_s, c^o_{ss'}) \pi_{ss'}}{\sum_{\tau \in \mathcal{S}} u^{s\tau}_1 (c^y_s, c^o_{s\tau}) \pi_{s\tau}},$$

or equivalently,

$$0 = -m_{ss'} \sum_{\tau \in \mathcal{S}} u^{s\tau}_1 (c^y_s, c^o_{s\tau}) \pi_{s\tau} + u^{ss'}_2 (c^y_s, c^o_{ss'}) \pi_{ss'}.$$

Then, it follows from Eq.(5) that M must be a supporting price matrix of c, which implies $M \in \mathcal{P}(c)$. This completes the proof. Q.E.D.

Proof of Theorem 2. It follows from Theorem 1 ad Proposition 2. Q.E.D.

Proof of Proposition 3. Let c be an interior stationary feasible allocation. In order to prove Proposition 3, we should show that $\mathcal{P}^*(c) \subset \mathcal{P}(c)$. Let $P = [p_{ss'}]_{s,s' \in \mathcal{S}} \in \mathcal{P}^*(c)$ and b be an arbitrary stationary feasible allocation satisfying that $b_s^y + \sum_{s' \in \mathcal{S}} b_{ss'}^o p_{ss'} \leq c_s^y + \sum_{s' \in \mathcal{S}} c_{ss'}^o p_{ss'}$ for each $s \in \mathcal{S}$. Then, it follows that $b_s^y + \sum_{s' \in \mathcal{S}} b_{ss'}^o p_{ss'} \leq \omega_s^y + \sum_{s' \in \mathcal{S}} \omega_{ss'}^o p_{ss'}$ for each $s \in \mathcal{S}$. Because (P, c) is a stationary equilibrium, it holds that $U^s(c_s) \geq U^s(b_s)$ for each $s \in \mathcal{S}$, which implies that $P \in \mathcal{P}(c)$. Q.E.D.

Proof of Theorem 3. By the sequential budget constraints of an agent, we can obtain the agent's lifetime budget constraint such that: for all $s \in \mathcal{S}$,

$$c_s^y + \sum_{s' \in \mathcal{S}} c_{ss'}^o p_{ss'} \leq \omega_s^y + \sum_{s' \in \mathcal{S}} \omega_{ss'}^o p_{ss'} + \left(\sum_{s' \in \mathcal{S}} q_{s'} p_{ss'} - q_s \right) m.$$

By this equation, we can obtain the no arbitrage condition when the money price is positive, i.e., $q = P \cdot q$ for any stationary equilibrium with circulating money, (q, P, c), with $c_s \gg 0$ for each $s \in \mathcal{S}$. In order to verify this, we should show that

$$(\forall s \in \mathcal{S}) \quad q_s = \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}.$$

Suppose the contrary that $q_s \neq \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}$ for some $s \in \mathcal{S}$. If $q_s < \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}$, then agent born at state s will choose ∞ as m and his/her optimization problem has no solution. On the other hand, if $q_s > \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}$, then agent born at state s will choose $-\infty$ as m and his/her optimization problem has no solution.[15] In any cases, we obtain a contradiction, so that $q_s = \sum_{s' \in \mathcal{S}} p_{ss'} q_{s'}$ for all $s \in \mathcal{S}$.

Suppose now that there exists at least one stationary equilibrium with circulating money, (q, P, c), satisfying that $c_s \gg 0$ for all $s \in \mathcal{S}$. We have obtained that $Pq = q$, at which the lifetime budget constraint coincides with that in the complete market. Because q_s is now positive for all $s \in \mathcal{S}$, it follows from the Perron-Frobenius theorem that the $\mathcal{S} \times \mathcal{S}$ matrix P with positive

[15] When one wished to impose the lower bound for possible m, $m \geq 0$ for example, the agent born state s chooses 0 as the amount of money holding. However, this contradicts the fact that m should be equal to 1 at a stationary equilibrium with circulating money.

coefficients has the dominant root equal to unity. Now it follows from Corollary 3 that the equilibrium allocation c is CPO. This completes the proof of Theorem 3.[16] Q.E.D.

Proof of Theorem 4. By the sequential budget constraints of an agent, we can obtain the agent's lifetime budget constraint such that: for all $s \in \mathcal{S}$,

$$c_s^y + \sum_{s' \in \mathcal{S}} c_{ss'}^o p_{ss'} \leq \omega_s^y + \sum_{s' \in \mathcal{S}} \omega_{ss'}^o p_{ss'} + \left(\sum_{s' \in \mathcal{S}} (q_{s'} + d_{s'}) p_{ss'} - q_s \right) z.$$

By this equation, we can obtain the no arbitrage condition such that $q = P \cdot (q + d)$ for any stationary equilibrium with equity, (q, P, c), with $c_s \gg 0$ for each $s \in \mathcal{S}$, where $q+d = (q_s+d_s)_{s \in \mathcal{S}}$. In order to verify this, we should show that

$$(\forall s \in \mathcal{S}) \quad q_s = \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'}).$$

Suppose the contrary that $q_s \neq \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'})$ for some $s \in \mathcal{S}$. If $q_s < \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'})$, then agent born at state s will choose ∞ as z and his/her optimization problem has no solution. On the other hand, if $q_s > \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'})$, then agent born at state s will choose $-\infty$ as z and his/her optimization problem has no solution. In any cases, we obtain a contradiction, so that $q_s = \sum_{s' \in \mathcal{S}} p_{ss'}(q_{s'} + d_{s'})$ for all $s \in \mathcal{S}$.

Suppose now that there exists at least one stationary equilibrium with equity, (q, P, c), satisfying that $c_s \gg 0$ for all $s \in \mathcal{S}$. We have obtained that $P(q + d) = q$, at which the lifetime budget constraint coincides with that in the complete market. By the fact that $d \in \Re_+^S \setminus \{0\}$, it holds that $Pq < P(q + d) = q$. Therefore, it follows from the Perron-Frobenius theorem that the $\mathcal{S} \times \mathcal{S}$ matrix P with positive coefficients has the dominant root less than unity. Now it follows from Corollary 3 that the equilibrium allocation c is not CPO. This completes the proof of Theorem 4. Q.E.D.

Appendix: Superdifferential and its Calculus

This appendix aims to introduce the definition of superdifferential and its calculus rules. We first define the concept of superdifferential following Rockafellar (1970, p.214) and Hiriart-Urruty and Lemaréchal (2004, Definition D.1.2.1).

[16] To prove Theorem 3, we have adopted an indirect way of applying the dominant root criterion under ambiguity. Applying the technique provided per Sakai (1988), one can provide a more direct proof of Theorem 3.

Definition A For each concave real-valued function f on $\Re^n$ and each $x \in \Re^n$, the set

$$\partial f(x) := \{s \in \Re^n : (\forall y \in \Re^n) \quad f(y) \leq f(x) + \langle s, y - x \rangle \}$$

and each of its elements are called the *superdifferantial* and a *supergradient of f at x*, respectively.

The following result follows from Hiriart-Urruty and Lemaréchal (2004, Theorem D.4.1.1).

Theorem A *For each concave real-valued functions f_1 and f_2 on $\Re^n$, each positive numbers a_1 and a_2, and each $x \in \Re^n$, $\partial(a_1 f_1 + a_2 f_2)(x) = a_1 \partial f_1(x) + a_2 \partial f_2(x)$.*

We should note that this observation does not necessarily hold for more general concave functions (Rockafellar, 1970, p.223).

The next result follows from Hiriart-Urruty and Lemaréchal (2004, Corollary D.4.4.4).

Theorem B *Let J be a compact set in some metric space and $\{f_j\}_{j \in J}$ be a family of differentiable concave real-valued functions on $\Re^n$. Define the real-valued function f on $\Re^n$ by*

$$f(x) := \inf_{j \in J} f_j(x)$$

and let $J(x) := \{j \in J : f_j(x) = f(x)\}$ for each $x \in \Re^n$. Then, it follows that

$$\partial f(x) = \mathrm{co}\{\nabla f_j(x) : j \in J(x)\}.$$

Finally, we provide a useful result following from Hiriart-Urruty and Lemaréchal (2004, Theorem D.2.2.1), for optimization.

Theorem C *For each concave real-valued function f on $\Re^n$ and each $x \in \Re^n$, $f(x) \geq f(y)$ for each $y \in \Re^n$ if and only if $0 \in \partial f(x)$.*

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