TCER Working Paper Series

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April 2014

Working Paper E-75
http://tcer.or.jp/wp/pdf/e75.pdf

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The Kernel of a Patent Licensing Game*

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April 18, 2014

Abstract

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Keywords: patent licensing, kernel, coalition structures
JEL Classification: C71, D43, D45

*The authors wish to thank Theo Driessen, Hideo Konishi, Shigeo Muto, Cheng-zhong Qin, and participants in 1st Caribbean Game Theory Conference (Netherlands Antilles), 17th Decentralization Conference (Japan), 4th World Congress of the Game Theory Society (Turkey), and 2013 Asian Meeting of the Econometric Society (Singapore) for helpful comments and suggestions. This research was supported by the JSPS Grant-in-Aid 24310110 and 24730163 (Kishimoto) and the MEXT Grant-in-Aid 20310086 (Watanabe).

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1 Introduction

In the traditional literature, patent licensing had been investigated mainly with non-cooperative mechanisms: fixed license fee, per-unit royalty, and auction.[1] Licensing agreements are, however, contract terms signed by patent holders and licensees that result from bargaining. From this viewpoint, Watanabe and Muto (2008) formulated a patent licensing game and investigated the general bargaining outcomes by using solution concepts for games with coalition structures (Aumann and Drèze, 1974).

Watanabe and Muto (2008) studied general bargaining outcomes under coalition structures formed by an external patent holder and firms in oligopoly markets. Their main proposition is that if the number of licensees that maximizes the licensees' total surplus is greater than the number of existing non-licensees, each symmetric bargaining set for a coalition structure is then a singleton. In this case, the number of licensees that most benefits the patent holder (in terms of his revenue) is determined. If the above condition is not satisfied, however, the bargaining set cannot suggest the optimal number of licensees to the patent holder, because it may not be a singleton for some coalition structure, although it is always non-empty.

A question left to us is thus whether the problem can be solved by applying a stronger solution concept in that it is a subset of the bargaining set. This paper gives the following answer to this question. For each permissible coalition structure, the kernel is a singleton; thus, the optimal number of licensees for the patent holder is determined by the kernel. For each coalition structure, the bargaining set contains the core, if the core is non-empty. In this sense, the core is also a stronger solution concept. Watanabe and Muto (2008), however, showed that the cores are empty for any permissible coalition structures except the grand coalition. We thus chose the kernel, which is always non-empty, as another solution concept.

Driessen et al. (1992) considered an information trading game, which is related to our patent licensing game, and showed that the kernel for the grand coalition of the game is a singleton by using the bisection property of the kernel (Maschler et al., 1979). Our patent licensing game, however, requires consideration for any permissible coalition structures. Chang (1991) extended the bisection property to games with coalition structures, but we prove our propositions by using the definition of the kernel directly, instead of using the extended bisection property. Moreover, we specify the upper and lower bounds of the kernel for each coalition structure and provide sufficient conditions for the number of licensees that maximizes their total surplus to be optimal for the patent holder, which enable us to regain an asymptotic result of Kishimoto et al. (2011) within a finite number of firms.

[1] See Sen and Tauman (2007) and the references therein.

Note that, in this paper, no particular or practical negotiation process is assumed in the bargaining among the patent holder and oligopolistic firms, but the patent holder might negotiate with each firm on a one-by-one basis repeatedly. Rather, more important than the negotiation process in this paper is the meaning that each solution concept gives to the licensing agreements. The bargaining set suggests stable profit sharing in patent licensing, and the kernel requires some sort of equity as well as stability to licensing agreements.

Moreover, we analyze a generalized patent licensing game that was formulated by Watanabe and Muto (2008). In the traditional literature, many assumptions are made on the underlying markets; linear demand and cost functions, Cournot or Bertrand oligopoly, cost-reducing or quality-improving technology, and so on. Our model is much less specified. Instead, we retain the traditional assumption that all firms have an identical production technology before patent licensing to measure the private value of a patented technology, given that an old technology is disseminated among all firms in the market. Lastly, we note that a patent licensing game is considered as an extension of a one-to-many assignment problem with sophisticated externalities generated in oligopoly markets.

The outline of this paper is as follows. Section 2 describes a patent licensing game and solution concepts. Section 3 analyzes the game and provides major results. Section 4 provides implications on the optimal number of licensees. We there provide sufficient conditions for the number of licensees that maximizes their total surplus to be optimal for the patent holder. Section 5 refers to the case of drastic innovations as a special case of our model. This case is not much significant to analyze as patent licensing, but we there mention that the kernel may not be in the interior of the bargaining set, when the core is empty. Section 6 notes a final remark.

2 Model

2.1 A Patent licensing game

Let $N = \{1, 2, \ldots, n\}$ be the set of firms that have an identical production technology before patent licensing, where $2 \leq n < \infty$. An agent, who is not a producer, holds a patent of a new technology. This agent is referred to as an external patent holder, and is denoted by player 0.[2] The set of players of this game is $\{0\} \cup N$. Assume that the patent is perfectly protected; namely, no firm can use the patented technology without the patent holder's permission. Thus, there is neither piracy nor resale of the patented technology to non-licensees.

[2]Research laboratories and engineering departments at universities are typical examples of such agents, because they have no production facilities.

Remark 1. We retain the assumption in the traditional literature that all firms have an identical production technology before a patented technology is licensed.

The game has three stages. At stage (i), the patent holder selects a subset $S \subseteq N$ and invites the firms in S to negotiate on license issues. Firms in $N \setminus S$ cannot participate in this negotiation, and thus they are not licensed. At stage (ii), every firm in S negotiates with the patent holder over how much it should pay to the patent holder. It is assumed that all the firms in S that were invited to bargain will buy a license, and consequently, we focus solely on the fees paid to the patent holder.[3] Players in $\{0\} \cup S$ can communicate among themselves in the negotiation, but non-licensees (players in $N \setminus S$) cannot observe how the negotiation runs. The payment to the patent holder is made at the end of this stage. At stage (iii), firms compete in the market, knowing that firms are licensed or not. Licensees use the patented technology, while non-licensees use the old technology. Firms are prohibited from forming any cartels to coordinate their production levels and market behaviors, because, as in the traditional literature, they cannot make binding agreements on such cartels.

Remark 2. No negotiation process is specified at stage (ii), but the patent holder might negotiate with each firm in S on a one-by-one basis repeatedly.[4]

In what follows, the model stated above is analyzed backwardly from stage (iii) to stage (i). Let $s = |S|$ for each $S \subseteq N$, where $s = 0$ for $S = \emptyset$. As noted, all firms are identical before patent licensing and all licensees use the same patented technology. Thus, when s firms are licensed, the equilibrium gross profits of each licensee and each non-licensee at stage (iii) are denoted by $W(s)$ and $L(s)$, respectively. (The equilibrium net profit of a licensee is the amount of the equilibrium gross profit minus the license fee.) We do not specify the market structure and require only the following assumption;

$$W(s) > L(0) \text{ for } s = 1, \ldots, n, \text{ and } L(0) > L(s) \geq 0 \text{ for } s = 1, \ldots, n-1. \qquad (1)$$

Any market structures, e.g., Cournot or Bertrand, homogeneous product or differentiated products, are considered in our general model, as far as $W(s)$, $L(0)$, and $L(s)$ are uniquely determined in the market competition and (1) is preserved.

[3] In our model, there always exists $S \subseteq N$ such that all firms in S buy the license, even if every invited firm chooses whether or not to buy it. Further, see the comments on Propositions 3 and 4.

[4] See, e.g., chapter 10 in Peleg and Sudhölter (2007) for such dynamic bargaining procedures that converge to relevant solutions, where at each time $t = 1, 2, \ldots$, a pair of players proceeds to make bilateral demands and transfer one's payoff to another.

Remark 3. The patent licensing game is considered as an example of one-to-many assignment problems with externalities characterized by (1), where the patent holder assigns identical technology to licensee firms.

We formalize the negotiation that is made at stage (ii) as a game with a coalition structure in the following way. Each non-empty subset of $\{0\} \cup N$ is called a coalition. At stage (ii), the firms that are not in S cannot participate in the negotiation on license issues, but play a relevant role in determining the outside options of players in $\{0\} \cup S$ in the negotiation. Thus, we need to provide the worth of each coalition, which is the total profit that the players belonging to the coalition can guarantee for themselves.

For each $t = 1, 2, \ldots, n$, define $\rho(t) \in \arg\min_{0 \le r \le n-t} L(r)$ (especially, $\rho(n) = 0$). The worth of a coalition $T' \subseteq \{0\} \cup N$ is represented by $v(T')$, which is called the characteristic function. As described above, all firms invited to the negotiation by the patent holder are licensed, and firms are not allowed to form any cartels both in production and in the market at stage (iii). Thus, the worth of each coalition is defined simply as the sum of equilibrium gross profits that players in the coalition obtain at stage (iii), i.e., the characteristic function $v : 2^{\{0\} \cup N} \to \mathbb{R}$ is given by

$$v(\{0\}) = v(\emptyset) = 0, \; v(\{0\} \cup T) = tW(t), \text{ and } v(T) = tL(\rho(t))$$

for each $T \subseteq N$ with $T \neq \emptyset$ and $|T| = t$.

The patent holder can gain nothing without licensing his patented technology because he is not a producer; thus $v(\{0\}) = 0$. Let $T \subseteq N$ with $T \neq \emptyset$ and $|T| = t$. The licensees' total equilibrium gross profit in T is given as $tW(t)$; thus $v(\{0\} \cup T) = tW(t)$. $v(T)$ is the total equilibrium gross profit that firms in T can guarantee for themselves in the worst anticipation when firms in T jointly break off the negotiation. We assume the worst case for coalition T in the spirit of von Neumann and Morgenstern (1944). In the worst case, $\rho(t) (\le n-t)$ firms are licensed; thus, $v(T) = tL(\rho(t))$.[5]

For a non-empty set $S \subseteq N$ of licensees selected by the patent holder at stage (i), a permissible coalition structure is denoted by $P^S = \{\{0\} \cup S\} \cup \{\{i\} | i \in N \setminus S\}$, because players in $\{0\} \cup S$ can communicate with one another but non-licensees are not allowed to communicate with any players.[6] Recall that all firms behave independently in the market at stage (iii). Thus, a coalition structure P^S is given

[5]For each $T \subseteq N$, the worth $v(T)$ of coalition T is defined from a pessimistic viewpoint. This definition does not play a major role in our main propositions. See Section 5 in Watanabe and Muto (2008) for more detail discussion on the characteristic functions.

[6]In this paper, a coalition structure is not the one that is determined as an equilibrium in a non-cooperative dynamic coalition formation game. The credibility of coalitional deviations is embedded in our solution concepts. See Remark 4 stated in the next subsection.

by the patent holder at stage (i) only for the negotiation at stage (ii). In what follows, for each set S of licensees, we analyze a bargaining game $(\{0\} \cup N, v, P^S)$ with a coalition structure, which we sometimes call a bargaining game if there is no confusion.

2.2 Solution concepts of a bargaining game

The licensees' total equilibrium gross profit is concerned in the negotiation over the license fees, because of the feasibility of payments for licenses. The set of imputations under a coalition structure P^S is defined as

$$I^S = \left\{ x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \middle| x_0 + \sum_{i \in S} x_i = sW(s),\ x_0 \geq v(\{0\}) = 0, \right.$$
$$\left. x_i \geq v(\{i\}) = L(\rho(1)) \text{ for all } i \in S,\ \text{and } x_i = L(s) \text{ for all } i \in N \setminus S \right\}.$$

Players in $\{0\} \cup S$ divide the licensees' total equilibrium gross profit, with each player $i \in \{0\} \cup S$ being guaranteed the worst payoff $v(\{i\})$.[7] Each non-licensee in $N \setminus S$ obtains the equilibrium gross profit $L(s)$, because s firms are licensed.[8] Every vector of payoffs for players should be in I^S. We consider only a subset S of licensees with $S \neq \emptyset$, because the patent holder can guarantee zero payoff by himself. The solutions for this bargaining game are defined as follows.

The definitions of our solutions follow Aumann and Drèze (1974). Let $T' \subseteq \{0\} \cup N$ and $x \in \mathbb{R}^{n+1}$. The excess of T' with respect to x in a bargaining game $(\{0\} \cup N, v, P^S)$ is defined as

$$e(T', x) = v(T') - \sum_{k \in T'} x_k.$$

A nonnegative (nonpositive) excess of T' with respect to x represents the gain (loss) to coalition T' when its members jointly withdraw from the payoff vector x. Let $i, j \in \{0\} \cup S$ and $i \neq j$. The maximum excess of i over j at x is represented by

$$\delta_{ij}(x) = \max_{T \subseteq \{0\} \cup N : i \in T, j \notin T} e(T, x).$$

Note that the maximum excess can be defined for each pair of players in $\{0\} \cup N$, but only players in $\{0\} \cup S$ can express it in the negotiation over the license fees.

[7]It is difficult to observe patent licensing by means of joint profit distribution in real practice. In our model, however, the patent holder negotiates over the license fee, not over how the joint profit is actually distributed.

[8]In cooperative games with coalition structures, an imputation x under P^S is usually defined as $x_i = v(\{i\})$ for all $i \in N \setminus S$. Our definition of imputations, however, preserves the properties of the solutions, which are introduced in this paper.

Then, the kernel for a coalition structure P^S is defined as

$$\mathcal{K}^S = \left\{ x \in I^S \middle| \delta_{ij}(x) \geq \delta_{ji}(x) \text{ or } x_i = v(\{i\}) \text{ for all } i, j \in \{0\} \cup S \text{ with } i \neq j \right\}.$$

For any imputation $x \in I^S$ in a bargaining game $(\{0\} \cup N, v, P^S)$, define the associated complaint vector $\theta(x)$ as the 2^{n+1}-tuple whose components are the excesses $e(T', x)$ for $T' \subseteq \{0\} \cup N$, arranged in nonincreasing order, i.e., $\theta_k(x) \geq \theta_{k+1}(x)$ for $k = 1, \ldots, 2^{n+1} - 1$. Let $\geq_{lex}$ denote the lexicographical ordering over $\mathbb{R}^{2^{n+1}}$; that is, $x \geq_{lex} y$, where $x, y \in \mathbb{R}^{2^{n+1}}$, if either $x = y$ or there exists l such that $1 \leq l \leq 2^{n+1}$, $x_k = y_k$ for $1 \leq k < l$, and $x_l > y_l$. Then, the nucleolus for a coalition structure P^S is defined as the set of imputations $x \in I^S$ satisfying that $\theta(y) \geq_{lex} \theta(x)$ for all $y \in I^S$. As a corollary of the results shown by Schmeidler (1969), the nucleolus for each coalition structure is always a singleton. Thus, the unique element of the nucleolus of a bargaining game $(\{0\} \cup N, v, P^S)$ is denoted by η^S, and we call it the nucleolus for a coalition structure P^S.

It is known in the literature that for each coalition structure, the nucleolus belongs to the kernel, the kernel is non-empty and a subset of the bargaining set (Davis and Maschler, 1965), and the core is a subset of the bargaining set if the core is non-empty.[9] As noted in Section 1, Watanabe and Muto (2008) showed that the core is empty unless the grand coalition forms. We thus chose the kernel as another stronger solution concept than the bargaining set.

Remark 4. We do not require for coalitional deviations to be binding. The credibility of coalitional deviations is, however, conceptually embedded in solutions in the bargaining set family, i.e., kernel and nucleolus.[10]

3 Major results

3.1 The kernel as a singleton

In our model, all firms are identical before the patented technology is licensed, and all licensees use the same patented technology. In any bargaining game $(\{0\} \cup N, v, P^S)$, it is then easy to see by the definition that for all $i, j \in S$, $x_i = x_j$ if $x \in \mathcal{K}^S$.[11] Thus, we only consider the symmetric payoff vectors defined by $x^S(\alpha) \in \mathbb{R}^{n+1}$ such that $x_0^S(\alpha) = sW(s) - s\alpha$, $x_i^S(\alpha) = \alpha$ for $i \in S$, and $x_j^S(\alpha) = L(s)$ for $j \in N \setminus S$.

[9] The other properties of these solutions are described in Peleg and Sudhölter (2007) in detail.

[10] The bargaining set is defined in such a way that an objection via a coalitional deviation from an imputation that is proposed in the negotiation should be justified against any counter objections via further coalitional deviations to the objection.

[11] For each permissible coalition structure, the kernel has this restricted equal treatment property. See Peleg (1986).

Let $i \in S$. By the above argument, it suffices to examine $\delta_{0i}(x^S(\alpha))$ and $\delta_{i0}(x^S(\alpha))$ for analyzing $\mathcal{K}^S$. The following lemma is useful to prove that for each coalition structure the kernel is a singleton. The proof is shown in the Appendix.

Lemma 1. *Suppose that $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s$, and let $i \in S$. Then, the following properties on the maximum excesses hold:*
(a) $\delta_{0i}(x^S(\alpha))$ is a continuous and strictly increasing function in α. Furthermore, $\delta_{i0}(x^S(\alpha))$ is a continuous and strictly decreasing function in α.
(b) There exists a unique $\alpha^*(s) \in \mathbb{R}$ such that $\delta_{0i}(x^S(\alpha^*(s))) = \delta_{i0}(x^S(\alpha^*(s)))$, and then $\alpha^*(s) < W(s)$.

Proposition 1. *Let $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s$. If $\alpha^*(s) > L(\rho(1))$, then $\mathcal{K}^S = \{x^S(\alpha^*(s))\}$. If $\alpha^*(s) \leq L(\rho(1))$, then $\mathcal{K}^S = \{x^S(L(\rho(1)))\}$.*

Proof. We first show that $x^S(\alpha^*(s)) \in \mathcal{K}^S$ if $\alpha^*(s) \geq L(\rho(1))$ and $x^S(L(\rho(1))) \in \mathcal{K}^S$ if $\alpha^*(s) < L(\rho(1))$. (Note that $x^S(\alpha^*(s)) = x^S(L(\rho(1)))$ if $\alpha^*(s) = L(\rho(1))$.) In the case where $\alpha^*(s) \geq L(\rho(1))$, $x^S(\alpha^*(s)) \in I^S$, and $\delta_{ij}(x^S(\alpha^*(s))) = \delta_{ji}(x^S(\alpha^*(s)))$ for each $i, j \in \{0\} \cup S$, because of Lemma 1 (b). Hence, $x^S(\alpha^*(s)) \in \mathcal{K}^S$. If $\alpha^*(s) < L(\rho(1))$, then $\delta_{0i}(x^S(L(\rho(1)))) > \delta_{i0}(x^S(L(\rho(1))))$ for each $i \in S$ by Lemma 1. For each $i \in S$, $x_i^S(L(\rho(1))) = L(\rho(1)) = v(\{i\})$. Thus, by the definition of $\mathcal{K}^S$, $x^S(L(\rho(1))) \in \mathcal{K}^S$.

We next show that the kernel for a coalition structure is a singleton. Suppose that $\alpha^*(s) > L(\rho(1))$. Let $x^S(\alpha') \neq x^S(\alpha^*(s))$. Then, if $x^S(\alpha') \in \mathcal{K}^S$, $\alpha' = W(s)$ or $\alpha' = L(\rho(1))$ (i.e., $x_0^S(W(s)) = 0 = v(\{0\})$ or $x_i^S(L(\rho(1))) = L(\rho(1)) = v(\{i\})$ for $i \in S$) by the definition of $\mathcal{K}^S$ and the uniqueness of $\alpha^*(s)$. Because $L(\rho(1)) < \alpha^*(s) < W(s)$ and by Lemma 1 (a), $\delta_{i0}(x^S(W(s))) < \delta_{0i}(x^S(W(s)))$ and $\delta_{0i}(x^S(L(\rho(1)))) < \delta_{i0}(x^S(L(\rho(1))))$. For all $i \in S$, however, $x_i^S(W(s)) = W(s) \neq v(\{i\})$ and $x_0^S(L(\rho(1))) = s(W(s) - L(\rho(1))) \neq v(\{0\})$. These facts contradict $x^S(\alpha') \in \mathcal{K}^S$. If $\alpha^*(s) \leq L(\rho(1))$, then, by Lemma 1, $\delta_{0i}(x^S(\alpha)) > \delta_{i0}(x^S(\alpha))$ for all $\alpha > L(\rho(1))$. Thus, for all α with $L(\rho(1)) < \alpha \leq W(s)$, $x^S(\alpha) \notin \mathcal{K}^S$ because $x_i^S(\alpha) = \alpha \neq L(\rho(1)) = v(\{i\})$. Therefore, the kernel for a coalition structure is also a singleton in this case. □

For each coalition structure, the kernel is a subset of the bargaining set, and thus the kernel can give a prediction of stable bargaining outcomes. Proposition 1 shows that for each permissible coalition structure, the kernel is a singleton; thus, the optimal number of licensees is determined. It is known that $\eta^S \in \mathcal{K}^S$ for any $S \subseteq N$. This fact and Proposition 1 jointly imply that for each permissible coalition structure, the kernel allocation coincides with the nucleolus in our bargaining game.

Corollary 1. *Let $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s$. If $\alpha^*(s) > L(\rho(1))$, then $\eta^S = x^S(\alpha^*(s))$. If $\alpha^*(s) \leq L(\rho(1))$, $\eta^S = x^S(L(\rho(1)))$.*

Driessen et al. (1992) considered an information trading game, which is related to our patent licensing game; in our notation, $W(t) \geq L(t)$, $W(t) \geq W(t+1)$, and $L(t) \geq L(t+1)$ for any $t = 1, 2, \ldots, n-1$, $W(1) > L(0) > 0$, $W(0) = L(n) = 0$. They showed that the kernel for the grand coalition of the game is a singleton consisting of the nucleolus by using the bisection property of the kernel (Maschler et al., 1979). Chang (1991) extended the bisection property to any games with coalition structures.[12] We could prove Proposition 1 by using the definition of the kernel directly, instead of using the bisection property. In the next subsection, we specify the upper and lower bounds of the kernel for each coalition structure.[13]

3.2 The upper and lower bounds of the kernel

Our model only assumes that the gross equilibrium profits of each licensee and each non-licensee satisfy (1); thus, it is hard to exactly characterize the kernel for each coalition structure. Instead, we examine the range of the kernel for each coalition structure, and identify the outcomes that cannot be realized through bargaining.[14] The results on the range of the kernel are used substantially to determine the optimal number of licensees that the patent holder selects at stage (i), which is investigated in the next section.

We begin with some notations that are used in what follows. For each $s = 1, 2, \ldots, n$, define $\alpha(s) \in \mathbb{R}$ by $\alpha(s) = \alpha^*(s)$ if $\alpha^*(s) > L(\rho(1))$, and by $\alpha(s) = L(\rho(1))$ if $\alpha^*(s) \leq L(\rho(1))$. Then, by Proposition 1, $x^S(\alpha(s))$ represents the kernel for coalition structure P^S. For notational ease, let $LTS(s) = s(W(s) - L(0))$. $LTS(s)$ is the licensees' total surplus when s firms hold the patent. Denote by s^* the number of licensees that maximizes their total surplus, i.e., $LTS(s^*) \geq LTS(s)$ for any $s = 1, 2, \ldots, n$.[15] Let $S^* \subseteq N$ with $|S^*| = s^*$. Note that the patent holder cannot form a coalition containing all licensees in S in order to withdraw from $x^S(\alpha)$. It is thus useful to define $s^*_{\text{-}n}$ as $s^*_{\text{-}n} \in \arg\max_{1 \leq s \leq n-1} LTS(s)$ in considering the upper and lower bounds of $\alpha(s)$.

For each $s = 1, 2, \ldots, n-1$, the range of $\alpha(s)$ is given in the next proposition. The proof is shown in the Appendix.

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[12] Roughly speaking, the bisection property of the kernel refers to that every payoff vector in the intersection of the kernel and ϵ-core is situated symmetrically with respect to certain bargaining range for any pair of players.

[13] Driessen et al. (1992) did not find those bounds in their information trading game.

[14] In Section 5, we restrict our general patent licensing game to the case of drastic innovations, and exactly characterize the kernel for each coalition structure.

[15] Note that, in general, the number s^* is not uniquely determined. In the following discussion, we take any $s^* \in \arg\max_{1 \leq s \leq n} LTS(s)$, and fix it.

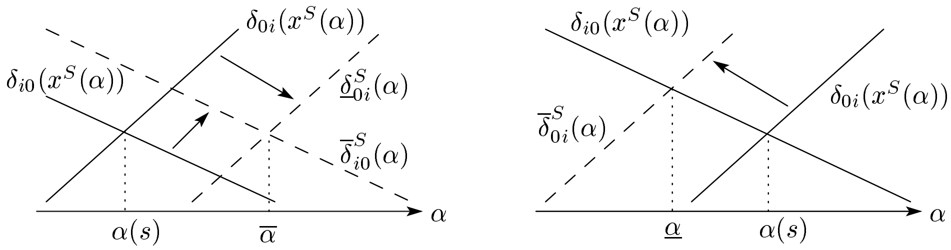

Figure 1: Upper and lower bounds of $\alpha(s)$

Proposition 2. *Take an arbitrary* $s^*_{\text{-}n} \in \arg\max_{1\le s\le n-1} LTS(s)$.
(a) For each $s = 1, 2, \ldots, n-1$*, the lower bound of* $\alpha(s)$ *is given as follows:*

$$\alpha(s) \geq \max\left\{L(\rho(1)), L(0) - \frac{LTS(s^*_{\text{-}n}) - LTS(s)}{s+1}\right\}.$$

(b) For each $s = 1, 2, \ldots, n-1$*, we have the following upper bound of* $\alpha(s)$*:*
(b-i) Suppose that $0 \le (LTS(s^*_{\text{-}n}) - LTS(s))/(L(0) - L(s)) \le n - s^*_{\text{-}n} + s$*. If* $1 \le s \le n - s^*_{\text{-}n}$ *and* $0 \le (LTS(s^*_{\text{-}n}) - LTS(s))/(L(0) - L(s)) < n - s^*_{\text{-}n} - s$*, then*

$$\alpha(s) \le L(0) + \frac{(n - s^*_{\text{-}n} - s)(L(0) - L(s)) - (LTS(s^*_{\text{-}n}) - LTS(s))}{s+1}.$$

If $1 \le s \le n - s^*_{\text{-}n}$ *and* $n - s^*_{\text{-}n} - s \le (LTS(s^*_{\text{-}n}) - LTS(s))/(L(0) - L(s)) \le n - s^*_{\text{-}n} + s$*, then*

$$\alpha(s) \le L(0) - \frac{LTS(s^*_{\text{-}n}) - LTS(s) - (n - s^*_{\text{-}n} - s)(L(0) - L(s))}{2s}.$$

If $n - s^*_{\text{-}n} \le s \le n-1$*, then*

$$\alpha(s) \le L(0) - \frac{LTS(s^*_{\text{-}n}) - LTS(s)}{n - s^*_{\text{-}n} + s}.$$

(b-ii) Suppose that $n - s^*_{\text{-}n} + s < (LTS(s^*_{\text{-}n}) - LTS(s))/(L(0) - L(s))$*. If* $1 \le s \le s^*_{\text{-}n} + 1$*, then*

$$\alpha(s) \le \max\left\{L(\rho(1)), L(0) - \frac{LTS(s^*_{\text{-}n}) - LTS(s) - (n - s^*_{\text{-}n} - 1)(L(0) - L(s))}{s+1}\right\}.$$

If $s^*_{\text{-}n} + 1 \le s \le n-1$*, then*

$$\alpha(s) \le \max\left\{L(\rho(1)), L(0) - \frac{LTS(s^*_{\text{-}n}) - LTS(s) - (n - s)(L(0) - L(s))}{2s - s^*_{\text{-}n}}\right\}.$$

A brief sketch of the proof is as follows. Let $\overline{\alpha}$ and $\underline{\alpha}$ be the upper and lower bounds of $\alpha(s)$, respectively. In order to prove that $\alpha(s) \le \overline{\alpha}$, we first find functions $\underline{\delta}_{0i}^S(\alpha)$ and $\overline{\delta}_{i0}^S(\alpha)$ (possibly $\underline{\delta}_{0i}^S(\alpha) = \delta_{0i}(x^S(\alpha))$ and $\overline{\delta}_{i0}^S(\alpha) = \delta_{i0}(x^S(\alpha))$) such that $\underline{\delta}_{0i}^S(\overline{\alpha}) \le \delta_{0i}(x^S(\overline{\alpha}))$ and $\overline{\delta}_{i0}^S(\overline{\alpha}) \ge \delta_{i0}(x^S(\overline{\alpha}))$. Then, by showing that $\underline{\delta}_{0i}^S(\overline{\alpha}) = \overline{\delta}_{i0}^S(\overline{\alpha})$, we confirm that $\delta_{0i}(x^S(\overline{\alpha})) \geq \delta_{i0}(x^S(\overline{\alpha}))$, which implies that $\alpha(s) \leq \overline{\alpha}$. (See the left-hand side of Figure 1.) It is similarly shown that $\underline{\alpha} \leq \alpha(s)$. We first find a function $\overline{\delta}^S_{0i}(\alpha)$ such that $\overline{\delta}^S_{0i}(\underline{\alpha}) \geq \delta_{0i}(x^S(\underline{\alpha}))$, and then we show that $\overline{\delta}^S_{0i}(\underline{\alpha}) \leq \delta_{i0}(x^S(\underline{\alpha}))$. This implies that $\delta_{0i}(x^S(\underline{\alpha})) \leq \delta_{i0}(x^S(\underline{\alpha}))$; thus $\underline{\alpha} \leq \alpha(s)$. (See the right-hand side of Figure 1.)

Suppose that $s^* < n$. Then, Proposition 2 implies the next proposition, which shows the range of kernel for coalition structure P^{S^*}.

Proposition 3. *If $1 \leq s^* < n/2$, then $L(0) \leq \alpha(s^*) \leq L(0) + [(n-2s^*)(L(0) - L(s^*))]/(s^*+1)$. If $n/2 \leq s^* \leq n-1$, then $\alpha(s^*) = L(0)$.*

Proof. Suppose that $s^* < n$. Then, $s^* \in \arg\max_{1\leq s\leq n-1} LTS(s)$, so $LTS(s^*_{\text{-}n}) - LTS(s^*) = 0$. Thus, $\alpha(s^*) \geq L(0)$ by Proposition 2 (a). When $1 \leq s^* < n/2$, substituting $s^*_{\text{-}n} = s^*$ into the equation in Proposition 2 (b-i), we have $\alpha(s^*) \leq L(0) + [(n-2s^*)(L(0) - L(s^*))]/(s^*+1)$. If $n/2 \leq s^* \leq n-1$, by Proposition 2 (b-i), we have $\alpha(s^*) \leq L(0)$, which implies that $\alpha(s^*) = L(0)$. $\square$

By Proposition 3, we can completely characterize the kernel for coalition structure P^{S^*} when the number of licensees that maximizes licensees' total surplus is greater than the number of existing non-licensees (i.e., $\mathcal{K}^{S^*} = \{x^{S^*}(L(0))\}$ if $n/2 \leq s^* \leq n-1$). This result is obtained not only from Proposition 2, but also from Proposition 5 of Watanabe and Muto (2008).

Proposition 3 also means that when $s^*(< n)$ firms are licensed through bargaining, each licensee can be always guaranteed $L(0)$, which is its profit before the technology is developed. Note that each licensee may gain at least as much as $L(0)$ as the bargaining outcome, even if $s^*(< n)$ firms are not licensed. The following example shows that there exists a case in which $\alpha(s) > L(0)$ when $s \neq s^*(< n)$.

Example 1. $n = 5$; $W(1) = 12$, $W(2) = 6$, $W(3) = 4$, $W(4) = 3$, $W(5) = 2.4$, $L(0) = 2$, $L(t) = 0$ for any $t = 1,2,3,4$. Then, $LTS(1) = 10 > LTS(2) = 8 > LTS(3) = 6 > LTS(4) = 4 > LTS(5) = 2$, and thus $s^* = 1$. When $S = \{1,2\}$, $\delta_{0i}(x^{\{1,2\}}(\alpha)) = 2\alpha$ if $\alpha \geq 0$ and $\delta_{i0}(x^{\{1,2\}}(\alpha)) = 10 - 2\alpha$ if $\alpha \leq 10$. (See Lemma 2 in the Appendix.) Thus, $\alpha(2) = 2.5 > 2 = L(0)$, but $s^* \neq 2$.

We next consider the range of $\alpha(n)$. The next proposition is proved in the same way as Proposition 2. The proof is shown in the Appendix.

Proposition 4. *Take any $s^*_{\text{-}n} \in \arg\max_{1\leq s\leq n-1} LTS(s)$.*
(a) If $s^* = n$, then

$$L(0) \leq \alpha(n) \leq L(0) + \frac{LTS(n) - LTS(s^*_{\text{-}n})}{n - s^*_{\text{-}n} + 1}.$$

(b) If $s^* \neq n$, then

$$\max\left\{L(\rho(1)), L(0) - \frac{LTS(s^*_{\text{-}n}) - LTS(n)}{n+1}\right\} \leq$$
$$\alpha(n) \leq \max\left\{L(\rho(1)), L(0) - \frac{LTS(s^*_{\text{-}n}) - LTS(n)}{2n - s^*_{\text{-}n}}\right\}.$$

Proposition 4 shows that, in the case where $s^* = n$, when the patent holder selects all firms at stage (i), each licensee always gains at least as much as $L(0)$ as the bargaining outcome. In addition, if $LTS(n) = LTS(s^*_{\text{-}n})$, then $\alpha(n) = L(0)$, which implies that the kernel for the grand coalition is completely characterized. Thus, Propositions 3 and 4 (a) jointly suggest that when s^* firms are licensed through bargaining, the licensee can be always guaranteed $L(0)$.

The next proposition gives the upper bound of the patent holder's revenue in the kernel for each coalition structure. This proposition implies that the patent holder can never gain more than $s^*(W(s^*) - L(0))$ as the bargaining outcome.

Proposition 5. *For each $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s$, $x_0^S(\alpha(s)) \leq s^*(W(s^*) - L(0))$.*

Proof. Propositions 2 to 4 jointly imply that $\alpha(s) \geq L(0) - (LTS(s^*_{\text{-}n}) - LTS(s))/(s+1)$ when $s \neq s^*$, and that $\alpha(s^*) \geq L(0)$. Thus, in the case where $s \neq s^*$,

$$x_0^S(\alpha(s)) = s(W(s) - \alpha(s)) \leq \frac{LTS(s) + sLTS(s^*_{\text{-}n})}{s+1} < s^*(W(s^*) - L(0)),$$

and $x_0^S(\alpha(s^*)) = s^*(W(s^*) - \alpha(s^*)) \leq s^*(W(s^*) - L(0))$. □

4 Optimal number of licensees

We next consider the optimal number of licensees for the patent holder. Recall that $s^*(W(s^*) - L(0)) \geq s(W(s) - L(0))$ for any $s = 1, 2, \ldots, n$. By Proposition 3, if $n/2 \leq s^* \leq n-1$, then the kernel for coalition structure P^{S^*} gives $s^*(W(s^*) - L(0))$ to the patent holder, which is, by Proposition 5, the maximum revenue that the patent holder can gain in the kernels for any permissible coalition structures. Thus, in this case, it is optimal for the patent holder to invite s^* firms at stage (i).

Proposition 6. *If $n/2 \leq s^* \leq n - 1$, then $s^* \in \arg\max_{1 \leq s \leq n} s(W(s) - \alpha(s))$.*

In the other cases, however, the following example shows that s^* is not always the optimal number of licensees in our patent licensing game.

Example 2. $n = 3$; $W(1) = 9$, $W(2) = 5$, $W(3) = 4$, $L(0) = 2$, $L(1) = L(2) = 0$. Then, $LTS(1) = 7 > LTS(2) = LTS(3) = 6$, and thus $s^* = 1$. When $S = \{1\}$,

$\delta_{0i}(x^{\{1\}}(\alpha)) = 1 + \alpha$ and $\delta_{i0}(x^{\{1\}}(\alpha)) = 6 - \alpha$ if $\alpha \geq 0$. Thus, $\alpha(1) = 2.5$ and $x_0^{\{1\}}(\alpha(1)) = 6.5$. On the other hand, when $S = N$, $\delta_{0i}(x^N(\alpha)) = -3 + 2\alpha$ if $\alpha \geq 1$ and $\delta_{i0}(x^N(\alpha)) = 6 - 3\alpha$ if $\alpha \leq 3$. Then, $\alpha(3) = 1.8$ and $x_0^N(\alpha(3)) = 6.6$. Because $x_0^N(\alpha(3)) - x_0^{\{1\}}(\alpha(1)) = 0.1$, $s^* \notin \arg\max_{1\leq s\leq n} s(W(s) - \alpha(s))$. (In this example, $\{3\} = \arg\max_{1\leq s\leq n} s(W(s) - \alpha(s))$. $\alpha(3) = 1.8 < 2 = L(0)$, and therefore, we do not generally have $\alpha(\hat{s}) \geq L(0)$ for all $\hat{s} \in \arg\max_{1\leq s\leq n} s(W(s) - \alpha(s))$.)

The next proposition, however, gives a set of conditions that is sufficient for s^* to be the optimal number of licensees also in the case where $1 \leq s^* < n/2$. The proof is shown in the Appendix.

Proposition 7. *If $1 \leq s^* < n/2$, then $s^* \in \arg\max_{1\leq s\leq n} s(W(s) - \alpha(s))$, under the following Conditions (a) to (d);*
(a) $s^*W(s^*) + (n - s^*)L(s^*) \geq sW(s) + (n - s)L(s)$ for each $s = 1, 2, \ldots, s^*$.
(b) $s^*W(s^*) \geq sW(s)$ for each $s = s^*, s^* + 1, \ldots, n$.
(c) $L(t) \geq L(t + 1)$ for each $t = 1, 2, \ldots, n - 2$.
(d) $n(L(0) - L(t)) \geq (n - s^* - s)(L(t) - L(s))$ for all $s = 1, 2, \ldots, n - s^*$ and for each $t = 1, 2, \ldots, s$.

Conditions (a) and (b) imply that when s^* firms are licensed, both the total industry profit and the licensees' total equilibrium gross profit are maximized, respectively. By the definition of s^* and (1), Conditions (a) and (b) are satisfied when $L(0)$ is sufficiently small. In many oligopoly markets, in fact, $L(0)$ tends to zero as the number of firms goes to infinity.[16] Condition (c) represents the monotonicity in equilibrium gross profits of each non-licensee, which is also satisfied in many oligopoly markets. Given this monotonicity, Condition (d) requires that $L(0) - L(1)$ is sufficiently large; that is, even when only one firm is licensed, non-licensees suffer a large amount of losses.

Kishimoto et al. (2011) showed that in the general Cournot market, the bargaining set for each permissible coalition structure converges to a point as the number of firms in the market goes to infinity, and at that time, the patent holder should license his patented technology to $s^*(< n)$ firms.[17] As far as the optimal number of licensees is concerned, Propositions 6 and 7 can jointly regain the asymptotic result of Kishimoto et al. (2011), within a finite number of firms.

[16] See, e.g., Kishimoto et al. (2011) for the general Cournot market.

[17] This asymptotic result that the bargaining finally reaches exactly coincides with the non-cooperative one, which is remarkable because the patent holder does not have full bargaining power in the cooperative approach. Tauman and Watanabe (2007) and Jelnov and Tauman (2009) obtained related asymptotic results, but they did not treat coalition structures that finally form, except the grand coalition.

In the case where $s^* = n$, $nW(n) \geq sW(s) + (n-s)L(s)$ for any $s = 1, 2, \ldots, n$. Thus, the total equilibrium gross profit that is divided among the patent holder and licensees is maximized under the grand coalition. The following example, however, suggests that the patent holder should not always invite all firms to the negotiation even if $s^* = n$.

Example 3. $n = 3$; $W(1) = 9$, $W(2) = 6$, $W(3) = 6$, $L(0) = 1$, $L(1) = L(2) = 0$. Then, $LTS(3) = 15 > LTS(2) = 10 > LTS(1) = 8$, and thus $s^* = 3$ and $s^*_{\text{-}n} = 2$. When $S = \{1, 2\}$, by Proposition 2 (b-i), $\alpha(2) = L(0) = 1$ and $x_0^{\{1,2\}}(\alpha(2)) = LTS(2) = 10$. On the other hand, when $S = N$, $\delta_{0i}(x^N(\alpha)) = -6 + \alpha$ if $0 \leq \alpha \leq 3$ and $\delta_{i0}(x^N(\alpha)) = -\alpha$ if $\alpha \geq 3/2$. Then, $\alpha(3) = 3$ and $x_0^N(\alpha(3)) = 9$. Therefore, $x_0^N(\alpha(3)) < x_0^{\{1,2\}}(\alpha(2))$ and $s^*(= n) \notin \arg\max_{1 \leq s \leq n} s(W(s) - \alpha(s))$.

The next proposition gives a sufficient condition under which n is the optimal number of licensees when $s^* = n$. The proof is shown in the Appendix.

Proposition 8. *Suppose that $s^* = n$. If there exists $s^*_{\text{-}n} \in \arg\max_{1 \leq s \leq n-1} LTS(s)$ such that $t(L(0) - L(\rho(t))) \geq (s^*_{\text{-}n} - t)(LTS(n) - LTS(s^*_{\text{-}n}))/n$ for all $t = 1, 2, \ldots, s^*_{\text{-}n}$, then $s^* \in \arg\max_{1 \leq s \leq n} s(W(s) - \alpha(s))$.*

The condition assumed in Proposition 8 can be interpreted as follows. By Propositions 4 (a) and 5, it is optimal for the patent holder to invite all firms to the negotiation, when there is no difference between $LTS(n)$ and $LTS(s^*_{\text{-}n})$. As this difference in licensees' total surplus increases, however, the upper bound of $\alpha(n)$ becomes large by Proposition 4 (a), which means that the lower bound of $x_0^N(\alpha(n))$ becomes small. On the other hand, it is easy to see that for each $S \subsetneq N$ with $S \neq \emptyset$ and $|S| = s$, $x_0^S(\alpha(s)) \leq LTS(s^*_{\text{-}n})$ by the lower bounds of $\alpha(s)$ shown in Proposition 2 (a). Thus, the optimal number of licensees is n unless the upper bound of $x_0^S(\alpha(s))$ (i.e., $LTS(s^*_{\text{-}n})$) exceeds the lower bounds of $x_0^N(\alpha(n))$; thus, the difference between $LTS(n)$ and $LTS(s^*_{\text{-}n})$ is bounded from above.

When $n = 2$, it is obvious that $s^*_{\text{-}n} = 1 \geq n/2$. Then, the sufficient condition supposed in Proposition 8 is satisfied, because $s^*_{\text{-}n} - t = 0$. Thus, Propositions 6 and 8 jointly imply that it is optimal for the patent holder to license his patented technology to s^* firms in duopoly markets (e.g., Muto, 1993).

Corollary 2. *If $n = 2$, then $s^* \in \arg\max_{1 \leq s \leq n} s(W(s) - \alpha(s))$.*

Sen and Tauman (2007) considered patent licensing of a cost-reducing technology in a linear Cournot market. They showed that the patented technology is licensed to almost all firms through a non-cooperative mechanism. In their model, we can easily show that $s^* \leq (n+1)/2$ and conjecture that the optimal number of licensees is

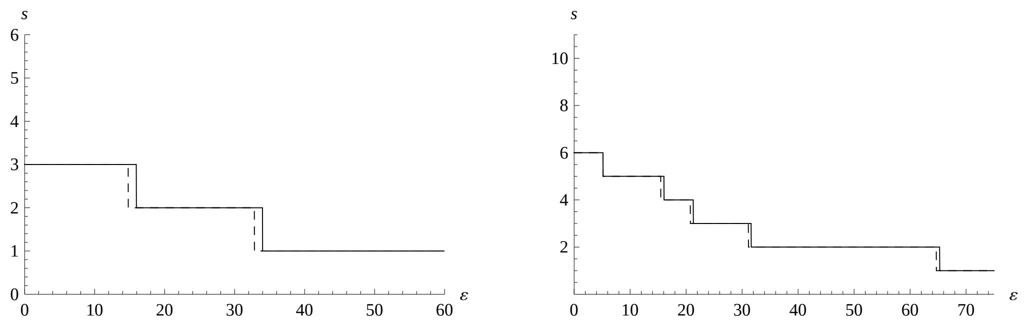

Figure 2: The optimal number of licensees in a linear Cournot market with a cost-reducing technology. Let $\hat{s}$ be the optimal number of licensees in the kernel. The notations a, c, and ε follow Sen and Tauman (2007). Horizontal axis: ε, Vertical axis: $\hat{s}$ and s^*, Solid line: $(\varepsilon, \hat{s})$, Dashed line: (ε, s^*), Left-hand side: $a = 100,\ c = 60,$ and $n = 6$, Right-hand side: $a = 150,\ c = 75,$ and $n = 11$.

equal to s^* or $s^* + 1$; that is, the patented technology is not diffused among all firms when licensing is carried out through bargaining with the kernel. Our conjecture comes from numerical examples depicted in Figure 2. This is left for future research.

5 Drastic Innovations

This section refers to the case of drastic innovations as a special case of our model. Under a drastic innovation, the total equilibrium gross profit of licensees is maximized when a single firm is licensed. Furthermore, when a drastic innovation is licensed, all non-licensees are driven out of the market. Formally, we say that an innovation is drastic if the equilibrium gross profits of each licensee and each non-licensee satisfy the following condition in addition to (1);

$$W(1) > tW(t) \text{ for } t = 2, 3, \ldots, n \text{ and } L(t) = 0 \text{ for } t = 1, 2, \ldots, n-1. \tag{2}$$

Then, each licensee's payoff in the kernel for each permissible coalition structure is characterized as follows. The proof is shown in the Appendix.

Proposition 9. *Suppose that an innovation is drastic. If $sW(s) + nL(0) > W(1)$ and $s \neq n$, then $\alpha(s) = [sW(s) - W(1) + nL(0)]/2s$. If $nW(n) + nL(0) > W(1)$, then $\alpha(n) = [nW(n) - W(1) + nL(0)]/(2n-1)$. In all the other cases, $\alpha(s) = 0$.*

By (2), $s^* = 1$.[18] It is easy to see that all conditions listed in Proposition 7 are

[18]Suppose not. Then, there would exist $s \geq 2$ such that $W(1) - L(0) < s(W(s) - L(0))$, i.e., $W(1) - sW(s) < -(s-1)L(0)$. By (2), the left-hand side is positive, but the right-hand side is negative. This is a contradiction.

satisfied under drastic innovations. Thus, Propositions 7 and 9 jointly imply the following corollary.

Corollary 3. *For a drastic innovation,* $s^* \in \arg\max_{1 \leq s \leq n} s(W(s) - \alpha(s))$. *The kernel* $\mathcal{K}^{S^*}$ *then gives payoffs to the players as follows:*

$$x_i^{S^*}(\alpha(s^*)) = \begin{cases} W(1) - nL(0)/2 & \textit{if } i = 0 \\ nL(0)/2 & \textit{if } i \in S^* \\ 0 & \textit{if } i \in N \setminus S^*. \end{cases}$$

In real practice, it is observed that only one firm is licensed. Corollary 3 means that such a situation occurs when the innovation is drastic. As stated in Proposition 7, however, it can be optimal for the patent holder to license only one firm even for non-drastic innovations. Actually, Figure 2 shows that, in the linear Cournot market analyzed by Sen and Tauman (2007), there exists a case where the optimal number of licensees is equal to one for non-drastic innovations.[19]

For drastic innovations, we can prove that under coalition structure P^{S^*}, the kernel gives the patent holder a payoff that is at the lower limit of his revenue in the bargaining set.[20] The bisection property provided by Chang (1991) implies that for each permissible coalition structure, the strictly individual imputations in the intersection of the kernel and the non-empty core are included in the interior of the non-empty core that is not a singleton, and it is known that the non-empty core is contained by the bargaining set for each permissible coalition structure.[21] Watanabe and Muto (2008), however, proved that the cores for any coalition structures are empty unless the grand coalition forms at stage (i). Thus, Corollary 3 shows that when the core is empty, the strictly individual imputation in the kernel is not always in the interior of the bargaining set.[22]

[19] In their model, the innovation is non-drastic if $a - c > \varepsilon$.

[20] If licensee $i \in S^*$ gains more than $nL(0)/2$, then the patent holder makes a justified objection $(y, \{0, j\})$ such that $j \neq i$, $y_0 = W(1) - nL(0)/2$ and $y_j = nL(0)/2$, because $v(N) = nL(0)$ and $v(T) = 0$ for every $T \subsetneq N$. For a drastic innovation, $s^* = 1$. Thus, by the inclusion of the kernel in the bargaining set and Proposition 3 (a) shown in Watanabe and Muto (2008), if x belongs to the (symmetric) bargaining set for coalition structure P^{S^*}, the range of the patent holder's revenue x_0 is $W(1) - nL(0)/2 \leq x_0 \leq W(1) - L(0)$.

[21] An imputation $x \in I^S$ is strictly individual if $x_i > v(\{i\})$ for all $i \in \{0\} \cup S$.

[22] The Aumann-Drèze value is an extension of the Shapley value to games with coalition structures. In the general Cournot market, Kishimoto et al. (2011) also showed that the Aumann-Drèze value is not included in the bargaining set as the number of firms goes to infinity. At that time, the Aumann-Drèze value allocates to the patent holder a half of the revenue that the kernel gives, even if the patent holder licenses his patented technology to s^* firms.

6 Final remark

The characteristic function given in Subsection 2.1 does not necessarily exhibit super-additivity that is often presumed in the cooperative analysis.[23] Super-additivity of characteristic functions is required in analyzing how to divide the total payoff in the grand coalition, because the grand coalition may not actually form without it.[24]

It would not be a pre-requisite in games where there is no need for players to form the grand coalition. In fact, Aumann and Drèze (1974) did not require the super-additivity for analysis of games with coalition structures. This paper assumes that there is no binding agreement on any types of cartel among firms to coordinate their production levels and market behaviors at stage (iii), because we wished to consider the same situation as in the non-cooperative analysis in the traditional literature on patent licensing. Thus, the worth of each coalition is defined simply as the sum of equilibrium gross profit that players in the coalition obtain at stage (iii). This is one of the reasons why our characteristic function does not necessarily satisfy super-additivity.

Appendix: Proofs

We give the proofs of the lemmas and propositions in this paper. For notational ease, let $L(n) = 0$ throughout this appendix, although $L(n)$ is not actually defined.

Proof of Lemma 1

Proof. (a) Suppose that $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s$, and let $i \in S$. Note that $L(n) = 0$. Then, for each $T \subseteq N \setminus \{i\}$,

$$e(\{0\} \cup T, x^S(\alpha)) = v(\{0\} \cup T) - v(\{0\} \cup S) + |S \setminus T|\alpha - |T \setminus S|L(s), \quad (3)$$

$$e(\{i\} \cup T, x^S(\alpha)) = v(\{i\} \cup T) - (|T \cap S| + 1)\alpha - |T \setminus S|L(s), \quad (4)$$

where $|R| = 0$ if $R = \emptyset$. For each $T \subseteq N \setminus \{i\}$, $e(\{0\} \cup T, x^S(\alpha))$ (resp. $e(\{i\} \cup T, x^S(\alpha))$) is a continuous and strictly increasing (decreasing) function of α, because $|S \setminus T| \geq 1$ and $|T \cap S| \geq 0$. By the definition of the maximum excess, $\delta_{0i}(x^S(\alpha))$ ($\delta_{i0}(x^S(\alpha))$) is given as the maximum of the finite number of continuous and strictly increasing (decreasing) functions of α. Thus, $\delta_{0i}(x^S(\alpha))$ ($\delta_{i0}(x^S(\alpha))$) is continuous and strictly increasing (decreasing) in α.

[23] A characteristic function v' is super-additive if $v'(S \cup T) \geq v'(S) + v'(T)$ for all coalitions S and T with $S \cap T = \emptyset$.

[24] In Example 3, the characteristic function v is super-additive. Nevertheless, the grand coalition is not selected by the patent holder in our patent licensing game.

(b) Note that $\delta_{0i}(x^S(\alpha))$ and $\delta_{i0}(x^S(\alpha))$ depend on $s(=|S|)$ and not on S, because all firms are identical. Let $h(\alpha, s) = \delta_{0i}(x^S(\alpha)) - \delta_{i0}(x^S(\alpha))$. For each $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s$, $h(\alpha, s)$ is a continuous and strictly increasing function of α by Lemma 1 (a), and $h(\alpha, s) \to \infty$ as $\alpha \to \infty$ and $h(\alpha, s) \to -\infty$ as $\alpha \to -\infty$ because of (3) and (4). Therefore, by applying the intermediate value theorem to $h(\alpha, s)$, there exists a unique $\alpha^*(s) \in \mathbb{R}$ such that $h(\alpha^*(s), s) = 0$.

In order to show that $\alpha^*(s) < W(s)$ for each $s = 1, 2, \ldots, n$, it suffices to confirm $\delta_{0i}(x^S(W(s))) > \delta_{i0}(x^S(W(s)))$, where $|S| = s$, because of Lemma 1 (a). Let $\hat{T} \subseteq N \setminus \{i\}$ such that $\delta_{i0}(x^S(W(s))) = e(\{i\} \cup \hat{T}, x^S(W(s)))$ and $\hat{t} = |\hat{T}|$. Then,

$$\begin{aligned}\delta_{0i}(x^S(W(s))) - \delta_{i0}(x^S(W(s))) &\geq e(\{0\} \cup \hat{T}, x^S(W(s))) - e(\{i\} \cup \hat{T}, x^S(W(s))) \\ &= \hat{t}W(\hat{t}) + W(s) - (\hat{t}+1)L(\rho(\hat{t}+1)) > 0,\end{aligned}$$

by (1), (3), and (4). Hence, for each $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s$, $\delta_{0i}(x^S(W(s))) > \delta_{i0}(x^S(W(s)))$. □

Proof of Proposition 2

Proof. Before proving this proposition, we specify the maximum excesses in the next lemma.

Lemma 2. *Suppose that $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s$, and let $i \in S$. Define $\tilde{W}^s(t) = W(t) - L(s)$, $\tilde{L}^s(t) = L(t) - L(s)$, and $\tilde{\alpha}^s = \alpha - L(s)$. Then, the maximum excesses $\delta_{0i}(x^S(\alpha))$ and $\delta_{i0}(x^S(\alpha))$ are given by the following equations:*

$$\delta_{0i}(x^S(\alpha)) = \begin{cases} \max\limits_{1 \leq t \leq n-1} t\tilde{W}^s(t) - s\tilde{W}^s(s) + (\max\{1, s-t\})\tilde{\alpha}^s & \text{if } \tilde{\alpha}^s \leq 0 \\ \max\limits_{1 \leq t \leq n-1} t\tilde{W}^s(t) - s\tilde{W}^s(s) + (\min\{s, n-t\})\tilde{\alpha}^s & \text{if } 0 \leq \tilde{\alpha}^s, \end{cases}$$

$$\delta_{i0}(x^S(\alpha)) = \begin{cases} n\tilde{L}^s(0) - s\tilde{\alpha}^s & \text{if } \tilde{\alpha}^s \leq \tilde{L}^s(0) \\ \max\limits_{1 \leq t \leq n} t\tilde{L}^s(\rho(t)) - (\max\{1, t-n+s\})\tilde{\alpha}^s & \text{if } \tilde{L}^s(0) \leq \tilde{\alpha}^s. \end{cases}$$

Proof. Consider the maximum excess of 0 over i at $x^S(\alpha)$. Let $t \in \{1, 2, \ldots, n-1\}$ and $T \in \arg\max_{T' \subseteq N \setminus \{i\}: |T'| = t} e(\{0\} \cup T', x^S(\alpha))$. Suppose that $\tilde{\alpha}^s \leq 0$ (i.e., $\alpha \leq L(s)$). Then, $T \subseteq S \setminus \{i\}$ if $t \leq s-1$ and $T \supseteq S \setminus \{i\}$ if $t \geq s$. So, by (3), if $t \leq s-1$, $e(\{0\} \cup T, x^S(\alpha)) = tW(t) - sW(s) + (s-t)\alpha = t\tilde{W}^s(t) - s\tilde{W}^s(s) + (s-t)\tilde{\alpha}^s$, and if $t \geq s$, $e(\{0\} \cup T, x^S(\alpha)) = tW(t) - sW(s) + \alpha - (t-s+1)L(s) = t\tilde{W}^s(t) - s\tilde{W}^s(s) + \tilde{\alpha}^s$. Unifying the two equations, we have $e(\{0\} \cup T, x^S(\alpha)) = t\tilde{W}^s(t) - s\tilde{W}^s(s) + (\max\{1, s-t\})\tilde{\alpha}^s$. Thus, when $\tilde{\alpha}^s \leq 0$,

$$\delta_{0i}(x^S(\alpha)) = \max_{1 \leq t \leq n-1} t\tilde{W}^s(t) - s\tilde{W}^s(s) + (\max\{1, s-t\})\tilde{\alpha}^s.$$

If $0 \leq \tilde{\alpha}^s$ (i.e., $L(s) \leq \alpha$), $T \subseteq N \setminus S$ when $t \leq n-s$, and $T \supseteq N \setminus S$ when $t \geq n-s+1$. Then, by (3), $e(\{0\} \cup T, x^S(\alpha)) = tW(t) - sW(s) + s\alpha - tL(s) = t\tilde{W}^s(t) - s\tilde{W}^s(s) + s\tilde{\alpha}^s$ if $t \leq n-s$, and $e(\{0\} \cup T, x^S(\alpha)) = tW(t) - sW(s) + (n-t)\alpha - (n-s)L(s) = t\tilde{W}^s(t) - s\tilde{W}^s(s) + (n-t)\tilde{\alpha}^s$ if $t \geq n-s+1$. So, $e(\{0\} \cup T, x^S(\alpha)) = t\tilde{W}^s(t) - s\tilde{W}^s(s) + (\min\{s, n-t\})\tilde{\alpha}^s$. Thus, when $0 \leq \tilde{\alpha}^s$,

$$\delta_{0i}(x^S(\alpha)) = \max_{1 \leq t \leq n-1} t\tilde{W}^s(t) - s\tilde{W}^s(s) + (\min\{s, n-t\})\tilde{\alpha}^s.$$

We next consider the maximum excess of i over 0 at $x^S(\alpha)$. If $\tilde{\alpha}^s \leq \tilde{L}^s(0)$ (i.e., $\alpha \leq L(0)$), then $L(0) - x_i^S(\alpha) = L(0) - \alpha \geq 0$ for all $i \in S$ and $L(0) - x_i^S(\alpha) = L(0) - L(s) > 0$ for all $i \in N \setminus S$. In addition, by (1), $L(0) > L(\rho(t))$ for each $t = 1, \ldots, n-1$. Thus, when $\tilde{\alpha}^s \leq \tilde{L}^s(0)$,

$$\delta_{i0}(x^S(\alpha)) = e(N, x^S(\alpha)) = nL(0) - s\alpha - (n-s)L(s) = n\tilde{L}^s(0) - s\tilde{\alpha}^s.$$

Let $t \in \{1, 2, \ldots, n\}$, and $T \in \arg\max_{T' \subseteq N : T' \ni i, |T'| = t} e(T', x^S(\alpha))$. If $\tilde{L}^s(0) \leq \tilde{\alpha}^s$ (i.e., $L(0) \leq \alpha$), then $T \setminus \{i\} \subseteq N \setminus S$ when $t \leq n-s+1$, and $T \supseteq \{i\} \cup (N \setminus S)$ when $t \geq n-s+2$. Thus, by (4), if $t \leq n-s+1$, $e(T, x^S(\alpha)) = tL(\rho(t)) - \alpha - (t-1)L(s) = t\tilde{L}^s(\rho(t)) - \tilde{\alpha}^s$, and if $t \geq n-s+2$, $e(T, x^S(\alpha)) = tL(\rho(t)) - (t-n+s)\alpha - (n-s)L(s) = t\tilde{L}^s(\rho(t)) - (t-n+s)\tilde{\alpha}^s$. Thus,

$$\delta_{i0}(x^S(\alpha)) = \max_{1 \leq t \leq n} t\tilde{L}^s(\rho(t)) - (\max\{1, t-n+s\})\tilde{\alpha}^s,$$

when $\tilde{L}^s(0) \leq \tilde{\alpha}^s$. □

By using this lemma, we specify the upper and lower bounds of $\alpha(s)$ for each $s = 1, 2, \ldots, n-1$. Take any $s = 1, 2, \ldots, n-1$ and $s^*_{\text{-}n} \in \arg\max_{1 \leq s \leq n-1} LTS(s)$, and fix them. For notational ease, define $\hat{k}^t_\alpha$ by $\hat{k}^t_\alpha = \max\{1, s-t\}$ if $\alpha \leq L(s)$, and by $\hat{k}^t_\alpha = \min\{s, n-t\}$ if $\alpha \geq L(s)$.

(a) Let $\alpha' = L(0) - (LTS(s^*_{\text{-}n}) - LTS(s))/(s+1)$, and $t_{\alpha'}$ be the maximizer that attains $\delta_{0i}(x^S(\alpha'))$. It suffices to show that $\delta_{i0}(x^S(\alpha')) \geq \delta_{0i}(x^S(\alpha'))$, which implies that $\alpha' \leq \alpha(s)$. Then,

$$\begin{aligned}
&\delta_{i0}(x^S(\alpha')) - \delta_{0i}(x^S(\alpha')) \\
&= (n - t_{\alpha'} - \hat{k}^{t_{\alpha'}}_{\alpha'})\tilde{L}^s(0) - LTS(t_{\alpha'}) + LTS(s) + \frac{s + \hat{k}^{t_{\alpha'}}_{\alpha'}}{s+1}(LTS(s^*_{\text{-}n}) - LTS(s)) \\
&\geq LTS(s^*_{\text{-}n}) - LTS(t_{\alpha'}) \geq 0,
\end{aligned}$$

because $\alpha' \leq L(0)$ and $1 \leq \hat{k}^{t_{\alpha'}}_{\alpha'} \leq n - t_{\alpha'}$. By the definition of $\alpha(s)$, $L(\rho(1)) \leq \alpha(s)$; hence, we have $\max\{L(\rho(1)), \alpha'\} \leq \alpha(s)$ for each $s = 1, \ldots, n-1$.

(b) Consider the upper bound of $\alpha(s)$ for each $s = 1, \ldots, n-1$. Suppose that $1 \leq s \leq n - s^*_{\text{-}n}$ and $0 \leq (LTS(s^*_{\text{-}n}) - LTS(s))/(L(0) - L(s)) < n - s^*_{\text{-}n} - s$, and let $\beta =$

$L(0)+[(n-s^*_{\text{-}n}-s)(L(0)-L(s))-LTS(s^*_{\text{-}n})+LTS(s)]/(s+1)$. Then, $\beta > L(0)$. Denote by t_β the maximizer that attains $\delta_{i0}(x^S(\beta))$, and define $\hat{t}_\beta = \max\{1, t_\beta - n + s\}$. In contrast to the lower bound, it suffices to prove that $\delta_{0i}(x^S(\beta)) \geq \delta_{i0}(x^S(\beta))$, in order to show that $\alpha(s) \leq \beta$. Then

$$\begin{aligned}
&\delta_{0i}(x^S(\beta)) - \delta_{i0}(x^S(\beta)) \\
&\quad \geq s^*_{\text{-}n}\tilde{W}^s(s^*_{\text{-}n}) - s\tilde{W}^s(s) + \hat{k}^{s^*_{\text{-}n}}_\beta(\beta - L(s)) - t_\beta\tilde{L}^s(\rho(t_\beta)) + \hat{t}_\beta(\beta - L(s)) \\
&\quad \geq LTS(s^*_{\text{-}n}) - LTS(s) + (s^*_{\text{-}n} - t_\beta + \hat{t}_\beta)\tilde{L}^s(0) \\
&\qquad\qquad + \frac{s+\hat{t}_\beta}{s+1}[(n - s^*_{\text{-}n} - s)\tilde{L}^s(0) - LTS(s^*_{\text{-}n}) + LTS(s)] \geq 0,
\end{aligned}$$

because $\hat{k}^{s^*_{\text{-}n}}_\beta = s$, $L(\rho(t_\beta)) \leq L(0)$, $1 \leq \hat{t}_\beta$, and $t_\beta - n + s \leq \hat{t}_\beta$. Thus, $\alpha(s) \leq \beta$ if $1 \leq s \leq n - s^*_{\text{-}n}$ and $0 \leq (LTS(s^*_{\text{-}n}) - LTS(s))/(L(0) - L(s)) < n - s^*_{\text{-}n} - s$.

In the remaining cases, let $\gamma(\omega) = L(0) - \omega$ for $\omega \geq 0$. By Lemma 2,

$$\begin{aligned}
&\delta_{0i}(x^S(\gamma(\omega))) - \delta_{i0}(x^S(\gamma(\omega))) \\
&\quad \geq s^*_{\text{-}n}\tilde{W}^s(s^*_{\text{-}n}) - s\tilde{W}^s(s) + \hat{k}^{s^*_{\text{-}n}}_{\gamma(\omega)}(\tilde{L}^s(0) - \omega) - n\tilde{L}^s(0) + s(\tilde{L}^s(0) - \omega) \\
&\quad = LTS(s^*_{\text{-}n}) - LTS(s) - (n - s^*_{\text{-}n} - \hat{k}^{s^*_{\text{-}n}}_{\gamma(\omega)})\tilde{L}^s(0) - (s + \hat{k}^{s^*_{\text{-}n}}_{\gamma(\omega)})\omega, \qquad (5)
\end{aligned}$$

We consider the case where $1 \leq s \leq n - s^*_{\text{-}n}$ and $n - s^*_{\text{-}n} - s \leq (LTS(s^*_{\text{-}n}) - LTS(s))/(L(0) - L(s)) \leq n - s^*_{\text{-}n} + s$. Let $\omega' = [LTS(s^*_{\text{-}n}) - LTS(s) - (n - s^*_{\text{-}n} - s)(L(0) - L(s))]/2s$. Then, $L(s) \leq \gamma(\omega') \leq L(0)$, and $\hat{k}^{s^*_{\text{-}n}}_{\gamma(\omega')} = s$. So, by (5),

$$\delta_{0i}(x^S(\gamma(\omega'))) - \delta_{i0}(x^S(\gamma(\omega'))) \geq LTS(s^*_{\text{-}n}) - LTS(s) - (n - s^*_{\text{-}n} - s)\tilde{L}^s(0) - 2s\omega' = 0.$$

Thus, in this case, $\alpha(s) \leq \gamma(\omega')$.

In the case where $n - s^*_{\text{-}n} \leq s \leq n - 1$ and $0 \leq (LTS(s^*_{\text{-}n}) - LTS(s))/(L(0) - L(s)) \leq n - s^*_{\text{-}n} + s$, let $\omega'' = (LTS(s^*_{\text{-}n}) - LTS(s))/(n - s^*_{\text{-}n} + s)$. Then, we have $L(s) \leq \gamma(\omega'') \leq L(0)$ and $\hat{k}^{s^*_{\text{-}n}}_{\gamma(\omega'')} = n - s^*_{\text{-}n}$, so

$$\delta_{0i}(x^S(\gamma(\omega''))) - \delta_{i0}(x^S(\gamma(\omega''))) \geq LTS(s^*_{\text{-}n}) - LTS(s) - (n - s^*_{\text{-}n} + s)\omega'' = 0,$$

because of (5). Hence, $\alpha(s) \leq \gamma(\omega'')$ in this case.

Finally, we consider the case where $n - s^*_{\text{-}n} + s < (LTS(s^*_{\text{-}n}) - LTS(s))/(L(0) - L(s))$. Let $\hat{\omega} = [LTS(s^*_{\text{-}n}) - LTS(s) - (n - s^*_{\text{-}n} - 1)(L(0) - L(s))]/(s+1)$ and $\tilde{\omega} = [LTS(s^*_{\text{-}n}) - LTS(s) - (n - s)(L(0) - L(s))]/(2s - s^*_{\text{-}n})$. Note that $\gamma(\hat{\omega}) < L(s)$ and $\gamma(\tilde{\omega}) < L(s)$ in this case. If $1 \leq s \leq s^*_{\text{-}n} + 1$, then $\hat{k}^{s^*_{\text{-}n}}_{\gamma(\hat{\omega})} = 1$. So,

$$\delta_{0i}(x^S(\gamma(\hat{\omega}))) - \delta_{i0}(x^S(\gamma(\hat{\omega}))) \geq LTS(s^*_{\text{-}n}) - LTS(s) - (n - s^*_{\text{-}n} - 1)\tilde{L}^s(0) - (s+1)\hat{\omega} = 0,$$

by (5). When $s^*_{\text{-}n} + 1 \leq s \leq n - 1$, $\hat{k}^{s^*_{\text{-}n}}_{\gamma(\tilde{\omega})} = s - s^*_{\text{-}n}$. By (5),

$$\delta_{0i}(x^S(\gamma(\tilde{\omega}))) - \delta_{i0}(x^S(\gamma(\tilde{\omega}))) \geq LTS(s^*_{\text{-}n}) - LTS(s) - (n - s)\tilde{L}^s(0) - (2s - s^*_{\text{-}n})\tilde{\omega} = 0.$$

By the definition of $\alpha(s)$, $\alpha(s) \geq L(\rho(1))$. Thus, the above arguments jointly imply that $\alpha(s) \leq \max\{L(\rho(1)), \gamma(\hat{\omega})\}$ if $1 \leq s \leq s^*_{\text{-}n} + 1$, and $\alpha(s) \leq \max\{L(\rho(1)), \gamma(\tilde{\omega})\}$ if $s^*_{\text{-}n} + 1 \leq s \leq n - 1$. □

Proof of Proposition 4

Proof. (a) Suppose that $s^* = n$. Note that $L(n) = 0$. By substituting $S = N$ $(s = n)$ in Lemma 2, we have

$$\delta_{i0}(x^N(L(0))) - \delta_{0i}(x^N(L(0))) = LTS(n) - LTS(s^*_{\text{-}n}) \geq 0,$$

which implies that $L(0) \leq \alpha(n)$. We next consider the upper bound of $\alpha(n)$. Let $\alpha' = L(0) + (LTS(n) - LTS(s^*_{\text{-}n}))/(n - s^*_{\text{-}n} + 1)$, and let $t_{\alpha'}$ be the maximizer that attains $\delta_{i0}(x^N(\alpha'))$. Because $\alpha' \geq L(0)$, $L(\rho(t_{\alpha'})) \leq L(0)$, and $1 \leq t_{\alpha'} \leq n$, we have

$$\begin{aligned}
&\delta_{0i}(x^N(\alpha')) - \delta_{i0}(x^N(\alpha')) \\
&\geq s^*_{\text{-}n}W(s^*_{\text{-}n}) - nW(n) + (n - s^*_{\text{-}n})\alpha' - t_{\alpha'}(L(\rho(t_{\alpha'})) - \alpha') \\
&= LTS(s^*_{\text{-}n}) - LTS(n) + \frac{n - s^*_{\text{-}n} + t_{\alpha'}}{n - s^*_{\text{-}n} + 1}(LTS(n) - LTS(s^*_{\text{-}n})) - t_{\alpha'}(L(\rho(t_{\alpha'})) - L(0)) \\
&\geq LTS(s^*_{\text{-}n}) - LTS(n) + \frac{n - s^*_{\text{-}n} + t_{\alpha'}}{n - s^*_{\text{-}n} + 1}(LTS(n) - LTS(s^*_{\text{-}n})) \geq 0.
\end{aligned}$$

Consequently, $L(0) \leq \alpha(n) \leq \alpha'$ if $s^* = n$.

(b) In the case where $s^* \neq n$, let $\beta = L(0) - (LTS(s^*_{\text{-}n}) - LTS(n))/(n + 1)$ and $\beta' = L(0) - (LTS(s^*_{\text{-}n}) - LTS(n))/(2n - s^*_{\text{-}n})$, and denote by t_β the maximizer that attains $\delta_{0i}(x^N(\beta))$. Note that $\beta \leq \beta' < L(0)$ because $s^*_{\text{-}n} \leq n - 1$ and $LTS(s^*_{\text{-}n}) > LTS(n)$ in this case. Then, because $1 \leq t_\beta \leq n - 1$,

$$\begin{aligned}
\delta_{i0}(x^N(\beta)) - \delta_{0i}(x^N(\beta)) &= \frac{2n - t_\beta}{n + 1}(LTS(s^*_{\text{-}n}) - LTS(n)) - LTS(t_\beta) + LTS(n) \\
&\geq LTS(s^*_{\text{-}n}) - LTS(t_\beta) \geq 0, \\
\delta_{0i}(x^N(\beta')) - \delta_{i0}(x^N(\beta')) &\geq s^*_{\text{-}n}W(s^*_{\text{-}n}) - nW(n) + (n - s^*_{\text{-}n})\beta' - n(L(0) - \beta') = 0.
\end{aligned}$$

Because $L(\rho(1)) \leq \alpha(n)$, $\max\{L(\rho(1)), \beta\} \leq \alpha(n) \leq \max\{L(\rho(1)), \beta'\}$ if $s^* \neq n$. □

Proof of Proposition 7

Proof. We first show the property of s^*, which is useful to prove this proposition.

Lemma 3. *If (a) to (c) of Proposition 7 hold, then for all α with $0 \leq \alpha \leq L(0)$, $s^*(W(s^*) - \alpha) \geq s(W(s) - \alpha)$ for each $s = 1, 2, \ldots, n$.*

Proof. By the definition of s^*, the statement of this lemma holds when $\alpha = L(0)$. We show that $s^*W(s^*) \geq sW(s)$ for each $s = 1, 2, \ldots, n$. Let $\tilde{s} \in \arg\max_{1 \leq s \leq s^*} sW(s)$. If $L(s^*) = 0$, then $s^*W(s^*) \geq \tilde{s}W(\tilde{s}) + (n - \tilde{s})L(\tilde{s}) \geq \tilde{s}W(\tilde{s})$ by (a), so $s^* \in \arg\max_{1 \leq s \leq s^*} sW(s)$. When $L(s^*) > 0$, $L(\tilde{s}) > 0$ by (c). Then, by (a) and the definition of $\tilde{s}$, $s^*W(s^*) + (n - s^*)L(s^*) \geq \tilde{s}W(\tilde{s}) + (n - \tilde{s})L(\tilde{s})$ and $\tilde{s}W(\tilde{s}) \geq s^*W(s^*)$, so $(n - s^*)L(s^*) \geq (n - \tilde{s})L(\tilde{s})$. If $s^* \neq \tilde{s}$, then $\tilde{s} < s^*$, which implies that $(n - \tilde{s})L(\tilde{s}) > (n - s^*)L(s^*)$ because of (c). This is a contradiction. Thus, $s^* = \tilde{s} \in \arg\max_{1 \leq s \leq s^*} sW(s)$ even when $L(s^*) > 0$. This result and (b) jointly imply that $s^*W(s^*) \geq sW(s)$ for each $s = 1, 2, \ldots, n$.

We next show that for all α with $0 < \alpha < L(0)$, $s^*(W(s^*) - \alpha) \geq s(W(s) - \alpha)$ for each $s = 1, 2, \ldots, n$. Let $\tilde{s}^\alpha \in \arg\max_{1 \leq s \leq n} s(W(s) - \alpha)$. By the above result and the definition of $\tilde{s}^\alpha$, $s^*W(s^*) \geq \tilde{s}^\alpha W(\tilde{s}^\alpha)$ and $\tilde{s}^\alpha(W(\tilde{s}^\alpha) - \alpha) \geq s^*(W(s^*) - \alpha)$, and thus $s^*\alpha \geq \tilde{s}^\alpha \alpha$, which implies that $s^* \geq \tilde{s}^\alpha$. On the other hand, $s^*(W(s^*) - L(0)) \geq \tilde{s}^\alpha(W(\tilde{s}^\alpha) - L(0))$ and $\tilde{s}^\alpha(W(\tilde{s}^\alpha) - \alpha) \geq s^*(W(s^*) - \alpha)$ by the definitions of s^* and $\tilde{s}^\alpha$. Then, because $\tilde{s}^\alpha(L(0) - \alpha) \geq s^*(L(0) - \alpha)$, $\tilde{s}^\alpha \geq s^*$. Therefore, $s^* = \tilde{s}^\alpha$ for all α with $0 < \alpha < L(0)$, because $s^* \geq \tilde{s}^\alpha \geq s^*$. □

We use $\tilde{W}^s(t)$, $\tilde{L}^s(t)$, and $\tilde{\alpha}^s$, which are defined in Lemma 2. Let $\tilde{\alpha}^s(s) = \alpha(s) - L(s)$, and define $\tilde{k}^t_{\tilde{\alpha}^s}$ by $\tilde{k}^t_{\tilde{\alpha}^s} = \max\{1, s - t\}$ if $\tilde{\alpha}^s \leq 0$, and by $\tilde{k}^t_{\tilde{\alpha}^s} = \min\{s, n - t\}$ if $\tilde{\alpha}^s \geq 0$. By using Lemma 3, we prove the following two lemmas, which jointly imply the statement of this proposition.

Lemma 4. *Suppose that $1 \leq s^* < n/2$. If (a) to (d) of Proposition 7 hold, then $s^* \in \arg\max_{s^* \leq s \leq n} s(W(s) - \alpha(s))$.*

Proof. To characterize $\tilde{\alpha}^s(s)$ for each $s = s^*, s^* + 1, \ldots, n$, we consider four cases. In each case, let $S \subseteq N$ with $s = |S| \geq s^*$ satisfying the condition of the case.

Case 1: $s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) \leq n\tilde{L}^s(0)$ and $s + s^* \leq n$. In this case, we show that $\tilde{\alpha}^s(s) = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/2s$. Let $\tilde{\beta}^s = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/2s \geq 0$. Then, $\tilde{k}^t_{\tilde{\beta}^s} = \min\{s, n - t\}$, so $\tilde{k}^t_{\tilde{\beta}^s} \leq s$ for each $t = 1, 2, \ldots, n - 1$. Thus, for each $t = 1, 2, \ldots, n - 1$,

$$[s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) + s\tilde{\beta}^s] - [t\tilde{W}^s(t) - s\tilde{W}^s(s) + \tilde{k}^t_{\tilde{\beta}^s}\tilde{\beta}^s] \geq s^*\tilde{W}^s(s^*) - t\tilde{W}^s(t) \geq 0,$$

because of Lemma 3. On the other hand, if $\tilde{L}^s(0) \leq \tilde{\alpha}^s$, then $\delta_{i0}(x^S(\alpha)) = \max_{0 \leq t \leq s-1}(n - t)\tilde{L}^s(t) - (s - t)\tilde{\alpha}^s$, because $n - t \in \arg\min_{0 \leq r \leq n-t} L(r)$ for each

$t = 1, 2, \ldots, n$ by (c). So, when $0 \le t \le s-1$,

$$\begin{aligned}
&[n\tilde{L}^s(0) - s\tilde{\beta}^s] - [(n-t)\tilde{L}^s(t) - (s-t)\tilde{\beta}^s] \\
&\quad = \frac{1}{2s}\Big[(2s-t)n\tilde{L}^s(0) - 2s(n-t)\tilde{L}^s(t) + t(s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s))\Big] \\
&\quad \ge \frac{1}{2s}\Big[(2s-t)n\tilde{L}^s(0) - 2s(n-t)\tilde{L}^s(t) + t(s^* - s)\tilde{L}^s(t)\Big] \\
&\quad \ge \frac{(s-t)(n-s^*-s)}{s}\tilde{L}^s(t) \ge 0,
\end{aligned}$$

because of Lemma 3 and (d). By the above inequalities, $\delta_{0i}(x^S(\tilde{\beta}^s + L(s))) = (s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) + n\tilde{L}^s(0))/2 = \delta_{i0}(x^S(\tilde{\beta}^s + L(s)))$. Therefore, $\tilde{\alpha}^s(s) = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/2s$ in this case.

Case 2: $s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) \le n\tilde{L}^s(0)$ and $s + s^* > n$. Let $\tilde{\gamma}^s = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/(n - s^* + s)$. Note that $0 \le \tilde{\gamma}^s \le \tilde{L}^s(0)$ by the definition of s^*, and $1 \le \tilde{k}^t_{\tilde{\gamma}^s} \le n - t$ for each $t = 1, 2, \ldots, n-1$. Then, $\delta_{i0}(x^S(\tilde{\gamma}^s + L(s))) = n\tilde{L}^s(0) - s\tilde{\gamma}^s = [s(s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s)) + (n - s^*)n\tilde{L}^s(0)]/(n - s^* + s)$ by Lemma 2. For each $t = 1, 2, \ldots, n-1$,

$$\begin{aligned}
&[s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) + (n-s^*)\tilde{\gamma}^s] - [t\tilde{W}^s(t) - s\tilde{W}^s(s) + \tilde{k}^t_{\tilde{\gamma}^s}\tilde{\gamma}^s] \\
&\qquad\qquad \ge s^*(\tilde{W}^s(s^*) - \tilde{\gamma}^s) - t(\tilde{W}^s(t) - \tilde{\gamma}^s) \ge 0,
\end{aligned}$$

because of Lemma 3. So, $\delta_{0i}(x^S(\tilde{\gamma}^s + L(s))) = s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) + (n - s^*)\tilde{\gamma}^s = [s(s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s)) + (n-s^*)n\tilde{L}^s(0)]/(n-s^*+s)$, which implies that $\delta_{0i}(x^S(\tilde{\gamma}^s + L(s))) = \delta_{i0}(x^S(\tilde{\gamma}^s + L(s)))$. Thus, $\tilde{\alpha}^s(s) = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/(n - s^* + s)$.

Case 3: $n\tilde{L}^s(0) \le s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) \le n\tilde{L}^s(0) + (2s - s^*)\tilde{L}^{n-1}(s)$. In this case, $s \ne s^*$. Let $\tilde{\omega}^s = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/(2s - s^*)$. Note that $-\tilde{L}^{n-1}(s) \le \tilde{\omega}^s \le 0$, and thus $\delta_{i0}(x^S(\tilde{\omega}^s + L(s))) = n\tilde{L}^s(0) - s\tilde{\omega}^s = [s(s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s)) + (s - s^*)n\tilde{L}^s(0)]/(2s - s^*)$. On the other hand, because $\tilde{k}^t_{\tilde{\omega}^s}(= \max\{1, s-t\}) \ge s - t$, $\tilde{\omega}^s \le 0$, and Lemma 3 holds, for each $t = 1, 2, \ldots, n-1$,

$$\begin{aligned}
&[s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) + (s-s^*)\tilde{\omega}^s] - [t\tilde{W}^s(t) - s\tilde{W}^s(s) + \tilde{k}^t_{\tilde{\omega}^s}\tilde{\omega}^s] \\
&\qquad\qquad \ge s^*(\tilde{W}^s(s^*) - \tilde{\omega}^s) - t(\tilde{W}^s(t) - \tilde{\omega}^s) \ge 0.
\end{aligned}$$

So, $\delta_{0i}(x^S(\tilde{\omega}^s + L(s))) = s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) + (s - s^*)\tilde{\omega}^s = [s(s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s)) + (s - s^*)n\tilde{L}^s(0)]/(2s - s^*)$. Therefore, $\tilde{\alpha}^s(s) = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/(2s - s^*)$, because $\delta_{0i}(x^S(\tilde{\omega}^s + L(s))) = \delta_{i0}(x^S(\tilde{\omega}^s + L(s)))$.

Case 4: $s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) \ge n\tilde{L}^s(0) + (2s - s^*)\tilde{L}^{n-1}(s)$. Note that $L(\rho(1)) = L(n-1)$ by (c). Let $\tilde{\lambda}^s = -\tilde{L}^{n-1}(s) \le 0$. In this case also, $s \ne s^*$. By the same argument as in Case 3, $\delta_{0i}(x^S(L(n-1))) = \delta_{0i}(x^S(\tilde{\lambda}^s + L(s))) = s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) - (s-s^*)\tilde{L}^{n-1}(s)$ and $\delta_{i0}(x^S(L(n-1))) = \delta_{i0}(x^S(\tilde{\lambda}^s + L(s))) = n\tilde{L}^s(0) + s\tilde{L}^{n-1}(s)$. Then, $\delta_{0i}(x^S(L(n-1))) \geq \delta_{i0}(x^S(L(n-1)))$, which implies that $\tilde{\alpha}^s(s) = -\tilde{L}^{n-1}(s)$ by Proposition 1.

Let $S^* \subseteq N$ with $|S^*| = s^*$. By Case 1, $x_0^{S^*}(\alpha(s^*)) = s^*\tilde{W}^{s^*}(s^*) - s^*\tilde{\alpha}^{s^*}(s^*) = (2s^*\tilde{W}^{s^*}(s^*) - n\tilde{L}^{s^*}(0))/2$, because $s^* < n/2$. To conclude the proof of this lemma, we show that $x_0^{S^*}(\alpha(s^*)) \geq x_0^S(\alpha(s))$ for each $S \subseteq N$ with $|S| = s \geq s^*$. In each case, $s^* < n/2$ and $s^* \leq s$, and then $L(s^*) \geq L(s)$ by (c). Furthermore, by Lemma 3, $s^*W(s^*) \geq sW(s)$ for each $s = 1, 2, \ldots, n$. In Case 1, $x_0^S(\alpha(s)) = (s^*\tilde{W}^s(s^*) + s\tilde{W}^s(s) - n\tilde{L}^s(0))/2$. Thus,

$$
\begin{aligned}
&x_0^{S^*}(\alpha(s^*)) - x_0^S(\alpha(s)) \\
&\qquad = \frac{1}{2}\left[s^*W(s^*) - sW(s) + (n-s^*)(L(s^*) - L(s)) + sL(s) - s^*L(s^*)\right] \\
&\qquad \geq \frac{1}{2}\left[s^*W(s^*) - sW(s) + (n-2s^*)(L(s^*) - L(s))\right] \geq 0.
\end{aligned}
$$

In Case 2, $x_0^S(\alpha(s)) = [(n-s^*)s\tilde{W}^s(s) + s(s^*\tilde{W}^s(s^*) - n\tilde{L}^s(0))]/(n-s^*+s)$. Thus,

$$
\begin{aligned}
&x_0^{S^*}(\alpha(s^*)) - x_0^S(\alpha(s)) \\
&\quad = \frac{1}{2(n-s^*+s)}\big[2(n-s^*)(s^*W(s^*) - sW(s)) \\
&\qquad\qquad + (s+s^*-n)nL(0) + (n-2s^*)(n-s^*+s)L(s^*)\big] \geq 0,
\end{aligned}
$$

because $s + s^* > n$ in Case 2. We consider $x_0^S(\alpha(s))$ in Case 3. Then, $x_0^S(\alpha(s)) = [(s-s^*)s\tilde{W}^s(s) + s(s^*\tilde{W}^s(s^*) - n\tilde{L}^s(0))]/(2s-s^*)$, and thus

$$
\begin{aligned}
&x_0^{S^*}(\alpha(s^*)) - x_0^S(\alpha(s)) \\
&\quad = \frac{1}{2(2s-s^*)}\big[2(s-s^*)(s^*W(s^*) - sW(s)) + s^*nL(0) \\
&\qquad\qquad + (2s-s^*)(n-2s^*)L(s^*) - 2s(n-s)L(s)\big] \\
&\quad \geq \frac{1}{2}\left[n(L(0) - L(s)) + (n-2s^*)(L(s^*) - L(s))\right] \geq 0,
\end{aligned}
$$

by the condition of Case 3. Finally, in Case 4, $x_0^S(\alpha(s)) = s(\tilde{W}^s(s) + \tilde{L}^{n-1}(s))$. Thus,

$$
\begin{aligned}
&x_0^{S^*}(\alpha(s^*)) - x_0^S(\alpha(s)) \\
&\quad = \frac{1}{2}\left[2(s^*W(s^*) - sW(s)) - 2s^*L(s^*) - nL(0) + nL(s^*) + 2sL(n-1)\right] \\
&\quad \geq \frac{1}{2}\big[n(L(0) - L(s)) + n(L(s^*) - L(s)) + 2s(L(s) - L(n-1)) \\
&\qquad\qquad - 2s^*(L(s^*) - L(n-1))\big] \\
&\quad \geq \frac{1}{2}\left[n(L(0) - L(s)) + (n-2s^*)(L(s^*) - L(s))\right] \geq 0,
\end{aligned}
$$

because of the condition of Case 4. $\square$

Lemma 5. *Suppose that $1 \leq s^* < n/2$. If (a) to (d) of Proposition 7 hold, then $s^* \in \arg\max_{1\leq s\leq s^*} s(W(s) - \alpha(s))$.*

Proof. Let $S^* \subseteq N$ with $|S^*| = s^*$. As in Lemma 4, we first characterize $\tilde{\alpha}^s(s)$ for each $s \leq s^*$, and then show that $x_0^{S^*}(\alpha(s^*)) \geq x_0^S(\alpha(s))$ for each $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s \leq s^*$. In order to specify $\tilde{\alpha}^s(s)$ for each $s \leq s^*$, there are three cases. In each case, consider $S \subseteq N$ with $S \neq \emptyset$ and $s = |S| \leq s^*$.

Case 1: $s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) \leq n\tilde{L}^s(0)$. By the same proof as in Case 1 of Lemma 4, we can show that $\tilde{\alpha}^s(s) = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/2s$, because $s + s^* \leq 2s^* < n$.

Case 2: $n\tilde{L}^s(0) \leq s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) \leq n\tilde{L}^s(0) + (s+1)\tilde{L}^{n-1}(s)$. Let $\tilde{\beta}^s = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/(s+1)$. Note that $-\tilde{L}^{n-1}(s) \leq \tilde{\beta}^s \leq 0$, and thus $\delta_{i0}(x^S(\tilde{\beta}^s + L(s))) = n\tilde{L}^s(0) - s\tilde{\beta}^s = [s(s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s)) + n\tilde{L}^s(0)]/(s+1)$. On the other hand, for each $t = 1, 2, \ldots, n-1$,

$$[s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) + \tilde{\beta}^s] - [t\tilde{W}^s(t) - s\tilde{W}^s(s) + \tilde{k}^t_{\tilde{\beta}^s}\tilde{\beta}^s] \geq s^*\tilde{W}^s(s^*) - t\tilde{W}^s(t) \geq 0,$$

because $\tilde{k}^t_{\tilde{\beta}^s}(= \max\{1, s-t\}) \geq 1$, $\tilde{\beta}^s \leq 0$, and Lemma 3 holds. So, $\delta_{0i}(x^S(\tilde{\beta}^s + L(s))) = s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) + \tilde{\beta}^s = [s(s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s)) + n\tilde{L}^s(0)]/(s+1)$. Therefore, $\tilde{\alpha}^s(s) = (s\tilde{W}^s(s) - s^*\tilde{W}^s(s^*) + n\tilde{L}^s(0))/(s+1)$, because $\delta_{0i}(x^S(\tilde{\beta}^s + L(s))) = \delta_{i0}(x^S(\tilde{\beta}^s + L(s)))$.

Case 3: $s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) \geq n\tilde{L}^s(0) + (s+1)\tilde{L}^{n-1}(s)$. Note that $L(\rho(1)) = L(n-1)$ by (c). Let $\tilde{\gamma}^s = -\tilde{L}^{n-1}(s) \leq 0$. By the same argument as in Case 2, $\delta_{0i}(x^S(L(n-1))) = \delta_{0i}(x^S(\tilde{\gamma}^s + L(s))) = s^*\tilde{W}^s(s^*) - s\tilde{W}^s(s) - \tilde{L}^{n-1}(s)$ and $\delta_{i0}(x^S(L(n-1))) = \delta_{i0}(x^S(\tilde{\gamma}^s + L(s))) = n\tilde{L}^s(0) + s\tilde{L}^{n-1}(s)$. Then, $\delta_{0i}(x^S(L(n-1))) \geq \delta_{i0}(x^S(L(n-1)))$, which implies that $\tilde{\alpha}^s(s) = -\tilde{L}^{n-1}(s)$ by Proposition 1.

Finally, we show that $x_0^{S^*}(\alpha(s^*)) \geq x_0^S(\alpha(s))$ for each $S \subseteq N$ with $S \neq \emptyset$ and $|S| = s \leq s^*$. Note that $L(s^*) \leq L(s)$ by (c). In Case 1, $x_0^{S^*}(\alpha(s^*)) = s^*\tilde{W}^{s^*}(s^*) - s^*\tilde{\alpha}^{s^*}(s^*) = (2s^*\tilde{W}^{s^*}(s^*) - n\tilde{L}^{s^*}(0))/2$ and $x_0^S(\alpha(s)) = (s^*\tilde{W}^s(s^*) + s\tilde{W}^s(s) - n\tilde{L}^s(0))/2$ if $s < s^*$. Thus,

$$\begin{aligned} x_0^{S^*}(\alpha(s^*)) - x_0^S(\alpha(s)) &= \frac{1}{2}\left[s^*W(s^*) - sW(s) - (n-s-s^*)L(s) + (n-2s^*)L(s^*)\right] \\ &\geq \frac{s^*}{2}(L(s) - L(s^*)) \geq 0, \end{aligned}$$

by (a) and (c). In Case 2, $x_0^S(\alpha(s)) = [s\tilde{W}^s(s) + s(s^*\tilde{W}^s(s^*) - n\tilde{L}^s(0))]/(s+1)$. Then,

$$\begin{aligned}
&x_0^{S^*}(\alpha(s^*)) - x_0^S(\alpha(s)) \\
&\quad = \frac{1}{2(s+1)}\big[2(s^*W(s^*) - sW(s)) + (s-1)nL(0) \\
&\qquad\qquad + (n-2s^*)(s+1)L(s^*) - 2s(n-s^*-1)L(s)\big] \\
&\quad \geq \frac{1}{2}\big[n(L(0)-L(s)) - (n-2s^*)(L(s)-L(s^*))\big] \geq 0,
\end{aligned}$$

by the condition of Case 2 and (d). In Case 3, $x_0^S(\alpha(s)) = s(\tilde{W}^s(s) + \tilde{L}^{n-1}(s))$. Then,

$$\begin{aligned}
&x_0^{S^*}(\alpha(s^*)) - x_0^S(\alpha(s)) \\
&\quad = \frac{1}{2}\left[2(s^*W(s^*) - sW(s)) - nL(0) + (n-2s^*)L(s^*) + 2sL(n-1)\right] \\
&\quad \geq \frac{1}{2}\left[n(L(0)-L(s)) - (n-2s^*)(L(s)-L(s^*)) + 2(L(s)-L(n-1))\right] \geq 0,
\end{aligned}$$

because of the condition of Case 3, (c) and (d). □

Lemmas 4 and 5 complete the proof of Proposition 7. □

Proof of Proposition 8

Proof. We first show that $\alpha(n) \leq L(0) + (LTS(n) - LTS(s^*_{\text{-}n}))/n$. Let $\alpha' = L(0) + (LTS(n) - LTS(s^*_{\text{-}n}))/n$. By the suppositions of this proposition and (1), for any $t = 1, 2, \ldots, n$,

$$\begin{aligned}
&\big[s^*_{\text{-}n}(L(0)-\alpha')\big] - \big[t(L(\rho(t))-\alpha')\big] \\
&\qquad\qquad = t(L(0) - L(\rho(t))) - \frac{(s^*_{\text{-}n} - t)(LTS(n) - LTS(s^*_{\text{-}n}))}{n} \geq 0,
\end{aligned}$$

which implies that $\delta_{i0}(x^N(\alpha')) \leq s^*_{\text{-}n}(L(0)-\alpha')$. Then,

$$\begin{aligned}
&\delta_{0i}(x^N(\alpha')) - \delta_{i0}(x^N(\alpha')) \\
&\qquad\qquad \geq \big[s^*_{\text{-}n}W(s^*_{\text{-}n}) - nW(n) + (n-s^*_{\text{-}n})\alpha'\big] - \big[s^*_{\text{-}n}(L(0)-\alpha')\big] = 0.
\end{aligned}$$

Thus, $\alpha(n) \leq \alpha' = L(0) + (LTS(n) - LTS(s^*_{\text{-}n}))/n$.

For any $S \subsetneq N$ with $S \neq \emptyset$ and $|S| = s$, the lower bound of $\alpha(s)$ in Proposition 2 (a) directly implies that $x_0^S(\alpha(s)) = s(W(s) - \alpha(s)) \leq LTS(s^*_{\text{-}n})$. Thus,

$$x_0^N(\alpha(n)) = nW(n) - n\alpha(n) \geq LTS(s^*_{\text{-}n}) \geq x_0^S(\alpha(s)).$$

Therefore, $s^*(=n) \in \arg\max_{1\leq s\leq n} s(W(s) - \alpha(s))$. □

Proof of Proposition 9

Proof. Note that $L(\rho(1)) = 0$ because of (2). Consider $S \subsetneq N$ such that $S \neq \emptyset$ and $|S| = s$. By Lemma 2 and (2), when $0 \leq \alpha \leq nL(0)/(s-1)$,

$$\delta_{0i}(x^S(\alpha)) = \max_{1 \leq t \leq n-1} tW(t) - sW(s) + (\min\{s, n-t\})\alpha = W(1) - sW(s) + s\alpha,$$
$$\delta_{i0}(x^S(\alpha)) = \max_{1 \leq t \leq n} tL(\rho(t)) - (\max\{1, t-n+s\})\alpha = nL(0) - s\alpha.$$

Let $\alpha' = [sW(s) - W(1) + nL(0)]/2s$. Then, because $\alpha' < nL(0)/(s-1)$ by (2),

$$\delta_{0i}(x^S(\alpha')) = \delta_{i0}(x^S(\alpha')) = \frac{W(1) - sW(s) + nL(0)}{2}.$$

Thus, by Proposition 1, if $sW(s) + nL(0) > W(1)$, then $\alpha(s) = [sW(s) - W(1) + nL(0)]/2s$; otherwise, $\alpha(s) = 0$.

We next consider the kernel for the grand coalition. Let $\alpha'' = [nW(n) - W(1) + nL(0)]/(2n-1)$. Then, $\alpha'' < L(0)$ by (2). Thus, if $0 \leq \alpha''$, then

$$\begin{aligned}\delta_{0i}(x^N(\alpha'')) &= \max_{1 \leq t \leq n-1} tW(t) - nW(n) + (n-t)\alpha'' \\ &= W(1) - nW(n) + (n-1)\alpha'' = \frac{n[W(1) - nW(n)] + n(n-1)L(0)}{2n-1}, \\ \delta_{i0}(x^N(\alpha'')) &= nL(0) - n\alpha'' = \frac{n[W(1) - nW(n)] + n(n-1)L(0)}{2n-1},\end{aligned}$$

because of Lemma 2 and (2). Therefore, because $\delta_{0i}(x^N(\alpha'')) = \delta_{i0}(x^N(\alpha''))$ when $\alpha'' \geq 0$, $\alpha(n) = [nW(n) - W(1) + nL(0)]/(2n-1)$ if $nW(n) + nL(0) > W(1)$; otherwise, $\alpha(n) = 0$. □

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